1 Trajector along valley

We consider

$$S = \int d^4x \sqrt{-g_{\mathrm{J}}} \left[\frac{M^2}{2} R_{\mathrm{J}} + \alpha R_{\mathrm{J}}^2 + \frac{1}{2}\xi\phi^2 R_{\mathrm{J}} - \frac{1}{2} g_{\mathrm{J}}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V_{\mathrm{J}}(\phi) \right] , \tag{1}$$

where the subscript J indicates that the quantities are given in the Jordan frame. In this work, we consider $V_{\mathrm{J}}(\phi) = (\lambda/4)(\phi^2 - v^2)^2$, with v being the vacuum expectation value of the ϕ field. One may Weyl-transform to the Einstein frame. The action in the Einstein-frame is then given by

$$S = \int d^4x \sqrt{-g_{\mathrm{E}}} \left[\frac{M_{\mathrm{P}}^2}{2} R_{\mathrm{E}} - \frac{1}{2} g_{\mathrm{E}}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - \frac{1}{2} g_{\mathrm{E}}^{\mu\nu} f(\chi) \partial_\mu \phi \partial_\nu \phi - V_{\mathrm{E}}(\phi, \chi) \right] , \tag{2}$$

with the subscript E indicating that we are in the Einstein frame. Here,

$$\chi = \frac{\sqrt{6}}{2} \ln \left(\frac{M + \xi\phi^2 + 4\alpha s}{M_P^2} \right) , \tag{3}$$

$$f(\chi) = e^{-\sqrt{\frac{2}{3}} \frac{\chi}{M_P}} , \tag{4}$$

$$V_{\mathrm{E}}(\phi, \chi) = \frac{1}{16} e^{-2\sqrt{\frac{2}{3}} \frac{\chi}{M_P}} \left[4\lambda(\phi^2 - v^2)^2 + \frac{M_P^4}{\alpha} \left(e^{\sqrt{\frac{2}{3}} \frac{\chi}{M_P}} - 1 - \frac{\xi(\phi^2 - v^2)}{M_P^2} \right)^2 \right] , \tag{5}$$

where s is the scalaron and ϕ is Higgs field. The potential is invariant under $\phi \to -\phi$. Considering (5), it has extrema

$$\frac{\partial V_{\mathrm{E}}}{\partial \phi} = e^{-\frac{2\sqrt{\frac{2}{3}}\chi}{M_p}} \left[\phi(\phi^2 - v^2)\lambda - \frac{\phi M_p^2 \xi \left(-1 + e^{\frac{\sqrt{\frac{2}{3}}\chi}{M_p}} - \frac{(\phi^2 - v^2)\xi}{M_p^2} \right)}{4\alpha} \right] \tag{6}$$

$$\phi = 0 \qquad \phi_{\min}^2 = M_P^2 \xi \frac{e^{\sqrt{\frac{2}{3}} \frac{\chi}{M_P}} - 1}{\xi^2 + 4\lambda\alpha} + v^2 \qquad \left. \frac{\partial^2 V_E}{\partial \phi^2} \right|_{\phi_{\min}^2} = e^{-\sqrt{\frac{8}{3}} \frac{\varphi}{M_P}} \left(2\lambda + \frac{\xi^2}{2\alpha} \right) \phi_{\min}^2 \tag{7}$$

We only consider ξ , λ and α are graet than 0. Therefore, $\phi_{\min}^2$ is minimum. It is not difficult to prove that $\phi = 0$ is max*. In this note, we only consider the field of real numbers.

$$V_E(\chi, \phi_{\min}^2) = \frac{\left(1 - e^{-\frac{\sqrt{\frac{2}{3}}\chi}{M_p}} \right)^2 M_p^4 \lambda}{4\left(4\alpha\lambda + \xi^2\right)} \tag{8}$$

In this note, we do not ignore v and consider positive parameters unless otherwise specified. Considering the slow roll along the valley, we get:

$$V_{,\phi} = 0 \to \phi_{\min}^2 = M_P^2 \xi \frac{e^{\sqrt{\frac{2}{3}} \frac{\chi}{M_P}} - 1}{\xi^2 + 4\lambda\alpha} + v^2 \tag{9}$$

$$V_E(\chi, \phi_{\min}^2) = \frac{\left(1 - e^{-\frac{\sqrt{\frac{2}{3}}\chi}{M_p}} \right)^2 M_p^4}{4\left(4\alpha + \xi^2/\lambda\right)} \tag{10}$$

where:

$$\begin{cases} 4\alpha\lambda \ll \xi^2 & \text{Higgs-Inlfation like} \\ 4\alpha\lambda \gg \xi^2 & R^2\text{- inlfation like} \end{cases} \tag{11}$$

