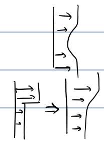
Mid Term.

Final Pres.



$\overline{ave} + flu'.$

$v = \bar{v} + v'$

Index Notation

$u = \sum_i u_i e_i \longrightarrow u_i$

$u \cdot v = u_i v_i$

$u v = u_i v_j$

$e_i e_j = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

$\nabla = \nabla_i = e_i \frac{\partial}{\partial x_i}$

WRONG:

$\cancel{u_i v_i v_i}$ only 2 dumy

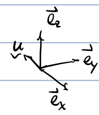
$\cancel{u_i v_j}$ same direction

Index notation & Cartesian Tensors

$\vec{i} \quad \vec{j} \quad \vec{k}$

$\vec{u}: \quad u = (u_x, u_y, u_z) \rightarrow u = u_x e_x + u_y e_y + u_z e_z$

(u, v, w)



e_x, e_y, e_z unit cartesian vectors (orthogonal)

$u = u_1 e_1 + u_2 e_2 + u_3 e_3$

$= \sum_{i=1}^{3} u_i e_i$

dot product v & u: $\quad u \cdot v = \left(\sum_{i=1}^{3} u_i e_i\right)\left(\sum_{i=1}^{3} v_i e_i\right) = \sum_{i=1}^{3} u_i v_i$

$e_i \cdot e_j = 0$ if $i \neq j$

$= 1$ if $i = j$

"Calculus": Gradient Operator $\nabla = \sum_{i=1}^{3} e_i \frac{\partial}{\partial x_i}$

$\nabla = (\partial_x, \partial_y, \partial_z)$ or $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

Scalar fn. $\phi(x) = \phi(x_1, x_2, x_3)$

Ex. $\nabla\phi = \sum_{i=1}^{3} e_i \frac{\partial}{\partial x_i}\phi = e_1 \frac{\partial\phi}{\partial x_1} + e_2 \frac{\partial\phi}{\partial x_2} + e_3 \frac{\partial\phi}{\partial x_3}$

Gradient Op. on vector $\quad \nabla \cdot u = \left(\sum_{i=1}^{3} e_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3} u_j e_j\right) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} u_i$

divergence

$u \cdot \nabla\phi = \left(\sum_{i=1}^{3} u_i e_i\right) \cdot \left(\sum_{j=1}^{3} e_j \frac{\partial}{\partial x_j}\phi\right) = \sum_{i=1}^{3} u_i \frac{\partial}{\partial x_i}\phi$ or $\left(u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3}\right)\phi$

Repeated Index Notation

No need $\sum_i$ when an index appear twice in a "product"

Einstein Summation rule

$u \cdot v = \sum_{i=1}^{3} u_i v_i = u_i v_i$ contraction over "i"

$u \cdot \nabla\phi = u_i \frac{\partial}{\partial x_i}\phi$

$\nabla \cdot u = \frac{\partial}{\partial x_i} u_i$

independent

$u_i v_i = u_j v_j$

"dummy" index

$\cancel{\sum_{i=1}^{3} u_i}$ $\quad u_i \ (i = 1, 2, 3)$ vector $\quad \cancel{v_i v_i v_i}$

only 2 indices repeated

Vectors: 1 index $\quad$ Ex. x or x_i position

u or u_i velocity

a or a_i acceleration

f or f_i force

$\nabla\phi$ or $\frac{\partial\phi}{\partial x_i}$ grad(ϕ) gradient

Gibbs Notation $\quad$ Index Notation

Ex. $((\nabla \times b) \times a - \sigma \times b)$

2nd Tensors: 2 indices (matrices) in 3D: 3×3 matrices

e.g. Stress tensor σ_{ij} $\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zy} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$ or $\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \ddots & \\ \sigma_{31} & & \ddots \end{pmatrix}$

$(i = 1,2,3 \quad j = 1,2,3)$

Means: $\sigma = \sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_{ij} e_i e_j \quad e_i, e_j$ tensor basis

$= \sigma_{ij} e_i e_j$

2nd Tensor

$\sigma = \sigma_{ij} e_i e_j$, use σ_{ij}

$\sigma = \begin{bmatrix} & \vdots & \\ \cdots & \sigma_{ij} & \cdots \\ & \vdots & \end{bmatrix}$ i, j $\quad \delta_{ij} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$

Dyadic products between two vectors $\quad (u \cdot v)$

$(u \times v)$

$(u v)$ or $(u \otimes v)$

$q = u \otimes v \quad q_{ij} = u_i v_j$

$\begin{pmatrix} q_{11} & q_{12} & \cdots \\ & \ddots & \\ q_{31} & & \ddots \end{pmatrix} = \begin{pmatrix} u_1 v_1 & \cdots & u_1 v_3 \\ \vdots & \ddots & \\ u_3 v_3 & & u_3 v_3 \end{pmatrix}$

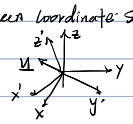
$\underset{\sim}{u}$, u_i $i=1,2,3$ (u_1, u_2, u_3) (3–

$\underset{\approx}{\sigma}$, σ_{ij} $i,j=1,2,3$

transformation between coordinate sys.

$u_i' = R_{ij} u_j$

Rotation matrix



tensor $\nabla_{ij}' = R_{ik} R_{jp} \nabla_{kp}$ ✓ generalizable to any higher order tensor

Rules

$\underset{\sim}{c} = \underset{\sim}{a} + \underset{\sim}{b} \to c_i = a_i + b_i$ $= \cancel{a_i + b_j}$

$\underset{\approx}{t} = \underset{\approx}{\nabla}^a + \underset{\approx}{\nabla}^b \to t_{ij} = \nabla_{ij}^a + \nabla_{ij}^b$ (or ∇_{ji}^b)

$\underset{\approx}{t} = \underset{\approx}{\nabla}^a + (\underset{\approx}{\nabla}^b)^T$

$\nabla(\underset{\sim}{a}\cdot\underset{\sim}{b}) = (\nabla\underset{\sim}{a})\cdot\underset{\sim}{b} + (\nabla\underset{\sim}{b})\cdot\underset{\sim}{a} \neq \underset{\sim}{a}\cdot\nabla\underset{\sim}{b} + \underset{\sim}{b}\cdot\nabla\underset{\sim}{a}$

$\frac{\partial}{\partial x_j}(a_i b_i) = \frac{\partial a_i}{\partial x_j} b_i + a_i \frac{\partial b_i}{\partial x_j} \left(= \frac{\partial b_i}{\partial x_j} a_i\right)$

$a \circ \nabla b = a_i \frac{\partial b_i}{\partial x_j} = \frac{\partial b_i}{\partial x_j} a_i$

$\neq$

$(a\cdot\nabla) b = a_i \frac{\partial b_j}{\partial x_i}$

Diadic product $\nabla \underset{\sim}{a} = \frac{\partial a_i}{\partial x_j}$ NOT symmetric

unit tensor (isotropic tensor)

Kronecker "delta"

$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} = \begin{bmatrix} \ddots & & 0 \\ & 1 & \\ 0 & & \ddots \end{bmatrix}$

Ex. $\delta_{ij} u_j \overset{?}{=} u_i$

prove. $C_i = \delta_{i1} u_1 + \delta_{i2} u_2 + \delta_{i3} u_3$, $(i=1,2,3)$

$= \begin{cases} \text{if } i=1, & C_1 = u_1 \\ i=2 & C_2 = u_2 \\ i=3 & C_3 = u_3 \end{cases}$

Others $\delta_{ij} u_i u_j = u_j u_j = u_j^2 = u_1^2 + u_2^2 + u_3^2 = |\underset{\sim}{u}|^2$

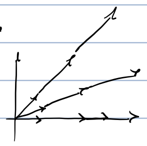
$\underset{\approx}{1} : (u \otimes u)$

$\underset{\approx}{1} : (\underset{\sim}{u}\,\underset{\sim}{u}) = |\underset{\sim}{u}|^2$

δ_{ij} is Gibbs : $\underset{\approx}{1}$

$\underset{\approx}{1} \circ \underset{\sim}{u} = \underset{\sim}{u}$

$a : b$



$\underset{\sim}{x}$ position vector

$\frac{\partial x_i}{\partial x_j} = \delta_{ij}$

$\nabla \underset{\sim}{x} = \underset{\approx}{1}$

trace $(\underset{\approx}{\nabla}) = \nabla_{jj} = \nabla_{11} + \nabla_{22} + \nabla_{33}$

trace $(\underset{\approx}{1}) = \delta_{ii} = \delta_{jj} = 3$ (3D)

Trace

$Tr(\underset{\approx}{\sigma}) = \sigma_{ii}$

$Tr(\underset{\approx}{\delta}) = \delta_{ii} = 3$

(dimension)

$\Sigma\,\circ$

$\epsilon_{ijk} = \begin{cases} 1 & ijk/kij/jki \\ -1 & ikj/kji/jik \\ 0 & i=j/j=k/i=k \end{cases}$

curl

$C_i = \underset{\sim}{u} \times \underset{\sim}{v} = \epsilon_{ijk} u_j v_k$

3^{rd} rank tensors C_{ijk} $\nabla\underset{\approx}{\nabla} = \frac{\partial \nabla_{ij}}{\partial x_k}$

unit 3^{rd} rank tensor

ϵ_{ijk} (Levy–Civita tensor)

Altrenating tensor

$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123 \text{ or } 231 \text{ or } 312 \\ -1 & \text{if } ijk = 213 \text{ or } 321 \text{ or } 132 \\ 0 & \text{if } i=j \text{ or } j=k \text{ or } i=k \quad \text{ex. } 112 \text{ or } 221 \end{cases}$

useful applications

$\underset{\sim}{c} = \underset{\sim}{u} \times \underset{\sim}{v}$ $(\underset{\sim}{u}\times\underset{\sim}{v})_i = \epsilon_{ijk} u_j v_k$

reminder : $(\underset{\sim}{u}\times\underset{\sim}{v}) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk} u_j v_k \underset{\sim}{e}_{(i)}$

eg. 1 component $(i=1)$ of $(\underset{\sim}{u}\times\underset{\sim}{v})$

$= \underset{1}{\epsilon_{123}} u_2 v_3 + \underset{-1}{\epsilon_{132}} u_3 v_2$

$= u_2 v_3 - u_3 v_2$

Can prove $\underset{\sim}{u}\times\underset{\sim}{v} = -\underset{\sim}{v}\times\underset{\sim}{u}$

Curl of vector field $\underset{\sim}{a}(\underset{\sim}{x}, t)$

Curl$(\underset{\sim}{a})$ $\nabla \times \underset{\sim}{a}$

$\underset{\sim}{c} = \nabla\times\underset{\sim}{a}$, $C_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} a_k = \epsilon_{ijk} \frac{\partial a_k}{\partial x_j}$

Vorticity : $\underset{\sim}{\omega} = \nabla \times \underset{\sim}{u}$: $\omega_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} u_k$

velocity

$\underset{\sim}{x}$: $\nabla \times \underset{\sim}{x} = 0$

$= \epsilon_{ijk} \frac{\partial x_k}{\partial x_j} = \epsilon_{ijk} \delta_{jk} = 0$

$\epsilon_{ijk}\epsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}$ (same)

r and $\vec{x}$ $f(r)$ in index Notation

$r = |x_i| = \sqrt{x_j x_j}$, $\frac{\partial}{\partial x_i}\sqrt{x_j x_j} = \frac{x_i}{r}$

$\left[\frac{\partial}{\partial x_i} f(r) = f' \frac{\partial r}{\partial x_i} = f' \frac{x_i}{r}\right]$

$\frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x_i}$

$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$

Invariance $|u|^2 = u_i u_i$

$I_t = t_{ii}$

$II_t = \frac{1}{2!}(t_{ii}t_{jj} - t_{ij}t_{ji})$

$III_t = \frac{1}{3!}(t_{ii}t_{jj}t_{kk} - 3t_{ii}t_{jk}t_{kj} + 2t_{ij}t_{jk}t_{ki})$

Cayleigh-Hamilton Theorem

$\underset{\sim}{t}^3 - I_t \underset{\sim}{t}^2 + II_t \underset{\sim}{t} + III_t \underset{\sim}{1} = 0$

$(\underset{\sim}{t})^3_{ij} = t_{im}t_{mn}t_{nj}$

Transport $\left(\frac{\partial}{\partial t} + \nabla\cdot\right)(\rho F) = \text{Source}$

$\left(\frac{D}{Dt}\right) = \left(\frac{\partial}{\partial t} + \underset{\sim}{u}\cdot\nabla\right)$

Conservation Laws

Mass conservation

$\frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\underset{\sim}{u}) = 0 \longrightarrow \frac{D}{Dt}\rho = -\rho(\nabla\cdot\underset{\sim}{u})$

$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_j}(\rho u_j) = 0$

Useful identity

(delta-delta identity)

$[\epsilon_{ijk}\epsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}]$

$\underset{\sim}{u}\times(\underset{\sim}{v}\times\underset{\sim}{w}) = (\underset{\sim}{u}\cdot\underset{\sim}{w})\underset{\sim}{v} - (\underset{\sim}{u}\cdot\underset{\sim}{v})\underset{\sim}{w}$

$= \epsilon_{ijk}u_j(\epsilon_{kpq}v_p w_q)$

$= \epsilon_{ijk}\epsilon_{kpq}u_j v_p w_q$

$= \epsilon_{kij}\epsilon_{kpq}u_j v_p w_q$

$= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})u_j v_p w_q$

$= \begin{matrix} i=p \\ j=q \end{matrix} \quad \begin{matrix} i=q \\ j=p \end{matrix}$

$= u_j v_i w_j - u_j v_j w_i$

$= (\underset{\sim}{u}\cdot\underset{\sim}{w})\underset{\sim}{v} - (\underset{\sim}{u}\cdot\underset{\sim}{v})\underset{\sim}{w}$

$\underset{\sim}{\nabla}\times(\underbrace{\underset{\sim}{\nabla}\times\underset{\sim}{u}}_{=\underset{\sim}{\omega}}) = (\underset{\sim}{\nabla}\cdot\underset{\sim}{u})\nabla$?

Scalar functions

radial functions 

radially symmetric functions $f(r)$

$r = |x_i| = \sqrt{x_i x_i} = \sqrt{\underset{\sim}{x}\cdot\underset{\sim}{x}}$

$\underset{\sim}{\nabla} f$ (vector) $\quad \frac{\partial}{\partial x_i} f = \frac{\partial}{\partial x_i} f(\sqrt{x_j x_j}) = \frac{\partial f}{\partial r}\cdot\frac{\partial}{\partial x_i}(\sqrt{x_j x_j}) = \frac{\partial f}{\partial r}\cdot\frac{1}{\sqrt{x_j x_j}}\cdot\frac{1}{2}\cdot x_p\frac{\partial x_p}{\partial x_i}$ (δ_{pi}, x_i)

$= f'\cdot\frac{x_i}{|x_i|} = \hat{x}_i\cdot f'$

unit vector in radial dir.

$\nabla\cdot\nabla f = \nabla^2 f(r) \neq \frac{\partial^2}{\partial r^2} f(r)$

$\nabla\cdot\left(f'\frac{x_i}{|x_i|}\right) = \left(\frac{\partial f'}{\partial x_i}\right)\frac{x_i}{\sqrt{x_i x_i}} + f'\frac{\partial}{\partial x_i}\left(\frac{x_i}{\sqrt{x_i x_i}}\right) = \ldots = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)f$ 3D.

Invariants: Vectors magnitude $|\underset{\sim}{u}|^2 \to (u_i u_i)$

2^{nd} rank tensor $\underset{\sim}{t}$: 3 invariant

t_{ij} $\quad I_t = t_{ii}$ (trace) $\quad \mathrm{Tr}(\underset{\sim}{t})$

$II_t = \frac{1}{2!}(t_{ii}t_{jj} - t_{ij}t_{ji})$ $\quad \mathrm{Tr}(\underset{\sim}{t}^2)$ $\quad \underset{\sim}{t}^2 = t_{ik}t_{kj}$

$III_t = \frac{1}{3!}(t_{ii}t_{jj}t_{kk} - 3t_{ii}t_{jk}t_{kj} + 2t_{ij}t_{jk}t_{ki})$ $\quad \mathrm{Tr}(\underset{\sim}{t}^2) = t_{ij}t_{ji}$

Cayleigh Hamilton Thm.

$\underset{\sim}{t}^3 - I_t\underset{\sim}{t}^2 + II_t\underset{\sim}{t} + III_t\underset{\sim}{1} = 0$

$(t_{ij})^3 = t_{im}t_{mn}t_{nj}$

$\underset{\sim}{a}(\underset{\sim}{b}) = \sum_{n=0}^{\infty} C_{ni}(\underset{\sim}{b})^n$ (only up to $n=2$)

Vector: $a(\underset{\sim}{b}) = \sum_{n=1}^{\infty} C_n(\underset{\sim}{b})^n = f(\underset{\sim}{b}\cdot\underset{\sim}{b})\underset{\sim}{b}$ only strechy $\underset{\sim}{b}$

$n=2$: $\underset{\sim}{b}\cdot\underset{\sim}{b} =$ scalar; $\underset{\sim}{b}\times\underset{\sim}{b} = 0$; $\underset{\sim}{b}\underset{\sim}{b} \to$ second rank tensor

n odd: $(\underset{\sim}{b})^3 = (\underset{\sim}{b}\cdot\underset{\sim}{b})\underset{\sim}{b}$ $\quad (\underset{\sim}{b}\underset{\sim}{b})\cdot\underset{\sim}{b} = |\underset{\sim}{b}|^2\underset{\sim}{b}$

$b_i b_j b_j$

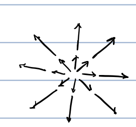
Conservation Laws

Mass conservation $\longrightarrow$ Conservative form

density $\rho(\underset{\sim}{x}, t)$ $\quad \frac{\partial}{\partial t}(\rho) + \frac{\partial}{\partial x_j}(\rho u_j) = 0$ $\quad \frac{\partial}{\partial t}\rho + u_j\frac{\partial}{\partial x_j}\rho = -\rho\frac{\partial u_j}{\partial x_j}$ Gauss Thm.

velocity $\underset{\sim}{v}(\underset{\sim}{x}, t)$ $\quad \frac{\partial}{\partial t}(\rho) + \nabla\cdot(\rho\underset{\sim}{u}) = 0$ $\quad \underbrace{\dot{\rho} + \underset{\sim}{u}\cdot\nabla\rho}_{\frac{D\rho}{Dt}} = -\rho\nabla\cdot\underset{\sim}{u}$

$u_i(\underset{\sim}{x}, t)$ $\quad \frac{D\rho}{Dt} = -\rho\nabla\cdot\underset{\sim}{u}$



$\|\hat{n}\| = 1$ unit vector

$\hat{n}\cdot\nabla\rho = \frac{\partial}{\partial n}\rho$

Constant ρ flows

if ρ constant along fluid tag. $\Leftrightarrow \nabla\cdot\underset{\sim}{u} = 0$

Conservative field: $\underset{\sim}{f} = -\nabla\phi$

Momentum Equation

$$\frac{\partial}{\partial t}(\rho u_i)+\underbrace{\frac{\partial}{\partial x_j}(\rho u_i u_j)}_{\text{Momentum flux}}=-\underbrace{\frac{\partial}{\partial x_j}\sigma_{ji}}_{\text{Total stress tensor}}+\underbrace{f_i}_{\text{Body force}}$$

$$\rho\frac{\partial u_i}{\partial t}+u_i\underbrace{\left(\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_j}(\rho u_j)\right)}_{\text{Mass conservatn}=0}+\rho u_j\frac{\partial u_i}{\partial x_j}$$

$$\left[\rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=-\frac{\partial}{\partial x_j}\sigma_{ji}\right] \quad \text{(General)}$$

Energy Equation $e_0 = \underset{\text{Total } E}{e}+\underset{\text{Internal } E}{\tfrac{1}{2}|u|^2}$

$$\frac{\partial}{\partial t}(\rho e_0)+\frac{\partial}{\partial x_j}(\rho u_j e_0)=\frac{\partial}{\partial x_j}(\sigma_{ji}u_i)+f_i u_i-\frac{\partial q_j}{\partial x_j}$$

