Assessment 2

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1 Problem 10.38

(a)
$aRb \Rightarrow L(a) \subset L(b)$: If aRb, by transitivity every $x \in A$ means that xRa and also xRb. Consequently, $x \in$ L(b), which means $L(a) \subseteq L(b)$. Also, $b \in$ L(b) but $b \notin L(a)$, which means $L(a) \subset L(b)$.
$L(a) \subset L(b) \Rightarrow aRb$: If $L(a) \subset L(b)$, $a \in L(a)$ and also $a \in L(b)$, which means aRb.
In conlusion, the statement(10.16) holds.
(b) PROOF:
If $L(a) = L(b)$, then for every $x \in L(a)$ also $x \in L(b)$. When $x = a$, there is $a \in L(a)$ also $a \in L(b)$. If $a \in L(b)$ and $a \neq b$, there must be aRb. For the same reason, when $x = b$ and $a \neq b$, there must be bRa. But R is asymetric, so it is a contradiction. So there must be $a = b$.
(c) Let A = {1,2,3} aRb ::= a is a subset of b. Let a = {1,2} and b = {2,3}, then L(a) = L(b) = {{1,2,3}}. But $a \neq b$, which means the conclusion of part(b) would not hold in this case.

2 Problem 14.37

PROOF: We know that $f = O(g)$ and $g = O(f)$, which means:

$$\lim_{x\to\infty} \sup \frac{f(x)}{g(x)} < \infty \ , \ \lim_{x\to\infty} \sup \frac{g(x)}{f(x)} < \infty$$

Thus, $\frac{ln\ g}{ln\ f} = \frac{ln\ g - ln\ f + ln\ f}{ln\ f} = \frac{ln \frac{f}{g}}{ln\ f} + 1$. Because $\lim_{x\to\infty} \frac{f(x)}{g(x)} < \infty$, we can get: $ln\frac{f}{g} < \infty$. And $\lim_{x\to\infty} f(x) = 0$. So $\frac{ln\frac{f}{g}}{ln\ f} \to 0$. Therefore, $\frac{ln\frac{f}{g}}{ln\ f} + 1 =$ 1.

3 Problem 15.16

(a) We know that there is an order of the four groups but there is no order of the members in a specific group. For a specific sequence of four groups like {{A, B, C},{D, E, F},{G, H, I},{J, K, L}}, consider the first group {A, B, C}, the following 6 lists of students can all map to it. ("—" is just to make it easilier to read.)

> ABC—DEFGHIJKL, ACB—DEFGHIJKL, BAC—DEFGHIJKL,
> BCA—DEFGHIJKL, CAB—DEFGHIJKL, CBA—DEFGHIJKL.

Thus, $k = 6^4 = 1296$.
(b) For a specific sequence of 4 groups: ({Group 1,Group 2,Group 3,Group 4}), if there is no order between them, it will be 4! sequences map one group assignment.
Thus, $j = 4! = 24$.
(c) There are 12! different lists of 12 students. And there is a $k - to - 1$ mapping from the lists of students to the sequences of groups. Also, there is a $j - to - 1$ mapping from the sequences of groups to the group assignments. Thus, assume there are t group assignments:

$$t = \frac{12!}{k \cdot j} = \frac{12!}{1296 \times 24} = 15400$$

(d)
Different lists of $3n$ students: $3n!$.
There is a $6^n - to - 1$ mapping from the lists of students to the sequences of groups.
There is a $n! - to - 1$ mapping from the sequences of groups to the non-order n groups of 3.
Thus, assume there are m ways:

$$m = \frac{3n!}{6^n \cdot n!}$$

4 Problem 15.47

All the integers in $\{1, 2, 3, \ldots, 2n\}$ can be written uniquely in the form $2^a b$, where $a \leq 0, b \leq 1$, and b is not a multiple of 2. Now apply the pigeonhole principle:
Pigeons: $2n$ numbers.
Pigeonholes: each value of b is a hole. There are n holes.
If there are $n + 1$ numbers selected, there must be at least two numbers

share the same value of b and their a value is different, which means their quotient is a power of 2.

5 Problem 15.67

Algebraic manipulication:

$$\binom{n}{r}\binom{r}{k} = \frac{n!}{(n-r)!r!}\frac{r!}{(r-k)!k!}$$
$$= \frac{n!}{(n-r)!(r-k)!k!}$$
$$= \frac{n!}{k!(n-k)!}\frac{(n-k)!}{(r-k)!(n-r)!}$$
$$= \frac{n!}{k!(n-k)!}\frac{(n-k)!}{(r-k)![(n-k)-(r-k)]!}$$
$$= \binom{n}{k}\binom{n-k}{r-k}$$

Combinatorial argument: Suppose we are forming a r-person ACM team with k people from the ACM team to join the competition.
Method 1: First choose the ACM team in $\binom{n}{r}$ ways. Then choose $\binom{r}{k}$ students to join the competition. So there are totally $\binom{n}{r}\binom{r}{k}$ ways.
Method 2: First choose the k students to join the competition in $\binom{n}{k}$ ways. Then choose the other (r-k) ACM team mumbers from the remaining pool of (n-k) people. So there are $\binom{n}{k}\binom{n-k}{r-k}$ ways.
Both two methods solve the same question, so there must be:

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

6 Problem 15.11

(a) a^k

(b) $a^k - a$

(c) Suppose z is $a_1a_2a_3\ldots a_k$. Then u will be

$$a_{n+1}a_{n+2}a_{n+3}\ldots a_ka_1a_2a_3\ldots a_n$$

and v will be like

$$a_{m+n+1}a_{m+n+2}a_{m+n+3}\ldots a_ka_1a_2a_3\ldots a_na_{n+1}a_{n+2}a_{n+3}\ldots a_{m+n}.$$

Therefore, we can find that v is the length-$(n-m)$ rotation of z.

