1   Conventions

Our are consistent with previous works. We work with the spatially flat FLRW universe described by theline-elemen:

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2 \tag{1}$$

We work with Plank system where M_{P} is reduced Plank mass. ....

2 $f(R)$ theory

Consider a $f(R)$ action:

$$S = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2} f(R) \right] \tag{2}$$

We neglect matter term in this note. The first step we need to do is to rewrite the term $f(R)$ into the linear form of R. This step requires the introduction of a field χ, whose dynamics are completely equivalent to R. Let's rewrite term $f(R)$as:

$$f(R) = (R - \chi)\frac{\partial f}{\partial \chi} + f(\chi) \tag{3}$$

Varing $f(R)$ with respect to χ, we obtain $(R - \chi)\frac{\partial f_{,\chi}}{\partial \chi} = 0$. Provided $\frac{\partial f_{,\chi}}{\partial \chi} = 0$, it follows that $\chi = R$.

Let's say $f_{,\chi} = \frac{\partial f}{\partial \chi}$ and define $U(\chi) = \chi f_{,\chi} - f(\chi)$. Action becomes:

$$S = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2}(f_{,\chi}R - U(\chi)) \right] \tag{4}$$

This is a liner term plus a χ funciton. Then We plant to eliminate the coefficient of R by conformal transformation.

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \qquad \tilde{g}_{\mu\nu} = f_{,\chi} g_{\mu\nu} \tag{5}$$

where Ω is conformal factor. In our case $\Omega = \sqrt{f_{,\chi}}$. The metric before conformal transformation $(g_{\mu\nu})$ called metric in Jordan frame. The metric after conformal transformation $(\tilde{g}_{\mu\nu})$ called metric in Einstein frame. Under conformal transformation, conformal Ricci scalar is*:

$$R = \Omega^2 \left[\tilde{R} + 2(D-1)\frac{\tilde{\Box}\Omega}{\Omega} - (D-1)(D-2)\tilde{g}_{\mu\nu}\frac{\Omega_{,\mu}\Omega_{,\nu}}{\Omega^2} \right] \tag{6}$$

$$R = f_{,\chi} \left[\tilde{R} + 6\tilde{\Box}\sigma - 6\tilde{g}_{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma \right] \tag{7}$$

where D is dimensions of spacetime and

$$\sigma = \ln\sqrt{f_{,\chi}} \qquad \partial_\mu\sigma = \frac{\partial\sigma}{\partial\tilde{x}^\mu} \qquad \tilde{\Box}\sigma = \frac{1}{\sqrt{-\tilde{g}}}\partial_\mu(\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu\sigma) \tag{8}$$

Under the conformal transformation, the metric relationship between two frames is:

$$\sqrt{-\tilde{g}} = \sqrt{-\Omega^{2D}g} = \Omega^D\sqrt{-g} \qquad \sqrt{-g} = f_{,\chi}^{-2}\sqrt{-\tilde{g}} \tag{9}$$

Combining (7) and (9), (4) becomes:

$$S = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2}(f_{,\chi}R - U(\chi)) \right] \tag{10}$$

$$= \int d^4x \sqrt{-\tilde{g}}f_{,\chi}^{-2} \left[\frac{M_P^2}{2}(f_{,\chi}^2(\tilde{R} + 6\tilde{\Box}\sigma - 6\tilde{g}_{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma) - U(\chi)) \right] \tag{11}$$

$$= \int d^4x \sqrt{-\tilde{g}} \left[\frac{M_P^2}{2}(\tilde{R} - 6\tilde{g}_{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - f_{,\chi}^{-2}U(\chi)) \right] \tag{12}$$

where $\tilde{\Box}\sigma$ disappear because of Gauss' law. Finally, theory has become the form of GR + auxiliary fields χ, σ.