Entropy Balance S

$$\frac{\partial}{\partial t}(\rho S)+\frac{\partial}{\partial x_j}(\rho u_j S)=$$

$$S\left(\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_j}(\rho u_j)\right)$$

$$\left[\rho\frac{\partial}{\partial t}S+\rho u_j\frac{\partial}{\partial x_j}S=\underbrace{\mathcal{N}}_{\text{Entropy generatn rate}}-\underbrace{\frac{\partial}{\partial x_j}\left(\frac{q_j}{T}\right)}_{\text{Entropy Surface flow}}\right]$$

Thermal Energy Eqn. e

$$\rho\left(\frac{\partial e}{\partial t}+u_j\frac{\partial e}{\partial x_j}\right)=\frac{\partial}{\partial x_j}(\sigma_{ji}u_i)-u_i\frac{\partial\sigma_{ji}}{\partial x_j}-\frac{\partial q_j}{\partial x_j}$$

$$\left[\rho\left(\frac{\partial e}{\partial t}+u_j\frac{\partial e}{\partial x_j}\right)=\sigma_{ji}\frac{\partial u_i}{\partial x_j}-\frac{\partial q_j}{\partial x_j}\right]$$

$$(\tau_{ji}-p\delta_{ij})(S_{ij}+\Omega_{ij}) \quad ?$$

$$\rho\frac{\partial e}{\partial t}+\underline{u}\cdot\nabla e=\underline{\underline{\sigma}}:\nabla\underline{u}-\nabla\cdot\underline{q}$$

$$\left[\rho\left(\frac{\partial e}{\partial t}+u_j\frac{\partial e}{\partial x_j}\right)=\tau_{ji}S_{ji}-pS_{jj}-\frac{\partial q_j}{\partial x_j}\right]$$

Stress tensor

$$\underline{\underline{\sigma}}=\underline{\underline{\sigma}}^{\text{"deviatoric"}}+\underline{\underline{\sigma}}^{\text{"isotropic"}}$$

$$\sigma_{ji}=\underbrace{\left(\sigma_{ji}-\tfrac{1}{3}\sigma_{kk}\delta_{ij}\right)}_{\tau_{ji}}+\underbrace{\tfrac{1}{3}\sigma_{kk}\delta_{ij}}_{-p\delta_{ij}}$$

Velocity gradient tensor

$$\frac{\partial u_i}{\partial x_j}=\underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}_{S_{ij}}+\underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right)}_{\Omega_{ij}}$$

$$(\nabla\underline{u})_{ij}=\underset{\text{Symmetric}}{S_{ij}}+\underset{\text{Anti-Symmetric}}{\Omega_{ij}}$$

Thermal Dynamics

Gibbs Eqn.

$$\frac{1}{dt}\left[Tds=de+pd\left(\tfrac{1}{\rho}\right)\right]$$

$$T\underbrace{\left(\frac{\partial s}{\partial t}+u_j\frac{\partial s}{\partial x_j}\right)}_{\text{Entropy Eqn.}}=\underbrace{\left(\frac{\partial e}{\partial t}+u_j\frac{\partial e}{\partial x_j}\right)}_{\text{Thermal E Eqn.}}+p\left(-\frac{1}{\rho^2}\frac{d\rho}{dt}\right) \quad \text{mass conser}$$

$$\rho\nabla\cdot u$$

$$T\mathcal{N}-T\frac{\partial}{\partial x_j}\left(\frac{q_j}{T}\right)=\tau_{ji}S_{ji}-\cancel{pS_{jj}}-\cancel{\frac{\partial q_j}{\partial x_j}}+\cancel{p\frac{\partial u_j}{\partial x_j}}$$

$$-\frac{\partial q_j}{\partial x_j}+\frac{1}{T}q_j\frac{\partial T}{\partial x_j} \qquad \frac{\partial}{\partial x_j}u_j$$

$$\left[T\mathcal{N}=-\frac{1}{T}q_j\frac{\partial T}{\partial x_j}+\tau_{ji}S_{ji}\right]$$

$$\frac{\partial}{\partial t}(\rho u_i)+\frac{\partial}{\partial x_j}(\underbrace{\rho u_i u_j}_{\text{Momentum flux}})=-\frac{\partial}{\partial x_j}\sigma_{ji}+\overset{\text{Body force}}{f_i} \text{ — force per unit volume}$$

σ_{ij}: total stress tensor, force on i^{th} direction, j-direction unit area.

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$$\Rightarrow \rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial}{\partial x_j}u_i\right)=\frac{\partial\sigma_{ji}}{\partial x_j}+f_i \quad \text{General, Not only for compressible flow.}$$

Energy $e_0=e+\tfrac{1}{2}|u|^2$

e_0: Total Energy; e: internal energy (Thermal) per unit mass

$$\frac{\partial}{\partial t}(\rho e_0)+\frac{\partial}{\partial x_j}(\rho u_j e_0)=\underbrace{\frac{\partial}{\partial x_j}(\sigma_{ji}u_i)}_{\text{Work done by Surface}}+\underbrace{f_i u_i}_{\text{body force}}-\underbrace{\frac{\partial q_j}{\partial x_j}}_{\text{heat flux}}$$

$$\left(\rho\frac{\partial e}{\partial t}+u_j\frac{\partial e}{\partial x_j}\right)$$

Entropy balance S: entropy per unit mass

$$\frac{\partial}{\partial t}(\rho S)+\frac{\partial}{\partial x_j}(\rho u_j S)=$$

$$\rho\left(\frac{\partial S}{\partial t}+u_j\frac{\partial}{\partial x_j}S\right)=\underbrace{\mathcal{N}}_{\text{Entropy generatn rate}}-\underbrace{\frac{\partial}{\partial x_j}\left(\frac{q_j}{T}\right)}_{\text{Entropy Surface flow}}$$

Entropy ~ heat flux $dS=\frac{\delta Q}{T}$

2^{nd} Law: $\mathcal{N}\geq 0$

Mechanical Energy u_i(momentum Eqn.)$_i$

Total energy eqn. $e_0=e+\tfrac{1}{2}(\underline{u}\cdot\underline{u})$

− mechanical Eqn.

Thermal Energy Eqn. $\rho\left(\frac{\partial e}{\partial t}+u_j\frac{\partial e}{\partial x_j}\right)=\frac{\partial}{\partial x_j}(\sigma_{ji}u_i)-u_i\frac{\partial\sigma_{ji}}{\partial x_j}-\frac{\partial q_j}{\partial x_j}$

$$=\sigma_{ji}\frac{\partial u_i}{\partial x_j}-\frac{\partial q_j}{\partial x_j}$$

$$\rho\frac{\partial e}{\partial t}+\underline{u}\cdot\nabla e=\underline{\underline{\sigma}}:\nabla\underline{u}-\nabla\cdot\underline{q}$$

Stress tensor $\underline{\underline{\sigma}}=\underline{\underline{\sigma}}^{\text{"isotropic"}}+\underline{\underline{\sigma}}^{\text{"deviatoric"}}$

$$\sigma_{ji}=\overbrace{\underbrace{\left(\sigma_{ji}-\tfrac{1}{3}\sigma_{kk}\delta_{ij}\right)}_{\sigma_{ji}^{\text{"dev"}}}}^{\tau_{ji}}+\overbrace{\underbrace{\tfrac{1}{3}\sigma_{kk}\delta_{ij}}_{\sigma_{ji}^{\text{"iso"}}}}^{-p\delta_{ij}}$$

$\sigma_{ij}=\sigma_{ji}$ Symmetric tensor

$$Tr\left(\underline{\underline{\sigma}}^{\text{"dev"}}\right)=0 \quad : \quad Tr(\tau_{ji})=\sigma_{jj}-\tfrac{1}{3}\sigma_{kk}\underbrace{\delta_{jj}}_{3}=0$$

Velocity gradient tensor $\frac{\partial u_i}{\partial x_j}=\underset{\text{Strain rate tensor}}{S_{ij}}+\underset{\text{rotation tensor}}{\Omega_{ij}}$

$$S_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \quad \Omega_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right)$$

$$\rho\left(\frac{\partial e}{\partial t}+u_j\frac{\partial e}{\partial x_j}\right)=-p\cancel{\delta_{ij}}^{\,S_{jj}}+\underbrace{\tau_{ij}S_{ij}}_{\text{viscous heating}}-\frac{\partial q_j}{\partial x_j}$$

Thermal Dynamics

Gibbs eqn. $Tds=de+\overset{\text{"pdv"}}{pd\left(\tfrac{1}{\rho}\right)}$ $\quad dt$

$$T\underbrace{\left(\frac{\partial S}{\partial t}+u_j\frac{\partial S}{\partial x_j}\right)}_{\text{RHS of }S}=\rho\underbrace{\left(\frac{\partial e}{\partial t}+u_j\frac{\partial e}{\partial x_j}\right)}_{\text{RHS of }e}+p\left(-\frac{1}{\rho}\right)\frac{d\rho}{dt}$$

$$\underbrace{\frac{\partial\rho}{\partial t}+u_j\frac{\partial\rho}{\partial x_j}}_{-\rho\cdot\nabla u}$$

$$\mathcal{N}-\frac{\partial}{\partial x_j}\left(\frac{q_j}{T}\right) \;\vdots\; -p\frac{\partial u_j}{\partial x_j}+\tau_{ij}S_{ij}-\frac{\partial q_j}{\partial x_j}-\frac{p}{\rho}\frac{\partial u_j}{\partial x_j}$$

$$\left[T\mathcal{N}=\underbrace{-\frac{1}{T}q_j\frac{\partial T}{\partial x_j}+\tau_{ij}S_{ij}}_{\geq 0}\right]$$

Stress oppose deformatn

Constitutive Eqns. (homogeneous materials / isotropic m.)

$T_{ij} = ?$ $q_j = ?$

heat flux:

$e \xrightarrow{?} T$
$q \xrightarrow{?} T$

$\underset{\sim}{q} \overset{?}{=} f(T) = f_1(\nabla T)$

$\underset{\sim}{q} = C(|\nabla T|) \cdot \nabla T$

$\underset{\sim}{q} = -k \nabla T$

$T_{ij}, \underset{\approx}{T} = f(? \underset{\sim}{u})$

NOT Galilean invariant $\underset{\approx}{T} = \cancel{\underset{\sim}{u}\underset{\sim}{u}}$ should not damage if switch frame

$\nabla \underset{\sim}{u}$: Galilean invarient ?

$\underset{\approx}{T} = f(\nabla \underset{\sim}{u}) = f(\underset{\approx}{S} + \underset{\approx}{\Omega}) = f(\underset{\approx}{S})$

$f(\underset{\approx}{\Omega}) = 0$ — Ω still change when rotate frame — material frame-indifference

$\underset{\approx}{T} = f(\underset{\approx}{S}) = A\underset{\approx}{I} + B\underset{\approx}{S} + C\underset{\approx}{S}^2$ A.B.C of I_s, II_s, III_s ?

C--- Thm. $S^3 = \Sigma S^2 + S ..$

$\underset{\approx}{T} = A\underset{\approx}{I} + B\underset{\approx}{S} + C\underset{\approx}{S}^2$

$T_{ij} = A\delta_{ij} + BS_{ij} + CS_{ik}S_{kj}$

$T_{ij} = f(\underset{\approx}{S})$ $\nabla \underset{\sim}{u} \to \underset{\approx}{S}$

Cayleigh Hemilton Thm.

$\underset{\approx}{t}^3 - I_t \underset{\approx}{t}^2 + II_t \underset{\approx}{t} + III_t \underset{\approx}{I} = 0$

$T_{ij} = A\delta_{ij} + BS_{ij} + CS_{ik}S_{kj}$ ABC: Scalar function of (I_s, II_s, III_s)

incompressible $\nabla \cdot u = 0 \to I_s = 0$

isotropic material

$\underset{\approx}{T} = f(\underset{\approx}{S}, \underset{\approx}{z}) = A + B\underset{\approx}{S} + C\underset{\approx}{S}^2 + D\underset{\approx}{z}\underset{\approx}{z}$

$(\underset{\approx}{S} \cdot \underset{\approx}{z})(\underset{\approx}{S} \cdot \underset{\approx}{z})$

Viscosities

$\nu_{air} = 1.5 \cdot 10^{-5} \frac{m^2}{s}$

$\nu_{water} = 10^{-6} \frac{m^2}{s}$

$\mu = \rho \cdot \nu$

air gas

kinetic the. of gass

Speed $\doteq$ O(c) speed of sound $\doteq 300$ m/s

mean free path $\lambda \doteq 10^{-7}$ m

Time between collisions $t_c = \frac{\lambda}{c} \doteq \frac{10^{-7}}{3 \times 10^2} \sim 10^{-9}$ sec.

$\underset{\approx}{T}$ units of $(\frac{m}{s})^2$ "kinetic stress"

$\underset{\approx}{T} = \frac{1}{\rho}(\underset{\approx}{\sigma} - \frac{1}{3}Tr(\underset{\approx}{\sigma})\underset{\approx}{I})$

$\underset{\approx}{S}$: rate of deformation of material

unit of $\underset{\approx}{S}$: $\frac{1}{Time}$

units of A: $(\frac{m}{s})^2$

B: $(\frac{m}{s})^2 \cdot s$

C: $(\frac{m}{s})^2 \cdot s^2$

depend on macroscopic property

$\sigma_{ij} = -P\delta_{ij} + T_{ij}\rho$

$A \sim O(c^2)$

$B \sim O(c^2) \cdot t_m$ — microscopic time scale — 10^{-9} sec

$\underset{\approx}{S} \sim 10 \sim 10^4 \frac{1}{s}$

$\underbrace{t_m |S|}_{\text{Small}} \sim O(10^{-5})$

10^{-9}

$C \cdot S^2 \doteq O(t_m|S|)^2 \sim O(10^{-10})$

$O(t_m|S|) \ll 1$

$|B\underset{\approx}{S}| \gg |C \cdot \underset{\approx}{S}^2| \to 0$

Neglect 2^{nd} terms

$t_m \ll |\underset{\approx}{S}|^{-1}$

(Seperation of time Scale)

$T_{ij} = A\delta_{ij} + BS_{ij} + \cancel{C...}$

recall $T_{ii} = 0 = A \cdot 3 + BS_{ii}$ if $S_{ii} = 0$, $A = 0$

$A = -\frac{1}{3}BS_{ii}$

$T_{ij} = -\frac{1}{3}BS_{kk} + BS_{ij}$

$= B(S_{ij} - \frac{1}{3}S_{kk}\delta_{ij})$

Single parameter $= 2\nu$ kinematic viscosity

$\nu = \nu(|\underset{\approx}{S}|^2)$

constant independent of invariants I_s, II_s, III_s. Newtonian fluid

Navier-Stokes Eqns. $\sigma_{ij} = -P\delta_{ij} + 2\nu\rho S_{ij}$

$\frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$

momentum Eqn.

$\nabla \cdot (\underset{\approx}{T}) \longrightarrow \nabla \cdot (2\nu \underset{\approx}{S}) \xrightarrow[\text{Only if } \nu \neq \nu(\underset{\sim}{x})]{} 2\nu \nabla \cdot \underset{\approx}{S}$

if $\nabla \cdot \underset{\sim}{u} = 0$

if $\nabla \cdot u = 0$ $\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2}{\partial x_j^2} u_i$

Navier-Stokes for Newtonian fluid.

$\frac{\partial u_i}{\partial x_i} = 0$ 4 PDEs for 4 Fields.

u_i, $i = 1, 2, 3$

$p(\underset{\sim}{x}, t)$

N-S Eqn.

$\sigma_{ij} = -P\delta_{ij} + 2\nu\rho S_{ij}$

$\nabla \cdot \underset{\sim}{u} = 0$: Newtonian fluid.

$[\frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} u_i = -\frac{1}{\rho}\frac{\partial}{\partial x_i} P + \nu \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j} u_i]$

Mom-Eqn.

$\frac{\partial}{\partial x_i}$(Mom. Eqn.);

$\frac{1}{\rho}\nabla^2 P = -\frac{\partial}{\partial x_i}(u_j \frac{\partial}{\partial x_j} u_i) = -\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}$

Green's Fn.

$$p = -\rho \iiint_V \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}(y,t)\, G(y,t)\, d^3 y$$

Pressure Poisson $\quad \frac{\partial}{\partial x_i}(\text{mom. Eq.})_i \quad$ for $\frac{\partial u_i}{\partial x_i} = 0$

$$\frac{1}{\rho}\nabla^2 p = -\frac{\partial}{\partial x_i}\left(u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}$$

Solved using Green's function

$$p = -\rho \iiint_V \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}(y,t)\, G(x,y)\, d^3 y$$

for free-space ...

then $p = \ldots$? why

$$\underbrace{\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}}_{\text{Non linear, local}} = \frac{1}{4\pi}\iiint_V \underbrace{\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}(y,t)}_{\text{Non linear, non-local}} \frac{x_i - y_i}{|x-y|^3}\, d^3 y + \nu \frac{\partial^2 u_i}{\partial x_j^2}$$

3 Non-linear integral differential partial PE for 3 velocity components

Vorticity $\underset{\sim}{\omega} = \nabla \times \underset{\sim}{u}$

$\Omega = \frac{1}{2}(\nabla u - \nabla u^T)$

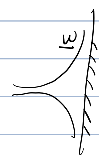
$\Omega_{qp} = -\frac{1}{2}\epsilon_{pqi}\,\omega_i$

Vorticity $\quad \underset{\sim}{\omega} = \nabla \times \underset{\sim}{u}$

$\omega_i \quad \Omega = \frac{1}{2}(\nabla u - \nabla u^T)$

$\omega_i = \epsilon_{ijk}\Omega_{kj} \qquad \nabla \cdot \underset{\sim}{\omega} = 0 \quad$ div. free.

$\Omega_{qp} = -\frac{1}{2}\epsilon_{pqi}\,\omega_i$



$\mathrm{Curl}(\underset{\sim}{\omega}) = \nabla\times(\nabla\times u) = -\nabla^2 u + \nabla(\nabla\cdot u)$ ($\nabla\times$; $\nabla\cdot u = 0$ div free)

$\nabla^2 u = -\nabla\times\omega$

$$\mathrm{Curl}(\underset{\sim}{\omega}) = \nabla\times(\nabla\times \underset{\sim}{u}) = -\nabla^2 \underset{\sim}{u} + \nabla(\underbrace{\nabla\cdot \underset{\sim}{u}}_{0 \text{ div free}})$$

$$\nabla^2 \underset{\sim}{u} = -\nabla\times\underset{\sim}{\omega}$$

free space. $\quad u_i(\underset{\sim}{x},t) = \frac{1}{4\pi}\iiint_V \epsilon_{ijk}\frac{x_k - x_k'}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3}\,\omega_j(\underset{\sim}{x}')\, d^3\underset{\sim}{x}' +$ potential part.

Vorticity Transport Eqn.

Vorticity Transport Eqn.

$\nabla\times(\text{mom. Eqn.}) \qquad \epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial u_k}{\partial t} = \frac{\partial \omega_i}{\partial t} \qquad (\nabla\cdot u = 0)$

$$\frac{\partial \omega_i}{\partial t} + \underbrace{u_j}_{\text{Still need to solve mom.}} \frac{\partial \omega_i}{\partial x_j} = \underbrace{\omega_k S_{ki}}_{\text{Vorticity Str. rate}} + \nu \frac{\partial^2 \omega_i}{\partial x_k^2}$$

Vortex Stretching & tilting

When Turbulence?

NOT all N-S stable

Perturbation.

$\nu_{air} \sim 10^{-5}\ \frac{m^2}{s}$

diffusion coeff. $\sim \mathcal{O}$(velocity $\times$ length scale) or energy $\times$ timescale.

Velocity & $c \quad \mathcal{O}(100\ m/s)$.

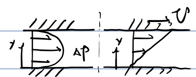
length scale & $\lambda \sim \mathcal{O}(10^{-7} m)$ free path

$c\,\lambda \sim \mathcal{O}(10^{-5}\ \frac{m^2}{s})$

* Brief Outline of Linear Stability Theory

* Turbulence

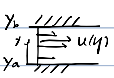
NOT all soln. to N-S Eqn. Stable. $\quad$ Steady:



Prob. or linear

Steady Soln. $u(y)$ are solutions at any Re.

Equilibrium. $\underset{\text{equilibm. Soln.}}{\bar{u}(y)}$ base flow. 2D.



Is it Stable?

?

