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5.2.3 (c)

Assume that $g(c) \neq 0$,

$$(f/g)'(c) = \lim_{x \to c} \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x-c} = \lim_{x \to c} \frac{f(x)g(c) - f(c)g(x)}{(x-c)\, g(x) \cdot g(c)}$$

$$= \lim_{x \to c} \frac{f(x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(x)}{(x-c) \cdot g(x) \cdot g(c)} = \lim_{x \to c} \frac{g(c)}{g(x)g(c)} \cdot \frac{f(x)-f(c)}{x-c} - \lim_{x \to c} \frac{f(c)}{g(x)g(c)} \cdot \frac{g(x)-g(c)}{x-c}$$

$$= \lim_{x \to c} \frac{g(c)}{g(x)g(c)} \cdot \lim_{x \to c} \frac{f(x)-f(c)}{x-c} - \lim_{x \to c} \frac{f(c)}{g(x)g(c)} \cdot \lim_{x \to c} \frac{g(x)-g(c)}{x-c}$$

$$= \frac{g(c)}{g^2(c)} \cdot f'(c) - \frac{f(c)}{g^2(c)} \cdot g'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g^2(c)}$$

5.3.1

(a) According to Extreme Value Thm, $\exists x_0 \in [a,b]$, s.t. $\forall x \in [a,b]$, $f'(x) \le f'(x_0)$

According to Lagrange MVT, for $x, y \in [a,b]$, $x \neq y$, it implies that:

$\left| \frac{f(x)-f(y)}{x-y} \right| = |f'(x_1)|$; ~~Also, $|f'|$ is continuous as f' is cts.~~

$\therefore |f'(x_1)| \le |f'(x_0)|$. $\therefore \left| \frac{f(x)-f(y)}{x-y} \right| \le |f'(x_0)|$

$\therefore f$ is Lipschitz and M can be $|f'(x_0)|$.

(b) If $|f'(x)| < 1$, then $\left| \frac{f(x)-f(y)}{x-y} \right| < |f'(x_0)| < 1$

f is contractive and $c = |f'(x_0)|$. (x_0 is under the definition in (a).)

5.3.2

$f: A \to B$. If f is 1-1, then if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

We prove by contradiction, if $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

According to Rolle's Thm, if $f(x_1) = f(x_2)$ and f is diff on $[x_1, x_2]$

then $\exists x_0 \in [x_1, x_2]$ s.t. $f'(x_0) = 0$. Contradiction!

Converse statement: If f is 1-1, then there is no $f'(x) = 0$.

$f(x) = x^3$ in $[-1, 1]$. $f(x)$ is 1-1, but $f'(0) = 0$.

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6.2.1

(a) $f(x) = \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{nx}{1+nx^2} = \frac{1}{x}$

(b) $|f(x) - f_n(x)| = \left|\frac{1}{x} - \frac{nx}{1+nx^2}\right| = \frac{1}{x+nx^3}$ $\lim\limits_{x\to 0} \frac{1}{x+nx^3} \to +\infty$.

$\therefore$ $\forall \varepsilon > 0$, we can always find x_0 small enough, that $\frac{1}{nx^3+x} > \varepsilon$.

so that $|f(x) - f_n(x)| > \varepsilon$. $\therefore$ The convergence is not uniform on $(0, +\infty)$.

(c) Same as (b). $|f(x) - f_n(x)| = \frac{1}{x+nx^3}$. $\forall \varepsilon > 0$, choose $0 < x < \frac{1}{n}$,

$\frac{1}{x+nx^3} > \frac{n^2}{n+1}$, we can choose a large n, s.t $\frac{n^2}{n+1} > \varepsilon$.

$\therefore$ The convergence is not uniform on $(0, 1)$

(d) If $x \in (1, +\infty)$, $|f_n(x) - f(x)| = \frac{1}{x+nx^3} < \frac{1}{n}$. By AP,

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $\frac{1}{n} < \varepsilon$ and $|f_n(x) - f(x)| < \varepsilon$.

$\therefore$ The convergence is uniform on $(1, +\infty)$.

