Decomposition of Covariance [3]

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1. Introduction

Given a normal distribution [5] with mean zero and a full covariance matrix [1], it can be known that the iso-contours take an ellipsoidal form with the major and minor axes at arbitrary orientations. Now consider viewing the distribution in a new coordinate frame where the axes are aligned with the axes of the normal (Figure 1): in this new frame of reference, the covariance matrix Σ_{diag} diag will be diagonal. Denote the data vector in the new coordinate system by x = $[x_1, x_2]^T$ where the frames of reference are related by x

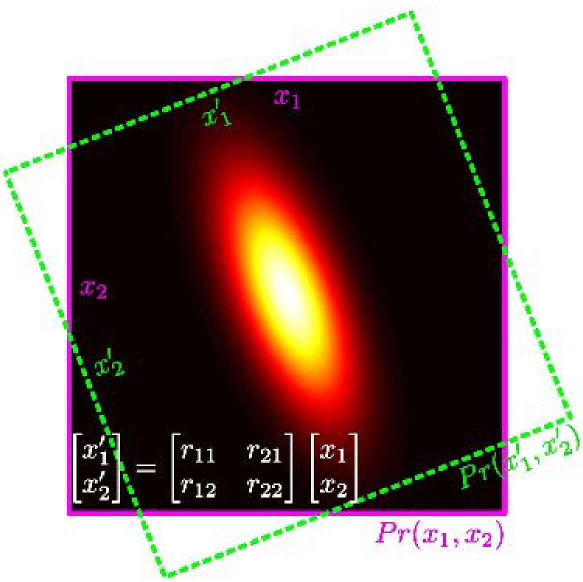

Figure 1. Decomposition of full co-variance. For every bivariate normal distribution in variables x_1 and x_2 with full covariance matrix, there exists a coordinate system with variables x_1' and x_2' where the covariance is diagonal: the ellipsoidal iso-contours align with the coordinate axes x_1' and x_2' in this canonical coordinate frame. The two frames of reference are related by the rotation matrix **R** which maps (x_1', x_2') to (x_1, x_2). From this it follows (see text) that any covariance matrix Σ can be broken down into the product $R^T\Sigma'_{diag}R$ of a rotation matrix **R** and a diagonal covariance matrix Σ'_{diag}.

$$Pr(x) = \frac{1}{2\pi^{\frac{D}{2}}\sqrt{|\Sigma'_{diag}|}} exp[-0.5x'^T\Sigma'^{-1}_{diag}x'] \quad (1)$$

If we convert back to the original axes by substituting in x 0 = Rx to get in Equation 2.

$$\begin{aligned} Pr(x) &= \frac{1}{2\pi^{\frac{D}{2}}\sqrt{|\Sigma'_{diag}|}} exp[-0.5(Rx)^T\Sigma'^{-1}_{diag}Rx] \\ &= \frac{1}{2\pi^{\frac{D}{2}}\sqrt{|R^T\Sigma'_{diag}R|}} exp[-0.5(x)^T(R^T\Sigma'_{diag}R)^{-1}x] \end{aligned} \quad (2)$$

where we have used $|R^T\Sigma R| = |R^T|.|\Sigma|.|R| = 1.|\Sigma|.1 = |\Sigma|$. Equation 3 is a multivariate normal with covariance.

$$\Sigma_{full} = R^T\Sigma'_{diag}R \quad (3)$$

2. Conclusions

It can be concluded that full covariance matrices are expressible as a product of this form involving a rotation matrix [2] R and a diagonal covariance matrix Σ, which is possible to retrieve these elements from an arbitrary valid covariance matrix Σ_{full} by decomposing it in this way using the singular value decomposition.

The matrix **R** contains the principal directions of the ellipsoid in its columns. The values on the diagonal of Σ_{diag} encode the variance (and hence the width of the distribution) along each of these axes. Hence the results of the eigen-decomposition can be used to answer questions about which directions in space are most and least certain.

References

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