$NS \begin{cases} \text{Stab} \\ \text{Unst.} \end{cases} \longrightarrow$ perturbation $\rightarrow$ Stability

Linear Stab. $\leftarrow$ Perturbation

NOT all N-S Eqn. Stable.

Perturbation.

Stream Fn. ψ $\begin{cases} u' = \frac{\partial \psi}{\partial y} \\ v' = -\frac{\partial \psi}{\partial x} \\ \omega_z' = -\nabla^2 \psi \end{cases}$

Perturbation $u'\ v'\ w'$ (small) [2D]

often useful Stream function $\psi'(x,y,z,t) \qquad u' = \frac{\partial \psi'}{\partial y},\ v' = -\frac{\partial \psi'}{\partial x}$

Prob. velocity $\quad \omega_z' = -\nabla^2 \psi'$

$$\psi' = \underset{\substack{\text{Small} \\ \epsilon \ll 1}}{\epsilon}\, \phi(y)\, e^{i\underset{\text{real}}{\alpha}(x - \underset{\text{complex phase-speed}}{c}t)} \qquad c = c_r + c_i$$

$\omega' = \epsilon \phi(y) e^{i\alpha(x-ct)}$

$\left[\frac{\partial \omega'}{\partial t} + \bar u \frac{\partial \omega'}{\partial x} + \bar v \frac{\partial \omega'}{\partial y} = \nu \nabla^2 \omega'\right]\left(u'\frac{\partial \omega'}{\partial} \text{.. neglect}\right)$

$\left[(\bar u(y)-c)\left(\frac{d^2\phi}{dy^2} - \alpha^2\phi\right) - \frac{d^2\bar u}{dy^2}\phi = -\frac{i\nu}{\alpha}\left(\frac{d^2}{dy^2}-\alpha^2\right)\phi\right]$

$\nu = 0$

$(\bar u(y)-c)\left(\frac{d^2\phi}{dy^2}-\alpha^2\phi\right) - \frac{d^2\bar u}{dy^2}\phi = 0$

$\bar\omega_z' = -\nabla^2\psi'$

$\psi' = \epsilon\phi(y)e^{i\alpha(x-ct)}$

wave number of perturbation

c_i growth rate of perturbation: $c_i>0$ perturbation ↑ unstable; $c_i<0$ perturbation ↓ stable

Perturbation Vorticity

$\hat z$: $\left[\frac{\partial \omega'}{\partial t} + \bar u \frac{\partial \omega'}{\partial x} + \bar v \frac{\partial \omega'}{\partial y} = \nu\nabla^2\omega'\right]$ Linear Eqn.

missing $u'\frac{\partial \omega'}{\partial x} + v'\frac{\partial \omega'}{\partial y} \sim O(\epsilon^2)$ neglected

$$\left[(\bar u(y)-c)\left(\frac{d^2\phi}{dy^2}-\alpha^2\phi\right) - \frac{d^2\bar u}{dy^2}\phi = -\frac{i\nu}{\alpha}\left(\frac{d^2}{dy^2}-\alpha^2\right)^2\phi\right]$$

Orr–Som Orth-ognal Sommerfeld.

$\Phi = 0$. Soln. Eigenvalue prob.

viscous effect neglected

Rayleigh Eqn. perturbation evolves to inviscid linear $\nu=0$

$$(\bar u(y)-c)\left(\frac{d^2\phi}{dy^2}-\alpha^2\phi\right) - \frac{d^2\bar u}{dy^2}\phi = 0$$

Set to 0 shows initial tend to stable/not. before viscosity damper

manipulations.

$$\phi'' = \left(\frac{\bar u''}{\bar u - c} + \alpha^2\right)\phi \quad / \ \Phi^*$$

$$\phi''\phi^* = (\%)\|\phi\|^2 \quad / \int_{y_A}^{y_B}$$

integration by parts.

$$\underbrace{-\int_{y_A}^{y_B}(|\phi'|^2+\alpha^2|\phi|^2)\,dy}_{\text{Real}} = \underbrace{\int_{y_A}^{y_B}|\phi|^2\frac{\bar u''(\bar u - c^*)}{\|\bar u - c\|^2}\,dy}$$

Real $\longrightarrow$ $\mathrm{Im}(\text{RHS}) = 0$

using complex conj.

$$\underbrace{(c-c^*)}_{\neq 0}\int_{y_A}^{y_B}\|\phi\|^2\frac{\bar u''}{\|\bar u - c\|^2}\,dy = 0$$ For soln. to exist

means c have im part.

Safe because $c = c_r + ic_i$, $\neq 0$

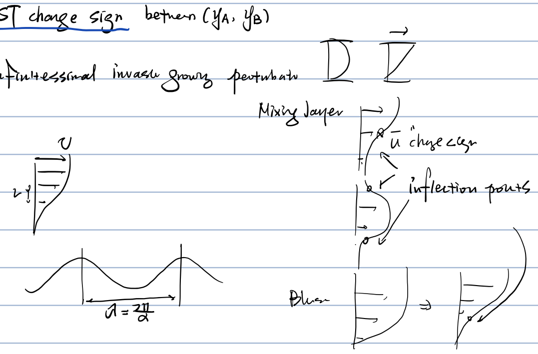
$$\int_{y_A}^{y_B}\|\phi\|^2\frac{\bar u''}{\|\bar u - c\|^2}\,dy = 0$$

Rayleh Instability creteria

"Nessessary condition for instability"

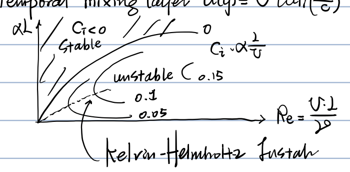
u'' MUST change sign between (y_A, y_B)"

Inflexion pt. unstable for infinitessimal inviscid growing perturbation

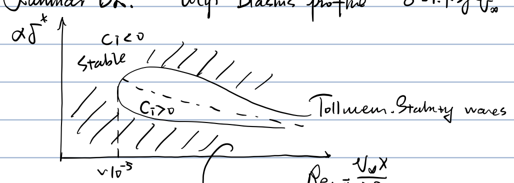


Solns to $\nu \neq 0$ (Orr Sommerfeld) ②

a) Temporal mixing layer $\bar u(y) = U \tanh\left(\frac{y}{\delta}\right)$



Laminar BL. $\bar u(y)$ Blasius profile $\delta^* = 1.73\sqrt{\frac{\nu x}{U_\infty}}$ displacement thickness



long wave length also stable

Round Pipe Flow

[Plane Couette flow. Linear stability analysis: Stable at ANY Re.

[Plane Poiseuille Flow. $Re_{trans} \gg Re_{trans}$ (from linear stab. / observed)

Need finite amp. stab. analysis.

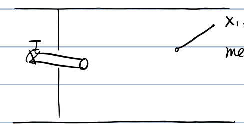
Amp. grow. $\rightarrow$ Complicated dynamics $\rightarrow$ Large fluctuations $\rightarrow$ Statistical analysis

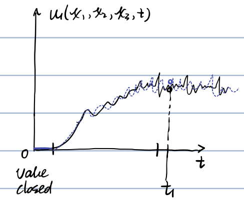
Make classes:

Wednesday Sep 25 th. $6^{pm}-7^{pm}$

Wed. Oct 4 th $6-7$

Krieger 205





Mean Velocity

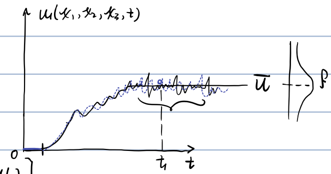
$$\overline{u}_1(x_1, x_2, x_3, t_1) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} u^{(k)}(x_1, x_2, x_3, t)$$

$\langle u_1 \rangle$

$u_1^{k}(\underbrace{x_1, x_2, x_3}_{x_0}, t) \sim$ R.V.

kth experiment

$$\overline{u}_1 = (x_1, x_2, x_3, t_1) = \frac{1}{n}\sum_{n=1}^{n} u_1^{(k)}$$



$$f(V_1)\,dV_1 = \text{Prob.}\left[u_1 \in (V_1, V_1 + dV_1)\right]$$

$f(V_1) = \frac{dF}{dV_1}$, F cummulative distribution $F(V_1) = \text{Prob}\left[u_1 < V_1\right]$

PDF

probability distribution function.

Dif. Turbulence — time average

- Stationary $\frac{\partial}{\partial t} = 0$ not related to time $\langle U \rangle = u$
- Homogeneous $\nabla_x = 0$ not related to location $\overline{u} = 0$
- Isotropic — not related to direction $u(\vec{r}) = u(r)$

$f(V_1, V_2, V_3)$ joint PDF of 3 components at (x_1, x_2, x_3) and time t_1

reality: $f(\underset{\sim}{V}; \underset{\sim}{x}, t)$

$$\text{prob}\Big[u_1 \in (V_1, V_1 + dV_1) \text{ and } u_2 \in (V_2, V_2 + dV_2) \text{ and } u_3 \in (V_3, V_3 + dV_3)\Big]$$

- Stationary $\frac{\partial f}{\partial t} = 0$, $f(\underset{\sim}{V}; x)$ only

"mean"

$+$

- Homogeneous $\nabla_x f = 0$ $f(\underset{\sim}{V})$
- Isotropic: Invariant respt. to frame location

Stationary Conditions

"ergodicity". $\frac{\text{avg (realization)}}{\text{avg (time)}}$

$$\overline{u}_i = \langle u_i \rangle = \int_{-\infty}^{\infty} V_i f(\underset{\sim}{V})\, d\underset{\sim}{V} \qquad \iiint_{-\infty}^{\infty} d^3\underset{\sim}{V}$$

NOT homog. on $\overline{u}_i(\underset{\sim}{x}) = \int_{-\infty}^{\infty} V_i f(\underset{\sim}{V}; \underset{\sim}{x})\, d^3\underset{\sim}{V}$ $\qquad \int_{-\infty}^{\infty} f(\underset{\sim}{V})\, d^3\underset{\sim}{V} = 1$

or

$f_{(i)}(V_{(i)})$ marginal PDF of "ith" components

Fluctuations.

$$\overline{u}_i = \int_{-\infty}^{\infty} V_i f_{(i)}(V_{(i)})\, dV_{(i)} \qquad \text{Ex. } f_1(V_1) = \iint_{-\infty}^{\infty} f(V_1, V_2, V_3)\, dV_2\, dV_3$$

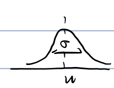
- Fluctuations $u_i'(\underset{\sim}{x}, t) = u_i(\underset{\sim}{x}, t) - \overline{u}_i(\underset{\sim}{x})$ — Confusion: mean various ways, many things.
- Variance of "j^{th}" component $\overline{u_{(j)}'^2} = \overline{u_{(j)}^2} - \overline{u}_{(j)}^2$

Covariance $\overline{\underset{\sim}{u}' \otimes \underset{\sim}{u}'}$

- Covariance Tensor: $\overline{u_i' u_j'} = \overline{u_i u_j} - \overline{u}_i \overline{u}_j$
- ~~Held~~. Trace $\overline{u_i'^2} = \overline{u_1'^2} + \overline{u_2'^2} + \overline{u_3'^2}$ $\left(= \overline{u_i^2} - \overline{u}_i^2\right)$
- RMS $\sqrt{\overline{u_{(i)}'^2}}$ $\quad u_{(i)\,RMS}$

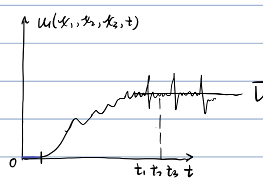
describe

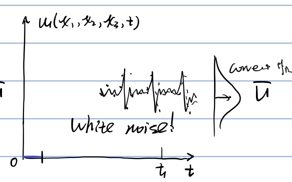
$$f(V) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(V-\overline{u})^2}{2\sigma^2}} \qquad \sigma: \text{RMS of } u.$$



- Review Sec. $f(V)$

Random Process





- Multi-time PDF

$\times$ $f(\underline{V}_1, t_1; \underline{V}_2, t_2, \ldots \underline{V}_m, t_m)$ joint PDF: $\underline{u}$ at find $\underline{V}_m$ and t_m (V_1, V_2, V_3)

Stationary: $f(\underline{V}_1, t_1, \ldots \underline{V}_m, t_m) = f(V_1, t_1+T, \ldots t_m+T)$ shift available

For Joint Gaussian Distribution

$\rightarrow$ Covariance matrices of all times $\begin{pmatrix} C_{11} & C_{12} & C_{\ldots} \\ & C_{22} & \\ \ldots & & \end{pmatrix}$

Stationary Correlation fn.

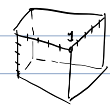
in time contains LOT information.

Random fields, single time.

$*$ N-point Statistics

$f(\underline{V}_1, \underline{x}_1, \underline{V}_2, \underline{x}_2, \ldots \underline{V}_n, \underline{x}_n, t)$

spacial controlled inter.



$*$ If Gaussian: 2-point inf. sufficient.

$*$ If Homogeneous: Only dependence on $\underline{x}_m - \underline{x}_n$

only depend on $\underline{x}$ difference (distance)

$*$ Distance Only: dependence only on $\|\underline{x}_m - \underline{x}_n\|$ (Covet vector $\underline{V}$)

Averaging is Linear OP.

$\overline{\nabla_x \underline{u}} = \nabla_x \overline{\underline{u}}$

$\overline{\dfrac{\partial u}{\partial x}} = \dfrac{\partial \overline{u}}{\partial x}$ BUT $\overline{uu} \neq \overline{u}\,\overline{u}$

Reynolds decomposition

$\underline{u} = \overline{\underline{u}} + \underline{u}'$ or $u_i = \overline{u}_i + u_i'$

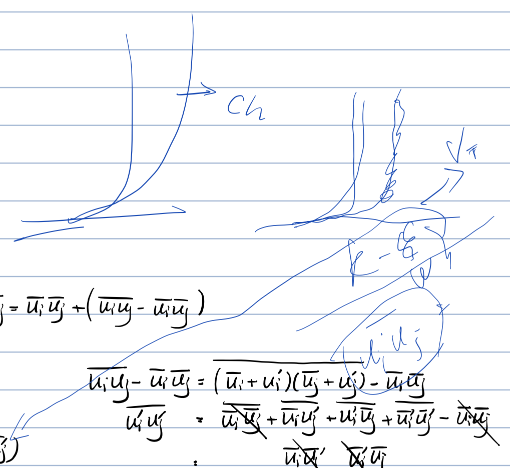
$\overline{\underline{u}'} = 0$

ρ = const. N-S $\rightarrow$ RANS

- continuity $\dfrac{\partial u_i}{\partial x_i} = 0$ $\quad \nabla \cdot \underline{u} = 0$, $\overline{\nabla \cdot \underline{u}} = \nabla \cdot \overline{\underline{u}} = 0$ $\Big\}$ $\nabla \cdot \underline{u}' = 0$

Mean flow div. free

- momentum.



Unsteady RANS:

$$\left(\frac{\partial u_i}{\partial t} + \underbrace{u_j \frac{\partial u_i}{\partial x_j}}_{\frac{\partial}{\partial x_j}(u_i u_j)} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k^2} + f_i\right)$$

$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\overline{u_i u_j}) = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_i} + \nu \frac{\partial^2 \overline{u}_i}{\partial x_k^2} + \overline{f}_i \qquad \overline{u_i u_j} = \overline{u}_i \overline{u}_j + (\overline{u_i u_j} - \overline{u}_i \overline{u}_j)$$

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial}{\partial x_j}\overline{u}_i = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_i} + \nu \frac{\partial^2 \overline{u}_i}{\partial x_k^2} + \overline{f}_i - \underbrace{\frac{\partial}{\partial x_j} T_{ij}}_{\frac{\partial}{\partial x_j}(\overline{u_i' u_j'})}$$

$$\overline{u_i u_j} - \overline{u}_i \overline{u}_j = \overline{(\overline{u}_i + u_i')(\overline{u}_j + u_j')} - \overline{u}_i \overline{u}_j$$

$$\overline{u_i' u_j'} = \overline{\overline{u}_i \overline{u}_j} + \overline{\overline{u}_i u_j'} + \overline{u_i' \overline{u}_j} + \overline{u_i' u_j'} - \overline{u}_i \overline{u}_j$$

$\overline{u}_i \overline{u_j'} = 0$, $\overline{u_i'} \overline{u}_j = 0$

$\Downarrow$

$u_i = \overline{u}_i + u_i'$

$T_{ij} = \overline{u_i' u_j'}$ "Reynolds Stress"

Reynold's Stress $\underbrace{\overline{u_i u_j}}_{\text{Ave of Momentum Flux}} - \underbrace{\overline{u}_i \overline{u}_j}_{\text{product of Ave momentum}} = \underbrace{\overline{u_i' u_j'}}_{\text{covariance of tensor of fluctuations}}$

$\sigma_{ij}^R = -\rho \overline{u_i' u_j'}$, consistant with name "tension"

preferable from processing data point view

Definitions (decompositions)

- Turbulent Kinetic Energy (TKE)

$k = \frac{1}{2}\overline{u_i' u_i'} = \frac{1}{2} T_{ii}$ $\left(= \frac{1}{2}\overline{q^2}\right)$

Scalar field $k(\underline{x}, t)$ $\quad \frac{1}{3} T_{ij}\delta_{ij}$

- Anisotropy $a_{ij} = \overline{u_i' u_j'} - \frac{2}{3} k \delta_{ij}$ $\quad Tr(\underline{a}) = 0$

component energy

Multi-time: $\frac{\partial \overline{\underline{u}}}{\partial t}$

Averaging is Linear operator

homogeneous $\overline{\dfrac{\partial}{\partial \underline{x}}\underline{u}} = \dfrac{\partial}{\partial \underline{x}}\overline{\underline{u}}$

Non-Stationary $\overline{\dfrac{\partial}{\partial t}\underline{u}} = \dfrac{\partial}{\partial t}\overline{\underline{u}}$

Mean is for spatial, not for time. So commucate

Reynold's decomposition

$\underline{u} = \overline{\underline{u}} + \underline{u}'$

continuum. $\dfrac{\partial u_i}{\partial x_i} = 0$

RANS

$$\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k^2} + f_i\right)$$

$\frac{\partial}{\partial x_j}(u_i u_j)$

$\frac{\partial}{\partial x_j}(\overline{u_i u_j}) = \frac{\partial}{\partial x_j}\left[\overline{(\overline{u}_i + u_i')(\overline{u}_j + u_j')}\right]$

$\overline{\overline{u}_i \overline{u}_j} + \overline{u_i' \overline{u}_j} + \overline{u_j' \overline{u}_i} + \overline{u_i' u_j'}$

$\overline{u}_i \overline{u}_j \quad 0 \quad 0 \quad \boxed{\overline{u_i' u_j'}}$

$$\left[\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial}{\partial x_j}\overline{u}_i = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_i} + \nu \frac{\partial^2 \overline{u}_i}{\partial x_k^2} + \overline{f}_i - \frac{\partial}{\partial x_j} T_{ij}\right]$$

TKE Turbulent kinetic energy

$k = \frac{1}{2}\overline{u_i' u_i'} = \frac{1}{2} T_{ii}$ $\left(= \frac{1}{2}\overline{q^2}\right)$

$a_{ij} = \overline{u_i' u_j'} - \frac{2}{3} k \delta_{ij}$

$b_{ij} = \dfrac{a_{ij}}{2k} = \dfrac{\overline{u_i' u_j'}}{\overline{u_i' u_i'}} - \frac{1}{3}\delta_{ij}$

$\overline{u_i' u_j'} = a_{ij} + \frac{2}{3} k \delta_{ij}$ $\quad \frac{\partial}{\partial x_j}\delta_{ij} = \frac{\partial}{\partial x_i}$

$-\rho\frac{\partial}{\partial x_j}(\tau_{ij})+(-\frac{\partial \bar{p}}{\partial x_i}) = -\rho\frac{\partial}{\partial x_j}a_{ij} - \frac{\partial}{\partial x_i}(\bar{p}+\frac{2}{3}\rho k)$

