Contents

1 Conventions and metric

We cosnider flat FRW spacetime

$$\bar{g}_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & a^2 & & \\ & & a^2 & \\ & & & a^2 \end{bmatrix} \qquad \bar{g}^{\mu\nu} = \begin{bmatrix} -1 & & & \\ & a^{-2} & & \\ & & a^{-2} & \\ & & & a^{-2} \end{bmatrix} \tag{1}$$

and a small perturbation

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \qquad g^{\mu\nu} = \bar{g}^{\mu\nu} + h^{\mu\nu} \tag{2}$$

We use bar to represent the amount of FRW, and $h_{\mu\nu}$ to represent the amount of perturbation.

We consider flat FRW metric and $h_{\mu\nu}$ is a small amount. We know $g_{\mu\rho}g^{\nu\rho} = \delta^{\nu}_{\mu}$ and $\bar{g}_{\mu\rho}\bar{g}^{\nu\rho} = \delta^{\nu}_{\mu}$. Then we can obtain

$$g_{\mu\rho}g^{\rho\nu} = \delta^{\nu}_{\mu} = (\bar{g}_{\mu\rho} + h_{\mu\rho})(\bar{g}^{\nu\rho} + h^{\nu\rho}) = \bar{g}_{\mu\rho}\bar{g}^{\nu\rho} + \bar{g}_{\mu\rho}h^{\nu\rho} + \bar{g}^{\nu\rho}h_{\mu\rho} + o(h^2) \tag{3}$$

$$\bar{g}_{\mu\rho}h^{\nu\rho} = -\bar{g}^{\nu\rho}h_{\mu\rho} \qquad h^{\nu\sigma} = -\bar{g}^{\nu\rho}\bar{g}^{\sigma\mu}h_{\mu\rho} \tag{4}$$

We use Greek letters to represent quantities of spacetime, and Latin letters to represent quantities of space.

$$\mu, \nu.. = 0, 1, 2, 3 \qquad i, j.. = 1, 2, 3 \tag{5}$$

In this paper we only consider first-order perturbation.

2 Christoffel symbols

Defination of Christoffel symbol is

$$\bar{\Gamma}^{\alpha}_{\mu\nu} = \frac{1}{2}\bar{g}^{\alpha\beta}(\bar{g}_{\beta\mu,\nu} + \bar{g}_{\nu\beta,\mu} - \bar{g}_{\mu\nu,\beta}) \qquad \delta\Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \bar{\Gamma}^{\alpha}_{\mu\nu} \tag{6}$$

$$\begin{aligned} \delta\Gamma^{\rho}_{\mu\nu} &= \frac{1}{2}h^{\rho\sigma}(\bar{g}_{\sigma\mu,\nu} + \bar{g}_{\nu\sigma,\mu} + \bar{g}_{\nu\mu,\sigma}) + \frac{1}{2}\bar{g}^{\rho\sigma}(h_{\sigma\mu,\nu} + h_{\nu\sigma,\mu} - h_{\nu\mu,\sigma}) + o(h^2) &(7)\\ &= -\frac{1}{2}\bar{g}^{\rho\alpha}\bar{g}^{\sigma\beta}h_{\alpha\beta}(\bar{g}_{\sigma\mu,\nu} + \bar{g}_{\nu\sigma,\mu} + \bar{g}_{\nu\mu,\sigma}) + \frac{1}{2}\bar{g}^{\rho\sigma}(h_{\sigma\mu,\nu} + h_{\nu\sigma,\mu} - h_{\nu\mu,\sigma}) &(8)\\ &= -\bar{g}^{\rho\alpha}h_{\alpha\beta}\left[\frac{1}{2}\bar{g}^{\beta\sigma}(\bar{g}_{\sigma\mu,\nu} + \bar{g}_{\nu\sigma,\mu} + \bar{g}_{\nu\mu,\sigma})\right] + \frac{1}{2}\bar{g}^{\rho\sigma}(h_{\sigma\mu,\nu} + h_{\nu\sigma,\mu} - h_{\nu\mu,\sigma}) &(9)\\ &= -\bar{g}^{\rho\alpha}h_{\alpha\beta}\bar{\Gamma}^{\beta}_{\mu\nu} + \frac{1}{2}\bar{g}^{\rho\sigma}(h_{\sigma\mu,\nu} + h_{\nu\sigma,\mu} - h_{\nu\mu,\sigma}) &(10)\\ &= \frac{1}{2}\left(-2h_{\sigma\beta}\bar{\Gamma}^{\beta}_{\mu\nu} + h_{\sigma\mu,\nu} + h_{\nu\sigma,\mu} - h_{\nu\mu,\sigma}\right) &(11) \end{aligned}$$

