Assessment 1

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January 24, 2019

1 Problem 1.11

Proof: Suppose $x = \sqrt[3]{2}^{\sqrt{3}}$.

Case 1: If x is rational, then let $x = a^{\sqrt{3}}$ and $a = \sqrt[3]{2}$. a is irrational and $a^{\sqrt{3}}$ is rational. So the theorem holds in this case.

Case 2: If x is irrational, then $x^{\sqrt{3}} = (\sqrt[3]{2}^{\sqrt{3}})^{\sqrt{3}} = (\sqrt[3]{2})^3 = 2$. So x is irrational and $x^{\sqrt{3}}$ is rational. So the theorem holds in this case.
Therefore, the theorem holds in every cases.

2 Problem 3.18

(a)(i)A IFF B $\Leftrightarrow A \leftrightarrow B \Leftrightarrow$
$(A \to B) \wedge (B \to A) \Leftrightarrow$
$(\bar{A} \vee B) \wedge (A \vee \bar{B})$
So A IFF B is equivalent to (NOT(A) OR B) AND (A OR NOT(B)).

(ii)A XOR B $\Leftrightarrow (A \wedge \bar{B}) \vee (\bar{A} \wedge B)$
So A XOR B is equivalent to (A AND NOT(B)) OR (B AND NOT(A)).

Conclusion: Every propositional formular is equivalent to an AND-OR-NOT formular. This is because AND-OR-NOT can represent IMPLIES, IFF and every logic operators.

(b) This is because the OR and NOT can represent AND. A AND B can be represented as NOT $(A$ OR $B)$... according to the De Morgan's law.

(c) First, we can get such formulars by truth table easily.
NOT $x \Leftrightarrow x$ NAND x
x AND $y \Leftrightarrow (x$ NAND $y)$ NAND $(x$ NAND $y)$
x OR $y \Leftrightarrow (x$ NAND $x)$ NAND $(y$ NAND $y)$
Using these formular we can get:

A IMPLIES $B \Leftrightarrow$
$(A$ NAND $A)$ OR $B \Leftrightarrow ((A$ NAND $A)$ NAND $(A$ NAND $A))$ NAND $(B$ NAND $B)$

3 Problem 3.23

(a) A XOR B XOR $C \Leftrightarrow$
$((A \wedge \bar{B}) \vee (\bar{A} \wedge B)) \vee \bar{C} \vee \neg((A \wedge \bar{B}) \vee (\bar{A} \wedge B)) \vee C \Leftrightarrow$
$(A \wedge \bar{B} \wedge \bar{C}) \vee (\bar{A} \wedge B \wedge \bar{C}) \vee (((A \wedge B) \vee (\bar{B} \wedge \bar{A})) \wedge C) \Leftrightarrow$
$(A \wedge \bar{B} \wedge \bar{C}) \vee (\bar{A} \wedge B \wedge \bar{C}) \vee (\bar{A} \wedge \bar{B} \wedge C) \vee (A \wedge B \wedge C)$

(b) A XOR B XOR $C \Leftrightarrow$
$((A \vee B) \wedge (\bar{A} \vee \bar{B})) \vee C \wedge \neg((A \vee B) \wedge (\bar{A} \vee \bar{B})) \vee \bar{C} \Leftrightarrow$
$(\bar{A} \vee \bar{B} \vee C) \wedge (A \vee B \vee C) \wedge (((A \vee \bar{B}) \wedge (\bar{A} \vee B)) \vee C) \Leftrightarrow$
$(A \vee \bar{B} \vee \bar{C}) \wedge (\bar{A} \vee B \vee \bar{C}) \wedge (\bar{A} \vee \bar{B} \vee C) \wedge (A \vee B \vee C)$

4 Problem 3.38

(a) $\exists k.(n = mk)$

(b) If n is a prime number, then it doesn't have any factor(k, m) except 1 and n.

$$\neg\exists k, m.\ (k \neq 1) \wedge (m \neq 1) \wedge (n = mk)$$

(c)We define that:

$$isPrime(a) ::= a\ is\ a\ prime\ number.$$
$$isFactor(n, a) ::= a\ is\ a\ factor\ of\ n.$$

So we can get isPrime(a) and $isFactor(n, a)$:

$$isPrime(a) \Leftrightarrow \neg\ \exists a_1, a_2.\ (a_1 \neq 1) \wedge (a_2 \neq 1) \wedge (a = a_1 a_2)$$
$$isFactor(n, a) \Leftrightarrow \exists k.(n = ak)$$

