# Solutions: Amortized Analysis

1. Consider inserting the keys 10, 22, 31, 4, 15, 28, 17, 88, 59 into a hash table of length $m = 11$ using open addressing with the auxiliary hash function $h^{'}(k) = k$.

Illustrate the result of inserting these keys using linear probing, using quadratic probing with $c_1 = 1$ and $c_2 = 3$, and using double hashing with $h_1(k) = k$, and $h_2(k) = 1 + (k \ mod(m-1))$.

**Solution:** We use $T_i$ to represent each time stamp and $i \in N$ start with $i = 0$, and if encountering a collision, then iterate $i$ from $i = 1$ until there is no collision.

For linear probing: Hash function is $h(k, i) = k + i$ and the procedure of inserting keys to hash table can be presented with the table below. In $T_8$, we got the final hash table.

| $h(k,i) = (k+i) \mod 11$ | $T_0$ | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ | $T_7$ | $T_8$ |
|---|---|---|---|---|---|---|---|---|---|
| $0 \mod 11$ | | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 |
| $1 \mod 11$ | | | | | | | | 88 | 88 |
| $2 \mod 11$ | | | | | | | | | |
| $3 \mod 11$ | | | | | | | | | |
| $4 \mod 11$ | | | | 4 | 4 | 4 | 4 | 4 | 4 |
| $5 \mod 11$ | | | | | 15 | 15 | 15 | 15 | 15 |
| $6 \mod 11$ | | | | | | 28 | 28 | 28 | 28 |
| $7 \mod 11$ | | | | | | | 17 | 17 | 17 |
| $8 \mod 11$ | | | | | | | | | 59 |
| $9 \mod 11$ | | | 31 | 31 | 31 | 31 | 31 | 31 | 31 |
| $10 \mod 11$ | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |

For quadratic probing: Hash function is $h(k,i) = (k + i + 3i^2)$ and the procedure of inserting keys to hash table can be presented with the table below. In $T_8$ , we got the final hash table.

| $h(k,i) = (k+i+3i^2)$ mod 11 | $T_0$ | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ | $T_7$ | $T_8$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 mod 11 | | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 |
| 1 mod 11 | | | | | | | | | |
| 2 mod 11 | | | | | | | | 88 | 88 |
| 3 mod 11 | | | | | | | 17 | 17 | 17 |
| 4 mod 11 | | | | 4 | 4 | 4 | 4 | 4 | 4 |
| 5 mod 11 | | | | | | | | | |
| 6 mod 11 | | | | | | 28 | 28 | 28 | 28 |
| 7 mod 11 | | | | | | | | | 59 |
| 8 mod 11 | | | | | 15 | 15 | 15 | 15 | 15 |
| 9 mod 11 | | | 31 | 31 | 31 | 31 | 31 | 31 | 31 |
| 10 mod 11 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |

For double hashing: Hash function is $h(k,i) = (k + i(1 + k \ mod \ 10))$ and the procedure of inserting keys to hash table can be presented with the table below. In $T_8$, we got the final hash table.

| $h(k,i) = (k + i(1 + k \mod 10)) \mod 11$ | $T_0$ | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ | $T_7$ | $T_8$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 mod 11 | | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 |
| 1 mod 11 | | | | | | | | | |
| 2 mod 11 | | | | | | | | | 59 |
| 3 mod 11 | | | | | | | 17 | 17 | 17 |
| 4 mod 11 | | | | 4 | 4 | 4 | 4 | 4 | 4 |
| 5 mod 11 | | | | | 15 | 15 | 15 | 15 | 15 |
| 6 mod 11 | | | | | | 28 | 28 | 28 | 28 |
| 7 mod 11 | | | | | | | | 88 | 88 |
| 8 mod 11 | | | | | | | | | |
| 9 mod 11 | | | 31 | 31 | 31 | 31 | 31 | 31 | 31 |
| 10 mod 11 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |

2. Suppose we perform a sequence of $n$ operations on a data structure in which the $i$th operation costs $i$ if $i$ is an exact power of 2, and 1 otherwise. Use aggregate analysis to determine the amortized cost per operation.

**Solution:**

For $n$ operations, there are $\lfloor log_2 n \rfloor + 1$ exact power of 2, e.g., 1, 2, 4, …, $2^{\lfloor log_2 n \rfloor}$.
So, the total cost of these operations is a geometric sum:

$$\sum_{i=0}^{\lfloor log_2 n \rfloor} 2^i = 2^{\lfloor log_2 n+1 \rfloor} - 1 \leq 2^{(log_2 n)+1} = 2n$$

The rest of the operations are cheap, each having a cost of *1*, and there are less than $n$ such operations.
So, the total cost $T(n) \leq 2n + n = 3n = O(n)$

It means that the amortized cost per operation is $\frac{O(n)}{n} = O(1)$

3. Redo the question.2 using a potential method of analysis.

**Solution:**

Define $\Phi$: $\Phi(D_0) = 0$; $\Phi(D_i) = \left(2i - 2^{\lfloor log_2 i \rfloor + 1}\right) + 1, for\ i > 0$

$$\Phi(D_i) \geq 0 = \Phi(D_0)$$

For $i$, which is an exact power of 2, the potential difference is:

$$\Phi(D_i) - \Phi(D_{i-1}) = (2i - 2i + 1) - (2(i-1) - i + 1) = 2 - i$$

So, $\hat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1}) = i + 2 - i = 2$

For $i$, which is not an exact power of 2: $2^{x-1} < i < 2^x, x \in N^+$

$$\Phi(D_i) - \Phi(D_{i-1}) = (2i - 2^x + 1) - (2(i-1) - 2^x + 1) = 2$$

So, $\hat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 2 = 3$

Which means, the amortized cost per operation is $O(1)$