(d) PROOF:First, to prove "$\approx$" is reflexive. $v \approx v$ holds when $n = 1$. Second, to prove "$\approx$" is symmetric. We can suppose length$|v| =$ length$|z| = k$. If v is a length-n rotation of z, then according to **(c)** z is a length-$k-n$ rotation of v, which means "$\approx$" is symmetric. Third, to prove it is transitive, we can use the conclusion in **(c)**. Thus, "$\approx$" is an equivalance relation.

(e)PROOF: We can prove by induction on $|xy|$. Suppose x and y are not empty strings.(If they are, when $|xy| = 0$, x, y, u are all empty strings. The theorem holds. When $|xy| = 1$, one of x and y is empty. The theorem also holds.) Let's start at $|xy| = 2$, then $x = y = u$. If $|xy| > 2$ and $|x| = |y|$, when $x = y = u$, the theorem holds. If $|x| < |y|$, then there must be a w that $xw = y = wx$, which means $xw = wx$. Since $|xw| < |xy|$, we end up with the same problem for smaller words($xw = wx$ is just like $xy = yx$), so we can conclude by induction. According to the induction hypothesis there exists u such that $x, w \in u*$. Since $y = wx$, $y \in u*$.

(f) If p is prime and z is a length-p string containing at least two different characters. Then after doing length-1 rotation for p times, we can get p different strings.(There is no two same strings because p is a prime number.) So z is equivalent under $\approx$ to exactly p strings (counting itself).

(g) We can get a^p strings for a length-p string from a-character alphabet. $a^p - a$ have at least 2 different characters. At the same time, the $\approx$ is a p-to-1 relation in all the $a^p - a$ strings, which means $a^p - a$ must be divisible by p.

7 Problem 16.64.

(a) Lets combine student B and C as student K, so there is $(7\times7!-6\times6!)\times2 = 61920$.
(b) Suppose all the lineups satisfy Rule I are in set A. Also, B is for Rule II and C is for Rule III.
$|A|$:$10! - 9! = 3265920$
$|B|$:$9! \times 2 = 725760$
$|C|$:$9! = 362880$
$|A \cap B|$:$(9! - 8!) \times 2 = 645120$
$|B \cap C|$:$(7 \times 7!) \times 2 = 70560$
$|A \cap C|$:$9! - 8! = 322560$
$|A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|B\cap C|-|A\cap C|+|A\cap B\cap C| = 3378240$

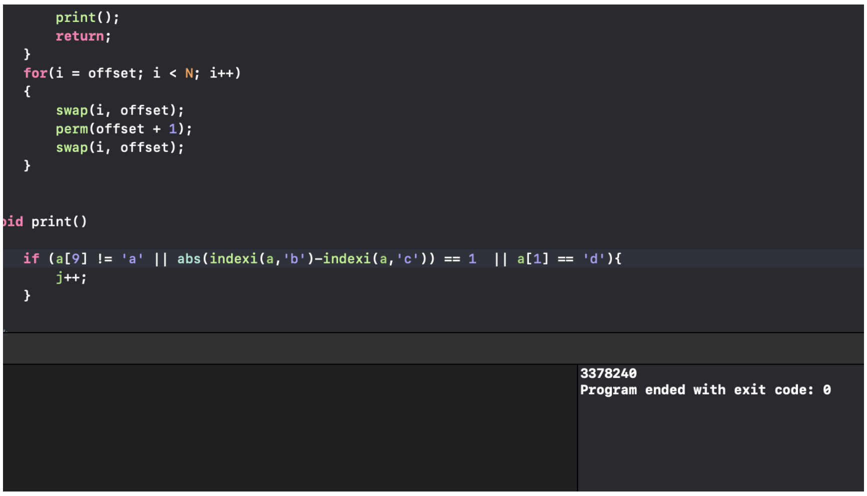

Figure 1: Using Xcode can get the same answer.

8 Problem 15.51.

(a)PROOF: We can use contradiction to prove. The length of the circle is c=1, so we only need to prove $k\sqrt{2} - m\sqrt{2} \neq nc(n = 0, 1, 2, \ldots)$ unless $m = k$. If $m \neq k$, then $(k - m)\sqrt{2}$ is irrational. (This is because $(k - m)$ is rationl and $\sqrt{2}$ is irrational.) But nc is rational. It is a contradiction. Only when $k = m$, then $k - m = 0$ and $k\sqrt{2} - m\sqrt{2} = nc(n = 0)$.

(b) We can prove by contradiction. Suppose any two of the n points are more than $1/n$ distance from each other. Then the clockwise distance from some point A to this point itself is more than 1. This is a contradiction. So there have to be two that are at most distance $1/n$ from each other.

(c) Let $\{an\}$ denote $an - \lfloor an \rfloor$. $S = \{an : a = \sqrt{2}, n = 0, 1, 2, 3, ...\}$. We need to prove there are at least one element of S is in $[0, 1/n]$. If we have proved this, then there must be at least one element in $[(k-1)/n, k/n]$.
Now we prove there are at least one element in S is in $[0, 1/n]$. First, we divide the $[0, 1]$ into n intervals. For set T = $\{\{\text{at}\}\}$,$(t = 1, 2, 3, \ldots, n + 1)$, according to the pigeonhole principle, there must be two elements of T are in the same interval. Suppose the two elements are $\{ap\}, \{aq\}$, then let $j = |p - q|$. There must be $0 < \{aj\} < 1/n$. So there are at least one element of S is in $[0, 1/n]$. In conclusion, the marked points are dense on the circle.