Note, here we assume all parameters are positive. The potential is a Straobinsky-like slow roll. When ξ^2 dominates, there is no α in the potential at all, that is, it can be regarded as Higg-like inflation. R^2-like inflation is the opposite. See (32) and (34). Then we consider the A_s and perturbative constraint, Figure1 in Ema's paper is easy to reproduce. Scalaron can be expressed as:

$$s^2(\chi, \phi) = \frac{M_P^2}{4\alpha} \cdot \left(e^{\sqrt{\frac{2}{3}} \frac{\chi}{M_P}} - 1 - \frac{\xi\phi^2}{M_P^2} \right) \tag{12}$$

At the extrema (ignore v):

$$s^2(\chi,\phi^2_{\min}) = \frac{M_P^2\lambda}{4\alpha\lambda+\xi^2}\left(e^{\sqrt{\frac{2}{3}}\frac{\chi}{M_P}}-1\right) \qquad \frac{s^2(\chi,\phi^2_{\min})}{\phi^2_{\min}} = \frac{\lambda}{\xi} \tag{13}$$

$$s^2(\chi,0) = \frac{M_P^2}{4\alpha}\left(e^{\sqrt{\frac{2}{3}}\frac{\chi}{M_P}}-1\right) \qquad \frac{s^2(\chi,0)}{\phi^2=0} = \infty \tag{14}$$

Pure R^2 or χ inflation is available, but pure ϕ is not. (A_S and perturbative constraints that s^2/ϕ^2 cannot meet zero)

2 A_S and UV constraints

Amplitude of the scalar perturbation spectrum and the observational also constraint there parameters:

$$A_s = \frac{1}{24\pi^2 M_P^4}\cdot\frac{V}{\epsilon_V} = \frac{\lambda}{8\pi^2}\frac{\sinh\left(\frac{\chi_{\mathrm{CMB}}}{\sqrt{6}M_P}\right)^4}{4\alpha\lambda+\xi^2} \sim \frac{5.57}{4\alpha+\xi^2/\lambda} \sim 2\times10^{-9} \tag{15}$$

$$4\alpha+\xi^2/\lambda \sim 10^9 \tag{16}$$

The theory to be perturbative require:

$$\lambda+\frac{\xi^2}{4\alpha} < 4\pi \tag{17}$$

3 Negative parameters

Firstly, let's show basic equation:

$$\phi^2_{\min} = M_P^2\xi\frac{e^{\sqrt{\frac{2}{3}}\frac{\chi}{M_P}}-1}{\xi^2+4\lambda\alpha}+v^2 \sim \frac{\xi}{\lambda}\cdot(\text{nonegative}) \qquad \mathrm{V_E}(\chi,\phi^2_{\min}) = \frac{\left(1-\mathrm{e}^{-\frac{\sqrt{\frac{2}{3}}\chi}{\mathrm{M_P}}}\right)^2\mathrm{M_p^4}\lambda}{4\left(4\alpha\lambda+\xi^2\right)} \tag{18}$$

$$\left.\frac{\partial^2 V_E}{\partial\phi^2}\right|_{\phi^2_{\min}} = e^{-\sqrt{\frac{8}{3}}\frac{\varphi}{M_P}}\left(2\lambda+\frac{\xi^2}{2\alpha}\right)\phi^2_{\min} \sim \frac{\xi M_P^2}{2\alpha}e^{-\sqrt{\frac{8}{3}}\frac{\chi}{M_P}}\left(e^{\sqrt{\frac{2}{3}}\frac{\chi}{M_P}}-1\right) \tag{19}$$

$$V_{,\phi\phi}(\chi,0) = -\frac{1}{16}e^{-\sqrt{\frac{8}{3}}\frac{\chi}{M_P}}\left(16v^2\lambda+\frac{4M_P^2\xi\left(-1+e^{\sqrt{\frac{2}{3}}\frac{\chi}{M_P}}+\frac{v^2\xi}{M_P^2}\right)}{\alpha}\right) \sim -\frac{M_P^2}{4}\frac{\xi}{\alpha}e^{-\sqrt{\frac{8}{3}}\frac{\chi}{M_P}}\left(-1+e^{\sqrt{\frac{2}{3}}\frac{\chi}{M_P}}\right) \tag{20}$$

$$V(\chi,0) = \frac{M_P^4}{16\alpha}\left(1-e^{-\sqrt{\frac{2}{3}}\frac{\chi}{M_P}}\right)^2 \tag{21}$$

Wavy Lines means we temporarily assume that v can be ignored. In the process of deriving the above equations, we did not assume any parameter is negative or postive.

