1 Conventions

We are consistent with previous works. We work with the spatially flat FLRW universe described by the line elemen:

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2 \tag{1}$$

We work with Plank system where $M_{\mathrm{P}} = (8\pi G)^{-1/2}$ is reduced Plank mass.

In different chapters, we use the same symbols to represent different functions, please be careful of the differences. Sometimes I include $M_P^2/2$ in the function, sometimes I don't, and this will be unified later (Maybe).

2 $f(R)$ theory

2.1 Basic equations and Conformal transformation

Consider a $f(R)$ action:

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}f(R)\right] \tag{2}$$

We neglect matter term in this note. The first step we need to do is to rewrite the term $f(R)$ into the linear form of R. This step requires the introduction of a field χ, whose dynamics are completely equivalent to R. Let's rewrite term $f(R)$as:

$$f(R) = (R-\chi)\frac{\partial f}{\partial \chi} + f(\chi) \tag{3}$$

Varing $f(R)$ with respect to χ, we obtain $(R-\chi)\frac{\partial f_{,\chi}}{\partial \chi} = 0$. Provided $\frac{\partial f_{,\chi}}{\partial \chi} = 0$, it follows that $\chi = R$.

Let's say $f_{,\chi} = \frac{\partial f}{\partial \chi}$ and define $U(\chi) = \chi f_{,\chi} - f(\chi)$. Action becomes:

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}(f_{,\chi}R - U(\chi))\right] \tag{4}$$

This is a liner term plus a χ funciton. If we substitute $\chi = R$ back to (4) we recover the original action. Then We plan to eliminate the coefficient of R by conformal transformation.

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \qquad \tilde{g}_{\mu\nu} = f_{,\chi}g_{\mu\nu} \tag{5}$$

where Ω is conformal factor. In our case $\Omega = \sqrt{f_{,\chi}}$. The metric before conformal transformation ($g_{\mu\nu}$) called metric in Jordan frame. The metric after conformal transformation ($\tilde{g}_{\mu\nu}$) called metric in Einstein frame. Under conformal transformation, conformal Ricci scalar is*:

$$R = \Omega^2\left[\tilde{R} + 2(D-1)\frac{\tilde{\Box}\Omega}{\Omega} - (D-1)(D-2)\tilde{g}^{\mu\nu}\frac{\Omega_{,\mu}\Omega_{,\nu}}{\Omega^2}\right] \tag{6}$$

$$R = f_{,\chi}\left[\tilde{R} + 6\tilde{\Box}\sigma - 6\tilde{g}^{\mu\nu}\tilde{\partial}_\mu\sigma\tilde{\partial}_\nu\sigma\right] \tag{7}$$

where D is dimensions of spacetime and

$$\sigma = \ln\sqrt{f_{,\chi}} \qquad \tilde{\partial}_\mu\sigma = \frac{\partial\sigma}{\partial\tilde{x}^\mu} \qquad \tilde{\Box}\sigma = \frac{1}{\sqrt{-\tilde{g}}}\partial_\mu(\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu\sigma) \tag{8}$$

Under the conformal transformation, the metric relationship between two frames is:

$$\sqrt{-\tilde{g}} = \sqrt{-\Omega^{2D}g} = \Omega^D\sqrt{-g} \qquad \sqrt{-g} = f_{,\chi}^{-2}\sqrt{-\tilde{g}} \tag{9}$$

Combining (7) and (9), (4) becomes:

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}(f_{,\chi}R - U(\chi))\right] &(10)\\ &= \int d^4x\sqrt{-\tilde{g}}f_{,\chi}^{-2}\left[\frac{M_P^2}{2}(f_{,\chi}^2(\tilde{R} + 6\tilde{\Box}\sigma - 6\tilde{g}^{\mu\nu}\tilde{\partial}_\mu\sigma\tilde{\partial}_\nu\sigma) - U(\chi))\right] &(11)\\ &= \int d^4x\sqrt{-\tilde{g}}\left[\frac{M_P^2}{2}(\tilde{R} - 6\tilde{g}^{\mu\nu}\tilde{\partial}_\mu\sigma\tilde{\partial}_\nu\sigma - f_{,\chi}^{-2}U(\chi))\right] &(12)\end{aligned}$$

where $\tilde{\Box}\sigma$ disappear because of Gauss' law. Finally, theory has become the form of GR + auxiliary fields χ, σ.