We know that flat FRW no-zero Christoffel symbols are

$$\bar{\Gamma}^0_{ij} = \delta_{ij} a\dot{a} \qquad \bar{\Gamma}^i_{0j} = \bar{\Gamma}^i_{j0} = \delta^i_j \frac{\dot{a}}{a} \tag{12}$$

Each component can be expressed as

$$\delta\Gamma^0_{00} = \frac{1}{2}\bar{g}^{00}(-2h_{00}\bar{\Gamma}^\beta_{00} + h_{00,0} + h_{00,0} - h_{00,0}) = -\frac{1}{2}h_{00,0} \tag{13}$$

$$\delta\Gamma^i_{00} = \frac{1}{2}\bar{g}^{ij}(-2h_{j\beta}\bar{\Gamma}^\beta_{00} + h_{i0,0} + h_{0j.0} - h_{00,j}) = \frac{1}{2}a^{-2}(2h_{i0,0} - h_{00,i}) \tag{14}$$

$$\delta\Gamma^0_{i0} = \delta\Gamma^0_{0i} = \frac{1}{2}\bar{g}^{00}(-2h_{0\beta}\bar{\Gamma}^\beta_{i0} + h_{0i,0} + h_{00,i} - h_{i0,0}) = -\frac{1}{2}(-2h_{0j}\delta^j_i\frac{\dot{a}}{a} + h_{00,i}) = \frac{\dot{a}}{a}h_{0i} - \frac{1}{2}h_{00,i} \tag{15}$$

$$\delta\Gamma^0_{ij} = \frac{1}{2}\bar{g}^{00}(-2h_{0\beta}\bar{\Gamma}^\beta_{ij} + h_{0i,j} + h_{j0,i} - h_{ij,0}) = \frac{1}{2}(2h_{00}\delta_{ij}\dot{a}a - h_{0i,j} - h_{j0,i} + h_{ij,0}) \tag{16}$$

$$\delta\Gamma^i_{0j} = \delta\Gamma^i_{j0} = \frac{1}{2}\bar{g}^{ik}(-2h_{k\beta}\bar{\Gamma}^\beta_{0j} + h_{k0,j} + h_{jk,0} - h_{0j,k}) \tag{17}$$

$$= \frac{1}{2}\delta^{ik}a^{-2}(-2h_{kl}\delta^l_j\frac{\dot{a}}{a} + h_{k0,j} + h_{jk,0} - h_{0j,k}) \tag{18}$$

$$= \frac{1}{2}a^{-2}(-2h_{ij}\frac{\dot{a}}{a} + h_{i0,j} + h_{ij,0} - h_{0j,i}) \tag{19}$$

$$\delta\Gamma^i_{jk} = \frac{1}{2}g^{il}(-2h_{l\beta}\bar{\Gamma}^\beta_{jk} + h_{lj,k} + h_{kl,j} - h_{jk,l}) \tag{20}$$

$$= \frac{1}{2}\delta^{il}a^{-2}(-2h_{l0}\delta_{jk}\dot{a}a + h_{lj,k} + h_{kl,j} - h_{jk,l}) \tag{21}$$

$$= \frac{1}{2}a^{-2}(-2h_{i0}\delta_{jk}\dot{a}a + h_{ij,k} + h_{ki,j} - h_{jk,i}) \tag{22}$$

3 Ricci tensors

Defination of Ricci tensor is

$$\bar{R}_{\mu\nu} = \bar{\Gamma}^\rho_{\mu\nu,\rho} - \bar{\Gamma}^\rho_{\mu\rho,\nu} + \bar{\Gamma}^\sigma_{\mu\nu}\bar{\Gamma}^\rho_{\sigma\rho} - \bar{\Gamma}^\sigma_{\mu\rho}\bar{\Gamma}^\rho_{\sigma\nu} \qquad \delta R_{\mu\nu} = R_{\mu\nu} - \bar{R}_{\mu\nu} \tag{23}$$

$$\delta R_{\mu\nu} = \delta\Gamma^\rho_{\mu\nu,\rho} - \delta\Gamma^\rho_{\mu\rho,\nu} + \delta\Gamma^\sigma_{\mu\nu}\bar{\Gamma}^\rho_{\sigma\rho} + \bar{\Gamma}^\sigma_{\mu\nu}\delta\Gamma^\rho_{\sigma\rho} - \delta\Gamma^\sigma_{\mu\rho}\bar{\Gamma}^\rho_{\sigma\nu} - \bar{\Gamma}^\sigma_{\mu\rho}\delta\Gamma^\rho_{\sigma\nu} + o(\delta^2) \tag{24}$$