Also, we can get isPrimeFactor(n, a):

$$isPrimeFactor(n, a) \Leftrightarrow isPrime(a) \wedge isFactor(n, a)$$

If n is a power of a prime, it will have only one prime factor a. We can get the final predicate using the uniqueness quantification.

$$\exists a \forall b.(isPrimeFactor(n, b) \leftrightarrow b = a).$$

5 Problem 3.40

Solution 1: (a) We can translate 'the student x has sent e-mail to n people' first. The n people are $\{d_1, d_2, \ldots, d_n\}$.

$$E(x, d_1) \wedge E(x, d_2) \wedge \ldots \wedge E(x, d_n)$$

Now we want to say that none of the n students is x. However, some students in n can be the same person. This is because we consider 'at most n students' in this case.

$$x \neq d_1 \wedge x \neq d_2 \wedge \ldots \wedge x \neq d_n$$

Now,we must think of a way to say that the only people x might have e-mailed are x,$d_1, d_2, \ldots, d_n$:

$$\forall s, E(x, s) \rightarrow (s = x) \vee (s = d_1) \vee (s = d_2) \vee \ldots \vee (s = d_n)$$

Finally, we can say there is some student who emailed at most n students by existentially quantifying x, $d_1, d_2, \ldots, d_n$. So the complete translation of $atMost(x, n)$ is:

$$\exists x \exists d_1 \exists d_2 \ldots \exists d_n . E(x, d_1) \wedge E(x, d_2) \wedge \ldots \wedge E(x, d_n) \wedge \tag{1}$$
$$x \neq d_1 \wedge x \neq d_2 \wedge \ldots \wedge x \neq d_n \wedge \tag{2}$$
$$\forall s, E(x, s) \rightarrow (s = x) \vee (s = d_1) \vee (s = d_2) \vee \ldots \vee (s = d_n) \tag{3}$$

(b)We can translate $Exactly(x, n)$ first. If we want to get $Exactly(x, n)$, we have to make sure $d_1, d_2, \ldots, d_n$ are different from each other:

$$\neg \exists d_a, d_b .\ d_a, d_b \in \{d_1, d_2, \ldots, d_n\}, (a \neq b) \wedge (d_a = d_b)$$

So the $Exactly(x, n)$ can be translate as:

$$\exists x \exists d_1 \exists d_2 \ldots \exists d_n . E(x, d_1) \wedge E(x, d_2) \wedge \ldots \wedge E(x, d_n) \wedge \tag{4}$$
$$x \neq d_1 \wedge x \neq d_2 \wedge \ldots \wedge x \neq d_n \wedge \tag{5}$$
$$\neg \exists d_a, d_b .\ d_a, d_b \in \{d_1, d_2, \ldots, d_n\}, (a \neq b) \wedge (d_a = d_b) \wedge \tag{6}$$
$$\forall s, E(x, s) \rightarrow (s = x) \vee (s = d_1) \vee (s = d_2) \vee \ldots \vee (s = d_n) \tag{7}$$

As to the $atLeast(x, n)$ we can get:

$$atLeast(x, n) \Leftrightarrow \neg(atMost(x, n) \wedge Exactly(x, n)) \tag{8}$$

Solution 2: We can first get $Exactly(x, m)$.

$$\exists x \exists d_1 \exists d_2 \ldots \exists d_m . E(x, d_1) \wedge E(x, d_2) \wedge \ldots \wedge E(x, d_m) \wedge$$
$$x \neq d_1 \wedge x \neq d_2 \wedge \ldots \wedge x \neq d_m \wedge$$
$$\neg \exists d_a, d_b .\ d_a, d_b \in \{d_1, d_2, \ldots, d_m\}, (a \neq b) \wedge (d_a = d_b) \wedge$$
$$\forall s, E(x, s) \rightarrow (s = x) \vee (s = d_1) \vee (s = d_2) \vee \ldots \vee (s = d_m)$$

Then we can get $Exactly(x, n)$, $atLeast(x, n)$ and $atMost(x, n)$ by discussing the value of n.

$$Exactly(x, n) \Leftrightarrow Exactly(x, m) \wedge (m = n) \tag{9}$$

$$atLeast(x, n) \Leftrightarrow Exactly(x, m) \wedge (m > n) \tag{10}$$

$$atMost(x, n) \Leftrightarrow Exactly(x, m) \wedge (m < n) \tag{11}$$