Let's define a new field to replace σ and χ.

$$\phi = \sqrt{\frac{3}{2}} M_P \ln f_{,\chi} \qquad \sigma = \frac{\phi}{\sqrt{6} M_P} \tag{13}$$

The second equation can be derived from the first equation and (8). Then the action can be written as:

$$S = \int d^4x \sqrt{-\tilde{g}} \left[\frac{M_P^2}{2} (\tilde{R} - 6\tilde{g}^{\mu\nu} \tilde{\partial}_\mu \sigma \tilde{\partial}_\nu \sigma - f_{,\chi}^{-2} U(\chi)) \right] \tag{14}$$

$$= \int d^4x \sqrt{-\tilde{g}} \left[\frac{M_P^2}{2} \tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\partial}_\mu \phi \tilde{\partial}_\nu \phi - V(\phi) \right] \tag{15}$$

$$V(\phi) = \frac{M_P^2}{2} e^{-\sqrt{\frac{8}{3}} \frac{\phi}{M_P}} \left(\chi(\phi) e^{\sqrt{\frac{2}{3}} \frac{\phi}{M_P}} - f(\phi) \right) \tag{16}$$

The action of $f(R)$ gravity theory is transformed into the form of the Einstein-Hilbert action plus a scalar field ϕ with self-interaction. Different $f(R)$ models provide different V. If an appropriate scalar potential is derived, it can serve as the potential function for inflation.

2.2 Example: $R + \alpha R^2$

We consider the models of the form $f(R) = R + \alpha R^2$. Taking is into (13), $f_{,\chi} = \partial f / \partial \chi$, and (16), we get:

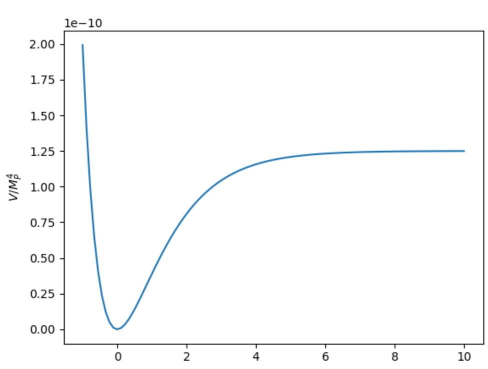
$$f_{,\chi} = 1 + 2\alpha\chi \qquad \chi = \frac{1}{2\alpha} \left(e^{\sqrt{\frac{2}{3}} \frac{\phi}{M_P}} - 1 \right) \tag{17}$$

$$V(\phi) = \frac{M_P^2}{2} e^{-\sqrt{\frac{8}{3}} \frac{\phi}{M_P}} \left(\chi(\phi) e^{\sqrt{\frac{2}{3}} \frac{\phi}{M_P}} - f(\phi) \right) \tag{18}$$

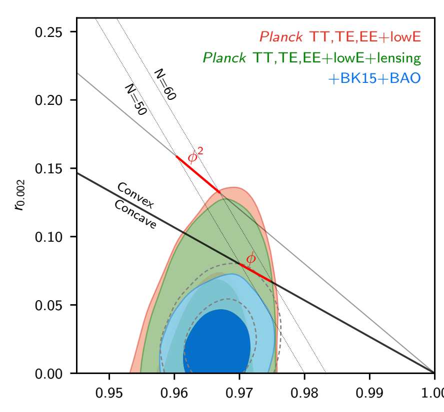
$$= \frac{M_P^2}{2} e^{-\sqrt{\frac{8}{3}} \frac{\phi}{M_P}} \left(\frac{1}{2\alpha} \left(e^{\sqrt{\frac{2}{3}} \frac{\phi}{M_P}} - 1 \right) e^{\sqrt{\frac{2}{3}} \frac{\phi}{M_P}} - \frac{1}{2\alpha} \left(e^{\sqrt{\frac{2}{3}} \frac{\phi}{M_P}} - 1 \right) - \frac{1}{4\alpha} \left(e^{\sqrt{\frac{2}{3}} \frac{\phi}{M_P}} - 1 \right)^2 \right) \tag{19}$$