Each component can be expressed as

$$\delta R_{00} = \delta\Gamma^\rho_{00,\rho} - \delta\Gamma^\rho_{0\rho,0} + \delta\Gamma^\sigma_{00}\bar{\Gamma}^\rho_{\sigma\rho} + \cancel{\bar{\Gamma}^\sigma_{00}\delta\Gamma^\rho_{\sigma\rho}} - \delta\Gamma^\sigma_{0\rho}\bar{\Gamma}^\rho_{\sigma 0} - \bar{\Gamma}^\sigma_{0\rho}\delta\Gamma^\rho_{\sigma 0} \tag{25}$$

$$= \delta\Gamma^i_{00,i} - \delta\Gamma^i_{0i,0} + \delta\Gamma^0_{00}\bar{\Gamma}^i_{0i} - 2\delta\Gamma^i_{0j}\bar{\Gamma}^j_{i0} \tag{26}$$

$$= \partial_i\left[\frac{1}{2}a^{-2}(2h_{i0,0} - h_{00,i})\right] - \partial_0\left[\frac{1}{2}a^{-2}(-2h_{ii}\frac{\dot{a}}{a} + h_{ii,0})\right] - \frac{1}{2}h_{00,0}\cdot 3\frac{\dot{a}}{a} - a^{-2}(-2h_{ii}\frac{\dot{a}}{a} + h_{ii,0})\frac{\dot{a}}{a} \tag{27}$$

$$= \frac{1}{2}a^{-2}(2h_{i0,0i} - h_{00,ii}) + \cancel{\frac{\dot{a}}{a^3}(-2h_{ii}\frac{\dot{a}}{a} + h_{ii,0})} - \frac{3}{2}\frac{\dot{a}}{a}h_{00,0} \tag{28}$$

$$-\frac{1}{2}a^{-2}\left[-2h_{ii,0}\frac{\dot{a}}{a} - 2h_{ii}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) + h_{ii,00}\right] - \cancel{\frac{\dot{a}}{a^3}\left(-2h_{ii}\frac{\dot{a}}{a} + h_{ii,0}\right)} \tag{29}$$

$$= \frac{1}{2}a^{-2}(2h_{0i,0i} - \nabla^2 h_{00}) - \frac{3}{2}\frac{\dot{a}}{a}h_{00,0} - \frac{1}{2}a^{-2}\left[-2h_{ii,0}\frac{\dot{a}}{a} - 2h_{ii}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) + h_{ii,00}\right] \tag{30}$$

$$\tag{31}$$

$$
\begin{aligned}
\delta R_{0i} = \delta R_{i0} = & \, \boxed{\delta\Gamma^{\rho}_{0i,\rho} - \delta\Gamma^{\rho}_{0\rho,i}} + \boxed{\delta\Gamma^{\sigma}_{0i}\bar{\Gamma}^{\rho}_{\sigma\rho}} + \boxed{\bar{\Gamma}^{\sigma}_{0i}\delta\Gamma^{\rho}_{\sigma\rho}} - \boxed{\delta\Gamma^{\sigma}_{0\rho}\bar{\Gamma}^{\rho}_{\sigma i}} - \boxed{\bar{\Gamma}^{\sigma}_{0\rho}\delta\Gamma^{\rho}_{\sigma i}} && (32) \\
= & \, \delta\Gamma^{0}_{0i,0} - \delta\Gamma^{0}_{00,i} + \delta\Gamma^{k}_{0i,k} - \delta\Gamma^{k}_{0k,i} + \delta\Gamma^{0}_{0i}\bar{\Gamma}^{j}_{0j} + \cancel{\bar{\Gamma}^{j}_{0i}\delta\Gamma^{0}_{j0}} + \cancel{\bar{\Gamma}^{j}_{0i}\delta\Gamma^{k}_{jk}} - \cancel{\delta\Gamma^{0}_{0j}\bar{\Gamma}^{j}_{0i}} - \delta\Gamma^{j}_{00}\bar{\Gamma}^{0}_{ji} - \cancel{\bar{\Gamma}^{k}_{0j}\delta\Gamma^{j}_{ki}} && (33) \\
= & \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) h_{0i} + \frac{\dot{a}}{a} h_{0i,0} + \left(\frac{\dot{a}}{a} h_{0i} - \frac{1}{2} h_{00,i}\right)\frac{3\dot{a}}{a} - a\dot{a}\frac{1}{2a^2}(2h_{i0,0} - h_{00,i}) && (34) \\
& + \frac{1}{2a^2}\left[-\frac{2\dot{a}}{a}(h_{ik,k} - h_{kk,i}) + h_{ik,k0} + h_{k0,ik} - h_{i0,kk} + h_{kk,0i} + \cancel{h_{k0,ki} - h_{k0,ki}}\right] && (35) \\
= & \left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2}\right) h_{0i} - \frac{\dot{a}}{a} h_{00,i} - \frac{1}{2a^2}(\nabla^2 h_{i0} - h_{j0,ji}) - \frac{\partial}{2\partial t}\left[\frac{1}{a^2}(h_{kk,i} - h_{ki,k})\right] && (36)
\end{aligned}
$$

