Exercises 3.2.7 (b), 3.2.11, 3.3.1, 3.3.5, 4.2.5 (c), 4.3.3 (a), 4.3.9, 4.4.3, 4.4.12, 4.5.7.

Exercise 3.2.7. Given $A \subseteq \mathbf{R}$, let L be the set of all limit points of A.

(a) Show that the set L is closed.

(b) Argue that if x is a limit point of $A \cup L$, then x is a limit point of A. Use this observation to furnish a proof for Theorem 3.2.12.

(b) Argue that if x is a limit point of $A \cup L$, then x is a limit point of A. Use this observation to furnish a proof for Theorem 3.2.12.

Proof. Suppose x is a limit point of $A \cup L$. Let $\epsilon > 0$. Then since x is a limit point of $A \cup L$, there exists $y \in A \cup L$ such that $y \neq x$ and

$$|x - y| < \frac{\epsilon}{2}.$$

If $y \in A$, then

$$|x - y| < \frac{\epsilon}{2} < \epsilon,$$

so x is a limit point of A.

If $y \notin A$, then $y \in L$. Therefore, y is a limit point of A, so there exists $a \in A$ such that

$$|a - y| < |x - y|.$$

Therefore, $a \neq x$ and

$$|a - x| = |a - y + y - x| \leq |a - y| + |y - x| < |x - y| + |x - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Again, this means that x is a limit point of A.

Thus, we see that if x is a limit point of $A \cup L$, then x is a limit point of A. Therefore, the set of limit points of $A \cup L$ is contained in L; in particular, $A \cup L$ contains all of its limit points, so $\overline{A} = A \cup L$ is closed, which was all that remained to complete the proof of Theorem 3.2.12. $\square$

b) If x is a limit point of $A \cup L$, then x is a limit point of A.

Proof. Suppose x is a limit point of $A \cup L$. Let (c_n) be a sequence converging to x. Either infinitely many elements of the sequence come from A or infinitely many come from L. In the first case, we can find a subsequence of (c_n) that lies in A. Since the sequence converges, the subsequence converges to x, and so x is a limit point. In the second case, we can find a subsequence that lies in L, but since L is closed, $x \in L$ and is a limit point of A. $\square$

Prove Theorem 3.2.12

Proof. From the observation above, the closure of A contains all of its limit points and is therefore closed.
Suppose B is a closed subset with $A \subseteq B \subseteq \overline{A}$. We will show $B = \overline{A}$.
Let $x \in \overline{A}$. Then, $x \in A$ or $x \in L$, where L is the set of limit points of A. If $x \in A$, then $x \in B$. If $x \in L$, then there is a sequence (x_n) in A that converges to x. However, the sequence is also in B, and since B is closed, it contains the limit. Thus, $x \in B$. This shows $\overline{A} \subseteq B$, and so $B = \overline{A}$. Thus, $\overline{A}$ is the smallest closed set that contains A. $\square$

Exercise 3.2.11. (a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(b) Does this result about closures extend to infinite unions of sets?

(a)

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

It is clear that $A \subseteq A \cup B \implies \overline{A} \subseteq \overline{A \cup B}$ so

$$\overline{A \cup B} \supseteq \overline{A} \cup \overline{B}$$

To show inclusion the other way, Let $x \in \overline{A \cup B}$ such that $x \notin A \cup B$ (otherwise it would be trivial). If $x \notin \overline{A}$ then for some $\epsilon > 0, V_\epsilon(x) \cap A = \emptyset$ but $V_\epsilon(x) \cap (A \cup B) \neq \emptyset$ implies $V_\epsilon(x) \cap B \neq \emptyset$ for all $\epsilon > 0$. This shows $x \in \overline{B}$. By same argument, if $x \notin \overline{B}$ then $x \in \overline{A}$ so $x \in \overline{A \cup B}$. This shows

$$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$$

So, we get the required result.

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

(b)

Take sets

$$A_n = \left[0, 1 - \frac{1}{n}\right]$$

where $n \geq 2$. Since each A_n is closed set. Then $\overline{A}_n = \left[0, 1 - \frac{1}{n}\right]$ so we get

$$\bigcup_{n=2}^{\infty} A_n = \bigcup_{n=2}^{\infty} \left[0, 1 - \frac{1}{n}\right] = [0, 1)$$

However

$$\overline{\bigcup_{n=2}^{\infty} A_n} = \overline{\bigcup_{n=2}^{\infty} \left[0, 1 - \frac{1}{n}\right]} = \overline{[0, 1)} = [0, 1]$$

It shows this does not extend to union of infinite sets.

