Questions & Solutions: Randomized Algorithms

(1)

You are given a function *Random()* such that:

- it returns 0 with probability 1/2,
- it returns 1 with probability 1/2.

By calling the function *Random()*, write an algorithm such that:

- it is a Las-Vegas algorithm,
- it takes a positive integer n as input,
- it returns a random integer between 1 and n (with the same probability).

Analyze the expected time complexity of your algorithm.

Referenced solution:

ALG_1(n)

$L \leftarrow \lceil \log_2 n \rceil$
Create an array $A[1..L]$
Repeat
    for $i \leftarrow 1$ to L
        $A[i] \leftarrow$ *Random()*
    Let X be the **binary number** formed by $A[1..L]$.
    $X \leftarrow X + 1$
Until $X \leq n$
Return X

Explanation:
- $A[1..L]$ is the binary number of X.
The binary representation of an integer between 1 and n at most has $\log n$ digits.

- For each position in $A[1..L]$, $A[i]$ is selected uniformly at random from {0,1}.
In this case, each integer is chosen with the same probability.

Analysis:
- In the loop, each iteration takes $O(L) = O(\log n)$ time
- Since $n > 2^{L-1}$, the probability of success at Line 8 is at least 1/2.
- By the geometric distribution formula, the expected number of iterations: $\frac{1}{(1/2)} = 2$
- Time complexity $= O(2 * logn) = O(\log n)$

(2)

You are given a function *Random-Bias()* such that:

- it returns 0 with probability p,
- it returns 1 with probability $1-p$.

where p is a constant and $0 < p < 1$. However, you do not know the value of p.
By calling the function *Random-Bias()*, write an algorithm such that:

- it is a Las-Vegas algorithm,
- it returns 0 with probability 1/2; it returns 1 with probability 1/2.

Express the expected time complexity of your algorithm in terms of p.

Referenced solution:

ALG_2()

Repeat

$\quad X \leftarrow Random - Bias()$

$\quad Y \leftarrow Random - Bias()$

Until $X \neq Y$

Return X

Analysis:

- In the loop, each iteration takes $O(1)$ time.
- The probability of success at Line 4 is $2p(1{-}p)$
- By the geometric distribution formula, the expected number of iterations is $\frac{1}{(2p(1-p))}$
- Time complexity: $O(\frac{1}{2p(1-p)}) = O(\frac{1}{p(1-p)})$

Proof:

Let A be the event "$X = 0$" and B the event "$X + Y = 1$".
By definition, the conditional probability $Pr\{A \mid B\} = \frac{Pr\{A \cap B\}}{Pr\{B\}}$.

$$Pr\{output = 0\}$$

$$= Pr\{X = 0 \mid X + Y = 1\}$$

$$= \frac{Pr\{X=0 \cap X+Y=1\}}{Pr\{X+Y=1\}}$$

$$= \frac{Pr\{X=0 \cap Y=1\}}{Pr\{X=0 \cap Y=1\} + Pr\{X=1 \cap Y=0\}}$$

$$= \frac{p(1-p)}{p(1-p)+(1-p)p}$$

$$= \frac{1}{2}$$

In the same way, $Pr\{output = 1\} = \frac{1}{2}$.