- Normalized - % $b_{ij} = \frac{a_{ij}}{2k} = \frac{\overline{u_i'u_j'}}{\overline{u_k'^2}} - \frac{1}{3}\delta_{ij}$

can obtain

$\tau_{ij} = \overline{u_i'u_j'}$ if know a_{ij} & k

$\overline{u_i'u_j'} = 2k(\frac{1}{3}\delta_{ij}+b_{ij})$

- Effect of τ_{ij} on $\bar{u}_i$

$-\rho\frac{\partial}{\partial x_j}(\overline{u_i'u_j'}) + (-\frac{\partial \bar{p}}{\partial x_i}) = -\rho\frac{\partial a_{ij}}{\partial x_j} - \frac{\partial}{\partial x_i}\underbrace{(\bar{p}+\frac{2}{3}\rho k)}_{\bar{p}^* \text{ modified pressure}}$

extra pressure of turbulence

Only a_{ij} has direct effect on $\bar{u}_i(\underline{x},t)$

Transport Eqn. for τ_{ij}

N-S $\frac{\partial}{\partial t}u_i + u_k\frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial}{\partial x_i}p + \nu\nabla^2 u_i + f_i$

$f = \bar{f} + f'$

$\frac{\partial}{\partial t}(\bar{u}_i+u_i') + \bar{u}_k\frac{\partial \bar{u}_i}{\partial x_k} + \bar{u}_k\frac{\partial u_i'}{\partial x_k} + u_k'\frac{\partial \bar{u}_i}{\partial x_k} + u_k'\frac{\partial u_i'}{\partial x_k}$

$\bar{u}_k\frac{\partial \bar{u}_i}{\partial x_k} + \bar{u}_k\frac{\partial u_i'}{\partial x_k} + u$

Transport eqn. for τ_{ij}

fluctuation div free

Total (N-S) $\frac{\partial}{\partial t}(\bar{u}_i+u_i') + \bar{u}_k\frac{\partial \bar{u}_i}{\partial x_k} + u_k'\frac{\partial \bar{u}_i}{\partial x_k} + \bar{u}_k\frac{\partial u_i'}{\partial x_k} + \frac{\partial}{\partial x_k}(u_i'u_k') = -\frac{1}{\rho}\frac{\partial}{\partial x_i}(\bar{p}+p') + \nu\nabla^2(\bar{u}_i+u_i')$

⊖ RANS for $\bar{u}_i$: $\frac{\partial}{\partial t}\bar{u}_i + \bar{u}_k\frac{\partial \bar{u}_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial}{\partial x_i}\bar{p} - \frac{\partial}{\partial x_k}(\overline{u_i'u_k'}) + \nu\nabla^2\bar{u}$

⊜ $\frac{\partial}{\partial t}u_i' + u_k'\frac{\partial \bar{u}_i}{\partial x_k} + \bar{u}_k\frac{\partial u_i'}{\partial x_k} + \frac{\partial}{\partial x_k}(u_i'u_k') = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_k}(\overline{u_i'u_k'}) + \nu\nabla^2 u_i'$

A] $\cdot u_j'$

rewrite for j $\frac{\partial}{\partial t}u_j' + u_k'\frac{\partial \bar{u}_j}{\partial x_k} + \bar{u}_k\frac{\partial u_j'}{\partial x_k} + \frac{\partial}{\partial x_k}(u_j'u_k') = -\frac{1}{\rho}\frac{\partial p'}{\partial x_j} + \frac{\partial}{\partial x_k}(\overline{u_j'u_k'}) + \nu\nabla^2 u_j'$

B] $\cdot u_i'$

$u_j'\frac{\partial}{\partial t}u_i' + u_i'\frac{\partial}{\partial t}u_j'$

$\frac{\partial}{\partial t}(u_i'u_j') + (u_i'u_k')\frac{\partial \bar{u}_j}{\partial x_k} + (u_j'u_k')\frac{\partial \bar{u}_i}{\partial x_k} + \bar{u}_k\frac{\partial}{\partial x_k}(u_i'u_j') + \frac{\partial}{\partial x_k}(u_i'u_j'u_k') = -\frac{1}{\rho}(u_j'\frac{\partial p'}{\partial x_i} + u_i'\frac{\partial p'}{\partial x_j}) + \nu(u_j'\nabla^2 u_i' + u_i'\nabla^2 u_j')$

$+ u_j'\frac{\partial}{\partial x_k}\tau_{ik} + u_i'\frac{\partial}{\partial x_k}\tau_{jk}$

average eqn: ⇓ AVG

$\frac{\partial}{\partial t}\underbrace{(\overline{u_i'u_j'})}_{\tau_{ij}} + \underbrace{(\overline{u_i'u_k'})}_{\tau_{ik}}\frac{\partial \bar{u}_j}{\partial x_k} + \underbrace{(\overline{u_j'u_k'})}_{\tau_{jk}}\frac{\partial \bar{u}_i}{\partial x_k} + \bar{u}_k\frac{\partial}{\partial x_k}\underbrace{(\overline{u_i'u_j'})}_{\tau_{ik}} + \frac{\partial}{\partial x_k}(\overline{u_i'u_j'u_k'}) = -\frac{1}{\rho}(\overline{u_j'\frac{\partial p'}{\partial x_i}} + \overline{u_i'\frac{\partial p'}{\partial x_j}}) + \nu(\overline{u_j'\nabla^2 u_i'} + \overline{u_i'\nabla^2 u_j'})$

$+ \cancel{\overline{u_j'}\frac{\partial}{\partial x_k}\tau_{ik}} + \cancel{\overline{u_i'}\frac{\partial}{\partial x_k}\tau_{jk}}$

avg of fluctuation

$\frac{\partial}{\partial t}\tau_{ij} + \bar{u}_k\frac{\partial}{\partial x_k}\tau_{ij} = \underbrace{-(\tau_{ik}\frac{\partial \bar{u}_j}{\partial x_k} + \tau_{jk}\frac{\partial \bar{u}_i}{\partial x_k})}_{\text{Symmetric}} - \frac{\partial}{\partial x_k}(\overline{u_i'u_j'u_k'}) - \frac{\partial}{\partial x_k}(\frac{1}{\rho}\overline{p'u_j'}\delta_{ik} + \frac{1}{\rho}\overline{p'u_i'}\delta_{jk}) + \frac{1}{\rho}\overline{p'(\underbrace{\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}}_{2S_{ij}})}$

$+ 2\nu[\frac{\partial}{\partial x_k}(\overline{u_j'\frac{\partial u_i'}{\partial x_k}}) + \frac{\partial}{\partial x_k}(\overline{u_i'\frac{\partial u_j'}{\partial x_k}}) - 2\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}]$

$\nu\frac{\partial^2}{\partial x_k^2}\tau_{ij} - 2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}$

closure

Reynolds Stress Transport Eqn.

$\frac{\partial}{\partial t}\tau_{ij} + \bar{u}_k\frac{\partial}{\partial x_k}\tau_{ik} = \underbrace{-(\tau_{ik}\frac{\partial \bar{u}_j}{\partial x_k} + \tau_{jk}\frac{\partial \bar{u}_i}{\partial x_k})}_{\text{Symmetric}} - \frac{\partial}{\partial x_k}(\overline{u_i'u_j'u_k'}) - \frac{\partial}{\partial x_k}(\frac{1}{\rho}\overline{p'u_j'}\delta_{ik} + \frac{1}{\rho}\overline{p'u_i'}\delta_{jk}) + \frac{1}{\rho}\overline{p'(\underbrace{\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}}_{2S_{ij}})} + 2\nu[\frac{\partial}{\partial x_k}(\overline{u_j'\frac{\partial u_i'}{\partial x_k}}) + \frac{\partial}{\partial x_k}(\overline{u_i'\frac{\partial u_j'}{\partial x_k}}) - 2\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}]$

$\nu\frac{\partial^2}{\partial x_k^2}\tau_{ij} - 2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}$

$$\frac{D\tau_{ij}}{Dt} = \frac{\partial \tau_{ij}}{\partial t} + \bar{u}_k\frac{\partial}{\partial x_k}\tau_{ik} = \underbrace{-(\tau_{ik}\frac{\partial \bar{u}_j}{\partial x_k} + \tau_{jk}\frac{\partial \bar{u}_i}{\partial x_k})}_{\text{Production of R.S.}} - \underbrace{\frac{\partial}{\partial x_k}Q_{kij}}_{\text{Turbulent + viscous diffusion}} + \underbrace{2\frac{1}{\rho}\overline{p'S_{ij}}}_{\text{pressure strain correlation}} - \underbrace{\epsilon^*_{ij}}_{\text{pseudo dissipation}}$$

$\epsilon^* = 2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}$

$Q_{kij} = \overline{u_i'u_j'u_k'} + \frac{1}{\rho}\overline{p'u_j'}\delta_{ik} + \frac{1}{\rho}\overline{p'u_i'}\delta_{jk} + \nu\frac{\partial}{\partial x_k}\tau_{ij}$

?

?

$\frac{D\tau_{ij}}{Dt} = \text{Prod}_{ij} - \nabla\cdot(\nu\nabla\tau_{ij}) + ??$

Most important part of τ_{ij} (TKE)

$i=j$ & $\cdot\frac{1}{2}$ $k = \frac{1}{2}\tau_{ii}$

$-\epsilon^*$: sink of T.K.E.

$$\frac{\partial k}{\partial t} + \bar{u}_k\frac{\partial k}{\partial x_k} = \underbrace{-\tau_{ik}\frac{\partial \bar{u}_i}{\partial x_k}}_{\text{production}} - \underbrace{\frac{\partial}{\partial x_k}(\frac{1}{2}\overline{u_i'u_i'u_k'} + \frac{1}{\rho}\overline{p'u_k'})}_{\text{turblant diffusion}} + \nu\nabla^2 k - \underbrace{\epsilon^*}_{\nu(\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}})}$$

"Psudo dissipatin"

NOTE: $\overline{p'S_{ii}} = 0$ due to $\frac{\partial u_i'}{\partial x_i} = 0$ everywhere

Turblance modeling

repeating "closure" arg. t_m $t_m|S| \ll 1 \rightarrow \tau_{ij}^{(d)\,\text{viscous}} = -2\nu S_{ij}$ a_{ij} $\overset{\shortparallel}{d}$

$\tau_{ij} = f(\cancel{\underline{u}}, \nabla\bar{u} \ldots) \rightarrow f(\underline{\underline{S}}, \cancel{\underline{\underline{\Omega}}} \ldots) = f(\underline{\underline{S}}) \rightarrow -2\nu_T\underline{\underline{S}} = \tau_{ij}$

Galilen invariance — not justify — eddy viscosity

$\tau_{ij} \cong G1.$

$\overline{u_iu_j}$ fluctuation not

For eddy viscosity

$\tau_{ik}^d S_{ik}$

RANS. $u = \bar{u} + u'$

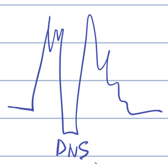
$\bar{u}$ ↑ RANS

$\overline{N\text{-}S} \longrightarrow [\text{PDE}(\bar{u}, p) + \nabla\tau(u')]$

$\left.\begin{array}{l} u_j'[(N\text{-}S) \ominus (RANS)]_i \\ u_i'[(N\text{-}S) \ominus (RANS)]_j \end{array}\right] \oplus \longrightarrow [\text{PDE}(\tau, \bar{u}, u', p')]$

Reynold Stress Transport Eqn.

C Eqn of Tur.



rely on mean

$Prod = + 2\nu_T \bar{S}_{ik}\bar{S}_{ik} \geq 0$ −

Use of Eddy viscosity NOT imply assumption of isotropic turbulence
$a_{ij}=0$ $\overline{u_i u_j} = \frac{2}{3}k\delta_{ij}$

$\tau_{ij} - \frac{2}{3}k\delta_{ij} = -2\nu_T \bar{S}_{ij}$ Eddy-viscosity
isotropic turbulence $\tau_{ij} = \frac{2}{3}k\delta_{ij}$
assuming "weak" deviation from isotropy

$\bar{u}_i(\underline{x},t)$
note $\nu_T(\underline{x},t)$ in RANS: $\nabla\cdot(2\nu_T(-2\bar{S}))$ inside div NOT $\cancel{\nu_T \nabla^2 u}$

isotropic homogeneous turbulence

V_∞ ... $\bar{\underline{u}} = U_\infty \vec{i}$ + u' mean velocity not change but add turbulence

Frame moving with V_∞: $\underline{\bar{u}}=0$, $\overline{u_1'^2} = \overline{u_2'^2} = \overline{u_3'^2}$ and $\overline{u_1' u_2'} = 0$
$\tau_{ij}^d = 0$

Description of hom-turbulence ($\bar{u} = U_\infty$, u')

Description of Spatial Momentums
Superposition of Fourier Modes.
Spatial Fourier Transpose

$$u_i(\underline{x},t) = \iiint \hat{u}_i(\underline{k},t)e^{i\underline{k}\cdot\underline{x}}d^3\underline{k}$$

Spatial Fourier Transf.

1D: $u(x) = \sum$ $k = \frac{2\pi}{L}$ → $\hat{u}$

$$\hat{u}_i(\underline{k},t) = \frac{1}{(2\pi)^3}\iiint u_i(\underline{k},t)e^{-i\underline{k}\cdot\underline{x}}d^3\underline{x}$$

3D: $\underline{k}$ k_3 k_1 $\underline{k}$ k_2

Sapital Fourier Transpose
3D wave number $\underline{k}$

$\hat{u}_i(\underline{k},t) \longleftrightarrow u_i(\underline{x},t)$

$u_i(\underline{x},t) = \iiint \hat{u}_i(\underline{k},t)e^{i\underline{k}\cdot\underline{x}}d^3\underline{k}$

$\hat{u}_i(\underline{k},t) = \iiint u_i(\underline{x},t)e^{i\underline{k}\cdot\underline{x}}d^3\underline{k}$

in 1D: $\frac{d}{dx}u(x,t) = \frac{d}{dx}\int \hat{u}(k,t)e^{ikx}dx = \int ik\hat{u}\,\hat{e}\,dx$ No x No x

$\mathcal{F}(\frac{d}{dx}u) = ik\hat{u}$

$\widehat{\frac{\partial u_i}{\partial x_j}} = ik_j\hat{u}$ ($\widehat{\nabla u} = i\underline{k}\hat{u}$)
$\widehat{\nabla\cdot u} = i\underline{k}\cdot\hat{u}$
$\widehat{\nabla^2 u} = -|\underline{k}|^2\hat{u}$
$\widehat{\frac{\partial u}{\partial t}} = \frac{\partial}{\partial t}\hat{u}$

differentiation

1-D. $\frac{d}{dx}u(x) = \frac{d}{dx}\left[\int \hat{u}(k)e^{ikx}dk\right]\frac{1}{2\pi} = \int ik\hat{u}e^{ikx}dk\cdot\frac{1}{2\pi}$

FT of $\frac{du}{dx}$ — $\left[\begin{array}{l}\widehat{\frac{du}{dx}} = ik\hat{u} \\ \widehat{\frac{d^2u}{dx^2}} = -k^2\hat{u}\end{array}\right.$

Vectors. $\widehat{\frac{\partial u_i}{\partial x_j}} = ik_j\hat{u}_i$ $\quad \widehat{\frac{\partial u}{\partial t}} = \frac{\partial}{\partial t}\hat{u}$

$\widehat{\nabla\underline{u}} = i\underline{k}\hat{\underline{u}}$

$\widehat{\nabla\cdot\underline{u}} = i\underline{k}\cdot\hat{\underline{u}}$

$\widehat{\nabla^2\underline{u}} = -|\underline{k}|^2\hat{\underline{u}}$

product in 1D. $a(x)b(x) = \left(\int \hat{a}(k)e^{ikx}dk\right)\left(\int \hat{b}(k')e^{ik'x}dk'\right) = \iint_{k\,k'} \hat{a}(k)\hat{b}(k')e^{i(k+k')x}dk\,dk'$

$\widehat{a(x)b(x)}(k'') = \iiint_{x\,k\,k'} dk\,dk'\,e^{-ik''x}dx\cdot\frac{1}{2\pi} = \iint_{k\,k'} \hat{a}(k)\hat{b}(k')\int_x e^{i(k+k'-k'')x}dx\,\frac{dk\,dk'}{2\pi}$

$\underbrace{\int_x e^{i(k+k'-k'')x}dx}_{2\pi\delta(k+k'-k'')}$ $\cancel{\frac{1}{2\pi}}$

$\widehat{a\,b}(k'') = \int_{k'} \hat{a}(k''-k')\hat{b}(k')dk'$

3D: $\widehat{ab}(\underline{k}'') = \iiint_{\underline{k}'} \hat{a}(\underline{k}''-\underline{k}')b(\underline{k}')d^3\underline{k}'$

3D F.T(N-S):

$\widehat{\frac{\partial u_i}{\partial t}} = \frac{1}{(2\pi)^3}\iiint d^3x\,\frac{\partial u_i}{\partial t}(\underline{x},t)e^{-i\underline{k}\cdot\underline{x}} = \frac{\partial}{\partial t}\hat{u}_i$

$\nu\widehat{\frac{\partial^2 u_i}{\partial x_k^2}} = -\nu k^2\hat{u}_i$

$\frac{1}{\rho}\widehat{\frac{\partial P}{\partial x_i}} = -\frac{1}{\rho}(i\underline{k})\hat{P}$

$u_j\widehat{\frac{\partial u_i}{\partial x_j}} = \widehat{\frac{\partial}{\partial x_j}(u_i u_j)} = ik_j\widehat{u_i u_j} = ik_j\int_{\underline{q}} u_i(\underline{k}-\underline{q},t)u_j(\underline{q},t)d\underline{q}$

Possion Eqn. $\frac{\partial}{\partial x_i}(\text{Mom. Eqn.})_i$

$-\frac{1}{\rho}\frac{\partial P}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}(u_i u_j)$

Fourier Transformed N-S Eqn.

$$\widehat{(N\text{-}S)} = \mathcal{F}\left(\frac{\partial u_i}{\partial t} + u_j \nabla_j u_i = -\frac{1}{\rho}\nabla_i p + \nu \nabla_k^2 u_i + f_i\right)$$

$\frac{\partial}{\partial t}\hat{u}$ $\quad$ $\widehat{u_j \nabla_j u_i}$ $\quad$ $\widehat{-\frac{1}{\rho}\frac{\partial p}{\partial x_i}}$ $\quad$ $-\nu k^2 \hat{u}$

$\widehat{\nabla_j(u_i u_j)} - \widehat{\nabla_j u_j}$ $\quad$ $-\frac{1}{\rho}(ik)\hat{p}$

$= ik_m \int \hat{u}_m(q)\hat{u}_i(k-q)\,d^3q$

Possion Eqn: $\frac{\partial}{\partial x_i}(NS)_i$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial t} + u_j \nabla_j u_i = -\frac{1}{\rho}\nabla_i p + \nu \nabla_k^2 u_i + f_i\right)_i$$

$\nabla_j(u_i u_j)$ $\quad$ $\frac{\partial}{\partial t}\nabla\cdot u = 0$ $\quad$ $\nu^2 \nabla\cdot u = 0$

$\left[\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}(u_i u_j) = -\frac{1}{\rho}\frac{\partial^2}{\partial x_i^2}p\right]$ Possion Eqn.