Exercise 3.3.1. Show that if K is compact and nonempty, then $\sup K$ and $\inf K$ both exist and are elements of K.

Let K be a non empty compact set. Then by Theorem 3.3.4 K must be closed and bounded. Since K is bounded, $\alpha = \sup K$ and $\beta = \inf K$ exists since K is non-empty.

For any $\epsilon > 0$, $\alpha - \epsilon$ is not supremum so there exists $x \in K$ such that $\alpha - \epsilon < x \leq \alpha$. If $x \neq a$ then α is limit point and K being closed contains α.

Similarly, $\beta + \epsilon$ is not infimum so there exists y such that $\beta \leq y < \beta - \epsilon$ and if $y \neq \beta$, then β is limit point so $\beta \in K$ as K is closed. .

Exercise 3.3.5. Decide whether the following propositions are true or false. If the claim is valid, supply a short proof, and if the claim is false, provide a counterexample.

(a) The arbitrary intersection of compact sets is compact.

(b) The arbitrary union of compact sets is compact.

(c) Let A be arbitrary, and let K be compact. Then, the intersection $A \cap K$ is compact.

(d) If $F_1 \supseteq F_2 \supseteq F_3 \supseteq F_4 \supseteq \cdots$ is a nested sequence of nonempty closed sets, then the intersection $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

The arbitrary intersection of compact set is compact.

Since arbitrary intersection of closed sets is closed, arbitrary intersection of compact sets are also closed. By definition, compact set is bounded and arbitrary intersection of compact is contained in each (bounded) compact set. Therefore being bounded and closed, it must be compact.

(b)

The arbitrary intersection of compact sets may not be compact.

Take $I_n = [n, n+1]$ which are compact for each n. Then

$$\bigcup_{n=1}^{\infty} I_n = [1, \infty)$$

which is not bounded so not compact.

(c)

Take $A = (0,1)$ and $K = [0,1]$ then $A \cap K = (0,1)$ which is not compact since it is not closed.

(d)

Take $F_n = [n, \infty)$ then F_n is closed as it contains it's every limit point. Also $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$ but

$$\bigcap_{n=1}^{\infty} F_n = \emptyset$$

as we can prove any $x \geq 1$ cannot lie on the set (using Archimedean property we may cho n)

Exercise 4.2.5. Use Definition 4.2.1 to supply a proper proof for the following limit statements.

(a) $\lim_{x\to 2}(3x+4) = 10$.

(b) $\lim_{x\to 0} x^3 = 0$.

(c) $\lim_{x\to 2}(x^2 + x - 1) = 5$.

(d) $\lim_{x\to 3} 1/x = 1/3$.

c) $\lim_{x \to 2}(x^2 + x - 1) = 5$.

Proof. Let $\epsilon > 0$ be given. Choose $\delta = min(1, \epsilon/6)$. If $0 < |x-2| < \delta$, then $|x-2| < 1$, or $1 < x < 3$. Thus, $4 < x+3 < 6$, and

$$\frac{1}{x+3} > \frac{1}{6}.$$

Also,

$$|x-2| < \frac{\epsilon}{6} < \frac{\epsilon}{|x+3|},$$

so

$$|x-2||x+3| < \epsilon.$$

However, $|x-2||x+3| = |x^2+x-6| = |x^2+x-1-5|$, so

$$|x^2+x-1-5| < \epsilon.$$

Exercise 4.3.3. (a) Supply a proof for Theorem 4.3.9 using the ϵ–δ characterization of continuity.

(b) Give another proof of this theorem using the sequential characterization of continuity (from Theorem 4.3.2 (iii)).

Theorem 4.3.9 (Composition of Continuous Functions). *Given $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$, assume that the range $f(A) = \{f(x) : x \in A\}$ is contained in the domain B so that the composition $g \circ f(x) = g(f(x))$ is defined on A.*

If f is continuous at $c \in A$, and if g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

Proof. Exercise 4.3.3. □

Exercise 4.3.9. Assume $h : \mathbf{R} \to \mathbf{R}$ is continuous on $\mathbf{R}$ and let $K = \{x : h(x) = 0\}$. Show that K is a closed set.

Let $h : \mathbb{R} \to \mathbb{R}$ be continuous function and $K = \{x \in \mathbb{R} : h(x) = 0\}$ then K is closed set.

Proof : For continuous function h, and $V_\delta(c) \cap K \neq \emptyset$ for all $\delta > 0$. For all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $x \in V_\delta(c)$ then $h(x) \in V_\epsilon(h(c))$.