$$
\begin{aligned}
\delta R_{jk} = & \frac{1}{2} h_{00,kj} + \left(2\dot{a}^2 + a\ddot{a}\right)\delta_{jk} h_{00} + \frac{1}{2} a\dot{a}\delta_{jk} h_{00,0} && (37) \\
& - \frac{1}{2a^2}\left(\nabla^2 h_{jk} - h_{ik,ji} - h_{ij,ki} + h_{ii,kj}\right) && (38) \\
& + \frac{1}{2} h_{jk,00} - \frac{\dot{a}}{2a}\left(h_{jk,0} - \delta_{jk} h_{ii,0}\right) - \frac{\dot{a}^2}{a^2}\left(-2h_{jk} + \delta_{jk} h_{ii}\right) - \frac{\dot{a}}{a}\delta_{jk} h_{i0,i} && (39) \\
& - \frac{1}{2}\left(h_{k0,0j} + h_{j0,0k}\right) - \frac{\dot{a}}{2a}\left(h_{k0,j} + h_{j0,k}\right) && (40)
\end{aligned}
$$

where $\nabla^2 A = A_{,ii}$ and dot means the derivative with respect to time.

4 Energy momentum tensor

Consider perfect fluid

$$\bar{T}_{\mu\nu} = \bar{p}\bar{g}_{\mu\nu} + (\bar{p} + \bar{\rho})\bar{u}_\mu\bar{u}_\nu \qquad \bar{u}^\mu = (1,0,0,0) \quad \bar{u}_\mu = (-1,0,0,0) \tag{41}$$

Consider the perturbation of energy-momentum tensor. We define

$$u^\mu = \bar{u}^\mu + \delta u^\mu \qquad \rho = \bar{\rho} + \delta\rho \qquad p = \bar{p} + \delta p \tag{42}$$

We do not have enough degrees of freedom because of $\bar{u}^\mu\bar{u}_\mu = -1$, and as a result

$$
\begin{aligned}
g^{\mu\nu}u_\mu u_\nu &= \bar{g}^{\mu\nu}\bar{u}_\mu\bar{u}_\nu + \bar{g}^{\mu\nu}\delta u_\mu\bar{u}_\nu + \bar{g}^{\mu\nu}\bar{u}_\mu\delta u_\nu + h^{\mu\nu}\bar{u}_\mu\bar{u}_\nu + o(\delta u^2) && (43) \\
\Rightarrow 0 &= \delta u_0 + \delta u_0 + h^{00} \Rightarrow \delta u_0 = -\frac{1}{2}h^{00} = \frac{1}{2}h_{00} = \delta u^0 && (44)
\end{aligned}
$$

and (Helmholtz decomposition)

$$\delta u_i = \partial_i\delta u + \delta u_i^V \qquad \partial_i(\delta u_i^V) = 0 \tag{45}$$

The perturbation of energy-momentum tensors and components become

$$\delta T_{\mu\nu} = (\delta\rho + \delta p)\bar{u}_\mu\bar{u}_\nu + (\bar{\rho} + \bar{p})(\delta u_\mu\bar{u}_\nu + \bar{u}_\mu\delta u_\nu) + \delta p\bar{g}_{\mu\nu} + \bar{p}h_{\mu\nu} \tag{46}$$

$$\delta T_{00} = \delta p - \bar{\rho}h_{00} \qquad \delta T_{0i} = -(\bar{\rho} + \bar{p})\delta u_i + \bar{p}h_{i0} \qquad \delta T_{ij} = \bar{p}h_{ij} + a^2\delta_{ij}\delta p \tag{47}$$

More generally, we can always put the perturbed energy-momentum tensor in a form like that of the perturbed metric. Details see Ref.[1] Eq.(5.1.39):

$$
\begin{aligned}
\delta T_{00} &= \delta p - \bar{\rho}h_{00} && (48) \\
\delta T_{0i} &= -(\bar{\rho} + \bar{p})(\partial_i\delta u + \delta u_i^V) + \bar{p}h_{i0} && (49) \\
\delta T_{ij} &= \bar{p}h_{ij} + a^2\left[\delta_{ij}\delta p + \pi^S_{,ij} + \pi^V_{i,j} + \pi^T_{ij}\right] && (50)
\end{aligned}
$$

where π terms represent dissipative corrections to the inertia tensor.