$$= \frac{M_P^2}{8\alpha} \left(1 - e^{-\sqrt{\frac{2}{3}} \frac{\phi}{M_P}} \right)^2 \tag{20}$$

α affects the shape of potential, observation of promordial powerspectrum shows that $\alpha \sim 10^9$. The potential shows in Fig1a. We clearly know that this includes inflation $\phi \gg M_P$, the end of inflation$\phi \sim M_P$, and oscillation $\phi \sim 0$.



(a) Potential of Starobinsky model



(b) Data from Planck 2018

During inflation, quantum fluctuations occur in both the scalar field ϕ and the gravitational field, giving rise to what are known as primordial perturbations. As inflation progresses, the wavelengths of these perturbations are stretched far beyond the scale of the horizon at the time. After inflation ends, as the horizon scale expands, these long-wavelength perturbations re-enter the horizon, eventually imprinting themselves on the CMB, thus becoming observable signatures of inflation. These primordial perturbations are characterized by scalar spectral tilt and tensor-to-scalar ratio:

$$n_s = 1 + 2\eta(\phi) - 6\epsilon(\phi) \qquad r = 16\epsilon(\phi) \qquad \epsilon(\phi) = \frac{M_P^2}{2}\left[\frac{V_{,\phi}(\phi)}{V(\phi)}\right]^2 \qquad \eta(\phi) = M_P^2 \frac{V_{,\phi\phi}(\phi)}{V(\phi)} \tag{21}$$

where $\epsilon \quad \eta$ are slow roll parameters under the slow roll codnition. Different models, or equivalently, different inflationary potential functions predict varying ranges for the n_s and r. Therefore, we can use these two observational indicators to differentiate among various inflation models, or to test which specific inflation model is responsible for driving the cosmic inflation.

Let's compute these parameters with potential (20).

$$\epsilon(\phi) = \frac{M_p^2}{2}\left[\frac{V_{,\phi}(\phi)}{V(\phi)}\right]^2 = \frac{4}{3}\frac{1}{\left(e^{\sqrt{2/3}\phi/M_p} - 1\right)^2} \qquad \eta(\phi) = M_p^2 \frac{V_{,\phi\phi}(\phi)}{V(\phi)} = -\frac{4}{3}\frac{e^{\sqrt{2/3}\phi/M_p} - 2}{\left(e^{\sqrt{2/3}\phi/M_p} - 1\right)^2} \tag{22}$$

Inflation will end at $\epsilon = 1 \quad \eta \sim 1$, where $\phi_{\text{end}-\text{of}-\text{inflation}}$ is (see. TASI eq 82)

$$\frac{\phi_{\text{e}}}{M_P} = \sqrt{\frac{3}{2}} \ln\left(1 + \frac{2}{\sqrt{3}}\right) \sim 0.94 \tag{23}$$

Considering the slow-roll condition, e-folding N can be wirtten as

$$N \equiv \ln\frac{a_e}{a_i} = \int_{\phi_i}^{\phi_e} \frac{d\phi/M_P}{\sqrt{2\epsilon}} = \left[\frac{3}{4}\left(e^{\sqrt{2/3}\phi/M_P} - \sqrt{\frac{2}{3}}\frac{\phi}{M_P}\right)\right]_{\phi_i}^{\phi_e} \tag{24}$$

$$= \frac{3}{4}\left[e^{\sqrt{2/3}\phi_i/M_P} - e^{\sqrt{2/3}\phi_e/M_P} - \sqrt{\frac{2}{3M_P^2}}(\phi_i - \phi_e)\right] \tag{25}$$

