5.2.3 c)

$$\frac{\left(\frac{f}{g}\right)(x)-\left(\frac{f}{g}\right)(c)}{x-c} = \frac{1}{x-c}\left(\frac{f(x)}{g(x)}-\frac{f(c)}{g(c)}\right)$$

$$= \frac{1}{x-c}\cdot\frac{f(x)\cdot g(c)-f(c)\,g(x)}{g(x)\cdot g(c)}$$

$$= \frac{1}{g(x)g(c)}\cdot\frac{f(x)g(c)-f(c)g(c)+f(c)g(c)-f(c)g(x)}{x-c}$$

$$= \frac{1}{g(x)g(c)}\cdot\left(\frac{f(x)-f(c)}{x-c}\cdot g(c)-f(c)\cdot\frac{g(x)-g(c)}{x-c}\right)$$

Take limit as $x\to c$ and use Algebraic Limit Thm, we have

$$\left(\frac{f}{g}\right)'(c) = \frac{1}{[g(c)]^2}\left(f'(c)\cdot g(c)-f(c)\cdot g'(c)\right)$$

Note that we also used $\lim_{x\to c} g(x)=g(c)$ as g is diff. at c implies g is cts at c.

5.3.1

a) Since f' is cts on compact set $[a,b]$, $\exists M>0$ s.t.

$\forall c \in [a,b]$, $|f'(c)| \leq M$.

Now $\forall a \leq x < y \leq b$, use MVT for f on $[x,y]$, $\exists c \in (x,y)$,

s.t. $$\frac{f(x)-f(y)}{x-y} = f'(c)$$

$$\left|\frac{f(x)-f(y)}{x-y}\right| = |f'(c)| \leq M.$$

So f is Lipschitz on $[a,b]$.

b) f' is cts on compact set $[a,b]$, it attains both max and min on $[a,b]$. So $\exists x_0 \in [a,b]$, s.t. $\forall c \in [a,b]$,

$$|f'(c)| \leq |f'(x_0)| < 1.$$

Again by MVT, $\forall a \leq x < y \leq b$,

$$\left|\frac{f(x)-f(y)}{x-y}\right| = |f'(c)| \leq |f'(x_0)| < 1.$$

$$|f(x)-f(y)| \leq |f'(x_0)|\,|x-y| \quad , |f'(x_0)| < 1.$$

f is contractive.

5.3.2

Assume f is not 1-to-1, $\exists$ $x<y$ in A s.t.

$$f(x)=f(y).$$

f is cts on $[x,y]$ and diff on (x,y), by Rolle's Thm, $\exists$ $c\in(x,y)\subseteq A$, s.t. $f'(c)=0$. Contradiction.

So f is 1-to-1 on A.

$f(x)=x^3$ is 1-to-1 on $[-1,1]$, but $f'(0)=0$.

6.2.1

a) $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{\frac{1}{n}+x^2} = \frac{1}{x}$.

The pointwise limit of $f_n(x)$ is $f(x) = \frac{1}{x}$.

b) The convergence is not uniform on $(0,\infty)$.

$$|f_n(x) - f(x)| = \left|\frac{nx}{1+nx^2} - \frac{1}{x}\right| = \frac{1}{x+nx^3}$$

$\forall \varepsilon > 0$, $\frac{1}{x+nx^3} < \varepsilon \iff n > \frac{1-\varepsilon x}{\varepsilon x^3}$.

As $\frac{1-\varepsilon x}{\varepsilon x^3}$ is not bounded as $x \to 0$, there is no fixed $N \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ and $x \in (0,\infty)$.

c) Use the same argument in b), the convergence is not uniform on $(0,1)$.

d) The convergence is uniform on $(1,\infty)$, as

$$|f_n(x) - f(x)| = \frac{1}{x+nx^3} < \frac{1}{n+1} \quad \text{on } (1,\infty).$$

$\forall \varepsilon > 0$, take $N \in \mathbb{N}$ s.t. $\frac{1}{N+1} < \varepsilon$, then $\forall n \geq N$ and $x \in (1,\infty)$, $|f_n(x) - f(x)| < \frac{1}{n+1} < \varepsilon$.

6.2.2 a)

$$f(x) = \begin{cases} 1, & x \in \{\frac{1}{n} : n \in \mathbb{N}\} \\ 0, & \text{otherwise.} \end{cases}$$

Each f_n is cts at 0 as $f_n(x) = 0$ on $(-\infty, \frac{1}{n})$.