5 Source tensor

Weinberg used a new method that avoided the calculation of the Einstein tensor of the Ricci scalar.

$$\begin{aligned}
R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} =& 8\pi G T_{\mu\nu} && (51)\\
g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}g_{\mu\nu} =& 8\pi G T_{\mu\nu}g^{\mu\nu} && (52)\\
-R =& 8\pi G T_{\rho\sigma}g^{\rho\sigma} && (53)\\
\Rightarrow R_{\mu\nu} =& 8\pi G T_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu} && (54)\\
=& 8\pi G T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\cdot 8\pi G T_{\rho\sigma}g^{\rho\sigma} && (55)\\
=& 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}T_{\rho\sigma}\right) && (56)\\
\equiv& 8\pi G S_{\mu\nu} && (57)
\end{aligned}$$

where $S_{\mu\nu}$ is source tensor and the perturbation of this term is

$$\delta S_{\mu\nu} = \delta T_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\delta T^{\lambda}{}_{\lambda} - \frac{1}{2}h_{\mu\nu}\bar{T^{\lambda}}_{\lambda} + o(\delta^2) \tag{58}$$

We know that (use Friedmann equations)

$$\bar{T}^{\lambda}_{\lambda} = -\rho + 3p = -\frac{3}{4\pi G}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \tag{59}$$

and the components can be written as

$$\begin{aligned}
\delta S_{00} =& \delta T_{00} + \frac{1}{2}\delta T^{\lambda}_{\lambda} + \frac{3}{8\pi G}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)h_{00} && (60)\\
\delta S_{0i} =& \delta T_{i0} + \frac{3}{8\pi G}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)h_{0i} && (61)\\
\delta S_{ij} =& \delta T_{ij} - \frac{a^2}{2}\delta_{ij}\delta T^{\lambda}_{\lambda} + \frac{3}{8\pi G}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)h_{ij} && (62)
\end{aligned}$$

Then one can get Einstein's equation.

6 Scalar perturbation

$$\begin{aligned}
h_{00} &= -2A && (63)\\
h_{0i} &= a\left(\frac{\partial B}{\partial x^i} + G_i\right) && (64)\\
h_{ij} &= a^2\left(-2\psi\delta_{ij} + \frac{\partial^2 E}{\partial x^i \partial x^j} + \frac{\partial C_i}{\partial x^j} + \frac{\partial C_j}{\partial x^i} + D_{ij}\right) && (65)
\end{aligned}$$

where A, B, ψ, E are scalar perturbations, C_i, G_i are vector perturbations and D_{ij} is tensor perturbation. Consider scalar perturbation as an example,

$$\frac{\partial_i C}{\partial x^i} = \frac{\partial G_i}{\partial x^i} = 0 \qquad \frac{\partial D_{ij}}{\partial x^i} = 0 \qquad D_{ii} = 0 \tag{66}$$

$$ds^2 = -(1+2A)dt^2 + 2aB_{,i}dx^i dt + a^2[(1-2\psi)\delta_{ij} + E_{,ij}]dx^i dx^j \tag{67}$$

$$h_{00} = -2A \qquad h_{0i} = aB_{,i} \qquad h_{ij} = -2a^2\psi\delta_{ij} + a^2 E_{,ij} \tag{68}$$

$$h^{00} = 2A \qquad h^{0i} = a^{-1}B^{,i} \qquad h^{ij} = 2a^{-2}\psi\delta^{ij} - a^{-2}E^{,ij} \tag{69}$$