6.2.2 (a)

① f_n is continuous everywhere except $x = 1, \frac{1}{2}, \cdots \frac{1}{n}$. If $\delta = \frac{1}{n}$, $f_n(x) = 0$ for all $x \in V_\delta(0)$. Therefore, f_n is cts at 0.

② We prove it is not uniformly continuous by contradiction:

If it is, then: $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, s.t. $|f_n(x) - f(x)| \leq \frac{1}{2}$.

Let $n = N$, $|f_N(x) - f(x)| \leq \frac{1}{2}$.

If $x = \frac{1}{N+1}$, $f_N(\frac{1}{N+1}) = 0$, $f(\frac{1}{N+1}) = 1$, $|f_N(\frac{1}{N+1}) - f(\frac{1}{N+1})| = 1 > \frac{1}{2}$

Contradiction!

③ If we choose $x_n = \frac{1}{n}$, then

$\lim\limits_{n\to\infty} f(\frac{1}{n}) = 1 \neq f(0)$, so f is not continuous at 0.

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6.3.1

(a) $g_n(x) = \frac{x^n}{n}$. By Cauchy criteria, $|g_n(x) - g_m(x)| = |\frac{x^n}{n} - \frac{x^m}{m}|$

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ and $\frac{1}{N} < \varepsilon$, s.t. $m, n > N$, $|g_n(x) - g_m(x)| < |\frac{x^n}{n}| \leq \frac{1}{N} < \varepsilon$.

$\therefore$ $g_n(x)$ uniformly converges on $[0,1]$.

And $x, n \geq 0$, $g(x) \geq 0$. Also, $g(x) = \lim_{n\to\infty} \frac{x^n}{n} \leq \lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore g(x) = 0$, $g'(x) = 0$

$\therefore$ g is differentiable, $g'(x) = 0$

(b) $g_n'(x) = x^{n-1}$. when $n \to \infty$, $h(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1. \end{cases}$

Take $x_n = \frac{1}{\sqrt[n]{2}}$, $g'_{n+1}(x_n) = \frac{1}{2}$, $h(x_n) = 0$

$\therefore$ $|g'_{n+1}(x_n) - h(x_n)| = \frac{1}{2} > 0$. So the convergence is not uniform.

And h is not the same as g'. $g'(1) = 0$. $h(1) = 1$.

6.4.5

(a) From $h(x)$, we know that $h_n(x) = \frac{x^n}{n^2}$. $h_n(x)$ is continuous on $[-1,1]$.

$|h_n(x)| = |\frac{x^n}{n^2}| \leq |\frac{1}{n^2}|$, and $\sum_{n=1}^{\infty} |\frac{1}{n^2}|$ converges. By Weierstras M-Test, $\sum_{n=1}^{\infty} h_n(x)$ unifory converges. By Term-by-term Continuity Thm, $h(x)$ is continuous on A.

(b) For a fixed x_0. let $\varepsilon > 0$ and $-1 < x_0 - \varepsilon < x_0 + \varepsilon < 1$. $V_\varepsilon(x_0) \in (-1,1)$.

Let $M_n = \sup\{x \mid x \in V_\varepsilon(x_0)\}$, then

$$\left|\frac{x^n}{n}\right| \leq \left|\frac{(M_n)^n}{n}\right| \leq (M_n)^n, \quad (M_n)^n \text{ is geometric series, } \sum_{n=1}^{\infty}(M_n)^n \text{ converge}$$

By W-M-Test, $\sum_{n=1}^{\infty}(x^n/n)$ converges uniformly on $V_\varepsilon(x_0)$.

By Term-by-term Continuity Thm, $\sum_{n=1}^{\infty}(x^n/n)$ continuous on $V_\varepsilon(x_0)$.

also continuous on x_0.

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6.5.3

Let $M_n = |a_n x_0^n|$, as $\sum_{n=0}^{\infty} a_n x^n$ absolutely converges at x_0. then $\sum_{n=0}^{\infty} |a_n x_0^n|$ converges, so M_n converges. When $x \in [-c, c]$, $|x| \le |x_0|$, so $|a_n x^n| \le |a_n x_0^n| = M_n$, By W-M-Test, $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-c, c]$.