The convergence $f_n \to f$ is not uniform on $\mathbb{R}$.

Take $\varepsilon = 1$, $\forall\ k \in \mathbb{N}$, $|f_n(\frac{1}{k}) - f(\frac{1}{k})|$

$$= |f_n(\tfrac{1}{k}) - 1| < 1 \iff n \geq k.$$

There is no fixed $N \in \mathbb{N}$ s.t. $\forall\ n \geq N$, $k \in \mathbb{N}$,

$$|f_n(\tfrac{1}{k}) - f(\tfrac{1}{k})| < \varepsilon.$$

f is not cts at 0 since $\lim \frac{1}{n} = 0$, $\lim\limits_{n \to \infty} f(\frac{1}{n}) = 1 \neq 0 = f(0)$.
Actually this shows that $f_n \to f$ is not uniform on $\mathbb{R}$.

6.3.1

a) $\forall x \in [0,1]$, $|g_n(x) - 0| = \left|\frac{x^n}{n}\right| \leq \frac{1}{n}$.

$\forall \varepsilon > 0$, take $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$.

$\forall n \geq N$, $x \in [0,1]$, $|g_n(x) - 0| \leq \frac{1}{n} < \varepsilon$.

$g_n(x)$ converges to $g(x) = 0$ uniformly on $[0,1]$.

$g(x) = 0$ is diff and $g'(x) = 0$ on $[0,1]$.

b) $g_n'(x) = x^{n-1}$.

$$h(x) = \lim_{n \to \infty} g_n'(x) = \begin{cases} 0, & x \in [0,1). \\ 1, & x = 1 \end{cases}.$$

$h(x) \neq g'(x)$ at $x = 1$.

The convergence $g_n' \to h$ is not uniform on $[0,1]$ as each g_n' is cts on $[0,1]$ but h is not.

6.4.5

a) $\forall x \in [-1,1]$, $\left|\frac{x^n}{n^2}\right| \le \frac{1}{n^2}$.

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by M-Test,

$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges uniformly on $[-1,1]$.

As each $\frac{x^n}{n^2}$ is cts on $[-1,1]$,

$h(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is cts on $[-1,1]$.

b) For fixed $x_0 \in (-1,1)$, take c s.t. $|x_0| < c < 1$.

$\forall x \in [-c,c]$, $\left|\frac{x^n}{n}\right| \le \frac{c^n}{n} \le c^n$.

Since $\sum_{n=1}^{\infty} c^n$ converges, by M-Test,

$\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges uniformly on $[-c,c]$,

so $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ is cts on $[-c,c]$ as each $\frac{x^n}{n}$ is,

$f(x)$ is cts at x_0 since $x_0 \in [-c,c]$.

6.5.3

Take $M_n = |a_n x_0^n|$, as $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at x_0, $\sum_{n=0}^{\infty} M_n$ converges.

Now $\forall x \in [-c, c]$, $|a_n x^n| \leq |a_n x_0^n| = M_n$.

By M-Test, $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-c, c]$.

6.6.5

a) $\forall\, n \geq 0,\ f^{(n)}(x) = e^x,\ a_n = \frac{e^0}{n!} = \frac{1}{n!}$.

The Taylor series is $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

The radius of convergence is $\lim_{n\to\infty} \left| \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} \right| = \lim_{n\to\infty} (n+1) = \infty$.

To show that the series converges to $f(x) = e^x$ uniformly on $[-R, R]$, use Lagrange's Remainder, $\forall\, x \in [-R, R]$,

$$|E_N(x)| = \left| \frac{e^c}{(N+1)!} x^{N+1} \right| \leq \frac{e^R}{(N+1)!} \cdot R^{N+1}$$

Note that $\sum_{N=0}^{\infty} \frac{R^{N+1}}{(N+1)!}$ converges so $\lim_{N\to\infty} \frac{R^{N+1}}{(N+1)!} = 0$.

$\forall\, \varepsilon > 0,\ \exists\, N_1 \in \mathbb{N}$ s.t. $\forall\, N \geq N_1,\ \frac{R^{N+1}}{(N+1)!} < e^{-R} \cdot \varepsilon$.