Christoffel symbols are easy to calculate

$$\delta\Gamma^0_{00} = -\frac{1}{2}h_{00,0} = A_{,0} \tag{70}$$

$$\delta\Gamma^i_{00} = \frac{1}{2}a^{-2}(2h_{i0,0} - h_{00,i}) = \dot{a}a^{-2}B_{,i} + a^{-1}B_{,i0} + a^{-2}A_{,i} \tag{71}$$

$$\delta\Gamma^0_{i0} = \frac{\dot{a}}{a}h_{0i} - \frac{1}{2}h_{00,i} = \dot{a}B_{,i} + A_{,i} \tag{72}$$

$$\delta\Gamma^0_{ij} = \frac{1}{2}(2h_{00}\delta_{ij}\dot{a}a - h_{0i,j} - h_{j0,i} + h_{ij,0}) = -2A\delta_{ij}a\dot{a} - aB_{,ij} - 2a\dot{a}\psi\delta_{ij} - a^2\psi_{,0}\delta_{ij} + a\dot{a}E_{,ij} + \frac{1}{2}a^2E_{,ij0} \tag{73}$$

$$\delta\Gamma^i_{0j} = \frac{1}{2}a^{-2}(-2h_{ij}\frac{\dot{a}}{a} + h_{i0,j} + h_{ij,0} - h_{0j,i}) = 2a^{-1}\dot{a}\psi\delta_{ij} - a^{-1}\dot{a}E_{,ij} - 2a^{-1}\dot{a}\psi\delta_{ij} - \dot{\psi}\delta_{ij} + a^{-1}\dot{a}E_{,ij} + \frac{1}{2}E_{,ij0} \tag{74}$$

$$\delta\Gamma^i_{jk} = \frac{1}{2}a^{-2}(-2h_{i0}\delta_{jk}\dot{a}a + h_{ij,k} + h_{ki,j} - h_{jk,i}) \tag{75}$$

$$= -B_{,i}\delta_{jk}\dot{a} - \psi_{,k}\delta_{ij} + \frac{1}{2}E_{,ijk} - \psi_{,j}\delta_{ki} + \frac{1}{2}E_{,kij} - \psi_{,i}\delta_{jk} + \frac{1}{2}E_{,jki} \tag{76}$$

Ricca tensors are easy to calculate

$$\delta R_{00} = \frac{1}{2}a^{-2}(2h_{0i,0i} - \nabla^2 h_{00}) - \frac{3}{2}\frac{\dot{a}}{a}h_{00,0} - \frac{1}{2}a^{-2}\left[-2h_{ii,0}\frac{\dot{a}}{a} - 2h_{ii}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) + h_{ii,00}\right] \tag{77}$$

$$= \frac{1}{a^2}\nabla^2 A + 3\frac{\dot{a}}{a}\dot{A} + \frac{1}{a}\nabla^2\dot{B} + \frac{\dot{a}}{a^2}\nabla^2 B - 3a^2\ddot{\psi} - \frac{6\dot{a}}{a}\dot{\psi} + \frac{\dot{a}}{a}\nabla^2\dot{E} + \frac{1}{2}\nabla^2\ddot{E} \tag{78}$$

$$\delta R_{0i} = \left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2}\right)h_{0i} - \frac{\dot{a}}{a}h_{00,i} - \frac{1}{2a^2}(\nabla^2 h_{i0} - h_{j0,ji}) - \frac{\partial}{2\partial t}\left[\frac{1}{a^2}(h_{kk,i} - h_{ki,k})\right] \tag{79}$$

$$= \left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2}\right)h_{0i} + 2A_{,i}\frac{\dot{a}}{a} + 2\psi_{,i0} \tag{80}$$

$$\delta R_{jk}(j \neq k) = \frac{1}{2}h_{00,kj} - \frac{1}{2a^2}\left(\nabla^2 h_{jk} - h_{ik,ji} - h_{ij,ki} + h_{ii,kj}\right) + \frac{1}{2}h_{jk,00} \tag{81}$$

$$- \frac{\dot{a}}{2a}h_{jk,0} + 2\frac{\dot{a}^2}{a^2}h_{jk} - \frac{1}{2}\left(h_{k0,0j} + h_{j0,0k}\right) - \frac{\dot{a}}{2a}\left(h_{k0,j} + h_{j0,k}\right) \tag{82}$$

$$= -\frac{1}{2}\partial_j\partial_k\left[2A - 2\psi - a^2\ddot{E} - 3a\dot{a}\dot{E} + 2a\dot{B} + 4\dot{a}F\right] + \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}}{a}\right)h_{jk} \tag{83}$$

$$-R_{jk}\delta_{jk} = -\frac{1}{2}\nabla^2 h_{00} - 3(2\dot{a}^2 + a\ddot{a})h_{00} - \frac{3}{2}a\dot{a}h_{00,0} + \frac{1}{a^2}(\nabla^2 h_{kk} - \partial_i\partial_k h_{ik}) \tag{84}$$