Let's define a new field to replace σ and χ.

$$\phi = \sqrt{\frac{3}{2}}M_P \ln f_{,\chi} \qquad \sigma = \frac{\phi}{\sqrt{6}M_P} \tag{13}$$

The second equation can be derived from the first equation and (8). Then the action can be written as:

$$S = \int d^4x \sqrt{-\tilde{g}} \left[\frac{M_P^2}{2}(\tilde{R} - 6\tilde{g}_{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - f_{,\chi}^{-2}U(\chi)) \right] \tag{14}$$

$$= \int d^4x \sqrt{-\tilde{g}} \left[\frac{M_P^2}{2}\tilde{R} - \frac{1}{2}\tilde{g}_{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \right] \tag{15}$$

$$V(\phi) = \frac{M_P^2}{2}e^{-\sqrt{\frac{8}{3}}\frac{\phi}{M_P}} \left(\chi(\phi)e^{\sqrt{\frac{2}{3}}\frac{\phi}{M_P}} - f(\phi) \right) \tag{16}$$

The action of $f(R)$ gravity theory is transformed into the form of the Einstein-Hilbert action plus a scalar field ϕ with self-interaction. Different $f(R)$ models provide different V. If an appropriate scalar potential is derived, it can serve as the potential function for inflation.

3 Inflation models

We consider the models of the form $f(R) = R + \alpha R^2$. Taking is into (13), $f_{,\chi} = \partial f/\partial\chi$, and (16), we get:
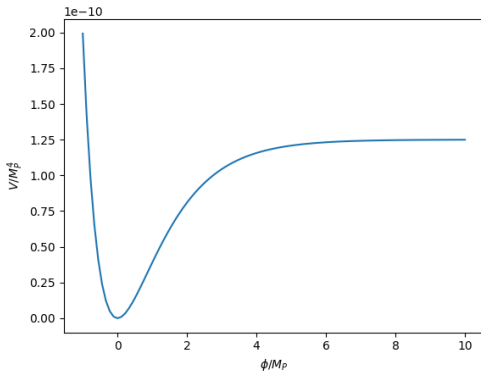
$$f_{,\chi} = 1 + 2\alpha\chi \qquad \chi = \frac{1}{2\alpha}\left(e^{\sqrt{\frac{2}{3}}\frac{\phi}{M_P}} - 1 \right) \tag{17}$$

$$V(\phi) = \frac{M_P^2}{2}e^{-\sqrt{\frac{8}{3}}\frac{\phi}{M_P}} \left(\chi(\phi)e^{\sqrt{\frac{2}{3}}\frac{\phi}{M_P}} - f(\phi) \right) \tag{18}$$
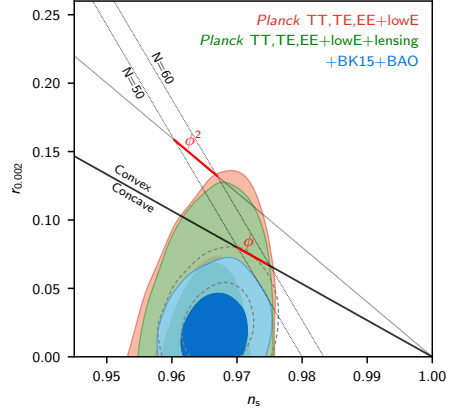
$$= \frac{M_P^2}{2}e^{-\sqrt{\frac{8}{3}}\frac{\phi}{M_P}} \left(\frac{1}{2\alpha}\left(e^{\sqrt{\frac{2}{3}}\frac{\phi}{M_P}} - 1 \right)e^{\sqrt{\frac{2}{3}}\frac{\phi}{M_P}} - \frac{1}{2\alpha}\left(e^{\sqrt{\frac{2}{3}}\frac{\phi}{M_P}} - 1 \right) - \frac{1}{4\alpha}\left(e^{\sqrt{\frac{2}{3}}\frac{\phi}{M_P}} - 1 \right)^2 \right) \tag{19}$$

$$= \frac{M_P^2}{8\alpha}\left(1 - e^{-\sqrt{\frac{2}{3}}\frac{\phi}{M_P}} \right)^2 \tag{20}$$

α affects the shape of potential, observation of promordial powerspectrum shows that $\alpha \sim 10^9$. The potential shows in Fig1a. We clearly know that this includes inflation $\phi \gg M_P$, the end of inflation$\phi \sim M_P$, and oscillation $\phi \sim 0$.

