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3.2.7 (b)

① Suppose x is a limit point of $A \cup L$. Find a sequence (a_n) form $A \cup L$, s.t. $(a_n) \to x$. (a_n) will either contains infinitely many terms from A or L.

If contains infinitely many terms from A, we can find (a_{n_k}) in A. $(a_n) \to x$ implies $(a_{n_k}) \to x$. So x is a limit point of A.

If contains infinitely many terms from L, we can find (a_{n_k}) in L, $(a_n) \to x$ implies $(a_{n_k}) \to x$. And L is closed. So $x \in L$, which means x is a limit point of A.

② From above, we know that $\bar{A} = A \cup L$ contains all the limit points of itself, so $\bar{A}$ is closed set.

Now, any closed set containing A must contain L as well. (If it does not contain any value k in L, then there is a sequence (a_n) in A s.t. $(a_n) \to k$ and k not in this closed set, which is a contradiction)

This shows that $\bar{A} = A \cup L$ is the smallest closed set containing A.

3.2.11

(a) We know that $A \subseteq A \cup B$ so $\bar{A} \subseteq \overline{A \cup B}$, similarly $\bar{B} \subseteq \overline{A \cup B}$, therefore, $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$.

Let L be all the limit points of $A \cup B$. For $\forall x \in \overline{A \cup B}$.

If $x \in A \cup B$, then $x \in \bar{A} \cup \bar{B}$.

If $x \in L$, then there is (c_n) in $A \cup B$ s.t. $(c_n) \to x$. Either A or B contains infinitely many terms of (c_n). In the first case, there is $(c_{n_k}) \to x \in \bar{A}$; in the second case, there is $(c_{n_k}) \to x \in \bar{B}$.

$\therefore \overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$

Therefore, $\bar{A} \cup \bar{B} = \overline{A \cup B}$

(b) Let $A_n = [\frac{1}{n}, 1]$, when $n \geq 2$, $\bar{A}_n = [\frac{1}{n}, 1]$

$\bigcup_{n=2}^{\infty} \bar{A}_n = (0, 1]$, $\bigcup_{n=2}^{\infty} A_n = (0, 1]$

$\overline{\bigcup_{n=2}^{\infty} A_n} = [0, 1]$ $\therefore \bigcup_{n=2}^{\infty} \bar{A}_n \neq \overline{\bigcup_{n=2}^{\infty} A_n}$. The result does not hold.

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3.3.1

If K is compact $\Leftrightarrow$ K is bounded and closed. By A.C, $\sup K$ and $\inf K$ both exist.

Suppose $a = \sup K$, $b = \inf K$. $\forall \varepsilon > 0$, $\exists\, k \in K$, s.t. $a - \varepsilon < k$. which means $\mathring{V}_\varepsilon(a) \cap A \neq \emptyset$. Therefore, "$a$" is a limit point of K. And K is closed, so $a = \sup K \in K$.

Similarly, $\forall \varepsilon > 0$, $\exists\, k \in K$, s.t. $b + \varepsilon > k$, which means: $\mathring{V}_\varepsilon(b) \cap A \neq \emptyset$. $\therefore$ b is a limit point of K. And $b \in K$.

Therefore, $\sup K$ and $\inf K$ are elements in K.

3.3.5

(a) True; First, arbitrary intersections of closed sets are closed. Second, arbitrary intersection of bounded sets are bounded because the elements of the intersections are in every set. Thus, arbitrary ~~etc~~ intersections of closed and bounded sets are closed and bounded.

(b) If $A_n = [0, n]$ $\quad \bigcup_{n=1}^{\infty} A_n = [0, +\infty)$

which is not compact. False!

(c) If $A = (0, 1)$, $K = [0, 1]$. $\quad A \cap K = (0, 1)$, not compact.
False!

(d) False. If $F_n = [n, +\infty)$ $\quad \bigcap_{n=1}^{\infty} F_n = \emptyset$.

For $\forall n$, $n \notin F_{n+1}$. Therefore, the statement is false.

4.2.5 (c)

$g(x) = x^2 + x - 1$, want to prove $\lim_{x \to 2} g(x) = 5$

$|g(x) - 5| = |x^2 + x - 6| = |x+3||x-2|$ $\quad \forall \varepsilon > 0$,

Choose $\delta = \min\{1, \varepsilon/6\}$. Then $|x+3| \leq |3+3| = 6$. If $0 < |x-2| < \delta$

$|x+3||x-2| < 6 \times \varepsilon/6 = \varepsilon$

Now the limit is proved.

