1.2.10

a) $a<b \Longleftrightarrow \forall \varepsilon>0, \quad a<b+\varepsilon.$

b) $\forall \varepsilon>0, a<b+\varepsilon \Rightarrow a<b.$

Both a) and b) are false by the counterexample

$\forall \varepsilon>0, a<a+\varepsilon$ but $a=a$.

c) $a\leq b \Leftrightarrow \forall \varepsilon>0, a<b+\varepsilon$ is true.

Proof. "$\Rightarrow$"

$a\leq b \Rightarrow \forall \varepsilon>0, a\leq b<b+\varepsilon.$

"$\Leftarrow$" proof by contradiction

Assume $a>b$, then take $\varepsilon=a-b>0$,

$$a<b+\varepsilon= b+(a-b)=a.$$

A contradiction, so $a\leq b$.

1.3.9

a) Take $\varepsilon = \sup B - \sup A > 0$, $\exists\, b \in B$

s.t. $\sup B - \varepsilon = \sup A < b$.

$\forall\, a \in A$, $a \leq \sup A < b$, so b is an upper bound of A.

b) Take $A = B = (0,1)$, $\sup A = \sup B = 1$.

There is no $b \in B$ that is an upper bound of A.

1.4.3

Proof. Assume $A = \bigcap_{n=1}^{\infty} (0, \frac{1}{n}) \neq \phi$, take $x \in A$.

$x > 0$, by Archimedean property,

$\exists\, m \in \mathbb{N}$, s.t. $\frac{1}{m} < x$, so $x \notin (0, \frac{1}{m})$.

Hence $x \notin \bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = A$, a contradiction.

So $A = \phi$.

1.5.6

a) $A_n = (n, n+1)$, $n \in \mathbb{Z}$.

b) No such collection exists.

Let $\Omega = \{O_\lambda = (a_\lambda, b_\lambda)\}_{\lambda \in \Lambda}$ be a collection of disjoint open intervals.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\forall \lambda \in \Lambda$, $\exists\ r_\lambda \in \mathbb{Q} \cap O_\lambda$.

$\forall \lambda_1 \neq \lambda_2 \in \Lambda$, $r_{\lambda_1} \neq r_{\lambda_2}$ since $O_{\lambda_1} \cap O_{\lambda_2} = \phi$.

Now we have a 1-1 correspondence between Ω and a subset of $\mathbb{Q}$ given by

$$O_\lambda \mapsto r_\lambda,$$

Since $\mathbb{Q}$ is countable and its subsets are countable or finite, Ω is countable or finite.

2.2.2 b)

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $\frac{1}{n} < \frac{\varepsilon}{2}$.

Now $\forall$ $n \geq N$,

$$\left| \frac{2n^2}{n^3+3} - 0 \right| < \frac{2n^2}{n^3} = \frac{2}{n} < 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

So $\lim \frac{2n^2}{n^3+3} = 0$.

2.3.1 b)

We may assume $x > 0$ since the case $x = 0$ is done in part a). Since $(x_n) \to x$,

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|x_n - x| < \varepsilon\sqrt{x}$.

Now $\forall n \geq N$,

$$|\sqrt{x_n} - \sqrt{x}| = \frac{|\sqrt{x_n} - \sqrt{x}| \cdot (\sqrt{x_n} + \sqrt{x})}{\sqrt{x_n} + \sqrt{x}}$$

$$= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}}$$

$$\leq \frac{|x_n - x|}{\sqrt{x}} < \frac{\varepsilon\sqrt{x}}{\sqrt{x}} = \varepsilon.$$

So $(\sqrt{x_n}) \to \sqrt{x}$.

2.3.3

Proof. $\forall \varepsilon > 0$, $\exists N_1 \in \mathbb{N}$, s.t. $\forall n \geq N_1$, $l-\varepsilon < x_n < l+\varepsilon$,

$\exists N_2 \in \mathbb{N}$, s.t. $\forall n \geq N_2$, $l-\varepsilon < z_n < l+\varepsilon$,

since $\lim x_n = \lim z_n = l$.