It seems difficult to solve. when we consider $N \sim 50, 60$, we can get a approximate solution:

$$N \sim \frac{4}{3}e^{\sqrt{2/3}\phi_i/M_P} \qquad \epsilon(\phi_i) \sim \frac{3}{4N^2} \qquad \eta(\phi_i) \sim \frac{1}{N} \tag{26}$$

Then we can compute scalar spectral tilt and tensor-to-scalar ratio directly under the potential (20). Bring (26) into (21)(assume $N \sim 60$):

$$n_s = 1 + 2\eta(\phi_i) - 6\epsilon(\phi_i) \sim 1 - \frac{2}{N} - \frac{9}{2N^2} \sim 0.9541 \tag{27}$$

$$r = 16\epsilon(\phi_i) \sim \frac{12}{N^2} \sim 3.3 \times 10^{-3} \tag{28}$$

We can find matching observations from Plank2018, see Fig.1b

3 Nonminimally-coupled model

In the single-field case the action is given by

$$S = \int d^4x\sqrt{-g}\left[f(\phi)R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \tag{29}$$

where ϕ is a scalar field and we will assume that function $f(\phi)$ is positive definite. Transforming to Einstein frame, the action becomes:

$$S = \int d^4x\sqrt{-\tilde{g}}\Omega^{-4}\left[f(\phi)\Omega^2\left(\tilde{R} - 6\tilde{g}^{\mu\nu}\frac{\Omega_{,\mu}\Omega_{,\nu}}{\Omega^2}\right) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \tag{30}$$

To obtain the canonical Einstein-Hilbert gravitational action in the transformed frame, we identify:

$$\Omega = \frac{\sqrt{2}}{M_P}\sqrt{f(\phi)} \tag{31}$$

Action can be expressed as:

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{M_p^2}{2}\tilde{R} - \frac{3M_P^2}{4f^2}\tilde{g}^{\mu\nu}\tilde{\partial}_\mu f\tilde{\partial}_\nu f - \frac{M_P^2}{4f}\tilde{g}^{\mu\nu}\tilde{\partial}_\mu\phi\tilde{\partial}_\nu\phi - \tilde{V}(\phi)\right] \tag{32}$$

where $\tilde{\partial}_\mu = \partial/\partial\tilde{x}^\mu$ and $\tilde{V} = V/\Omega^4$. In the single-field case, we may next rescale the field $\phi \to \tilde{\phi}$:

$$\frac{d\tilde{\phi}}{d\phi} = \sqrt{\frac{M_P^2}{2f^2}(f + 3f_{,\phi}^2)} \tag{33}$$

$$-\frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\partial}_\mu\tilde{\phi}\tilde{\partial}_\nu\tilde{\phi} = -\frac{3M_P^2}{4f^2}\tilde{g}^{\mu\nu}\tilde{\partial}_\mu f\tilde{\partial}_\nu f - \frac{M_P^2}{4f}\tilde{g}^{\mu\nu}\tilde{\partial}_\mu\phi\tilde{\partial}_\nu\phi \tag{34}$$

Action becomes:

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{M_p^2}{2}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\partial}_\mu\tilde{\phi}\tilde{\partial}_\nu\tilde{\phi} - \tilde{V}(\tilde{\phi})\right] \tag{35}$$

Once $f(\phi)$ and $V(\phi)$ is determined, scalar spectral tilt and tensor-to-scalar ratio can be derived from (21).