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4.3.3 (a)

g is cts at $f(c) \in B$: $\forall \varepsilon > 0$, $\exists \lambda > 0$, s.t. $0 < |f(x) - f(c)| < \lambda$ and $f(x) \in B$
$\Rightarrow |g(f(x)) - g(f(c))| < \varepsilon$
f is cts at $c \in A$: $\forall \lambda > 0$, $\exists \delta > 0$, s.t. $0 < |x - c| < \delta$ and $x \in A$
$\Rightarrow |f(x) - f(c)| < \lambda$.
Therefore, $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $0 < |x - c| < \delta$ and $x \in A \Rightarrow$
$|g(f(x)) - g(f(c))| < \varepsilon$, then $g \circ f$ is cts at c.

4.3.9

h is cts in $\mathbb{R}$. $\forall (a_n) \subseteq \mathbb{R}$ s.t. $(a_n) \to c$, $\lim f(a_n) = f(c) = f(\lim a_n)$
If $k \notin K$, and k is a limit point of K, then
$\exists (a_n) \subseteq K \setminus \{k\}$, s.t. $(a_n) \to k$. $\lim h(a_n) = 0 = h(k)$, which means
$k \in K$. Contradiction! (Also, if no limit pts, it is definitely closed)
$\therefore$ K contains all of its limit points, K is closed.

4.4.3

Suppose $(x_n)(y_n) \in (0,1]$ $x_n = \frac{2}{n}$, $y_n = \frac{1}{n}$ $n \in \mathbb{N}$
$|x_n - y_n| = \frac{1}{n} \to 0$, $|f(x_n) - f(y_n)| = |\frac{n^2}{4} - n^2| = \frac{3}{4} n^2 \geq \frac{3}{4}$

Therefore, f is not uniformly continuous on $(0, 1]$

$|f(x) - f(y)| = |\frac{1}{x^2} - \frac{1}{y^2}| = \frac{|x^2 - y^2|}{x^2 y^2}$ $x, y \in [1, +\infty)$, $\frac{1}{xy^2} < 1$, $\frac{1}{x^2 y} < 1$
$\forall \varepsilon > 0$, Choose $\delta = \varepsilon/2$. $\therefore$ $|x - y| < \delta$
$$\frac{|x^2 - y^2|}{x^2 y^2} = |x - y| \frac{x + y}{x^2 y^2} < |x - y| \left(\frac{1}{xy^2} + \frac{1}{x^2 y} \right) < 2|x - y| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$$

Therefore, f is uniformly continuous on $[1, +\infty)$

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4.4.12

If $f(x) = 1$, then (a)(b)(c) is false:

(a) $B = \{1\}$, which is finite; If $f: \mathbb{N} \to \{1\}$, then $f^{-1}(B)$ is infinite.
False

(b) $K = \{1\}$, which is compact; If $f: \mathbb{R} \to \{1\}$, then $f^{-1}(K)$ is not compact
False

(c) $A = \{1\}$, which is bounded; If $f: \mathbb{R} \to \{1\}$, then $f^{-1}(A)$ is unbounded.

(d) True; f is continuous; If F is closed, then F^c is open.
By continuity, $f^{-1}(F^c)$ is open. We now show $f^{-1}(F^c) = f^{-1}(F)^c$
$\forall a \in f^{-1}(F^c)$: $f(a) \notin F \Rightarrow a \notin f^{-1}(F) \Rightarrow a \in f^{-1}(F)^c$
$\forall a \in f^{-1}(F)^c$, $a \notin f^{-1}(F) \Rightarrow f(a) \notin F \Rightarrow a \in f^{-1}(F^c)$
Therefore, $f^{-1}(F^c) = f^{-1}(F)^c$. $f^{-1}(F)^c$ is open.
$f^{-1}(F)$ is closed whenever F is closed.

4.5.7

Suppose $g(x) = f(x) - x$. If $f(0) = 0$ or $f(1) = 1$, then 0 or 1 is fixed point.

If $f(0) \neq 0$, then $f(0) > 0$; If $f(1) \neq 1$, $f(1) < 1$.
$\therefore g(0) = f(0) > 0$; $g(1) = f(1) - 1 < 0$
$g(x): [0,1] \to R$ $g(1) < 0 < g(0)$, by IVP, $\exists\, c \in [0,1]$ s.t.
$g(c) = 0$, which means $g(c) = f(c) - c = 0$.
So, $f(x) = x$ must have a fixed point.