If $h(c) \neq 0$ then there exists some $\epsilon > 0$ such that $h(x) \in V_\epsilon(h(c)) \implies f(x) \neq 0$. This contradicts that $V_\delta(c) \cap K \neq \emptyset$ since it says for every δ there exists $x \in V_\delta(c)$ such that $f(x) = 0$.

So $h(c) = 0 \implies c \in K$ so K is closed as it contains its every limit point.

Exercise 4.4.3. Show that $f(x) = 1/x^2$ is uniformly continuous on the set $[1, \infty)$ but not on the set $(0, 1]$.

Proof. Let $\epsilon > 0$ be given. If $x, y \in [1, \infty)$ then $\dfrac{1}{xy^2} < 1$ and $\dfrac{1}{x^2y} < 1$. Choose $\delta = \epsilon/2$. Then, for $x, y \in [1, \infty)$, if $|x - y| < \delta$, then

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \frac{|x^2 - y^2|}{x^2y^2} = |x - y|\frac{x + y}{x^2y^2}$$

$$= |x - y|\left(\frac{1}{xy^2} + \frac{1}{x^2y}\right) \leq 2|x - y| < 2\delta = 2\epsilon/2 = \epsilon.$$

Thus, f is uniformly continuous on $[1, \infty)$.

To show f is not uniformly continuous on $(0, 1]$, we use the Sequential Criterion for Absence of Uniform Continuity. Let $a_n = \dfrac{1}{n}$, and $b_n = \dfrac{1}{2n}$. Then, $|a_n - b_n| = \dfrac{3}{4n^2} \to 0$. However,

$$|f(a_n) - f(b_n)| = |n^2 - 4n^2| = 3n^2 \geq 3.$$

Hence, f is not uniformly continuous on $(0, 1]$. $\square$

Exercise 4.4.12. Review Exercise 4.4.11, and then determine which of the following statements is true about a continuous function defined on $\mathbf{R}$:

(a) $f^{-1}(B)$ is finite whenever B is finite.

(b) $f^{-1}(K)$ is compact whenever K is compact.

(c) $f^{-1}(A)$ is bounded whenever A is bounded.

(d) $f^{-1}(F)$ is closed whenever F is closed.

1 (a) $f^{-1}(B)$ is finite whenever B is finite. **False**. Take function $f : \mathbb{N} \to \{0, 1\}$ by $f(x) = x \pmod 2$ then $f^{-1}(\{0, 1\}) = \mathbb{N}$ is infinite.	(b) If K is compact then $f^{-1}(K)$ is also compact. **False**. Take $f(x) = \sin x$ then $f : \mathbb{R} \to [-1, 1]$ where $[-1, 1]$ is compact and $\mathbb{R}$ is not compact.
2 (c) $f^{-1}(A)$ is bounded whenever A is bounded. **False**. Take above example, $f(x) = \sin x$.	(d) $f^{-1}(F)$ is closed whenever F is closed. **True**. By continuity if $\mathbb{R}\backslash F$ is open so $f^{-1}(\mathbb{R}\backslash F) = \mathbb{R}\backslash f^{-1}(F)$ is also open. This shows $f^{-1}(F)$ is closed.

(a) (b) (c) may be false. Consider the example $f(x) = 1$ for all $x \in \mathbf{R}$. Let $B = K = A$, then $f^{-1}(B) = f^{-1}(K) = f^{-1}(A) = \mathbf{R}$ which is clear not finite, not compact, not bounded.

(d) is true. To show $f^{-1}(F)$ is closed, it suffices to show that $f^{-1}(F)^c$ is open. We show that $f^{-1}(F)^c = f^{-1}(F^c)$. Then because F^c is open, by definition of Exercise 4.4.11, $f^{-1}(F)^c = f^{-1}(F^c)$ is also open.

Now we show $f^{-1}(F)^c = f^{-1}(F^c)$. First we show $f^{-1}(F)^c \subset f^{-1}(F^c)$. Let $a \in f^{-1}(F)^c$, then $a \notin f^{-1}(F)$, i.e. $f(a) \notin F$. Therefore $f(a) \in F^c$, hence $a \in f^{-1}(F^c)$.

Then we show $f^{-1}(F^c) \subset f^{-1}(F)^c$. Let $a \in f^{-1}(F^c)$, then $f(a) \in F^c$. Thus $f(a) \notin F$ and $a \notin f^{-1}(F)$. Therefore $a \in f^{-1}(F)^c$, this implies $f^{-1}(F^c) \subset f^{-1}(F)^c$.

Now we have $f^{-1}(F)^c = f^{-1}(F^c)$.

Exercise 4.5.7. Let f be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that f must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0, 1]$.