3.2.7

b) Assume that x is a limit pt of $A \cup L$ but x is not a limit pt of A.

$\exists$ sequence $(x_n) \subseteq A \cup L \setminus \{x\}$, s.t. $(x_n) \to x$,

but $\exists\, \varepsilon > 0$, s.t. $\mathring{V}_\varepsilon(x) \cap A = \phi$.

$\exists\, N \in \mathbb{N}$, s.t. $\forall\, n \geq N$, $x_n \in \mathring{V}_\varepsilon(x)$.

$x_n \notin A$ and hence $x_n \in L \setminus \{x\}$

$(x_N, x_{N+1}, \cdots) \subseteq L \setminus \{x\}$ converges to x,

x is a limit pt of L.

From 3.2.7 a), L is closed. So $x \in L$ and hence is a limit pt of A. Contradiction.

Now if x is a limit pt of $A \cup L$, then $x \in L \subseteq A \cup L$.

So $\overline{A} = A \cup L$ is closed.

3.2.11

a) Since $A \subseteq A \cup B$ and limit points of A are also limit points of $A \cup B$, $\bar{A} \subseteq \overline{A \cup B}$

Same argument shows $\bar{B} \subseteq \overline{A \cup B}$, and hence $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$.

By Thm 3.2.12, $\bar{A}$ and $\bar{B}$ are closed so $\bar{A} \cup \bar{B}$ is closed.

Also $A \subseteq \bar{A}$ and $B \subseteq \bar{B}$, hence $A \cup B \subseteq \bar{A} \cup \bar{B}$.

Since $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$, we have

$\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$.

So $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

b) No. Let $A_n = \{\frac{1}{n}\}$. $\bigcup_{n=1}^{\infty} \bar{A}_n = \{\frac{1}{n} : n \in \mathbb{N}\}$ and

$\overline{\bigcup_{n=1}^{\infty} A_n} = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$.

3.3.1

Proof. Since $K \neq \phi$ is compact, it is bounded and closed.

By AoC, $\alpha = \sup K$, $\beta = \inf K$ both exist.

$\forall n \in \mathbb{N}$, $\exists x_n \in K$ s.t. $\alpha - \frac{1}{n} < x_n \leq \alpha$.

$(x_n) \subseteq K$, $(x_n) \to \alpha$, so $\alpha \in K$ since K is closed.

$\forall n \in \mathbb{N}$, $\exists y_n \in K$ s.t. $\beta \leq y_n < \beta + \frac{1}{n}$.

$(y_n) \subseteq K$, $(y_n) \to \beta$, so $\beta \in K$ since K is closed.

3.3.5

a) True. Arbitrary intersection of closed set is closed, arbitrary intersection of bounded set is bounded.

b) False. $A_n = [n, n+1]$, $n \in \mathbb{N}$

$$\bigcup_{n=1}^{\infty} A_n = [1, \infty).$$

c) False. $A = (0,1)$, $K = [0,1]$, $A \cap K = (0,1)$.

d) False. $F_n = [n, \infty)$, $\bigcap_{n=1}^{\infty} F_n = \phi$.

4.2.5

c) $|(x^2+x-1)-5| = |x^2-x-6| = |x+3||x-2|$

$|x-2| < 1 \Rightarrow 1 < x < 3 \Rightarrow |x+3| < 6$.

$\forall \varepsilon > 0$, take $\delta = \min\{1, \frac{\varepsilon}{6}\}$, if $0 < |x-2| < \delta$,

then $|(x^2+x-1)-5| = |x+3||x-2| < 6 \cdot \frac{\varepsilon}{6} = \varepsilon$.

So $\lim_{x \to 2} (x^2+x-1) = 5$.

4.3.3

a) g is cts at $f(c) \in B$.

$\forall \varepsilon > 0, \exists \alpha > 0$ s.t. $|y - f(c)| < \alpha \Rightarrow |g(y) - g(f(c))| < \varepsilon$.

f is cts at $c \in A$.

For this $\alpha > 0$, $\exists \delta > 0$, s.t. $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \alpha$.

Now $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \alpha \Rightarrow |g(f(x)) - g(f(c))| < \varepsilon.$$

So $g \circ f$ is cts at c.

4.3.9

Proof. Let a be a limit pt of K.

$\exists$ sequence $(x_n) \subseteq K$ s.t. $(x_n) \to a$.

$\forall n \in \mathbb{N}$, $x_n \in K$, $h(x_n) = 0$.

Since $h(x)$ is cts on $\mathbb{R}$, $h(a) = h(\lim x_n) = \lim h(x_n) = 0$,

$a \in K$.

K contains all limit pts of K, so K is closed.

4.4.3

Proof. $\left|f(x)-f(y)\right| = \left|\frac{1}{x^2}-\frac{1}{y^2}\right| = \left|\frac{y^2-x^2}{x^2y^2}\right| = |y-x|\left|\frac{y+x}{x^2y^2}\right|$

If $x,y \in [1,\infty)$, $\left|\frac{y+x}{x^2y^2}\right| = \frac{1}{x^2y} + \frac{1}{xy^2} \leq 1+1 = 2.$

$\forall \varepsilon > 0$, take $\delta = \frac{\varepsilon}{2} > 0$, then

$$|x-y| < \delta \Rightarrow |f(x)-f(y)| < \delta \cdot 2 = \frac{\varepsilon}{2}\cdot 2 = \varepsilon.$$

So $f(x)$ is uniformly cts on $[1,\infty)$

Take $(x_n) = \left(\frac{1}{\sqrt{n}}\right)$ and $(y_n) = \left(\frac{1}{\sqrt{n+1}}\right)$, then

$|x_n - y_n| = \left|\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right| \to 0$ but

$|f(x_n) - f(y_n)| = |n-(n+1)| = 1.$

So $f(x)$ is not uniformly cts on $(0,1]$.

4.4.12

a), b), c) are all false.

Take $f(x) = 0$ cts on $\mathbb{R}$.

$\{0\}$ is finite, compact, bounded,

but $f^{-1}(\{0\}) = \mathbb{R}$ is not finite, not compact and not bounded.

d) True. For f cts on $\mathbb{R}$ and F closed,

let x be a limit pt of $f^{-1}(F)$, then

$\exists$ sequence $(x_n) \subseteq f^{-1}(F)$ s.t. $(x_n) \to x$.

The sequence $(f(x_n)) \subseteq F$ and since f is cts,

$$\lim f(x_n) = f(\lim x_n) = f(x).$$

So $f(x)$ is a limit pt of F, $f(x) \in F$ since F is closed. So $x \in f^{-1}(F)$.

$f^{-1}(F)$ is closed since it contains all limit pts of itself.

Note: We can also use the result in Ex 4.4.11 and the fact that A is open $\Leftrightarrow$ A^c is closed.

4.5.7

Proof. Let $g(x) = f(x) - x$, $g(x)$ is cts on $[0,1]$

and $g(0) = f(0) \geqslant 0$

$g(1) = f(1) - 1 \leqslant 0$ since $f(0), f(1) \in [0,1]$.

If $g(0) = 0$, $f(0) = 0$. If $g(1) = 0$, $f(1) = 1$.

If $g(0) > 0$ and $g(1) < 0$, by IVT, $\exists\, c \in (0,1)$ s.t.

$g(c) = 0$ and hence $f(c) = c$.