Date: / /

1.2.10

(a) False:

If $a<b$, then there is $a<b+\varepsilon$ for every $\varepsilon>0$.

But if $a<b+\varepsilon$ for every $\varepsilon>0$, it can be $a<b$ or $a=b$.

For example, $2<2+\varepsilon$ for every $\varepsilon>0$.

(b) False;

Like the example in (a). $a=2$, $b=2$.

$a<b+\varepsilon$ for every $\varepsilon>0$. However, $a=b$.

(c) True; The statement is:

$\forall a,b\in\mathbb{R},\ a\le b \iff \forall a,b\in\mathbb{R},\ a<b+\varepsilon$ for every $\varepsilon>0$.

$\Rightarrow$: $b\ge a \Rightarrow b+\varepsilon\ge a+\varepsilon$, $\varepsilon>0 \Rightarrow \forall\varepsilon>0,\ b+\varepsilon\ge a+\varepsilon>a$

$\Rightarrow a<b+\varepsilon$ for every $\varepsilon>0$.

$\Leftarrow$: $\forall\varepsilon>0,\ a<b+\varepsilon \Rightarrow \forall\varepsilon>0,\ \varepsilon>a-b$

if $a<b$, $a-b<0<\varepsilon$. The statement holds.

if $a=b$, $a-b=0<\varepsilon$. The statement holds.

if $a>b$. Assume $a-b=\varepsilon_0>0$. Then if $\varepsilon=\frac{\varepsilon_0}{2}>0$, $\varepsilon<a-b$.

$\therefore \forall a,b\in\mathbb{R},\ a<b+\varepsilon \Rightarrow \forall a,b\in\mathbb{R},\ a\le b$

$\therefore$ The statement is true.

Date: / /

1.3.9

(a) Firstly, we know that $S = \sup A \Leftrightarrow \forall \varepsilon > 0$, $\exists a \in A$, s.t. $S - \varepsilon < a$.

Assume $m = \sup A$, $n = \sup B$. $m < n$. Let $\varepsilon_0 = n - m > 0$

By the previous definition, let $\varepsilon = \varepsilon_0$. So, $\exists b \in B$, s.t. $n - \varepsilon_0 < b$. $\Rightarrow$ $n - (n-m) < b \Rightarrow m < b$. Therefore, $m = \sup A < b$, b is an upper bound of A ($\forall a \in A$, $a < b$).

$\therefore$ There exists $b \in B$ s.t. b is an upper bound of A.

(b) Assume $A = (0,1)$, $B = (0,1)$. So, $\sup A = \sup B = 1$. satisfies $\sup A \leq \sup B$. Assume one of the upper bound of A is k, $k \geq \sup A = 1$. $\forall b \in B$, $b < k$. Therefore, there does not exist $b \in B$ and b is an upper bound of A.

1.4.3

$\because$ Assume that $x \in \bigcap_{n=1}^{\infty} (0, \frac{1}{n})$, which means that $\forall n \in \mathbb{N}$, $x \in (0, \frac{1}{n})$. $\Rightarrow$ $\forall n \in \mathbb{N}$, $0 < x < \frac{1}{n}$. However, according to the Archimedean Property, $\forall x > 0$, $\exists m \in \mathbb{N}$, s.t. $\frac{1}{m} < x$. Contradiction!

$\therefore \bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \emptyset$.

Date: / /

1.5.6

(a) $\{(n, n+\frac{1}{2}) : n \in \mathbb{N}\}$, which is countable and consist of disjoint open intervals.

(b) Firstly, we know that: If $A \subseteq B$ and B countable, then A is countable or finite. In this question, let B be the $\mathbb{Q}$. For any interval, we can find an "a_x" and $a_x \in \mathbb{Q}$. For all the a_x, because all the intervals are disjoint, they can be arranged in an increasing order. A is the collection of all the a_x. $A \subseteq \mathbb{Q}$ and A is infinite. So, A is countable. There is no uncountable disjoint open intervals collection.

2.2.2 (b) $\forall \varepsilon > 0$, if $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $a_n \in V_\varepsilon(0)$, then the statement holds. $\left|\frac{2n^2}{n^3+3} - 0\right| < \varepsilon$. which is $\frac{1}{\varepsilon} < \frac{n}{2} + \frac{3}{2n^2}$. This will hold if $n > \frac{2}{\varepsilon}$. By Archimedean Property, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, s.t. $N > \frac{2}{\varepsilon}$, which means $\frac{1}{\varepsilon} < \frac{N}{2} + \frac{3}{2N}$. Then, $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, it follows: $\left|\frac{2n^2}{n^3+3} - 0\right| < \varepsilon$.

Therefore, $\lim \frac{2n^2}{n^3+3} = 0$.

2.3.1 (b) As $x_n \geq 0$ for all $n \in \mathbb{N}$, $x \geq 0$. If $x = 0$, then in part (a) we already showed this. If $x > 0$, then:

$\varepsilon > 0$, $\sqrt{x} > 0$, $\varepsilon \cdot \sqrt{x} > 0$. As $(x_n) \to x$, there $\exists N \in \mathbb{N}$, for $n > N$,

$|x_n - x| < \varepsilon \cdot \sqrt{x}$. So, $|\sqrt{x_n} - \sqrt{x}| = \dfrac{|x_n - x|}{|\sqrt{x_n} + \sqrt{x}|} < \dfrac{|x_n - x|}{\sqrt{x}} < \dfrac{\varepsilon \cdot \sqrt{x}}{\sqrt{x}} = \varepsilon$.