$$- \frac{1}{2}h_{kk,00} - \frac{\dot{a}}{a}h_{kk,0} + \frac{\dot{a}^2}{a^2}h_{kk} + \frac{3\dot{a}}{a}h_{k0,k} + h_{k0,k0} + \frac{\dot{a}}{a}h_{k0,k} \tag{85}$$

$$= 3a\dot{a}\dot{A} + 6(2\dot{a} + a\ddot{a})A - 3\nabla^2\psi - 3a^2\ddot{\psi} + 12a\dot{a}\psi - \frac{3}{2}a\dot{a}\nabla^2\dot{E} + 3\dot{a}\nabla B + \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}\right)h_{kk} \tag{86}$$

and the perturbated source tensors (consider scalar perturbation only):

$$8\pi G\cdot\delta S_{00} = 8\pi G\left[\delta T_{00} + \frac{1}{2}\delta T^\lambda_\lambda + \frac{3}{8\pi G}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)h_{00}\right] \tag{87}$$

$$= 4\pi G(\delta\rho + 3\delta p + \nabla^2\pi^S) + 6\frac{\ddot{a}}{a}A \tag{88}$$

$$8\pi G\cdot\delta S_{0i} = 8\pi G\left[\delta T_{0i} + \frac{3}{8\pi G}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)h_{0i}\right] \tag{89}$$

$$= -8\pi G\cdot(\bar{\rho} + \bar{p})\delta u_{,i} + \left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2}\right)h_{0i} \tag{90}$$

$$8\pi G\cdot\delta S_{jk}(j \neq k) = 8\pi G\left[\delta T_{jk} + \frac{3}{8\pi G}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)h_{jk}\right] \tag{91}$$

$$= 8\pi G\cdot\partial_j\partial_k a^2\pi^S + \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}\right)h_{jk} \tag{92}$$

$$8\pi G\cdot\delta S_{jk}\delta_{jk} = 8\pi G\cdot\left[\delta T_{kk} - \frac{a^2}{2}\delta_{kk}\delta T^\lambda_\lambda + \frac{3}{8\pi G}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)h_{kk}\right] \tag{93}$$

$$= -4\pi G\cdot 3a^2(\delta p - \delta\rho + \nabla^2\pi^S) + \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}\right)h_{kk} \tag{94}$$

7 Einstein Equations

We have obtained the expressions of the Ricci tensor and source tensor. Writing them out separately, it is not difficult to get the Einstein equation

$$R_{\mu\nu} = 8\pi G S_{\mu\nu} \tag{95}$$

The $(0,0)\,,(0,i)\,,(j\neq k)$ and $\delta_{\rm jk}$ compoents are:

$$4\pi G(\delta\rho + 3\delta p + \nabla^2\pi^S) = \frac{1}{a^2}\nabla^2 A + 3\frac{\dot a}{a}\dot A + \frac{1}{a}\nabla^2\dot B + \frac{\dot a}{a^2}\nabla^2 B + 3\ddot\psi + \frac{6\dot a}{a}\dot\psi - \frac{\dot a}{a}\nabla^2\dot E - \frac{1}{2}\nabla^2\ddot E + 6\frac{\ddot a}{a}A \tag{96}$$

$$8\pi G\cdot(\bar\rho + \bar p)\delta u_{,i} = -2A_{,i}\frac{\dot a}{a} - 2\psi_{,i0} \tag{97}$$

$$\partial_j\partial_k\left[16\pi G a^2\pi^S + 2A - 2\psi - a^2\ddot E - 3a\dot a\dot E + 2a\dot B + 4\dot a B\right] = 0 \tag{98}$$

$$-4\pi G a^2[\delta\rho - \delta p - \nabla^2\pi^S] = a\dot a\dot A + 2(2\dot a + a\ddot a)A - \nabla^2\psi + a^2\ddot\psi + 6a\dot a\dot\psi - \frac{1}{2}a\dot a\nabla^2\dot E + \dot a\nabla B \tag{99}$$

Another way to calculate and get those equations is:

$$\delta R = h^{\mu\nu}\bar R_{\mu\nu} + \bar g^{\mu\nu}\delta R_{\mu\nu} \tag{100}$$

and then calculate

$$\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2}\bar g_{\mu\nu}\bar R - \frac{1}{2}\bar g_{\mu\nu}\delta R \tag{101}$$

I won't elaborate too much here.