(a) Potential of Starobinsky model

(b) Data from Planck 2018

During inflation, quantum fluctuations occur in both the scalar field ϕ and the gravitational field, giving rise to what are known as primordial perturbations. As inflation progresses, the wavelengths of these perturbations are stretched far beyond the scale of the horizon at the time. After inflation ends, as the horizon scale expands, these long-wavelength perturbations re-enter the horizon, eventually imprinting themselves on the CMB, thus becoming observable signatures of inflation. These primordial perturbations are characterized by scalar spectral tilt and tensor-to-scalar ratio:

$$n_s = 1 + 2\eta(\phi) - 6\epsilon(\phi) \qquad r = 16\epsilon(\phi) \qquad \epsilon(\phi) = \frac{M_P^2}{2}\left[\frac{V_{,\phi}(\phi)}{V(\phi)} \right]^2 \qquad \eta(\phi) = M_P^2\frac{V_{,\phi\phi}(\phi)}{V(\phi)} \tag{21}$$

where ϵ η are slow roll parameters under the slow roll codnition. Different models, or equivalently, different inflationary potential functions predict varying ranges for the n_s and r. Therefore, we can use these two observational indicators to differentiate among various inflation models, or to test which specific inflation model is responsible for driving the cosmic inflation.

Let's compute these parameters with potential (20).

$$\epsilon(\phi) = \frac{M_p^2}{2} \left[\frac{V_{,\phi}(\phi)}{V(\phi)} \right]^2 = \frac{4}{3} \frac{M_p^2}{\left(e^{\sqrt{2/3}\phi/M_P} - 1 \right)^2} \qquad \eta(\phi) = M_p^2 \frac{V_{,\phi\phi}(\phi)}{V(\phi)} = -\frac{4M_p^2}{3} \frac{e^{\sqrt{2/3}\phi/M_P} - 2}{\left(e^{\sqrt{2/3}\phi/M_P} - 1 \right)^2} \tag{22}$$

Inflation will end at $\epsilon = 1$ $\eta \sim 1$, where $\phi_{\mathrm{end-of-inflation}}$ is (see. TASI eq 82)

$$\frac{\phi_e}{M_P} = \sqrt{\frac{3}{2}} \ln \left(1 + \frac{2}{\sqrt{3}} \right) \sim 0.94 \tag{23}$$

Considering the slow-roll condition, e-folding N can be wirtten as

$$N \equiv \ln \frac{a_e}{a_i} = \int_{\phi_i}^{\phi_e} \frac{d\phi/M_P}{\sqrt{2\epsilon}} = \left[\frac{3}{4} \left(e^{\sqrt{2/3}\phi/M_P} - \sqrt{\frac{2}{3}} \frac{\phi}{M_P} \right) \right]_{\phi_i}^{\phi_e} \tag{24}$$

$$= \frac{3}{4} \left[e^{\sqrt{2/3}\phi_i/M_P} - e^{\sqrt{2/3}\phi_e/M_P} - \sqrt{\frac{2}{3M_P^2}} (\phi_i - \phi_e) \right] \tag{25}$$

It seems difficult to solve. when we consider $N \sim 50, 60$, we can get a approximate solution:

$$N \sim \frac{4}{3} e^{\sqrt{2/3}\phi_i/M_P} \qquad \epsilon(\phi_i) \sim \frac{3}{4N^2} \qquad \eta(\phi_i) \sim \frac{1}{N} \tag{26}$$

Then we can compute scalar spectral tilt and tensor-to-scalar ratio directly under the potential (20). Bring (26) into (21)(assume $N \sim 60$):

$$n_s = 1 + 2\eta(\phi_i) - 6\epsilon(\phi_i) = 1 - \frac{2}{N} - \frac{9}{2N^2} \sim 0.9541 \tag{27}$$