$\mathcal{F}$

$-k_m k_n \int \hat{u}(q)\hat{u}(k-q)\,d^3q = \frac{1}{\rho}k^2\hat{p}$

$-\frac{k_m k_n}{k^2}(\hat{u}\otimes\hat{u}) = \frac{1}{\rho}\hat{p}$

$$\mathcal{P}_{in}(k) = \delta_{in} - \frac{k_i k_n}{k^2}$$

Projection Operator

Fourier Transformed N-S Eqn.

$$\boxed{\left(\frac{\partial}{\partial t} + \nu k^2\right)\hat{u}_i(k,t) = -ik_m \mathcal{P}_{in}\int_q \hat{u}_m(q)\,\hat{u}_n(k-q)\,d^3q}$$

project n direction to i

$\widehat{[N\text{-}S]}\left(\hat{u}, \frac{\partial}{\partial t}, \hat{u}\otimes\hat{u}, \hat{p}\right) \to \widehat{[N\text{-}S]}(\hat{u}, \mathcal{P}, \hat{u}\otimes\hat{u})$

$\widehat{[\nabla\cdot[N\text{-}S]]}(\hat{u}\otimes\hat{u}, \hat{p})$

possion Eqn $\quad$ project $(\widehat{u\otimes u})_n$ to $\hat{u}_i$

Cut off:

$\mathcal{F}.T.$

$$\frac{1}{\rho}\hat{p}k^2 = -k_m k_n \int_q \hat{u}_m(q,t)\,\hat{u}_n(k-q,t)\,d^3q$$

$$\frac{1}{\rho}\hat{p} = -\frac{k_m k_n}{k^2}\int_q (\;)\,d^3q$$

$$\widehat{\frac{\partial}{\partial x_j}u_i u_j} = ik_j \int_q \hat{u}_j(q)\,\hat{u}_i(k-q)\,dq$$

$$= ik_m \int_q \hat{u}_m(q)\,\hat{u}_n(k-q)\,dq \cdot \delta_{in}$$

NL together

$$\widehat{\frac{\partial}{\partial x_j}(u_i u_j)} + \widehat{\frac{1}{\rho}\frac{\partial p}{\partial x_i}} = ik_m\left(\delta_{im} - \frac{k_m k_n}{k^2}\right)\int_q \underbrace{\hat{u}\;\hat{u}\,d^3q}_{\text{3D Convolution.}}$$

$\frac{\partial}{\partial t}\hat{u} = (A) - \nu k^2\hat{u}$

$u = e^{(A) - \nu k^2}$

Same direction but decreasing exp

All Together

$$\left[\left(\frac{\partial}{\partial t} + \nu k^2\right)\hat{u}_i(k,t) = -ik_m \mathcal{P}_{in}(k)\int_q \hat{u}_m(q)\,\hat{u}_n(k-q)\,d^3q\right]$$

the convolution $\int \hat{u}\,\hat{u}$ been $\mathcal{P}$ projected to u_i direction

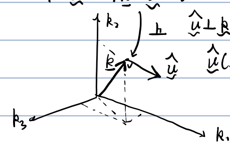
where, $\mathcal{P}_{in}(k) = \delta_{in} - \frac{k_i k_n}{k^2}$ (Projection Operator)

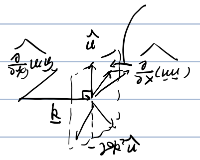
$\mathcal{P}$ projector

Continuity

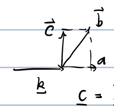
$\nabla\cdot\hat{u} = ik\cdot\hat{u} = 0$

$\perp$ $\quad$ $\hat{u}\perp k$ $\quad$ $\hat{u}(k,t)$





$\frac{\partial}{\partial t}\hat{u}$ has to be remain plane $\perp$ k

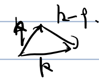


$c = b - a$

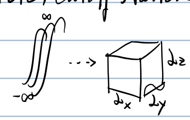
$= b - b\cdot\frac{k}{|k|}\cdot\frac{k}{|k|}$

$= \underbrace{\left(I - \frac{k\cdot k}{|k|^2}\right)}_{\text{Projection vector}}\cdot b$

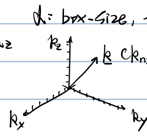
- Viscosity selectively damps higher Fourier modes $\quad \frac{d\hat{u}}{dt} = -\nu k^2\hat{u}$, $\quad \hat{u} = \hat{u}(0)e^{-\nu k^2 t}$
- Pressure Gradient and $u\cdot\nabla u$
  both NL of same order
- $\mathcal{P}$ field maintain incompressibility by projecting NL to plane $\perp$ to k
- Triadic interactions (3 wave vectors)



Discrete, Cutoff Navier-Stokes & Euler Eqns.



$\mathcal{L}$: box-size, periodic BC.



$k_{nx} = \frac{2\pi}{\mathcal{L}}\cdot n_x$ $\quad$ $n_x \sim (-\infty, \ldots -2, -1, 0, 1, \ldots \infty)$

need small wave $\lambda < \mathcal{L}$

$\lambda_{max} \to k_{min}$.

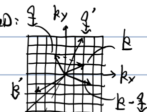
Cut off: $k_{max} = \frac{2\pi}{\lambda_{min}}$ (Smallest wave) $\quad$ $k_{max} = \frac{2\pi}{2\Delta x_{min}} = \frac{\pi}{\Delta x_{min}} = \frac{\pi}{\Delta_{min}} = \frac{2\pi}{\mathcal{L}}\cdot N_{max}$

Discrete N-S Eqn. in Fourier Space, truncated

$$\left(\frac{\partial}{\partial t} + \nu k^2\right)\hat{u}_i(n_x, n_y, n_z, t) = i\mathcal{P}_{in}(k)\,k_m \sum_q \hat{u}_m(q,t)\,\hat{u}_n(k-q,t)$$

$k = \frac{2\pi}{\mathcal{L}}(n_x\vec{i} + n_y\vec{j} + n_z\vec{k})$

2D: q



k, q, $k-q$ on grid.

restricted in $k, q, k-q$: $\sim$ $k - q'$ Non-linear interaction:

$|q|, |k-q| \leq k_{max}$ $\quad$ mode/scale smaller than included in truncated sys.

Since $u_i(x)$ real, $\hat{u}_i(k) = \hat{u}_i^*(-k)$

Complex Conjugate

Imagine programming ODE Solver $\quad$ Ex. $n_x = 8 = n_z = n_y$, $8^3 = 512$ ODE

NL: $(n_x^3)^2$ op.

Cutoff Euler Eqn. $\nu = 0$

$$\frac{\partial \hat{u}_i}{\partial t} = -i\mathcal{P}_{in}k_m \sum_q \hat{u}_m(q)\,\hat{u}_n(k-q,t)$$

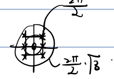
Know Energy Conserving:

$$\hat{u}_i^*\frac{\partial \hat{u}_i}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\hat{u}_i^*\hat{u}_i\right) = \frac{\partial}{\partial t}\frac{1}{2}\hat{u}_i(k)\hat{u}_i^*(k)$$

$$= \frac{\partial}{\partial t}\sum_k \frac{1}{2}|\hat{u}_i|^2 = 0.$$

Initialization: all $\hat{u} = 0$ (complex)

But few modes at small $|k|$

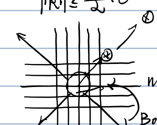


total Energy (initial)

$\frac{1}{2}\sum_k \hat{u}_i\,\hat{u}_i^*(k,0) = \frac{3}{2}u'^2$ $\quad$ put E in large $\mathcal{L}$

$|k| \leq \frac{2\pi}{\mathcal{L}}\cdot c$

$\nu = 0$



modes at larger k start receiving energy.

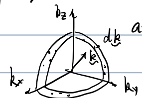
$t = \sigma\left(\frac{\mathcal{L}}{u'}\right)$ notably not dependent on N_{max} / k_{max}

reach $|k_{max}|$

long time.

equpartician of Energy

Def. $E(k)\,dk$ $\sim$ Energy of modes shall thickness dk



at $|\text{wavenumber}| = k$.

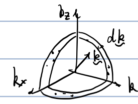
$E(k)\,dk = \frac{1}{2}\sum \hat{u}_i(k)\,\hat{u}_i^*(k)$ $\quad$ $k \leq |k| \leq k + dk$

radial E Spectum.

All modes on average have same E

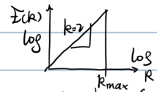
$E(k)$ Energy density.

For equipartition 3D.



area $\sim k^2$

$E(k) \sim k^2$ $\begin{pmatrix} 1D: & k^0 \\ 2D & k^1 \\ 3D & k^2 \end{pmatrix}$



NOT in Turbulence

in N-S (real fluid, $\nu > 0$) energy reaches k_{max} $(\sim \frac{\pi}{\Delta x})$

is dissipated into heat.

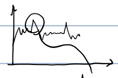
Initial $E. \sim \frac{3}{2}u'^2$, $t \sim o(\frac{l}{u'})$ time to get to small scales

Rate $= \frac{\text{energy}}{\text{time}} \sim \frac{u'^2}{(\frac{l}{u'})} \sim \frac{u'^3}{l}$ injection of energy. $= \varepsilon$ visious dissipation $\nu\left(\frac{\partial u_i'}{\partial x_j}\right)^2$

If. $\frac{u'^3}{l} \sim O(\varepsilon)$ "turbulence equlibrium" $\neq$ "Thermal Equlibium"

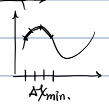
$k_{max} = ?$ $(\Delta x_{min} = ?)$

$\frac{u'^3}{l} \sim O(\varepsilon)$, $\varepsilon = \nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \doteq O\left[\nu\left(\frac{\delta u_{\Delta x_{min}}}{\Delta x_{min}}\right)^2\right]$



Structure Function.

$\delta u_r = [\langle(u(x+r) - u(x))^2\rangle]^{\frac{1}{2}}$ average.



Velocity implement at scale r

* nam. argument.

In turbulence.

eddies of size $r \rightarrow Re_r \sim \frac{\delta u_r \cdot r}{\nu}$

$r = \Delta x_{min}$, $Re_{\Delta x_{min}} = O(1)$ Stable.

$\frac{\delta u_{\Delta x_{min}} \cdot \Delta x_{min}}{\nu} = 1$

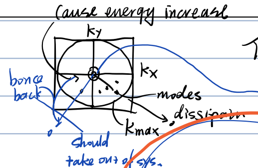
$\delta u_{\Delta x_{min}} = \frac{\nu}{\Delta x_{min}}$

$\epsilon \simeq \nu \cdot \left(\frac{\nu / \Delta x_{min}}{\Delta x_{min}}\right)^2 \doteq \nu^3 \frac{1}{\Delta x_{min}^4}$

$\left[\Delta x_{min} \doteq \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}} \simeq \left(\frac{\nu^3}{u'^3 \cdot \frac{1}{l}}\right)^{\frac{1}{4}}\right] \rightarrow k_{max} = \frac{\pi}{\left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}}$

$\left[\frac{\Delta x_{min}}{l} \sim \left(\frac{\nu}{u' l}\right)^{\frac{3}{4}}\right]$ $\left[k_{max} = \frac{\pi}{\nu^{\frac{3}{4}}} \epsilon^{\frac{1}{4}}\right]$

cause energy increase



$T \sim \frac{l}{u'}$ independent on k_{max}

min $k \sim$ large Eddy

if u' irrotational (or $\omega' = 0$):

$\frac{\partial}{\partial x_i}(\overline{u_i' u_j'}) = \frac{\partial}{\partial x_i} k$

Can be absorbed into P $\sim$ No effect on $\bar{u}$

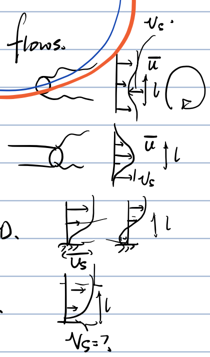
Turbulent shear flows.

wakes $<$ 2D, 3D

jets $<$ 2D, 3D

mixing layers – 2D.

Wall-bonded flows.



TKE. Eqn.

$\frac{\partial k}{\partial t} + u_i \frac{\partial k}{\partial x_i} = \underbrace{-\overline{u_i' u_j'} \frac{\partial \bar{u}_i}{\partial x_j}}_{\text{producten}} - \frac{\partial}{\partial x_i} Q_i - \underbrace{\epsilon}_{2\nu \overline{S_{ij}' S_{ij}'}}$

l: Characteristic length scale. of $\bar{u}$ profile.

V_s: Characteristic mean velocity scale. across scale "l"

$\nabla \bar{u}$

$\frac{\partial \bar{u}_i}{\partial x_j} \sim o\left(\frac{V_s}{l}\right)$ slope

Turbulent $\sim$ fluctuatn r.m.s $u' \sim o$ ($\frac{\text{change of } \bar{u}}{\text{scale } l.}$ i.e. V_s

largest / sufficen Eddy of turbulene are of size l.

in T.K.E. Eqn.

Production $-\overline{u_i' u_j'} \frac{\partial \bar{u}_i}{\partial x_j} \longrightarrow P \sim O\left(\frac{u'^3}{l}\right)$ or $\left(\frac{u'^2}{T}\right)$, $T \sim \frac{l}{u'}$

assume same order. (correlatn) u'^2 $\frac{V_s}{l} \sim \frac{u'}{l}$

Dominate terms P & ϵ

(Caveat: in ~~viscosity~~ many flows other also important)

$P \sim \frac{u'^3}{l}$ & $\epsilon = 2\nu \overline{S_{ij}' S_{ij}'}$

Similar

Turbulent Signal

Canotical turblen

HIT. homg Iso. $\bar{U} = 0$

free shear $\bar{U} \neq 0$ U_s l (1)

wall-bonded shear (2)

No class Thursday.

Turbulence Energy Dissipation

$\epsilon \sim \frac{u'^3}{l}$ Energy in. Large Eddy energy injected

$\epsilon \sim \frac{\eta^4}{\nu^3}$ energy dissipation. Smallest eddy energy diffusion

$\epsilon = \epsilon = \epsilon$

$\frac{u^3}{l} = \frac{\eta^4}{\nu^3}$

[L] $l > r > \eta = \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}$

[V] $u > U = (\epsilon r)^{\frac{1}{3}} > v = \left(\frac{\nu \epsilon}{\cdot}\right)^{\frac{1}{4}}$

[T] $\frac{l}{u} > T = \left(\frac{r^2}{\epsilon}\right)^{\frac{1}{3}} > \tau = \left(\frac{\nu}{\epsilon}\right)^{\frac{1}{2}}$

$\vec{x}$ $\vec{r}$

homogeneous $\overline{x_1' x_2'} = \overline{x_3' x_4'}$ not depend on $\vec{x}$

isotropic $\hat{r}$ $\overline{r_1 r_2} = \overline{r_3 r_4}$ not depend on $\vec{r}$

$E(k) =$

Structure Fn.
$\delta u_r = \langle [u(x+r) - u(x)]^2 \rangle^{\frac{1}{2}}$

① Statistics. dissipation + viscosity

$\frac{\text{Internal}}{\text{Viscosity}} = Re_\eta = 1$: small scale dissipate all input (internal energy)

$\| \frac{\delta u_\eta \cdot \eta}{\nu}$

$\epsilon = \frac{u'^3}{\ell} \, \frac{\quad}{\quad} \, \frac{(\delta u_\eta)^3}{\eta}$

Defination of "velocity incremental at scale r"

$\delta u_r = \langle [u(x+r) - u(x)]^2 \rangle^{\frac{1}{2}}$

Structure function of velocity

For $r \gg$ correlation scale signal. ②

$\delta u_r = \overline{u^2(x+r)} - 2\overline{u(x+r)u(x)} + \overline{u^2(x)}$

if $r \gg \rho$: $u(x+r)u'(x) \approx 0$ uncorrelated

$\delta u_r = \sqrt{2\overline{u'^2}}$

δu_r: "typical velocity difference over l-scale r"

Last time δu_η $\frac{\delta u_\eta \cdot \eta}{\nu} = Re_\eta = 1$ $\frac{[u\ell]}{[\nu]} = 1$ 惯性 v 黏性. balance

$\frac{u'^3}{\ell} = \epsilon$ $\epsilon = \nu \left(\frac{\delta u_\eta}{\eta}\right)^3$

$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}$ why

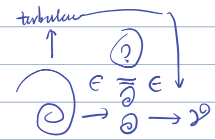
Zeroth-Flow turbulence:
$\epsilon = o\left(\frac{u'^3}{\ell}\right)$
regardless of ν

$\frac{u'^3}{\ell} \sim \nu \left(\frac{\delta u_\eta}{\eta}\right)^2$
fixed

$Re\uparrow \to \nu\downarrow$

Singular limit
$\nu \to 0 \neq \nu = 0$
ν close to 0 BUT NOT 0.
$0 \cdot \left(\frac{\delta u}{0}\right)^3 =$ constant

Large Eddy



Non-linearity of N-S Eqn. $\to$ generate smaller, smaller scales $\to$ tiny ν to generate "enough" dissipation

dissipate rate of largest eddies are feeding turbulence

Characteristics Velocities

δu_r for $r \gtrsim$ correlation scale "ℓ"

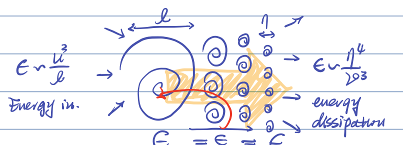
$\delta u_r \sim u'$

δu_η for $r \sim \eta$ Kolmogorov's scale

$\frac{\delta u_\eta \cdot \eta}{\nu} = 1 \to \delta u_\eta = \frac{\nu}{\eta} = \frac{\nu}{\nu^{\frac{3}{4}}}\epsilon^{\frac{1}{4}} \to \delta u_\eta = (\nu \cdot \epsilon)^{\frac{1}{4}}$ K. Velocity.

K-time scale. $\tau_\eta = \frac{\eta}{\delta u_\eta} = \frac{\frac{\nu^{\frac{3}{4}}}{\epsilon^{\frac{1}{4}}}}{\nu^{\frac{1}{4}}\epsilon^{\frac{1}{4}}} = \sqrt{\frac{\nu}{\epsilon}}$ How quickly smallest turbulence evolve

Large eddy transfer to small eddies, when reached K-length scale, energy dissipated due to viscosity. (Kolmogorov, smallest eddy)



(Dr. Yao's research) could happen in small eddy have large rotation speed transfer energy back to large eddy (but only local, global still $\to$)

Scales between. $\eta < r < \ell$:

"energy of eddies at scale r" $\sim \delta u_r^2$

"Kurnover time-scale of eddies of size r: $\frac{r}{\delta u_r}$

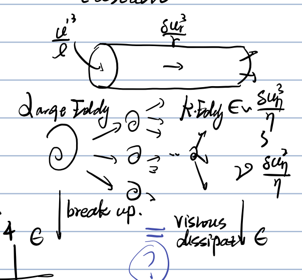
rate of decrease of energy. $\frac{\delta u_r^2}{\left(\frac{r}{\delta u_r}\right)} \sim \frac{\delta u_r^3}{r}$

K-scale
$\epsilon \sim \nu \frac{\delta u_\eta^2}{\eta^2} \sim \left[\frac{\nu}{\delta u_\eta \cdot \eta}\right] \frac{\delta u_\eta^3}{\eta}$

$\epsilon \sim \frac{\delta u_r^3}{r}$

Large scale
$\epsilon \sim \frac{u'^3}{\ell}$

Cascade:



$\frac{\delta u_r^3}{r} \sim \epsilon$

$\delta u_r \sim (r\epsilon)^{\frac{1}{3}}$

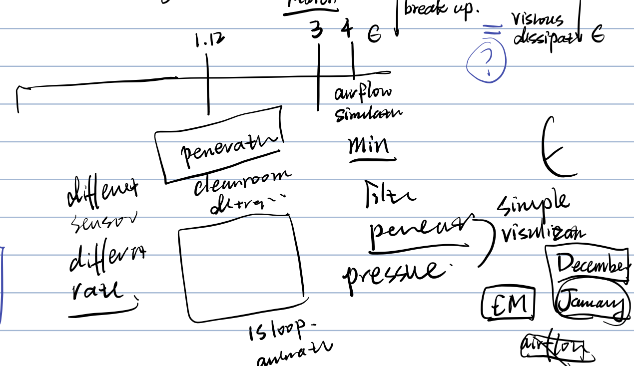
$\delta u_r^2 \sim \langle [u(x+r) - u(x)]^2 \rangle \sim r^{\frac{2}{3}} \epsilon^{\frac{2}{3}}$

In words: $E(k) \sim$ avgs at scale $r = \frac{1}{k}$

Energy spectrum per unit wave number $\left(\frac{1}{k}\right)$

☆ $\left[\frac{\epsilon^{\frac{2}{3}} r^{\frac{2}{3}}}{k} \simeq \frac{\epsilon^{\frac{2}{3}} k^{-\frac{2}{3}}}{k} \sim \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}} \right]$

Make up. Oct 9. 6 p.m.



$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}$, $\epsilon = \frac{u'^3}{\ell} \longrightarrow \frac{\eta}{\ell} \sim Re_\ell^{-\frac{3}{4}}$ $Re_\ell = \frac{u'\ell}{\nu}$ Re_ℓ for turbulence

$\frac{\tau_K}{(\ell/u')} \sim Re_\ell^{-\frac{1}{2}}$

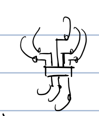
$\frac{\nu}{u'} \sim Re_\ell^{-\frac{1}{4}}$

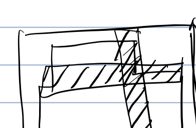
GPT API

Standardized

prompt Engineering.

? Formean





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Cleanroom

DNS cost: by degree of freedom.

DNS. cost.