Let's consider an example. The usual Higgs inflation corresponds to:

$$f(\phi) = \frac{M_P^2}{2} + \frac{1}{2}\xi\phi^2 \qquad V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2 \tag{36}$$

where λ, ξ are constants. $v \sim 246\ rmGeV$ is the vacuum expectation value of ϕ. (33) can be expressed as:

$$\frac{d\tilde{\phi}}{d\phi} = \sqrt{\frac{1 + (1+6\xi)\xi\phi^2/M_P^2}{(1+\xi\phi^2/M_P^2)^2}} \tag{37}$$

If $\xi\phi^2/M_P^2 \gg 1$ and $6\xi \gg 1$, equation becomes:

$$\frac{d\tilde{\phi}}{d\phi} = \frac{\sqrt{6}M_P}{\phi} \to \tilde{\phi} = \frac{\sqrt{6}}{2}M_P \ln\frac{\phi^2}{M_P^2} \to \phi^2 = M_P^2 e^{\sqrt{2/3}\tilde{\phi}/M_P} \tag{38}$$

The potential in Einstein frame is:

$$\tilde{V}(\tilde{\phi}) = \frac{V}{\Omega^4} = \frac{VM_P^4}{4f^2} = \frac{\lambda M_P^4}{4\xi^2}(1 - v^2\phi^{-2})^2 = \frac{\lambda M_P^4}{4\xi^2}\left(1 - \frac{v^2}{M_P^2}e^{-\sqrt{2/3}\tilde{\phi}/M_P}\right)^2 \tag{39}$$

By redefining $\tilde{\phi}$, v can be absorbed. Then, according to Starobinsky's technique (20), ns and r can be calculated and consistent results can be obtained:

$$n_s = 1 + 2\eta(\phi_i) - 6\epsilon(\phi_i) \sim 1 - \frac{2}{N} - \frac{9}{2N^2} \sim 0.9541 \tag{40}$$

$$r = 16\epsilon(\phi_i) \sim \frac{12}{N^2} \sim 3.3 \times 10^{-3} \tag{41}$$

4 Higgs-R^2 model

We begin from the following action for scalar field ϕ and metric $g_{\mu\nu}$

$$S = \int d^4x\sqrt{-g_J}\left[\frac{1}{2}F(R_J) + \frac{1}{2}G(\phi)R_J - \frac{1}{2}g_J^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V_J(\phi)\right] \tag{42}$$

where subscripts indicating different frames, ξ is a constant, V_J is the potential term and $F \sim M_P^4, G \sim M_P^2$ are functions of R and ϕ. Action can be expressed as where $f_{,\chi\chi} \neq 0$:

$$S = \int d^4x\sqrt{-g_J}\left[\frac{1}{2}G(\phi)R_J + \frac{1}{2}\left(F + F_{,\chi}(R-\chi)\right) - \frac{1}{2}g_J^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V_J(\phi)\right] \tag{43}$$

$$= \int d^4x\sqrt{-g_J}\left[\frac{1}{2}\left(G + F_{,\chi}\right)R_J + \frac{1}{2}\left(F - \chi F_{,\chi}\right) - \frac{1}{2}g_J^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V_J(\phi)\right] \tag{44}$$

Consider conformal transformation

$$g_E = \Omega^2 g_J \qquad \Omega^2 = \frac{F_{,x} + G}{M_P^2} \tag{45}$$

Action becomes:

$$S = \int d^4x \sqrt{-g_E} \left[\frac{M_P^2}{2} R_E - 3M_P^2 g_E^{\mu\nu} \frac{\partial_\mu \Omega \partial_\nu \Omega}{\Omega^2} - \frac{1}{2\Omega^2} g_E^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{\Omega^4} \left(V_J(\phi) - \frac{1}{2}F + \frac{1}{2}\chi F_{,\chi} \right) \right] \tag{46}$$

We redefine:

$$\varphi = \sqrt{6} M_P \ln \Omega \,, \qquad \sigma = \phi \,, \qquad V_J(\varphi, \sigma) = V_J(\phi, \chi) = \frac{1}{\Omega^4} \left(V_J(\phi) - \frac{1}{2}F + \frac{1}{2}\chi F_{,\chi} \right) \tag{47}$$

Action becomes:

$$S = \int d^4x \sqrt{-g_E} \left[\frac{M_P^2}{2} R_E - \frac{1}{2} g_E^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} f(\varphi) g_E^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - V_E(\varphi, \sigma) \right] \tag{48}$$

