Assessment 3

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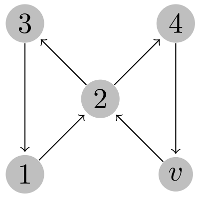
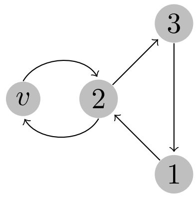
1 Problem 10.6

(a)  **(b)** 

(c) Proof:
Consider a shortest odd length walk W. If the length of W is 1, then the theory holds. When the length of W is bigger than 1: If W is a cycle then the theory holds. If W is not a cycle, then it has a repeated vertex x that is not the first/last one, i.e., W is $u, \ldots, x, \ldots, x, \ldots, u$. Notice there are two closed walks shorter than W (namely, x,. . . , x on inside and $u, \ldots, x, \ldots, u$ using outside portions). And because the sum vertices of u, v is a odd number, there must be a odd-length cycle in u, v.

2 Problem 10.28

(a) Let R be a strict partial order on A. Define a graph G with vertices using elements of A by setting (a $\rightarrow$ b) iff (a R b), which means R is a DAG. If G had a cycle and $a_0, a_1, a_2, a_3, a_4, \ldots, a_n$ are in this cycle, by transitivity, a_0 R a_1, a_1 R a_2,. . . ,a_n R a_0. So by transitivity, we get a_0 R a_0. This is against irreflexivity, which means there is no cycle in G and G is a DAG.

(b) The following DAG is not a strict partial order, because the relation (a R b) iff (a $\rightarrow$ b) is not transitive: there is an edge from a to b and one from b to c, but not from a to c.

a $\longrightarrow$ b $\longrightarrow$ c

(c) Let G be a DAG. From the above proof, we can know that there is no cycle in G, so any node in G is irreflexive. In terms of transitivity, considering three nodes A, B, C in G, if there is a x length path between A and B and a y length path between B and C, there is a $x + y$ length path between A and C. So the positive walk relation of a DAG a strict partial order.

3 Problem 12.25

2^k. We consume the two sets U and V of the bipartite graph and color the graph with two colors: if one colors all nodes in U blue, and all nodes in V green, each edge has endpoints of differing colors, as is required in the graph coloring problem. So considering each components, if the color of the first node is fixed then all nodes in this component sre fixed. So there are k components each has two choices of color. There are totally 2^k ways.

4 Problem 12.57

(a) If G is 2-colorable and contains a odd length closed walk. It must have a odd length cycle at the same time.(We have proved this in previous proof.) Considering this cycle, it has odd number of vertices so it's obviously cannot be colored by two colors.

(b) If every component of G can be 2-colored, then G can be 2-colored. Thus, we can focus on a single connected component H within G.

(c) Firstly, we choose a node as a root and give it a color. Then, give every adjacent nodes another color. We know that a tree has no cycle, so we can do this procedure until all the nodes are colored.

(d) Firstly, H contains no odd length closed walk. We let $a - b$ be an edge not in T but in H, we can prove it by contradiction. If a and b are in the same color, then it must have an even-length path P in T between a and b. Thus, $a - b$ and P form a cycle in H and this cycle is odd-length. This is a contradiction because H contains no odd length closed walk. So a and b are in the diffent color. In this way, any edge not in T must also connect different-colored vertices.

5 Problem 12.60

Proof: Base case, $V(G_1) = 1$. The theorem holds. Suppose $V(G_k) = k$ and it has exactly m components. $|V(G_k) - E(G_k)| = k - E(G_k) = m$. And we know that G is a forest.
When $V(G_{k+1}) = k + 1$, it has $|V(G_{k+1}) - E(G_{k+1})| = k + 1 - E(G_{k+1})$ components. Firstly we know that G_{k+1} must have at least 1 vertex of 1 degree (otherwise it is a set of isolated vertices and it's a forest.) Suppose

a is a 1 degree node in G_{k+1}. If we remove a, we get a new graph T and T also has $k+1-E(G_{k+1})$ components and $E(G_{k+1})-1$ edges. For T, there is $k+1-E(G_{k+1}) = k-(E(G_{k+1})-1) = k-(E(T))$. It means T satisfies induction hypothesis. So T is a forest. We add a back to T, because of a is of degree 1 and cannot form a cycle, we get G_{k+1}, which is also a forest.
From the induction, we proved that if a finite simple graph G has exactly $|V(G)-E(G)|$ components, then G is a forest.

6 Problem 12.67

(i) f is weakly edge-increasing. If we add an edge to a tree, f will not change. And because we add more edges to the graph, f cannot decrease.

(ii) f is weakly edge-increasing. After adding an edge, it may not change the connectivity but it also may connect two components and make f increased.

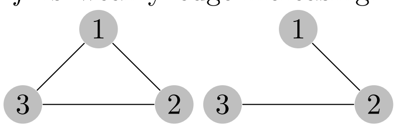
(iii) f is weakly edge-increasing. In this situation, f cannot decrease because adding a edges cannot change the maximum path length, it still there.

(iv) f is weakly edge-decreasing. In this situation, f may decrease because adding a edges may creat a shortcut to the graph.

(v) f is weakly edge-decreasing. In this situation, f may decrease because adding a edges may creat a shortcut to the graph.

(vi) f is weakly edge-decreasing. We may connect two different components and may not. But we can never creat more components.

(vii) f is weakly edge-increasing. Following is an example of f increasing.



(viii) f is strictly edge-increasing.
(ix) f is weakly edge-increasing.
(x) f has no property. If we add an edges and form a cycle, the cut edges will decrease. If we add a edges to connect two components, the cut edges will increase.

7 Problem 12.31

(a) $\wedge\vee$