DNS. cost:
τ_k small eddies

Number of degree of freedom (DNS)

1 dim. $N_x \sim \frac{\ell}{\eta}$ 3D. $N_{tot} \sim \left(\frac{\ell}{\eta}\right)^3 \sim Re_\ell^{\frac{9}{4}}$

Computational cost.

intrinsic evolution time of small eddies. τ_k

Capture all evolution of $\left(\frac{\ell}{u'}\right) \sim T_L$

number of time scales $\frac{T_L}{\tau_k} \sim \frac{\ell/u'}{\tau_k} \sim Re_\ell^{\frac{1}{2}}$

Total computational cost of DNS $\sim Re_\ell^{\frac{9}{4}+\frac{1}{2}} = Re_\ell^{\frac{11}{4}}$

Total computational cost.

Caveat: for most Eulerian N-M: τ_k η time.

Small scale eddy been advected by large-scale velocity

sweeping of velocity u'

advection time scale $T_a \sim \frac{\eta}{u'}$ $\frac{T_a}{(\ell/u')} \sim Re_\ell^{-\frac{3}{4}}$

$\left(\frac{\ell}{\eta}\right)^4 \sim Re_\ell^3$

Data Center test factory test

ch cmf cml

Expecatly

Ansys Startup

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Taylor microscale. δu_r

$\delta u_\ell \sim u'$
$\delta u_\eta \sim v_k$
$\delta u_r \to \epsilon = \frac{\delta u_r^3}{r} \sim \frac{u'^3}{\ell} \sim \frac{v_k^3}{\eta} = \epsilon$

viscous dissipation due to eddy size r:

$\epsilon^{(r)} = \nu\left(\frac{\partial u_i}{\partial x_j}\right)^2$ filtered scale r $\sim \nu \frac{\delta u_r^2}{r^2}$

$\epsilon^{(r)} \sim \nu \frac{(\epsilon r)^{\frac{2}{3}}}{r} \sim \nu \epsilon^{\frac{2}{3}} r^{-\frac{4}{3}}$

$\sim \frac{\nu}{u'\ell} u' \left(\frac{u'}{\ell}\right)^{\frac{2}{3}} \nu^{-\frac{4}{3}} \ell$

Taylor microscale

$\overline{\left(\frac{\partial u}{\partial x}\right)^2} = \frac{u'^2}{\lambda^2} \to \epsilon = \nu \overline{\frac{\partial u_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} = 15\nu\overline{\left(\frac{\partial u}{\partial x}\right)^2} = 15\nu S_{11}^2 = 15\nu \frac{u'^2}{\lambda^2}$

isotropic turbulence

$\frac{\lambda}{\ell} = \sqrt{15}\, Re_\ell^{-\frac{1}{2}}$ $Re_\ell^{\frac{1}{2}}$

Taylor microscale

$Re_\lambda = \frac{u'\lambda}{\nu}$ Re_λ

η λ ℓ

Turbulent thin Shear flows. in Boundary Layer Approx.

Free shear flow. Tennekes & Lumley

Slow evolution. x: $\frac{\partial}{\partial y} \gg \frac{\partial}{\partial x}$

y x $\bar{V}$ $\bar{u}$ $\bar{V} \ll \bar{u}$ v' w' u'

a) Wake. U_s (velocity deficit) ℓ u' L or x U_c (convection velocity)

b) jet U_s ℓ U_c $U_s \sim U_c$

c) mixing layer $U_s \sim U_c$ ℓ

RANS.
$\frac{\partial \bar{u}}{\partial x} \sim \frac{U_s}{L}$
how much $\bar{u}$ change
$\frac{\partial \bar{u}}{\partial y} \sim \frac{U_s}{\ell}$
$\frac{\partial}{\partial x} \ll \frac{\partial}{\partial y} \Leftrightarrow \ell \ll L$

$-\overline{u'v'} \sim O(u'^2)$
u' RMS.
$\overline{u'^2} \sim O(u'^2)$
$\overline{v'^2} \sim O(u'^2)$
$\overline{w'^2} \sim O(u'^2)$

$\bar{V} \sim$ from continuity. $\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{V}}{\partial y} = 0$

$\bar{U} \gg \bar{V} \leftarrow \bar{V} \sim \frac{\ell}{L} U_s$ $\frac{U_s}{L} \sim \frac{\bar{V}}{\ell}$

Major Problems
wakes
V_0 U_0 U_c $\bar{u}$ ℓ U_c Convection velocity
Jets

U_c: Convection velocity
U_s: velocity deficit.

Wake Modeling
$\frac{y}{\ell(x)}$ self-similar.
PDE $\longrightarrow$ ODE(ξ)
$\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \cdots = 0. \longrightarrow \frac{y}{\ell(x)} \to \ell$

Large Scale Small Scale
$\bar{U}, \bar{\theta}, L(x) \longrightarrow \epsilon \sim \frac{v'^2}{L}$

Scaling

Start with transverse mean momentum Eqn.

$$\bar{u}\frac{\partial \bar{V}}{\partial x} + \bar{V}\frac{\partial \bar{V}}{\partial y} + \frac{\partial}{\partial x}(\overline{u'v'}) + \frac{\partial}{\partial y}(\overline{v'^2}) = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial y} + \nu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{V}}{\partial y^2}\right)$$

$\cancel{\frac{\partial}{\partial z}(\overline{w'v'})}$ z direction homogeneous. 0 and not correlated.

choose an leading parameter ℓ on length scale
ℓ u' on velocity scale.

$U_s,\ \ell.$

$\bar{u}\frac{\partial \bar{V}}{\partial x} : \left[\frac{V_c}{u'}\frac{V_s}{u'}\left(\frac{\ell}{L}\right)^2\right]\frac{u'^2}{\ell}$ assume $\frac{V_c}{u'} \to \infty$ more slowly than $\left(\frac{\ell}{L}\right) \to 0$ $\to 0$

$\bar{V}\frac{\partial \bar{V}}{\partial y} : \left[\left(\frac{U_s}{u'}\right)^2\left(\frac{\ell}{L}\right)^2\right]\frac{u'^2}{\ell} \to 0$

$\frac{\ell}{L} \to 0.$

$\frac{\partial}{\partial x}\overline{u'v'} : \left[\frac{u'^2}{L}\right] = \left[\frac{\ell}{L}\right]\frac{u'^2}{\ell} \to 0$

$\left\{\begin{array}{ll} \frac{\partial}{\partial y}\overline{v'^2} : & [1]\frac{u'^2}{\ell} \\ \frac{1}{\rho}\frac{\partial \bar{P}}{\partial y} : & [?]\frac{u'}{\ell} \end{array}\right\} \longrightarrow \frac{\partial}{\partial y}\overline{v'^2} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial y}$ or $\frac{\partial}{\partial y}\left[\frac{\bar{P}}{\rho} + \overline{v'^2}\right] = 0$

outside turbulent sys

$\frac{1}{\rho}\bar{P} + \overline{v'^2} = \text{constant} = \frac{1}{\rho}\bar{P}_0 = \text{constant}$ among x domain

$\nu\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) \sim \frac{1}{Re_\ell}[\cdot\cdot]\frac{u'^2}{\ell} \longrightarrow 0.$

$\frac{\ell}{L} \to 0.$ $\downarrow 0$ $Re_\ell \to \infty$

neglect viscosity

Streamwise mean mom. Eqn.

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{V}\frac{\partial \bar{u}}{\partial y} + \frac{\partial}{\partial x}(\overline{u'^2} - \overline{v'^2}) + \frac{\partial}{\partial y}\overline{u'v'} = \nu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2}\right)$$

$\frac{\partial \bar{P}}{\partial x}$

$\bar{u}\frac{\partial \bar{u}}{\partial x} \sim \left[\frac{V_c}{u'}\frac{V_s}{u'}\cdot\frac{\ell}{L}\right]\frac{u'^2}{\ell}$

$\bar{V}\frac{\partial \bar{u}}{\partial x} \sim \left[\left(\frac{V_s}{u'}\right)^2\frac{\ell}{L}\right]\frac{u'^2}{\ell}$

$\frac{\partial}{\partial x}(\overline{u'^2} - \overline{v'^2}) \sim \left[\frac{\ell}{L}\right]\frac{u'}{\ell} \longrightarrow 0$

$\frac{\partial}{\partial y}\overline{u'v'} \sim [1]\frac{u'^2}{\ell}$

$\nu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2}\right) \sim \frac{1}{Re_\ell}[\cdot\cdot]\frac{u'^2}{\ell} \to 0$

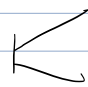
As $Re_\ell \to \infty$, keep. $\bar{u}\frac{\partial \bar{u}}{\partial x}$ $\bar{V}\frac{\partial \bar{u}}{\partial x}$ and $\frac{V_c}{u'}\frac{V_s}{u'}\frac{\ell}{L}$ or $\frac{V_s^2}{u'^2}\frac{\ell}{L}$ must be $O(1)$

a) wakes: $V_c \gg V_s$ $\frac{V_c}{u'}\frac{V_s}{u'}\frac{\ell}{L} \sim O(1)$

b) jets & mixing layer: $\left(\frac{V_s}{u'}\right)^2\frac{\ell}{L} \sim O(1)$

+. Empirical knowledge.

$\frac{u'}{V_s} \approx \text{constant} \approx 0.3$



$\frac{\ell}{L} \sim 0.1$

linear growth

$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{V}\frac{\partial \bar{u}}{\partial y} = -\frac{\partial}{\partial y}\overline{u'v'}$

Dominant Term.

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{V}\frac{\partial \bar{u}}{\partial y} = -\frac{\partial}{\partial y}\overline{u'v'}$$

$$\left[\frac{V_c V_s}{u' u'}\frac{\ell}{L} \quad \frac{V_s^2}{u'^2}\frac{\ell}{L} \quad 1\right]\frac{u'^2}{\ell}$$

Wakes Jets.

a) Wakes. $\bar{u}\frac{\partial \bar{u}}{\partial x} \gg \bar{V}\frac{\partial \bar{u}}{\partial y}$: $\bar{u}\frac{\partial \bar{u}}{\partial x} = -\frac{\partial}{\partial y}\overline{u'v'}$

$V_c \gg V_s$

$\bar{u} = U_0 - \Delta\bar{u}$

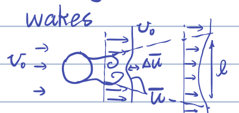
$\Delta\bar{u}$

$(U_0 - \Delta\bar{u})\frac{\partial(\bar{u} - U_0)}{\partial x}$

$-(U_0 - \Delta\bar{u})\frac{\partial(\Delta\bar{u})}{\partial x} = -\frac{\partial}{\partial y}\overline{u'v'}$

Major Problems

wakes



Jets

Linearize $U_0 \gg \Delta\bar{u}$ $U_0\frac{\partial \Delta\bar{u}}{\partial x} = \frac{\partial}{\partial y}\overline{u'v'}$

$$\boxed{U_0\frac{\partial \bar{u}}{\partial x} = -\frac{\partial}{\partial y}\overline{u'v'}}$$

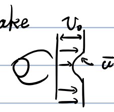
Mixing layers

Jets & Mixing layers.

$V_c \approx U_s$ $\left[\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{V}\frac{\partial \bar{u}}{\partial x} = -\frac{\partial}{\partial y}\overline{u'v'}\right]$

Integral invariants

wake



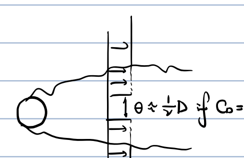
$\theta = \frac{1}{U_0^2}\int_{-\infty}^{\infty}\bar{u}(U_0 - \bar{u})\,dy$ Momentum thickness.

drag coefficient

$F_D = \rho\int_{\infty}^{\infty}\bar{u}(U_0 - \bar{u})\,dy = \rho U_0^2\theta = \frac{1}{2}\rho C_D U_0^2 D$

Drag per unit length.

$\theta = \frac{1}{2}C_D D$

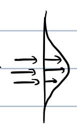


$\theta \approx \frac{1}{2}D$ if $C_D = 1$

Momentum thickness rep. by C_D.

θ independent of x

Jets.



$M = \rho\int_{\infty}^{\infty}\bar{u}\,dy$ constant on x

For wakes. planar

$\frac{U_0}{u'}\frac{V_s}{u'}\frac{\ell}{L} \sim 1$ $+$ $\int_{-\infty}^{\infty}\bar{u}(U_0 - \bar{u})\,dy = \text{constant}$

$U_0 \quad U_s \quad \ell \to U_0 U_s \ell \sim \text{constant}$

For Jets

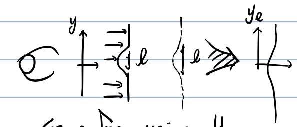
$\int \bar{u}^2 dy = \text{constant}$

& Mixing Layers.

$U_s^2 \ell \sim \text{constant}$.

Similarity Analysis: $\Delta \bar{U} = U_0 - \bar{u}$

$\Delta \bar{u} = U_s f\left(\frac{y}{\ell(x)}\right)$



Same form using y_e

$$\left[\begin{array}{l} U_0 \frac{\partial \bar{u}}{\partial x} = -\frac{\partial}{\partial y}\overline{u'v'} \\ \\ -\overline{u'v'} = U_s^2 g\left(\frac{y}{\ell}\right) \end{array}\right.$$

R_{stress} can be scaled by $U_s^2 g$

$$\frac{\partial \bar{u}}{\partial x} = \frac{\partial}{\partial x}(U_0 - \Delta\bar{u}) = -\frac{\partial U_s}{\partial x} f - U_s f' \frac{\partial\left(\frac{y}{\ell}\right)}{\partial x}$$

$\left[\xi = \frac{y}{\ell}\right]$ Similarity variable.

$$\frac{\partial \bar{u}}{\partial x} = -\frac{\partial U_s}{\partial x} f(\xi) + \frac{U_s}{\ell}\frac{d\ell}{dx}\xi$$

$$\frac{\partial}{\partial y}\overline{u'v'} = -\frac{U_s^2}{\ell} g'$$

Replacing Momentum Eqn. $U_s(x)$ $\ell(x)$

$$-\frac{U_0 \ell}{U_s^2}\frac{dU_s}{dx} f(\xi) + \frac{U_0}{U_s}\frac{d\ell}{dx}\xi f' = g'$$

if Similarity exist: f, g only on ξ, NOT x

then Not depend on x

$\left[\frac{U_0}{U_s^2}\ell\frac{dU_s}{dx}\right]$ & $\left[\frac{U_0}{U_s}\frac{d\ell}{dx}\right]$

Try power law solns.

$\ell = C_1 x^n$

$U_s = C_2 x^m$

$$\frac{\ell}{U_s^2}\frac{dU_s}{dx} \sim X^0 \qquad \frac{1}{U_s}\frac{d\ell}{dx} \sim X^0$$

$$X^{n-2m} X^{m-1} \sim X^0 \qquad X^{-m} X^{n-1}$$

$$\boxed{n-m-1=0 \longleftrightarrow m=n-1 \quad \text{Same}}$$

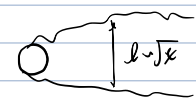
plane wakes $\int \bar{u}(U_0 - \bar{u})dy$

$U_s \quad \ell \sim X^0$

$X^m X^n \sim X^0$

$m = -n$.

$\left[n = \frac{1}{2}\right]$ $m = -\frac{1}{2}$ (or $U_0 \int U_s f\, dy$, $U_0 U_s \ell \int f d\xi$) some num.



?

Lets use. $U_s(x) = A x^{-\frac{1}{2}}$

$\ell(x) = B x^{\frac{1}{2}}$

$$\frac{1}{2}\left(U_0 \frac{B}{A}\right)(\xi f' + f) = g'$$

close eqn: express g interms of f

Use turbulence eddy viscosity

$-\overline{u'v'} = \nu_T \frac{\partial \bar{u}}{\partial y}$

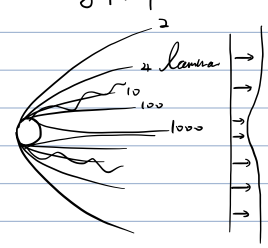
need ν_T assume ν_T is product of (U_s) velocity scale and length scale. (ℓ)

$\nu_T(x) = c\, U_s(x)\, \ell(x)$

$\frac{1}{Re_T} \sim Re_T = \frac{U_s \ell}{\nu_T} \neq Re_\ell$

eddy viscosity

mixing of large eddies.



$$U_s^2 g = -\frac{1}{Re_T} U_s \ell f' \frac{U_s}{\ell}$$

$$g = \frac{1}{Re_T} f'$$

$$g' = \frac{1}{Re_T} f''$$

$$\underbrace{\frac{1}{2} Re_T U_0 \frac{B}{A}}_{\alpha}(\xi f' + f) + f'' = 0$$

$$\int \left(\alpha \frac{d}{d\xi}(\xi f) + \frac{d}{d\xi} f' = 0\right)$$

$\alpha \xi f + f' = \text{constant}$ but $\xi = 0$: $f'(0) = 0$, $f(\pm\infty) = 0$

$f' = -\alpha \xi f \leftarrow C = 0$

$$\frac{df}{f} = -\alpha \xi d\xi$$

$$\ln f = -\alpha \frac{\xi^2}{2} + C_3$$

$$f(\xi) = f_0 e^{-\alpha\frac{\xi^2}{2}}$$

Since. $\bar{u}(0) = U_0 - U_s \rightarrow f(0) = 1$.

$$f(\xi) = e^{-\alpha\xi^2}$$

if $\alpha = 1$, $f(1) = e^{-\frac{1}{2}} \approx 0.6$
$y = \ell$

$\ell \quad \xi \quad e^{-\frac{1}{2}}$

$$\Downarrow$$

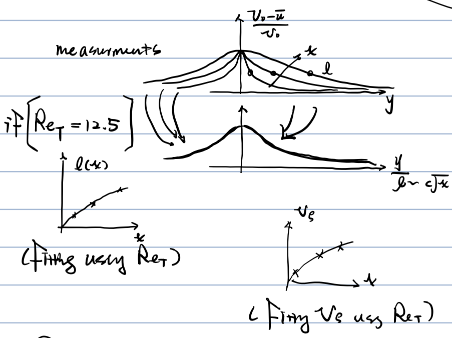
$$A = \frac{U_0 B}{2} Re_T$$

$$U_0^2\theta = \int_{-\infty}^{\infty} U_0(U_0 - \bar{u})\,dy = U_0 U_s \ell \int_{-\infty}^{\infty} f\,d\xi = U_0 U_s \ell \sqrt{2\pi}$$

$$U_s \sim Ax^{-\frac{1}{2}}, \quad \ell \sim Bx^{\frac{1}{2}}$$

$$U_0^2\theta = U_0 AB\sqrt{2\pi}$$

$$B = \sqrt{\frac{2}{\sqrt{2\pi}\,Re_T}}\,\theta^{\frac{1}{2}}$$



$$\ell(x) = \sqrt{\frac{2}{\sqrt{2\pi}\,Re_T}}\,\theta^{\frac{1}{2}} x^{\frac{1}{2}}$$

$$\left[\frac{\ell}{\theta} = \sqrt{\frac{2}{\sqrt{2\pi}\,Re_T}}\left(\frac{x}{\theta}\right)^{\frac{1}{2}}\right]$$

$$A = \frac{U_0\sqrt{Re_T}}{\sqrt{2\sqrt{2\pi}}}\,\theta^{\frac{1}{2}}$$

$$\left[\frac{U_s}{U_0} = \left(\frac{\theta}{x}\right)^{\frac{1}{2}}\frac{\sqrt{Re_T}}{\sqrt{2\sqrt{2\pi}}}\right]$$

Planer Turbulent wakes

$$\begin{cases} \frac{\ell}{\theta} = 0.252\sqrt{\frac{x}{\theta}} \\ \frac{U_s}{U_0} = 1.58\sqrt{\frac{\theta}{x}} \end{cases}$$

$u' \sim o(U_s)$

$$Re_T = \frac{U_s \ell}{\nu_T}$$

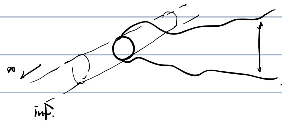
$$Re_\ell = \frac{u'\ell}{\nu} \sim \frac{U_s \ell}{\nu} = \frac{\nu_T}{\nu} Re_T$$

molecular viscosity

$$Re_\ell = \frac{Ax^{-\frac{1}{2}} Bx^{\frac{1}{2}}}{\nu} \sim \frac{AB}{\nu} x^0$$

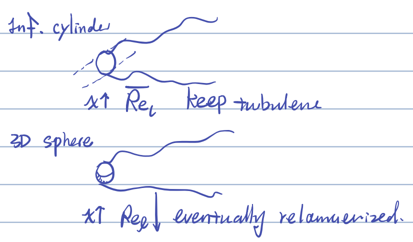
Reynolds # of turb. indep. of x

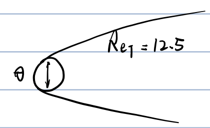
Turbulence $\overline{Re}$, keep turbulence



$Re \sim \ell \uparrow U \downarrow / \nu$

Turbulence keep turbulence

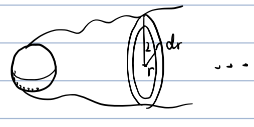


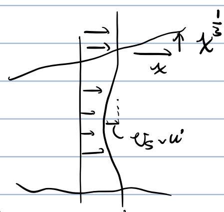


$$\int_{-\infty}^{\infty} \bar{u}(U_0 - \bar{u})\,dy$$

$U_0 \quad U_s \quad \ell \sim x^0.$

Axis-symmetric. wake





Momentum deficiency flux int.

everything else same

$$\int_{-\infty}^{\infty} \bar{u}(U_0 - \bar{u})\,2\pi r\,dr$$

$U_0 \quad U_s \quad \ell^2 \sim x^0$

$$+\begin{cases} \ell \sim x^n \\ U_s \sim x^{n-1} \end{cases} \rightarrow x^{n-1}x^{2n} \sim x^0$$

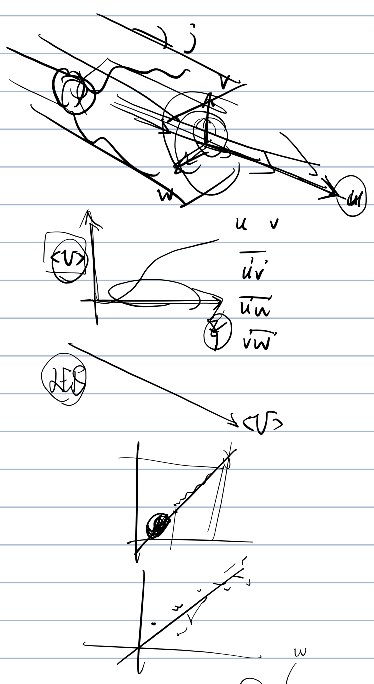
$$n = \frac{1}{3}$$

$$\nu_T = \frac{1}{Re_T} U_s \ell \sim x^{\frac{1}{3}}$$

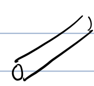
$$Re_\ell \sim \frac{U_s \ell}{\nu} \sim x^{-\frac{1}{3}}$$

Re_ℓ drops as $x \to \infty$.