$$r = 16\epsilon(\phi_i) = \frac{12}{N^2} \sim 3.3 \times 10^{-3} \tag{28}$$

We can find matching observations from Plank2018, see Fig.1b

4 Appendix

$$\tilde{R}^a_{\ bcd} = R^a_{\ bcd} + \frac{1}{\Omega} \left[\delta^a_d \Omega_{;bc} - \delta^a_c \Omega_{;bd} + g_{bc} \Omega^{;a}_{\ ;d} - g_{bd} \Omega^{;a}_{\ ;c} \right] \tag{29}$$
$$+ \frac{2}{\Omega^2} \left[\delta^a_c \Omega_{,b}\Omega_{,d} - \delta^a_d \Omega_{,b}\Omega_{,c} + g_{bd}\Omega^{,a}\Omega_{,c} - g_{bc}\Omega^{,a}\Omega_{,d} \right] + \frac{1}{\Omega^2} \left[\delta^a_d g_{bc} - \delta^a_c g_{bd} \right] g_{ef} \Omega^{,e}\Omega^{,f} \ ,$$

$$R^a_{\ bcd} = \tilde{R}^a_{\ bcd} - \frac{1}{\Omega} \left[\delta^a_d \Omega_{\tilde{;}bc} - \delta^a_c \Omega_{\tilde{;}bd} + \tilde{g}_{bc} \Omega^{\tilde{;}a}_{\ \tilde{;}d} - \tilde{g}_{bd} \Omega^{\tilde{;}a}_{\ \tilde{;}c} \right] \tag{30}$$
$$+ \frac{1}{\Omega^2} \left[\delta^a_d \tilde{g}_{bc} - \delta^a_c \tilde{g}_{bd} \right] \tilde{g}_{ef} \Omega^{,e}\Omega^{,f} \ ,$$

$$\tilde{R}_{ab} = R_{ab} + \frac{1}{\Omega^2} \left[2(D-2)\Omega_{,a}\Omega_{,b} - (D-3)\Omega_{,c}\Omega^{,c} g_{ab} \right] - \frac{1}{\Omega} \left[(D-2)\Omega_{;ab} + g_{ab}\Box\Omega \right] \ , \tag{31}$$

$$R_{ab} = \tilde{R}_{ab} - \frac{1}{\Omega^2} (D-1)\tilde{g}_{ab}\Omega_{,c}\Omega^{,c} + \frac{1}{\Omega} \left[(D-2)\Omega_{\tilde{;}ab} + \tilde{g}_{ab} \tilde{\Box} \Omega \right] \ , \tag{32}$$

$$\tilde{R} = \Omega^{-2} \left[R - 2(D-1)\frac{\Box\Omega}{\Omega} - (D-1)(D-4)g^{ab}\frac{\Omega_{,a}\Omega_{,b}}{\Omega^2} \right] \ , \tag{33}$$

$$R = \Omega^2 \left[\tilde{R} + 2(D-1)\frac{\tilde{\Box}\Omega}{\Omega} - D(D-1)\tilde{g}^{ab}\frac{\Omega_{,a}\Omega_{,b}}{\Omega^2} \right] \ , \tag{34}$$

$$\tilde{G}_{ab} = G_{ab} + \frac{D-2}{2\Omega^2} \left[4\Omega_{,a}\Omega_{,b} + (D-5)\Omega_{,c}\Omega^{,c} g_{ab} \right] - \frac{D-2}{\Omega} \left[\Omega_{;ab} - g_{ab}\Box\Omega \right] \ , \tag{35}$$

$$G_{ab} = \tilde{G}_{ab} + \frac{D-2}{2\Omega^2} (D-1)\Omega_{,e}\Omega^{,e}\tilde{g}_{ab} + \frac{D-2}{\Omega} \left[\Omega_{\tilde{;}ab} - \tilde{g}_{ab} \tilde{\Box} \Omega \right] \ , \tag{36}$$

References

[1] D. Baumann, doi:10.1142/9789814327183_0010 [arXiv:0907.5424 [hep-th]].

[2] A. De Felice and S. Tsujikawa, Living Rev. Rel. **13** (2010), 3 doi:10.12942/lrr-2010-3 [arXiv:1002.4928 [gr-qc]].

[3] N. Aghanim *et al.* [Planck], Astron. Astrophys. **641** (2020), A6 [erratum: Astron. Astrophys. **652** (2021), C4] doi:10.1051/0004-6361/201833910 [arXiv:1807.06209 [astro-ph.CO]].