$\forall\, N \geq N_1,\ x \in [-R, R],\ |E_N(x)| \leq e^R \cdot \frac{R^{N+1}}{(N+1)!} < \varepsilon$.

$E_N(x) \to 0$ uniformly on $[-R, R]$, $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to $f(x) = e^x$ uniformly on $[-R, R]$.

b) $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Differentiate term by term to get

$$(e^x)' = 0 + 1 + 2\cdot\frac{x}{2!} + 3\cdot\frac{x^2}{3!} + \cdots$$
$$= 1 + x + \frac{x^2}{2!} + \cdots = e^x.$$

c) $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$e^x \cdot e^{-x} = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)$$
$$= 1 + (1-1)x + \left(\frac{1}{2!} + \frac{1}{2!} - 1\right)x^2 + \left(\frac{1}{2!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{3!}\right)x^3 + \cdots$$
$$= 1 + 0x + 0x^2 + 0x^3 + \cdots = 1$$

6.6.6

a) $g'(x) = \frac{2}{x^3} e^{-\frac{1}{x^2}}$, $x \neq 0$

$g'(0) = 0$

$$g''(0) = \lim_{x \to 0} \frac{\frac{2}{x^3} e^{-\frac{1}{x^2}} - 0}{x - 0} = \lim_{x \to 0} \frac{2}{x^4} e^{-\frac{1}{x^2}} = \lim_{x \to 0} \frac{\frac{2}{x^4}}{e^{\frac{1}{x^2}}}$$

$$\stackrel{L'H}{=} \lim_{x \to 0} \frac{-\frac{8}{x^5}}{-\frac{2}{x^3} e^{\frac{1}{x^2}}}$$

$$= \lim_{x \to 0} \frac{\frac{4}{x^2}}{e^{\frac{1}{x^2}}}$$

$$\stackrel{L'H}{=} \lim_{x \to 0} \frac{-\frac{8}{x^3}}{-\frac{2}{x^3} e^{\frac{1}{x^2}}}$$

$$= \lim_{x \to 0} \frac{4}{e^{\frac{1}{x^2}}} = 0$$

b) $g''(x) = \left(\frac{4}{x^6} - \frac{6}{x^4} \right) e^{-\frac{1}{x^2}}$, $x \neq 0$

$g'''(x) = \left(\frac{8}{x^9} - \frac{36}{x^7} + \frac{24}{x^5} \right) e^{-\frac{1}{x^2}}$, $x \neq 0$.

$g^{(n)}(x) = \sum_{k=1}^{3n} \frac{C_{n,k}}{x^k} \cdot e^{-\frac{1}{x^2}}$, $x \neq 0$.

c) $g'(0) = g''(0) = 0$, assume $g^{(n)}(0) = 0$,

$$g^{(n+1)}(0) = \lim_{x \to 0} \frac{g^{(n)}(x) - 0}{x - 0}$$

$$= \lim_{x \to 0} \sum_{k=1}^{3n} \frac{C_{n,k}}{x^{k+1}} \cdot e^{-\frac{1}{x^2}}$$

To show $g^{(n+1)}(0) = 0$, we prove that

$\forall m \in \mathbb{N} \cup \{0\}$, $\lim_{x \to 0} \frac{1}{x^m} \cdot e^{-\frac{1}{x^2}} = 0$.

The statement is true for $m = 0, 1$, now assume $\lim_{x \to 0} \frac{1}{x^m} \cdot e^{-\frac{1}{x^2}} = 0$ for all $0 \le m \le N$, $N \ge 1$

$$\lim_{x \to 0} \frac{1}{x^{N+1}} \cdot e^{-\frac{1}{x^2}} = \lim_{x \to 0} \frac{\frac{1}{x^{N+1}}}{e^{\frac{1}{x^2}}}$$

$$\overset{L'H}{=} \lim_{x \to 0} \frac{-\frac{N+1}{x^{N+2}}}{-\frac{2}{x^3} e^{\frac{1}{x^2}}}$$

$$= \lim_{x \to 0} \frac{\frac{N+1}{2} \cdot \frac{1}{x^{N-1}}}{e^{\frac{1}{x^2}}}$$

$$= \frac{N+1}{2} \lim_{x \to 0} \frac{1}{x^{N-1}} \cdot e^{-\frac{1}{x^2}} = 0$$

So the statement is true for all $m \in \mathbb{N} \cup \{0\}$ by induction.