Therefore, $\lim(\sqrt{x_n}) = \sqrt{x}$.

2.3.3. As $\lim x_n = \lim z_n = l$, which means

$\forall \varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ s.t. $\forall n > N_1$, $|x_n - l| < \varepsilon$

$\forall \varepsilon > 0$, $\exists N_2 \in \mathbb{N}$ s.t. $\forall n > N_2$, $|z_n - l| < \varepsilon$.

Let $N = \max\{N_1, N_2\}$, $\forall \varepsilon > 0$, $n > N$,

$$l - \varepsilon < x_n \leq y_n \leq z_n < l + \varepsilon$$

$\therefore$ $|y_n - l| < \varepsilon$. for $n > N$. $\therefore$ $\lim y_n = l$.

2.4.3 (b) We can find that $a_{n+1} = \sqrt{2a_n}$.

① Prove (a_n) is monotone: $a_1 = \sqrt{2}$, $a_2 = \sqrt{2\sqrt{2}}$ $a_2 > a_1$. By induction, Assume $a_{n+1} > a_n$, $a_{n+2} = \sqrt{2a_{n+1}} > \sqrt{2a_n} = a_{n+1}$. (a_n) is increasing.

② Prove (a_n) is bounded. By induction, $0 < a_1 = \sqrt{2} < 2$, assume $0 < a_n < 2$. Then $a_{n+1} > 0$ and $a_{n+1} = \sqrt{2a_n} < \sqrt{2 \cdot 2} = 2$. $\therefore 0 < a_n < 2$

By MCT, (a_n) converges. Suppose $(a_n) \to a$.

$a = \sqrt{2a}$ $a = 0$ or $a = 2$. (a_n) is increasing, $a_1 > 0$.

$\therefore$ $(a_n) \to 2$.

Date: / /

2.5.5

Proof. We know that if $\lim(a_n) = a$, then each $V_\varepsilon(a)$ contains all but finitely many terms of a_n.

Assume (a_n) does not converge to a, then $\exists\ \varepsilon_0 > 0$, s.t. there are infinite terms of (a_n) not in $V_{\varepsilon_0}(a)$. Let (a_{n_k}) be a subsequence converges to a, containing all the infinite many terms of (a_n) not in $V_{\varepsilon_0}(a)$. By BW, (a_{n_k}) is convergent. But (a_{n_k}) does not converge to a because (a_{n_k}) has infinite terms outside of $V_{\varepsilon_0}(a)$. Contradiction!

$\therefore$ (a_n) must converge to a.

2.6.5

(i) False; Counterexample, let $S_n = \sum_{i=1}^{n} \frac{1}{i}$. So, $S_n - S_{n-1} = \frac{1}{n}$. By AP, $\forall\ \varepsilon > 0$, $\exists\ n \geq N$ s.t. $\frac{1}{n} < \varepsilon$. But (S_n) is not bounded.

(ii) $\forall\ \varepsilon > 0$, $\exists\ N_1 \in \mathbb{N}$, s.t. if $n \geq N_1$, $|x_{n+1} - x_n| < \frac{\varepsilon}{2}$.

$\forall\ \varepsilon > 0$, $\exists\ N_2 \in \mathbb{N}$, s.t. if $n \geq N_2$, $|y_{n+1} - y_n| < \frac{\varepsilon}{2}$.

$N = \max\{N_1, N_2\}$

$\forall\ \varepsilon > 0$, if $n \geq N$, then

$$|(x_{n+1} + y_{n+1}) - (x_n + y_n)| \leq |x_{n+1} - x_n| + |y_{n+1} - y_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore$ $(x_n + y_n)$ is pseudo-Cauchy as well.

Date: / /

2.7.2

(c) We know that if $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$.

Here, $a_k = \frac{(-1)^{n+1} \cdot (n+1)}{2n}$ $\lim \frac{n+1}{2n} = \frac{1}{2}$.

Therefore, (c) diverges.

(e) The series can be written as $\sum(a_n - b_n)$ where:

$\sum(a_n) = 1 + \frac{1}{3} + \frac{1}{5} + \cdots$ $\sum(b_n) = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots$

For $\sum_{i=1}^{n}(b_i) = \frac{1}{4} \times (1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \frac{1}{n^2}) < \frac{1}{4} \times (1 + \frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \cdots + \frac{1}{(n-1)n})$

$< \frac{1}{4} \times (2 - \frac{1}{n})$, which means that $\sum_{n=1}^{\infty}(b_n)$ is bounded.

By MCT, $\sum_{n=1}^{\infty}(b_n)$ converges.

For $\sum_{i=1}^{n}(a_i) = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1} < \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n+2}$

$= \frac{1}{2} \times (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}) < \frac{1}{2} \times (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \cdots) < \frac{1}{2} \times (1 + \frac{k}{2})$

which means $\sum_{n=1}^{\infty}(a_n)$ diverges.

$\therefore \sum_{n=1}^{\infty}(a_n)$ diverges and $\sum_{n=1}^{\infty}(b_n)$ converges.

Hence, the series $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \cdots$ diverges.