- $\xi,\lambda,\alpha > 0$. There are pure R^2 inflation, R^2-like inlfation and Higgs-like inflation, but no pure Higgs inflation.
- $\xi < 0$ and $\lambda,\alpha > 0$. For the quartic equation, mathematically speaking, it is reasonable to have complex numbers. In this case, $\phi^2_{\min}$ becomes maximum, $\phi = 0$ becomes minimum. See (21) which is nothing but the potential of pure R^2 Inflation.
- $\alpha < 0$, it's weird, Won't discuss it here.
- $\xi,\lambda < 0$ and $\alpha > 0$. $\phi^2_{\min}$ is maximum, $\phi = 0$ is minimum correspond to R^2 inflation. Some studies have pointed out after the inflation, these negative parameters lead to unstable electroweak vacuum. In order to prevent such a catastrophe, we should require $\phi < O(10)$

4 Initial values of inflation and preheating

4.1 Analytical calculation

Slow roll parameter ϵ can be expressed as:

$$\epsilon = \frac{M_P^2}{2}\left[\frac{(\partial_\chi V)^2 + f^{-1}(\partial_\phi V)^2}{V^2}\right] \sim \frac{M_P^2}{2}\frac{[\partial_\chi V(\chi, \phi_{\rm min}^2)]^2}{V^2(\chi, \phi_{\rm min}^2)} = \frac{4}{3}\left(e^{\sqrt{\frac{2}{3}}\frac{\chi}{M_P}} - 1\right)^{-2} \tag{22}$$

where one consider valley case, the $\partial_\phi V = 0$. It not difficult to get $\chi_{\rm end} \sim 0.94018 M_P$ at $\epsilon = 1$.** The number of e-folds before inflation end is:

$$N = \ln\frac{a_{\rm end}}{a} = \int_t^{t_{\rm end}} H {\rm d}t = \int_\chi^{\chi_{\rm end}} \frac{H}{\dot\chi}{\rm d}\chi \sim \frac{1}{M_P^2}\int_\chi^{\chi_{\rm end}} \frac{V}{V_{,\chi}} d\chi \sim \int_\chi^{\chi_{\rm end}} \frac{\sqrt{\frac{3}{2}}\left(e^{\sqrt{\frac{2}{3}}\frac{\chi}{M_P}} - 1\right)}{2M_P} d\chi \tag{23}$$

Note, we consider the valley case. We obtain $\chi_{\rm CMB}$ numerically by selecting $N = 60$. ***

$$\chi_{\rm CMB} \sim 5.45315 M_P \qquad \Delta\chi = 4.51297 M_P \tag{24}$$

One can easily get:

$$\Delta\phi = M_P\sqrt{\xi\frac{e^{\sqrt{\frac{2}{3}}\frac{\chi_{\rm CMB}}{M_P}} - 1}{\xi^2 + 4\lambda\alpha} + v^2} - M_P\sqrt{\xi\frac{e^{\sqrt{\frac{2}{3}}\frac{\chi_{\rm end}}{M_P}} - 1}{\xi^2 + 4\lambda\alpha} + v^2} \tag{25}$$

$$= \frac{\sqrt{85 M_P^2\xi + v^2(4\alpha\lambda + \xi^2)}}{\sqrt{4\alpha\lambda + \xi^2}} - \frac{\sqrt{1.154 M_P^2\xi + v^2(4\alpha\lambda + \xi^2)}}{\sqrt{4\alpha\lambda + \xi^2}} \tag{26}$$

4.2 Numerical calculation

Using our previous code, it is not difficult to get the values at the end of inflation. Let's set $\chi_i = 5.4 M_{\rm P}$, $\phi = \dot\phi = \dot\chi = 0$, $\xi^2 = 5\times 10^6$, $\alpha = 2\times 10^9$, $v = 0$ and $\lambda = 0.01$. which corresponds to R^2 inflation. We get:

Variables	$N_e = 0$	$N_e - 0.5$	$N_e - 1$
ϕ_e	0	0	0
χ_e	0.6163	1.142	1.483
$\dot\chi_e(\chi'_e)$	$-4.4\times 10^{-6}(-1.411)$	$-3.362\times 10^{-6}(-0.8104)$	$-2.7\times 10^{-6}(-0.579)$
$\dot\phi_e(\phi'_e)$	0	0	0

Table 1: intial conditions, we omit the $M_{\rm P}$, and the dot represents the derivative with respect to time, and the prime represents the derivative with respect to e-folds.

This is different from our theoretical calculations, and it seems quite large. This is because we completely ignored the kinetic energy in our theoretical calculations. However, if you look at Fig1a, you will find that the difference is only about 0.1 e-fold, so we think both are correct, but the numerical solution will be more accurate. Note, in the calculation examples above, we set $v = 0$. For other values, one can use my code Higgs-R2.ipynb to calculate.