6.6.5

(a) $a_n = \frac{f^n(0)}{n!} = \frac{e^0}{n!} = \frac{1}{n!}$ $\therefore f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

By Ratio Test, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n!}{(n+1)!}\right| = \left|\frac{1}{n+1}\right| = 0 < 1$,

Therefore, the series converges absolutely in $\mathbb{R}$. Take $x = R$, by Abel's Theorem, it converges uniformly on $[-R, R]$.

(b) From (a), we do diff on $f(x)$. $f'(x) = \left(\sum_{n=0}^{\infty} \frac{1}{n!} \cdot x^n\right)' = \left(\sum_{n=1}^{\infty} \frac{1}{(n-1)!} \cdot x^{n-1}\right)$
$= \sum_{n=0}^{\infty}\left(\frac{1}{n!} \cdot x^n\right) = f(x)$. Therefore, $f'(x) = e^x$.

(c) $e^{-x} = f(-x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot (-x)^n$

$e^x \cdot e^{-x} = 1 + x(1-1) + x^2 \cdot \left(\frac{1}{2!} + \frac{1}{2!} - 1\right) + x^3 \cdot \left(\frac{1}{3!} - \frac{1}{3!} + \frac{1}{2!} - \frac{1}{2!}\right) \cdots$

$= 1$

6.6.6

(a) If $x \ne 0$, $g'(x) = \frac{2e^{-\frac{1}{x^2}}}{x^3}$. $g''(0) = \lim_{x\to 0} \frac{g'(x) - g'(0)}{x - 0} = \lim_{x\to 0} \frac{\frac{2}{x^4}}{e^{\frac{1}{x^2}}}$, ∞/∞

By L'Hospital's rule, $\lim_{x\to 0} \frac{\frac{2}{x^4}}{e^{\frac{1}{x^2}}} = \lim_{x\to 0} \frac{-8 \cdot \frac{1}{x^5}}{e^{\frac{1}{x^2}} \cdot (-2) \cdot \frac{1}{x^3}} = \lim_{x\to 0} \frac{4 \cdot \frac{1}{x^2}}{e^{\frac{1}{x^2}}} = \lim_{t\to\infty} \frac{4t}{e^t}$

$= \lim_{t\to\infty} \frac{4}{e^t} = 0$ $\therefore g''(0) = 0$

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(b) $g''(x) = \left(\frac{-6}{x^4} + \frac{4}{x^6}\right) \cdot e^{-\frac{1}{x^2}}$; $g'''(x) = \left(\frac{24}{x^5} - \frac{36}{x^7} + \frac{8}{x^9}\right) \cdot e^{-\frac{1}{x^2}}$

We can find $g^{(n)}(x) = P\left(\frac{1}{x}\right) \cdot g(x)$

For $n=k$, $g^{k}(x) = P\left(\frac{1}{x}\right) \cdot g(x)$; For $n=k+1$,

$g^{k+1}(x) = \left[P'\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) + P\left(\frac{1}{x}\right) \cdot \left(-\frac{2}{x^3}\right)\right] g(x) = Q\left(\frac{1}{x}\right) g(x)$.

where $Q(t) = -P'(t) \cdot t^2 - 2P(t) \cdot t^3$, a polynomial.

Then, we can generate $g^{(n)}(x)$.

(c) We prove by induction, $g(0)=0$, $g'(0)=0$

suppose $g^{n-1}(0)=0$, then : $g^{(n)}(0) = \lim_{x \to 0} \frac{g^{n-1}(x) - 0}{x - 0} = \lim_{x \to 0} \left(\frac{1}{x}\right) P\left(\frac{1}{x}\right) \left(e^{-1/x^2}\right)$

$= \lim_{t \to \infty} \frac{tP(t)}{e^{t^2}}$, by repeating LH's rule, $\lim_{t \to \infty} \frac{tP(t)}{e^{t^2}} = 0$, as polynomial is not increasing as fast as e^{t^2}.

Therefore, $g^{(n)}(0) = 0$