5 Appendix

$$\begin{aligned} \tilde{R}^a{}_{bcd} &= R^a{}_{bcd} + \frac{1}{\Omega} \left[\delta^a_d \Omega_{;bc} - \delta^a_c \Omega_{;bd} + g_{bc} \Omega^{;a}{}_{;d} - g_{bd} \Omega^{;a}{}_{;c} \right] \qquad (49) \\ &+ \frac{2}{\Omega^2} \left[\delta^a_c \Omega_{,b} \Omega_{,d} - \delta^a_d \Omega_{,b} \Omega_{,c} + g_{bd} \Omega^{,a} \Omega_{,c} - g_{bc} \Omega^{,a} \Omega_{,d} \right] + \frac{1}{\Omega^2} \left[\delta^a_d g_{bc} - \delta^a_c g_{bd} \right] g_{ef} \Omega^{,e} \Omega^{,f} \quad , \\ R^a{}_{bcd} &= \tilde{R}^a{}_{bcd} - \frac{1}{\Omega} \left[\delta^a_d \Omega_{\tilde{;}bc} - \delta^a_c \Omega_{\tilde{;}bd} + \tilde{g}_{bc} \Omega^{\tilde{;}a}{}_{\tilde{;}d} - \tilde{g}_{bd} \Omega^{\tilde{;}a}{}_{\tilde{;}c} \right] \qquad (50) \\ &+ \frac{1}{\Omega^2} \left[\delta^a_d \tilde{g}_{bc} - \delta^a_c \tilde{g}_{bd} \right] \tilde{g}_{ef} \Omega^{,e} \Omega^{,f} \quad , \end{aligned}$$

$$\begin{aligned} \tilde{R}_{ab} &= R_{ab} + \frac{1}{\Omega^2} \left[2(D-2) \Omega_{,a} \Omega_{,b} - (D-3) \Omega_{,c} \Omega^{,c} g_{ab} \right] - \frac{1}{\Omega} \left[(D-2) \Omega_{;ab} + g_{ab} \Box \Omega \right] \,, \qquad (51) \\ R_{ab} &= \tilde{R}_{ab} - \frac{1}{\Omega^2} (D-1) \tilde{g}_{ab} \Omega_{,c} \Omega^{,c} + \frac{1}{\Omega} \left[(D-2) \Omega_{\tilde{;}ab} + \tilde{g}_{ab} \,\tilde{\Box}\, \Omega \right] \,, \qquad (52) \end{aligned}$$

$$\begin{aligned} \tilde{R} &= \Omega^{-2} \left[R - 2(D-1) \frac{\Box \Omega}{\Omega} - (D-1)(D-4) g^{ab} \frac{\Omega_{,a} \Omega_{,b}}{\Omega^2} \right] \,, \qquad (53) \\ R &= \Omega^2 \left[\tilde{R} + 2(D-1) \frac{\tilde{\Box}\, \Omega}{\Omega} - D(D-1) \tilde{g}^{ab} \frac{\Omega_{,a} \Omega_{,b}}{\Omega^2} \right] \,, \qquad (54) \end{aligned}$$

$$\begin{aligned} \tilde{G}_{ab} &= G_{ab} + \frac{D-2}{2\Omega^2} \left[4 \Omega_{,a} \Omega_{,b} + (D-5) \Omega_{,c} \Omega^{,c} g_{ab} \right] - \frac{D-2}{\Omega} \left[\Omega_{;ab} - g_{ab} \Box \Omega \right] \,, \qquad (55) \\ G_{ab} &= \tilde{G}_{ab} + \frac{D-2}{2\Omega^2} (D-1) \Omega_{,e} \Omega^{,e} \tilde{g}_{ab} + \frac{D-2}{\Omega} \left[\Omega_{\tilde{;}ab} - \tilde{g}_{ab} \,\tilde{\Box}\, \Omega \right] \,, \qquad (56) \end{aligned}$$

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