8 Gauge

Consider a spacetime coordinate transformation

$$x^\mu = x'^\mu \Rightarrow x^\mu + \epsilon^\mu(x) \tag{102}$$

Under this transfotmation, the metric tensor becomes

$$g'_{\mu\nu}(x') = g_{\rho\sigma}(x)\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} \tag{103}$$

We can attribute the whole change in $g_{\mu\nu}(x)$ to a change in the perturbation $h_{\mu\nu}(x)$:

$$\begin{aligned}
\Delta h_{\mu\nu}(x) &= g'_{\mu\nu}(x) - g_{\mu\nu}(x) && (104)\\
&= g'_{\mu\nu}(x') - [g'_{\mu\nu}(x') - g'_{\mu\nu}(x)] - g_{\mu\nu}(x) && (105)\\
&= g_{\rho\sigma}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} - \partial_\rho g_{\mu\nu}(x)\epsilon^\rho - g_{\mu\nu}(x) && (106)\\
&= -\bar g_{\lambda\mu}\frac{\partial\epsilon^\lambda(x)}{\partial x^\nu} - \bar g_{\lambda\nu}\frac{\partial\epsilon^\lambda(x)}{\partial x^\mu} - \frac{\partial\bar g_{\mu\nu}(x)}{\partial x^\lambda}\epsilon^\lambda(x) && (107)
\end{aligned}$$

or in more detail($\epsilon_0 = -\epsilon^0, \epsilon_i = a^2\epsilon^i$, $\epsilon_i = \epsilon^S_{,i} + \epsilon^V_i$ and $\bar g_{\mu\nu} = \mathrm{diag}(-1, a^2, a^2, a^2)$)

$$\begin{aligned}
\Delta h_{00} &= 2\epsilon_{0,0} && (108)\\
\Delta h_{0i} &= -\epsilon_{i,0} - \epsilon_{0,i} + 2\frac{\dot a}{a}\epsilon_i = -\epsilon^S_{,i0} - \epsilon^V_{,i0} - \epsilon_{0,i} + 2\frac{\dot a}{a}\epsilon^S_{,i} + 2\frac{\dot a}{a}\epsilon^V_i && (109)\\
\Delta h_{ij} &= -\epsilon_{i,j} - \epsilon_{j,i} + 2a\dot a\delta_{ij}\epsilon_0 = -2\epsilon^S_{,ij} - \epsilon^V_{j,i} - \epsilon^V_{i,j} + 2\dot a a\delta_{ij}\epsilon_0 && (110)
\end{aligned}$$

combine with (63)

$$\Delta\psi = \frac{\dot a}{a}\epsilon_0 \quad \Delta E = -\frac{2}{a^2}\epsilon^S \quad \Delta C_i = -\frac{1}{a^2}\epsilon^V_i \quad \Delta D_{ij} = 0 \quad \Delta A = \epsilon_{0,0} \tag{111}$$

$$\Delta B = \frac{1}{a}\left(-\epsilon_0 - \epsilon^S_{,0} + \frac{2\dot a}{a}\epsilon^S\right) \qquad \Delta G_i = \frac{1}{a}\left(-\epsilon^V_{i,0} + \frac{2\dot a}{a}\epsilon^V_i\right) \tag{112}$$

Noting that D is gauge invariant, we can also create other gauge invariants, such as $G_i - a\dot{C}_i$.

when consider scalar perturbation. We can set F and B equal to zero by choose ϵ^S and ϵ_0. This is called the Newtonian gauge. Synchronous gauge which we choose E and F equal to zero. Then we can simply write the Einstein equation, and we can use a similar method to get the perturbation of the energy-momentum tensor

$$\Delta\delta T_{\mu\nu} = -\bar{T}_{\lambda\mu}(x)\frac{\partial\epsilon^\lambda(x)}{\partial x^\nu} - \bar{T}_{\lambda\mu}(x)\frac{\partial\epsilon^\lambda(x)}{\partial x^\mu} - \frac{\partial\bar{T}_{\mu\nu}(x)}{\partial x^\lambda}\epsilon^\lambda(x) \tag{113}$$

$$\Delta\delta T_{00} = 2\bar{\rho}\epsilon_{0,0} + \dot{\bar{\rho}}\epsilon_0 \tag{114}$$

$$\Delta\delta T_{i0} = -\bar{p}\epsilon_{i,0} + \bar{\rho}\epsilon_{0,i} + 2\bar{p}\frac{\dot{a}}{a}\epsilon_i \tag{115}$$

$$\Delta\delta T_{ij} = -\bar{p}(\epsilon_{i,j} - \epsilon_{j,i}) + \frac{\partial}{\partial t}(a^2\bar{p})\delta_{ij}\epsilon_0 \tag{116}$$

9 References

[1] Weinberg, Steven. Cosmology. OUP Oxford, 2008.