3D wake decays and relaminerized.

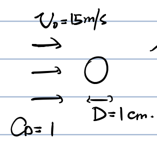


Gemllar



Plane wake: Velocity scale $U_s(x)$

time scale. of evolution $\begin{cases} \text{mean v.} \\ \text{turbulence} \end{cases}$



$U_s(5m) = ?$

$u' = (0.3 \sim 0.35)\,U_s$ (empirical)

Velocity Scale

$Re = \frac{\partial U_z}{\nu}$

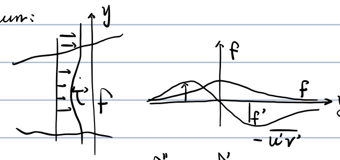
distribution. $A = A_0 e^{-\left(\frac{x}{\delta}\right)^2}$ $\bar{\mu}$ Re_D Re_x Re_z D x z

$\frac{z U_z}{\nu}$

Peak Shear Stress.

$-\overline{u'v'}\Big|_{peak} = -\frac{U_s^2}{Re_T} \cdot f'$ g

maximum:



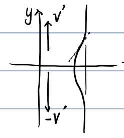
$f'' = 0,\ f'_{max}$

$\frac{d^2}{d\xi^2}\left(e^{-\frac{\xi^2}{2}}\right) = 0 \rightarrow \xi = \pm 1$

$\frac{-\overline{u'v'}}{U_s^2} = \frac{1}{Re_T} e^{-\frac{1}{2}} = \frac{0.6}{12.5} = 0.05$

Empirically. ρ correlatin and u', v':

$\rho_{u'v'} \simeq -0.4$

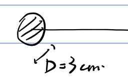


$\frac{-\overline{u'v'}}{\overline{u'^2}} = 0.4 \longrightarrow \frac{0.05 U_s^2}{\overline{u'^2}} = 0.4$

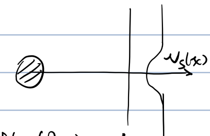
$\left(\frac{U_s}{u'}\right)^2 = \frac{0.4}{0.05}$

$\left[\frac{u'}{U_s} = 0.35\right]$

Ex. $V_0 = 10\,\frac{m}{s}$ ν from material



$\eta(x=8m) : ?$

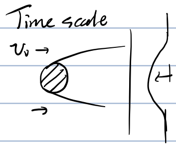


$U_s(x)(8m) \rightarrow u' = 0.35 U_s$

$\ell(8m)$ $\epsilon = \frac{u'^3}{\ell}$

$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}$

Time scale



$\frac{d\ell}{dt} = V_0 \frac{d\ell}{dx} = V_0 \cdot 0.25 \nu \frac{dx^{\frac{1}{2}}}{dx}\theta^{\frac{1}{2}} = \frac{0.25}{2}\underbrace{V_0\left(\frac{\theta}{x}\right)^{\frac{1}{2}}}_{U_s \frac{1}{158}} = 0.08\, U_s(x)$

smaller than u'

mean flow evolve time scale

Turnover time scale of eddies. $\frac{\ell}{u'} \sim \frac{\ell}{0.25 U_s}$

$\epsilon = \frac{u'^3}{\ell} = \frac{u'^2}{\tau_\ell} = \text{also} = \text{Prod.} = -\overline{u'v'}\frac{\partial \bar{u}}{\partial y}$

$\tau_\ell = \frac{\overline{u'^2}}{-\overline{u'v'}\frac{\partial \bar{u}}{\partial y}}\Big|_{max} = -3.75 \frac{\ell}{U_s f'_{max}} = 6.2 \frac{\ell}{U_s}$

Mean velocity time scale

$\frac{d\ell}{dt} = 0.08 U_s$

$\frac{d\ell}{dt} = \frac{\ell}{\tau_\ell^*} \rightarrow \tau_\ell^* = \frac{\ell}{\left(\frac{d\ell}{dt}\right)} = \frac{\ell}{U_s}\, 125$

Meanflow change significantly time scale $\frac{\ell}{U_s}\, 125$

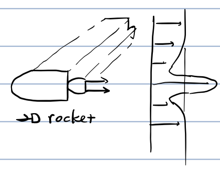
while turbulence change/evolves $6.2 \frac{\ell}{U_s}$

$\tau_\ell \approx \frac{1}{20}\tau_\ell^*$

"No real strong separation of time scales"

Self Propelled wake

planer case



$\int_{-\infty}^{\infty} \bar{u}(V_0 - \bar{u})\,dy = 0.$ Total Force. momentum integral (θ) vanishes

$\ell \sim x^n$ BUT $\ell \cdot U_s \sim x^0$ No Longer holds

$U_s \sim x^{n-1}$

higher moments.

momentum Eqn. .. (plane wake) $\frac{\partial}{\partial x}\left[V_0(\bar{u} - V_0)\right] = -\frac{\partial}{\partial y}\overline{u'v'} = \nu_T \frac{\partial \bar{u}}{\partial y}$

$\times y^2$

$\frac{\partial}{\partial x}\left[y^2 V_0(\bar{u} - V_0)\right] = \nu_T\, y^2 \frac{\partial \bar{u}}{\partial y}$ $= 0$ int. by parts. twice.

$\int$

$\frac{\partial}{\partial x}\int_{-\infty}^{\infty} y^2 V_0(\bar{u} - V_0)\,dy = 2\nu_T \int_{\infty}(\bar{u} - V_0)\,dy = 0.$

$\int_{-\infty}^{\infty} y^2 V_0(\bar{u} - V_0)\,dy = 0.$

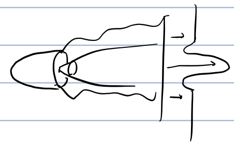
$\ell^2 \quad U_s \quad \ell$ $\quad \ell^3 U_s \sim x^0$

$x^{3n} x^{n-1} \sim 0 \rightarrow n = \frac{1}{4}$

$\ell \sim x^{\frac{1}{4}}$ $U_s \sim x^{-\frac{3}{4}}$

decay more quickly

Axis-Symmetric Self Propelled wake



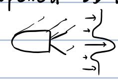
$\int r^2(\bar{u} - V_0)\,2\pi r\,dr = 0$

$x^{4n} x^{n-1} \sim x^0 \rightarrow n = \frac{1}{5}$ $\begin{cases} \ell \sim x^{\frac{1}{5}} \\ U_s \sim x^{-\frac{4}{5}} \end{cases}$ $Re_\ell = \frac{U_s \ell}{\nu} \sim x^{-\frac{3}{5}}$ decay very quick.

as opposed to wake / jet

For wake flows. similarity behaviour works OK for $\begin{cases} \frac{x}{D} \geq 80 & \text{for } \bar{u} \\ \frac{x}{D} \geq 200 & \text{for stresses} \end{cases}$? HW.

Self-propelled 2D wake $\quad l \sim x^n, \; U_s \sim x^{n-1}$



$U_s l^3 \int_{-\infty}^{\infty} \xi^2 f(\xi) d\xi = \text{constant}$

$n = \frac{1}{4} \begin{cases} U_s \sim A x^{-\frac{3}{4}} \\ l \sim B x^{\frac{1}{4}} \end{cases}$

$-\frac{U_0 l}{U_s} \frac{dU_s}{dx} f + \frac{U_0}{U_s} \frac{dl}{dx} \xi f' = g'$

eddy visc.

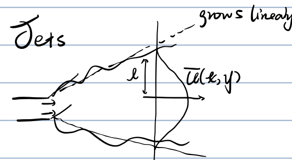
$\alpha(\xi f + \xi f') + f'' = 0 \quad , \quad \alpha = U_0 B \, Re_T \frac{1}{A}$, set $\alpha = 1$

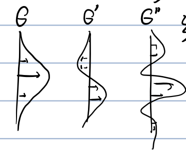
$f(\xi) = \frac{d^2}{d\xi^2} e^{-\frac{\xi^2}{2}}$ $\quad$ f is such that $\alpha = 1$

Can fit Re_T value to data

Turbulent Jets

planner





$\frac{U_c}{u'} \frac{U_s}{u'} \frac{l}{L} \sim O(1)$ $\quad$ wakes: $U_c = U_0 \gg U_s$

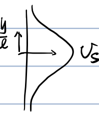
$\left(\frac{U_s}{u'}\right)^2 \frac{l}{L} \sim O(1)$ $\quad$ jets (mixing layer) $U_c \sim U_0$

$\frac{l}{L} \to 0 \quad \frac{u'}{U_s} \sim \text{constant} \sim 0.3 \sim 0.35 \longrightarrow \frac{u'}{U_s} \sim \text{constant} \to \frac{l}{L}$ constant

(u' — RMS fluctuate; U_s — mean velo. diff.)

$\frac{l}{L} \sim \left(\frac{u'}{U_s}\right)^2 \sim 0.1$ $\quad$ Still thin flow. "OK"?

$\bar{u} = U_s f(\xi) \quad \xi = \frac{y}{l}$



$f(0) = 1$

Mass conse.

mom. eqn. $\quad \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = -\frac{\partial}{\partial y}\underbrace{(\overline{u'v'})}_{-U_s^2 g(\xi)}$ $\quad$ Starting with. $\frac{\partial \bar{u}}{\partial x} = -\frac{\partial \bar{u}}{\partial y}$

$\bar{v} = -\int_0^y \frac{\partial \bar{u}}{\partial x} dy'$

$= -l \int_0^{\xi} \left(\frac{\partial U_s}{\partial x} f - \frac{U_s}{l} \frac{dl}{dx} \xi f \right) d\xi'$

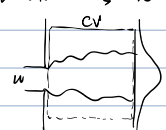
$\bar{v} = -l \frac{\partial U_s}{\partial x} \int_0^{\xi} (f(\xi) d\xi) + U_s \frac{dl}{dx} \int_0^{\xi} \xi' f d\xi'$

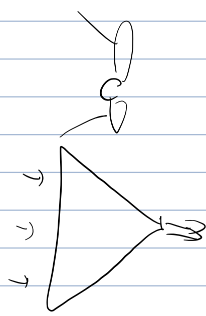
$\frac{l}{U_s} \frac{dU_s}{dx} f^2 - \frac{dl}{dx} \xi f f' - \frac{l}{U_s} \frac{dU_s}{dx} f' \int_0^{\xi} f d\xi' + \frac{dl}{dx} f' \int_0^{\xi} \xi' f d\xi' = g'$

$\frac{dl}{dx} = \text{const.} \quad \frac{l}{U_s} \frac{dU_s}{dx} = \text{const.}$

$\frac{x}{U_s} \frac{dU_s}{dx} = \text{const.} \to U_s \sim x^n$ for any n.

Mom. integral for jet





$h U_j^2 = \int_{-\infty}^{\infty} \bar{u}^2 dy \sim x^0$ $\quad$ Mass conserv. $\int_{-\infty}^{\infty} \bar{u} dy$?

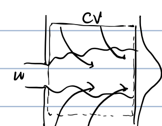
CANNOT conserve both.

$U_s^2 l \sim x^0$

$x^{2n+1} \sim x^0 \to n = -\frac{1}{2}$

$\begin{cases} U_s \sim x^{-\frac{1}{2}} \\ l \sim x^1 \end{cases}$

$Re_l \sim \frac{u' l}{\nu} \sim x^{\frac{1}{2}}$



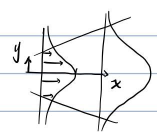
use. $g = f' \frac{\nu_T}{U_s l}$ $\quad$ and $f' \int_0^{\xi} \xi' f d\xi' = f' \left[\xi' f \Big|_0^{\xi} - \int_0^{\xi} f d\xi' \right]$

$-\frac{1}{2} \underbrace{\frac{dl}{dx}}_{\text{const.}} \underbrace{\frac{U_s l}{\nu_T}}_{Re_T} \left(f^2 + f' \int_0^{\xi} f d\xi' \right) = f''$ $\quad$ nonlinear ODE

α

1

$f(\xi) = \text{sech}^2 \left(\frac{\xi^2}{2} \right)^{\frac{1}{2}}$

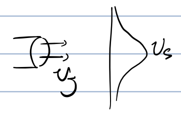


$f(\xi) = \text{sech}^2 \left(\frac{\xi^2}{2} \right)^{\frac{1}{2}}$

to fit data: $\frac{U_s l}{\nu_T} = Re_T = 25.7$

$\frac{l}{x} = 0.078, \quad U_s = U_j \, 2.7 \left(\frac{x}{w} \right)^{-\frac{1}{2}}$

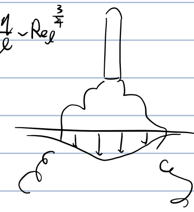
Round jet momentum int. $\int_0^\infty \bar{u}^2 2\pi r\,dr = U_J \pi \frac{d^2}{4}$ (A_J)



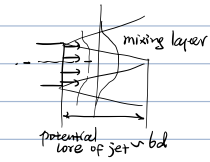
$U_s^2 l^2 \sim x^0$, $l \sim x$, $U_s \sim x^n$ } $n=-1$ { $U_s \sim x^{-1}$, $l \sim x$ } $\xrightarrow{\text{fit to data}}$ { $Re_T = 32$, $l = 0.067x$, $U_s = U_J \cdot 6.4 \left(\frac{d}{x}\right)$ }

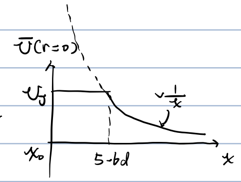
$Re_l = \frac{U_s l}{\nu} = 0.067 \cdot 6.4 \cdot \frac{U_J d}{\nu} = 0.43 \frac{U_J d}{\nu}$ independent of x $\frac{\eta}{l} \sim Re_l^{-\frac{3}{4}}$

NOT relaminarize

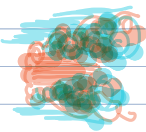


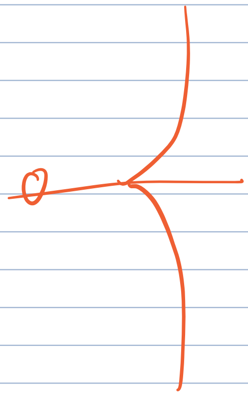
Near Field



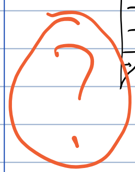


Many of scaling results agree better with data of we included "virtual origin" $U_s \sim (X - X_0)^{-1}$





Mixing layer



no integral constrain $\int_0^\infty u\,dy$ diverges

BUT $\frac{dU_s}{dx} = 0$, $\frac{dl}{dx} = \text{const.}$

$-\frac{dl}{dx} f' \int_0^\xi f\,d\xi = g'$ $\xrightarrow{\text{data}}$ $l = 0.057x$

$\frac{U_s l}{\nu_T} \frac{dl}{dx} f' \int_0^\xi f\,d\xi = f''$

Turbulent Kinetic E. in free shear flow.

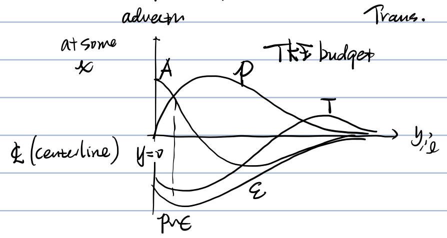
→ wake (ex.) TKE dominant term:

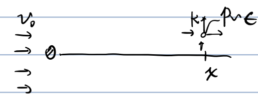
$$\underbrace{U_0 \frac{\partial k}{\partial x}}_{A} = \underbrace{-\overline{u'v'}\frac{\partial \bar{u}}{\partial y}}_{\text{Prod.}} - \frac{\partial}{\partial y}\Big(\overbrace{\underbrace{\overline{v' \tfrac{1}{2} u_i' u_i'} + \frac{\overline{p'v'}}{\rho}}_{\text{Turbulent Trans.}}}^{\text{flux}}\Big) - \varepsilon$$

advection

$\varepsilon = 2\nu \overline{s_{ij}' s_{ij}'}$

TKE budget





Wall-Bounded Turbulent Flow.

Wall bounded Turbulent Flow.

- Momentum.
- Kinetic Energy. — injection rate. $\frac{u'^2}{l}$
  to understand where dissipated : Second set of scale.

Free shear flow. Single dominant scale.

$l(x)$

$W(x) \sim U_s(x)$

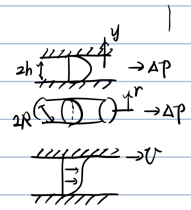
η_K, V_K, T_K

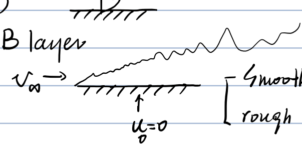
$\varepsilon \neq \frac{\delta^3 u(r)}{r}$

Momentum: 2 Scales.

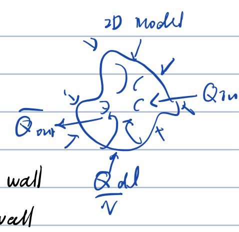
Canonical flows: Fully developed

- Chanel flow
- pipe flow
- Couette flow (plane)
- Developing B layer





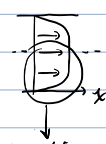
[Smooth wall
[rough wall



Wall bounded Turbulent flow

Fully developed case

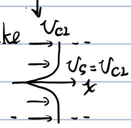
$\frac{\partial}{\partial x}(\,) = 0$



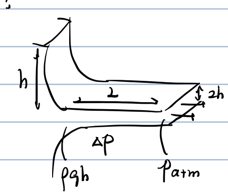
"l(x)" what is l?