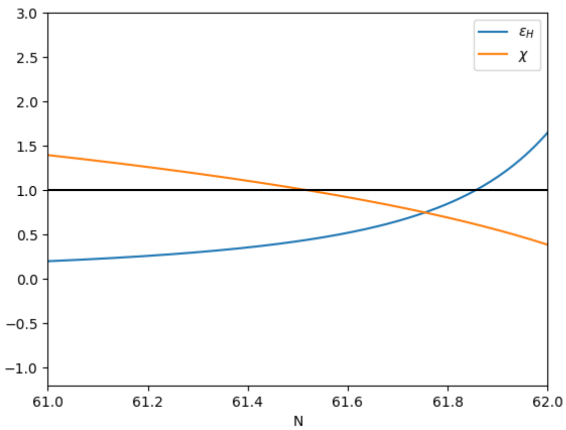
Different parameter selections(λ , ξ , α) will not affect the initial values, becasue the coefficient of potential will be eliminated.

5 Appendix

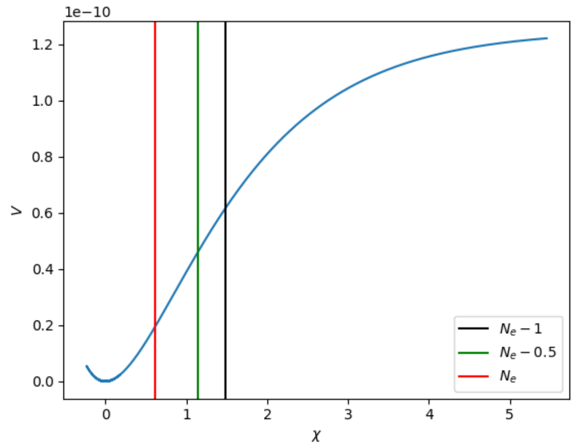
5.1 The derivative of V

$$\frac{\partial V_{\rm E}}{\partial\phi} = e^{-\frac{2\sqrt{\frac{2}{3}}\chi}{M_p}}\left[\phi(\phi^2 - v^2)\lambda - \frac{\phi M_p^2\xi\left(-1 + e^{\frac{\sqrt{\frac{2}{3}}\chi}{M_p}} - \frac{(\phi^2 - v^2)\xi}{M_p^2}\right)}{4\alpha}\right] \tag{27}$$

$$V_{,\chi} = 0 \quad\to\quad \chi_{\rm min} = \sqrt{\frac{3}{2}} M_P \ln\left[\frac{M_P^4 + (4\alpha\lambda + \xi^2)(\phi^2 - v^2)^2 + 2M_P^2\xi(\phi^2 - v^2)}{M_P^4 + M_P^2\xi(\phi^2 - v^2)}\right] \tag{28}$$



(a) When slow parameter ϵ meet one



(b) The point at the end of inflation N_e

$$V_{,\chi\chi}(\chi_{\min}, \phi) = \frac{\left(M_P^3 + M_P\xi(\phi^2 - v^2)\right)^2}{12\alpha\left(M_P^4 + (4\alpha\lambda + \xi^2)(\phi^2 - v^2)^2 + 2M_P^2\xi(\phi^2 - v^2)\right)} \tag{29}$$

5.2 Higgs and R^2 potential form

R^2 inflation action is

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R_J + \alpha R_J^2\right] \tag{30}$$

Introducing auxiliary fields $s^2 = R_J$ and transform to Einstein frame:

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{M_P^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - V(\chi)\right] \tag{31}$$

$$\chi = \frac{\sqrt{6}}{2}\ln\left(1 + \frac{4\alpha s^2}{M_P^2}\right) \qquad V(\chi) = \frac{M_P^4}{16\alpha}\left(1 - e^{-\sqrt{\frac{2}{3}}\frac{\chi}{M_P}}\right)^2 \tag{32}$$

Higgs inflation action is

$$S = \int d^4x\sqrt{-g}\left[\frac{M}{2}R_J + \xi\phi^2R_J - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V_J(\phi)\right] \qquad V_J = \frac{\lambda}{4}(\phi^2 - v^2)^2 \tag{33}$$

Transform to Einstein frame and redefine the field:

$$\chi = \frac{\sqrt{6}}{2}M_P\ln\left[1 + \frac{\xi(\phi^2 - v^2)}{M_P^2}\right] \qquad V(\chi) = \frac{\lambda M_P^4}{4\xi^2}\left(1 - \frac{1}{M_P^2}e^{-\sqrt{2/3}\chi/M_P}\right)^2 \tag{34}$$

There has some assumptions, read privious work for detials.