Exercise 5.2.3. (a) Use Definition 5.2.1 to produce the proper formula for the derivative of $h(x) = 1/x$.

(b) Combine the result in part (a) with the Chain Rule (Theorem 5.2.5) to supply a proof for part (iv) of Theorem 5.2.4.

(c) Supply a direct proof of Theorem 5.2.4 (iv) by algebraically manipulating the difference quotient for (f/g) in a style similar to the proof of Theorem 5.2.4 (iii).

Exercise 5.3.1. Recall from Exercise 4.4.9 that a function $f : A \to \mathbf{R}$ is Lipschitz on A if there exists an $M > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq M$$

for all $x \neq y$ in A.

(a) Show that if f is differentiable on a closed interval $[a, b]$ and if f' is continuous on $[a, b]$, then f is Lipschitz on $[a, b]$.

(b) Review the definition of a contractive function in Exercise 4.3.11. If we add the assumption that $|f'(x)| < 1$ on $[a, b]$, does it follow that f is contractive on this set?

Exercise 5.3.2. Let f be differentiable on an interval A. If $f'(x) \neq 0$ on A, show that f is one-to-one on A. Provide an example to show that the converse statement need not be true.

Exercise 6.2.2. (a) Define a sequence of functions on $\mathbf{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

and let f be the pointwise limit of f_n.

Is each f_n continuous at zero? Does $f_n \to f$ uniformly on $\mathbf{R}$? Is f continuous at zero?

Exercise 6.3.1. Consider the sequence of functions defined by

$$g_n(x) = \frac{x^n}{n}.$$

(a) Show (g_n) converges uniformly on $[0, 1]$ and find $g = \lim g_n$. Show that g is differentiable and compute $g'(x)$ for all $x \in [0, 1]$.

(b) Now, show that (g'_n) converges on $[0, 1]$. Is the convergence uniform? Set $h = \lim g'_n$ and compare h and g'. Are they the same?

Exercise 6.4.5. (a) Prove that

$$h(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \cdots$$

is continuous on $[-1, 1]$.

(b) The series

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

converges for every x in the half-open interval $[-1, 1)$ but does not converge when $x = 1$. For a fixed $x_0 \in (-1, 1)$, explain how we can still use the Weierstrass M-Test to prove that f is continuous at x_0.

Exercise 6.6.5. (a) Generate the Taylor coefficients for the exponential function $f(x) = e^x$, and then prove that the corresponding Taylor series converges uniformly to e^x on any interval of the form $[-R, R]$.

(b) Verify the formula $f'(x) = e^x$.

(c) Use a substitution to generate the series for e^{-x}, and then informally calculate $e^x \cdot e^{-x}$ by multiplying together the two series and collecting common powers of x.

Exercise 6.6.6. Review the proof that $g'(0) = 0$ for the function

$$g(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

introduced at the end of this section.

(a) Compute $g'(x)$ for $x \neq 0$. Then use the definition of the derivative to find $g''(0)$.

(b) Compute $g''(x)$ and $g'''(x)$ for $x \neq 0$. Use these observations and invent whatever notation is needed to give a general description for the nth derivative $g^{(n)}(x)$ at points different from zero.

(c) Construct a general argument for why $g^{(n)}(0) = 0$ for all $n \in \mathbf{N}$.