Take $N = \max\{N_1, N_2\}$, $\forall n \geq N$,

$$l-\varepsilon < x_n \leq y_n \leq z_n < l+\varepsilon.$$

So $\lim y_n = l$.

* Note that we can not use algebraic limit theorem here since it is not given that (y_n) converges.

2.4.3 b)

The sequence a_n is given by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2a_n}$, $n \in \mathbb{N}$.

i) $a_n \uparrow$

$a_2 = \sqrt{2\sqrt{2}} > \sqrt{2} = a_1$.

If $a_{n+1} > a_n$, then $a_{n+2} = \sqrt{2a_{n+1}} > \sqrt{2a_n} = a_{n+1}$.

ii) $\sqrt{2} \le a_n < 2$.

$\sqrt{2} \le a_1 = \sqrt{2} < 2$.

If $\sqrt{2} \le a_n < 2$, then $\sqrt{2} \le a_n < a_{n+1} = \sqrt{2a_n} < \sqrt{2 \cdot 2} = 2$.

Now by Monotone Convergence Theorem, (a_n) converges.

Let $\lim(a_n) = a$, take limits of $a_{n+1} = \sqrt{2a_n}$, we have $a = \sqrt{2a}$ (Note that we used the result of 2.3.1 here), $a = 0$ or $a = 2$.

Since $\sqrt{2} \le a_n < 2$, $\sqrt{2} \le a \le 2$, hence $a = 2$.

2.5.5

Proof. Assume that all convergent subsequence of a bounded sequence (a_n) converge to $a \in \mathbb{R}$ but (a_n) does not converge to a.

$\exists\, \varepsilon > 0$ s.t. $\exists$ infinitely many terms of (a_n) not in $V_\varepsilon(a)$.

So (a_n) has subsequence (a_{n_k}) s.t.

$\forall\, k \in \mathbb{N},\ a_{n_k} \notin V_\varepsilon(a)$.

Now (a_{n_k}) is a bounded sequence and hence by Bolzano-Weierstrass Thm it has a convergent subsequence $(a_{n_{k_i}})$. From the assumption

$(a_{n_{k_i}}) \to a$ but $\forall\, i \in \mathbb{N},\ a_{n_{k_i}} \notin V_\varepsilon(a)$,

a contradiction.

So $(a_n) \to a$.

2.6.5

i False.

Take $S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, $n \in \mathbb{N}$.

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, s.t. $\forall n \geq N$,

$$|S_{n+1} - S_n| = \frac{1}{n+1} < \varepsilon.$$

(S_n) is not bounded

ii True.

If (x_n), (y_n) are pseudo-Cauchy,

$\forall \varepsilon > 0$, $\exists N_1 \in \mathbb{N}$, s.t. $\forall n \geq N_1$, $|x_{n+1} - x_n| < \frac{\varepsilon}{2}$.

$\exists N_2 \in \mathbb{N}$, s.t. $\forall n \geq N_2$, $|y_{n+1} - y_n| < \frac{\varepsilon}{2}$.

Take $N = \max\{N_1, N_2\}$, $\forall n \geq N$,

$$\begin{aligned}
&|(x_{n+1} + y_{n+1}) - (x_n + y_n)| \\
&\leq |x_{n+1} - x_n| + |y_{n+1} - y_n| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

So $(x_n + y_n)$ is pseudo-Cauchy.

2.7.2

c) $a_n = (-1)^{n+1} \cdot \frac{n+1}{2n}$.

$$|a_n| = \frac{n+1}{2n} > \frac{n}{2n} = \frac{1}{2}.$$

So (a_n) does not converge to 0 and $\sum a_n$ diverges.

e) The partial sum

$$\begin{aligned} S_{2n} &= 1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \cdots + \frac{1}{2n-1} - \frac{1}{(2n)^2} \\ &> 1 - \frac{1}{1\cdot 2} + \frac{1}{3} - \frac{1}{3\cdot 4} + \cdots + \frac{1}{2n-1} - \frac{1}{(2n-1)(2n)} \\ &= \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n} \\ &= \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \end{aligned}$$

(S_{2n}) is unbounded, so the series diverges.