$U_s(x)$ what is U_s?

like wake



input: ∇P



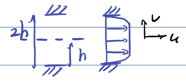
$-\nabla p = \frac{\rho g h - P_{atm}}{L}$

Laminar pipe flow: $\mu, \Delta P, L \to U_b$ velocity scale

Too Large

Channel Flow:

Channel flow. driven by $\frac{\partial P_0}{\partial x} = c$.



x-RANS

$$\left[0 = -\frac{1}{\rho}\frac{\partial P}{\partial x} - \frac{\partial}{\partial y}\overline{u'v'} + \nu \frac{\partial^2 \overline{u}}{\partial y^2} \right]$$

BCs $\left\{ 0, 2h: \; u'=0, v'=0, \; \overline{u'v'}=0 \right.$

$\int$ include viscous

$\int$ ignore viscous $\rightarrow \overline{u'v'} = C_1 + C_2 y$ NOT fullfill BC.

$$0 = -\frac{y}{\rho}\frac{\partial P}{\partial x} - \overline{u'v'}\Big|_y + \nu\frac{\partial \overline{u}}{\partial y}\Big|_y - \nu\frac{\partial \overline{u}}{\partial y}\Big|_0$$

$$\left[u_*^2 = \nu \frac{\partial \overline{u}}{\partial y}\Big|_0 \right]$$

$$0 = \frac{y u_*^2}{h} - \overline{u'v'} + \nu\frac{\partial \overline{u}}{\partial y} - u_*^2$$

$y=h$: $-\frac{h}{\rho}\frac{\partial P}{\partial x} = u_*^2$

$u_* = \sqrt{-\frac{h}{\rho}\frac{\partial P}{\partial x}}$

$$\left[-\overline{u'v'} + \nu\frac{\partial \overline{u}}{\partial y} = u_*^2\left(1 - \frac{y}{h}\right) \right]$$

$\frac{1}{u_*}$

$\left\{ \xi = \frac{y}{h} \text{ char. of } T \text{ length} \right.$; u_* char. of T u.

$$-\frac{\overline{u'v'}}{u_*^2} + \left(\frac{\nu}{u_* h}\right)\frac{\partial\left(\frac{\overline{u}}{u_*}\right)}{\partial\left(\frac{y}{h}\right)} = 1 - \left(\frac{y}{h}\right)$$

or

$$-\frac{\overline{u'v'}}{u_*^2} + \frac{\partial\left(\frac{\overline{u}}{u_*}\right)}{\partial\left(\frac{y u_*}{\nu}\right)} = 1 - \frac{y u_*}{\nu}\frac{\nu}{h u_*}$$

$$-\overline{u'v'}^+ + Re_h^{-1}\frac{\partial \overline{u}^+}{\partial \xi} = 1 - \xi$$

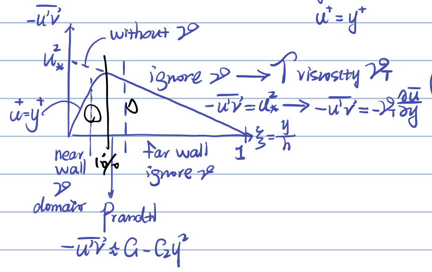
$$-\overline{u'v'}^+ + \frac{\partial \overline{u}^+}{\partial y^+} = 1 - \frac{y^+}{Re_\tau}$$

? how to analysis

$\xi \gg 0$, $Re_h \to \infty$: at wall $-\overline{u'v'}^+ = 1$, $-\overline{u'v'} = u_*^2$

$\overline{u'v'}^+ = \frac{\overline{u'v'}}{u_*^2}$, $Re_h = \frac{u_* h}{\nu}$, $\overline{u}^+ = \frac{\overline{u}}{u_*}$

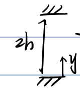
at wall: $\frac{\partial \overline{u}^+}{\partial y^+} = 1$, $u^+ = y^+$



?

?

$\overline{u_i^2} \to \overline{u_i u_i'}$



x-dir-mean momentum (RANS) Eqn. $\frac{\partial}{\partial t} = 0$, $\frac{\partial}{\partial x} = 0$, $\overline{v} = 0$, $\overline{w} = 0$

$$0 = -\frac{1}{\rho}\frac{\partial}{\partial x}P - \frac{\partial}{\partial y}(\overline{u'v'}) \xrightarrow{\int} \overline{u'v'} = C_1 + C_2 y$$

imposed constant

BC: $y = 0, 2h$ $u'=0, v'=0, \overline{u'v'}=0$ at wall

CANNOT be true

Must include viscous

$$\int\left[0 = -\frac{1}{\rho}\frac{\partial}{\partial x}P_0 - \frac{\partial}{\partial y}(\overline{u'v'}) + \nu\frac{\partial^2 \overline{u}}{\partial y^2} \right]$$

$\overline{u'v'}(0) = 0$

$$0 = -\frac{y}{\rho}\frac{\partial P_0}{\partial x} - \overline{u'v'}\Big|_y + \nu\left(\frac{\partial \overline{u}}{\partial y}\Big|_y - \frac{\partial \overline{u}}{\partial y}\Big|_0\right)$$

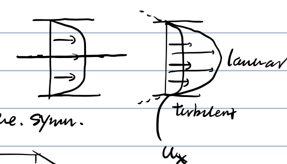
define u_* (or u_τ)

$u_*^2 = \nu \frac{\partial \overline{u}}{\partial y}\Big|_0 = \frac{1}{\rho}\tau_w$

$u_* \cdots \frac{dP_0}{dx}$

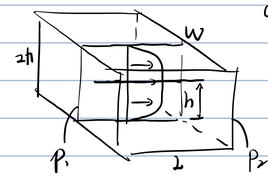
centerline $y = h$: $0 = -\frac{h}{\rho}\frac{dP_0}{dx} - u_*^2$

$y=h$: $\left\{ \frac{\partial u}{\partial y} = 0, \; \overline{u'v'} = 0 \right.$ along centerline. symm.



$u_* = \sqrt{-\frac{h}{\rho}\frac{dP_0}{dx}}$

Or control volume



$\frac{y}{h}u_*^2$

$$0 = -\frac{1}{\rho}\frac{\partial}{\partial x}P_0 - \frac{\partial}{\partial y}(\overline{u'v'}) + \nu\frac{\partial^2 \overline{u}}{\partial y^2}$$

$$\left[-\overline{u'v'} + \nu\frac{\partial \overline{u}}{\partial y} = u_*^2\left(1 - \frac{y}{h}\right) \right]$$

high $Re/\nu \to 0$: neglect viscous

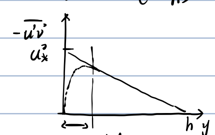
$-\overline{u'v'} = u_*^{*2}\left(1 - \frac{y}{h}\right)$

Viscous force = Pressure force

$2WL \cdot \mu \frac{\partial \overline{u}}{\partial y}\Big|_0 = (P_1 - P_2)\cdot W \cdot 2h$

$u_*^2 = \nu\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\frac{dP_0}{dx}h$

$\left\{ \begin{array}{l} u_*: \text{characteristic of turbulence fluctuation} \\ h: \text{length scale } \xi = \frac{y}{h} \end{array} \right.$



Cannot hold near $y=0$

$$-\frac{\overline{u'v'}}{u_*^2} + \frac{\nu}{u_* h}\frac{\partial\left(\frac{\overline{u}}{u_*}\right)}{\partial\left(\frac{y}{h}\right)} = \left(1 - \left(\frac{y}{h}\right)\right)$$

plus units $\left\{ -\overline{u'v'}^+ = -\frac{\overline{u'v'}}{u_*^2}, \; \overline{u}^+ = \frac{\overline{u}}{u_*} \right.$

$$-\overline{u'v'}^+ + \underbrace{\frac{\nu}{u_* h}}_{Re^{-1}}\frac{\partial \overline{u}^+}{\partial \xi} = 1 - \xi$$

$\frac{\overline{u}}{u^*} = \frac{y u_*}{\nu}$

$\overline{u} = \frac{y}{\nu}u_*^2$

$Re \to 0$, problems for $\overline{u'v'}$ at wall

$$-\frac{\overline{u'v'}}{u_*^2} + \frac{\partial\left(\frac{\overline{u}}{u_*}\right)}{\partial\left(\frac{y u_*}{\nu}\right)} = 1 - \frac{y^+}{Re_\tau}$$

$\frac{y}{h} = \frac{y u_*}{\nu}\cdot\frac{\nu}{u^* h} = \frac{y^+}{Re_\tau}$

$y^+ = \frac{y u_*}{\nu}$

$\delta_\nu = \frac{\nu}{u_*}$

$y^+ = \frac{y}{\delta_\nu}$

$$-\frac{\overline{u'v'}}{u_*^2} + \frac{\partial \overline{u}^+}{\partial y^+} = 1 - \frac{y^+}{Re_\tau} - h$$

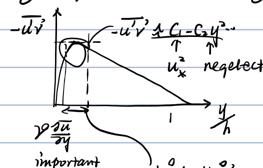
at wall $\frac{\partial u^+}{\partial u_*} = 1$, $\overline{u}^+ = y^+$

or $\overline{u} = \cancel{\overline{u}(0)} + \frac{\partial \overline{u}}{\partial y}\Big|_0 y + \ldots$, $\nu\frac{\partial \overline{u}}{\partial y}\Big|_0 = u_*^2$

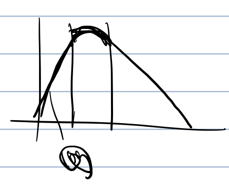
$\frac{\overline{u}}{u_*} = \frac{y u_*}{\nu}$

$u^+ = y^+$

Mixing length (Prandtl argument)

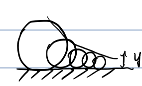


10%, 15% constant stress layer



$-\overline{u'v'} \approx u_*^2$

$-\overline{u'v'} = \nu_T \frac{\partial \overline{u}}{\partial y}$ $\nu_T \left\langle \begin{array}{l} \text{velocity} - u_* \\ \text{length scale} - ? \; ky \end{array} \right.$

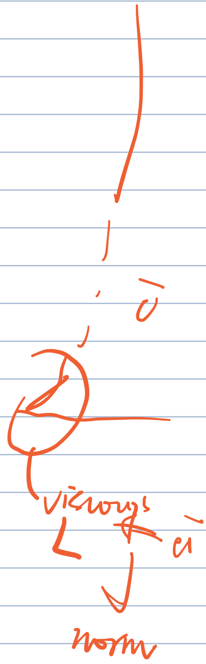


$u_*^2 = ky u_* \frac{\partial \overline{u}}{\partial y}$

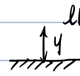
$\frac{\partial \overline{u}}{\partial y} = \frac{u_*}{ky}$

$\overline{u}(y) = \frac{u_*}{k}\ln y + C$

Boundary layer

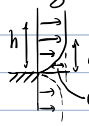


$l(x) \sim y : u_*$



$\hookrightarrow \bar{u}(y) = \frac{u_*}{k}\ln y + C$

Matching of inner-outer similarity solns.



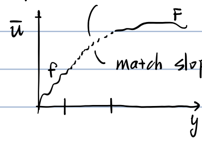
wake portion of Boundy layer

velocity deficit $-[U_{cl} - \bar{u}(y)] = u_* F(\xi) \quad \xi = \frac{y}{h}$

} Outer part

inner part: $\frac{y}{\delta_\nu} = \frac{y u_*}{\nu} = y^+, \quad \bar{u}(y^+) = u_* f(y^+) \leftarrow$ b.c. $f(0) = 0$

Two layers should match $\xi \to 0$ & $y^+ \to \infty$



match slope ~ rang of ξ, y^+ values.

$\frac{d\bar{u}}{dy} = \frac{d}{dy}[u_* F(\xi)]$ outer $\to u_* F' \frac{d\xi}{dy} \to \frac{u_*}{h} F'$ ($\frac{d\xi}{dy} = \frac{1}{h}$) $\to$ should be valid for

$\frac{d\bar{u}}{dy} = \frac{d}{dy}(u_* f(y^+))$ inner $\to u_* f' \frac{dy^+}{dy} \to u_* \frac{u_*}{\nu} f'$ ($\frac{dy^+}{dy} = \frac{u_*}{\nu}$)

ξ, y^+

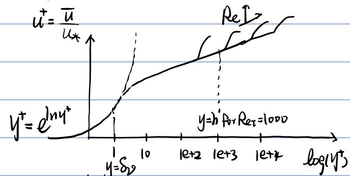
$y \frac{u_*}{h} F' = u_* \frac{u_*}{\nu} f' \, y$

$\xi F' = y^+ f'$

$\int \quad F(\xi) \quad f_1(y^+)$ Must be constant $= \frac{1}{k}$

Von Karman constant

Empirical data



valid $y^+ \to \infty, \xi \to 0$ $\left\{ \begin{array}{l} F(\xi) = \frac{1}{k}\ln\xi + C_1 \\ f(y^+) = \frac{1}{k}\ln y^+ + C_2 \end{array} \right\}$

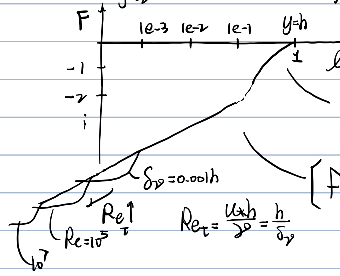
$y^+ \to 0: \bar{u} \to u_* y^+$

$f(y^+) \to y^+$

$F(\xi) \quad \bar{u} - U_{cl} = u_* F$

C_1, C_2 Must be picked

$\bar{u}$ continuity



$\ln(\xi) = \ln(\frac{y}{h})$

$F(\xi)$: NOT universal

[$f(y^+)$: universal] (same for chanel, pipe, couette, BL.

$Re_\tau = \frac{u_* h}{\nu} = \frac{h}{\delta_\nu}$

Kinetic Energy Production

log-layer

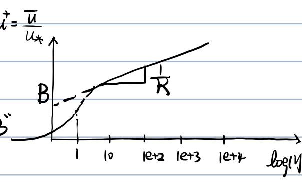
$-\overline{u'v'} \frac{\partial \bar{u}}{\partial y}$

$y^+ \to \infty \quad -\overline{u'v'} \simeq u_*^2$

$\frac{\partial \bar{u}}{\partial y} = \frac{\partial}{\partial y}\left[\frac{1}{k} u_* \ln(y^+) + C_2\right]$

$$\boxed{P \simeq \frac{u_*^3}{ky}} \longleftarrow = \frac{u_*}{k y^+} \cdot \frac{u_*}{\nu} = \frac{u_*}{ky}$$

in literature: "B"



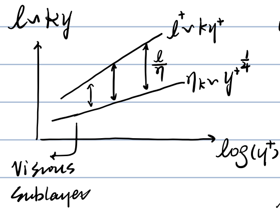
$k = 0.4, \; B \simeq 5$

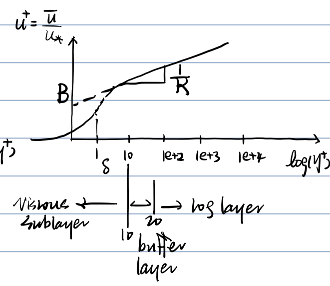
consistant with $u' \sim u_*$, $l \sim ky$

$P \sim \varepsilon \Rightarrow \varepsilon = \frac{u_*^3}{ky}$

$\downarrow$

$\eta_k(y)$





Given u_*: calculate lots sth. for $\bar{T}$

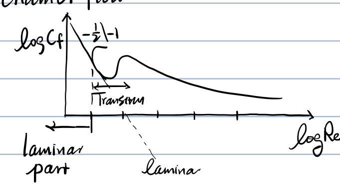
BUT: chanel flow, given $\nabla p \to u_*$

what is $U_{cl} = \bar{u}(y=h)$

empirical coeff. $C_f = \frac{\tau_w}{\frac{1}{2}\rho U_b^2} = \frac{\rho u_*^2}{\frac{1}{2}\rho U_b^2} = 2\left(\frac{u_*}{U_b}\right)^2$

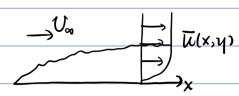
Friction Factor

channel flow



To use: $Re \to C_f \to u_*$

Developing ? Boundy layer



$$\frac{\partial}{\partial x}[\bar{u}(U_\infty-\bar{u})]+\frac{\partial}{\partial y}[\bar{v}(U_\infty-\bar{u})]+(U_\infty-\bar{u})\frac{dU_\infty}{dx}=-\frac{1}{\rho}\frac{\partial}{\partial y}\left[\nu\frac{\partial\bar{u}}{\partial y}-\overline{u'v'}\right]$$

integral $y=0\to\infty$

$$\frac{d}{dx}[U_\infty^2\theta]+\delta^* U_\infty\frac{dU_\infty}{dx}=\frac{\tau_w}{\rho}$$

$$\theta=\int_0^\infty\frac{\bar{u}}{U_\infty}\left(1-\frac{\bar{u}}{U_\infty}\right)dy$$

$$\delta^*=\int_0^\infty\left(1-\frac{\bar{u}}{U_\infty}\right)dy$$

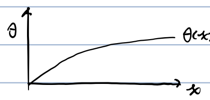
$$C_f=\frac{\tau_w}{\frac{1}{2}\rho U_\infty^2}=2\left(\frac{u_*}{U_\infty}\right)^2$$

$0\ \nabla P$: $\frac{dP}{dx}=\frac{dU_\infty}{dx}=0$

Mom:

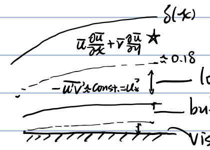
$$U_\infty^2\frac{d\theta}{dx}=u_*^2=\frac{\tau_w}{\rho}$$

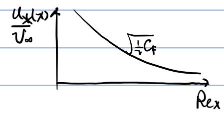
$$\frac{d\theta}{dx}=\left(\frac{u_*}{U_\infty}\right)^2=\frac{1}{2}C_f$$

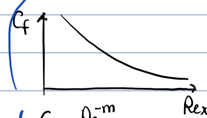


$\frac{d\theta}{dx}\to C_f\to u_*$

$C_f=f(Re_x)\to u_*(x)$







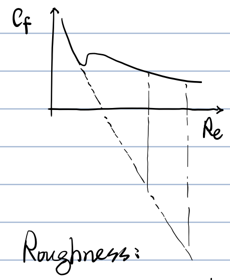
$C_f\sim Re^{-m}$

$m=\frac{1}{2}$ laminar (Blasus)

$\sim\frac{1}{4},\frac{1}{5}\sim\ldots\frac{1}{9}Re$

Power law velocity profile for T. BL. $\bar{u}=U_\infty\left(\frac{y}{\delta_{99}}\right)^{\frac{1}{n}}$

$n=5,6,7\ldots$



Roughness.

Roughness:

(RMS. of height)

new length scale k length.

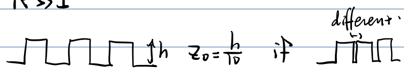
addition to $\delta_\nu=\frac{\nu}{u_*}$

Fully rough: $k\gg\delta_\nu$ or $k^+\gg1$

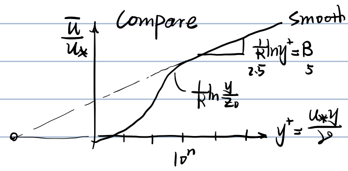
transiatnically rough $k^+\approx1$

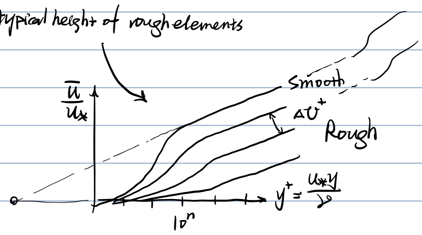
hydrodynamically smooth $k\ll\delta_\nu$ or $k^+\ll1\to$ red smooth

Fully rough: $k\gg\delta_\nu$ or $k^+\gg1$



$\bar{u}=\frac{u_*}{k}\ln\left(\frac{y}{z_0}\right)$, z_0 roughness letter $\approx\frac{1}{10}$ of typical height of rough elements





Re↑ ∴ every goes rough.

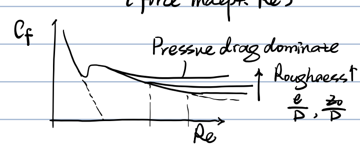
very rough: P drag dominate indep+ to Re

Fully rough: dominant drag due to Pressure drage (force indep. Re)

$$\frac{\bar{u}}{u_*}=\frac{1}{k}\ln y^+ + B-\Delta U^+$$ z_0 or ΔU^+ characterize roughness

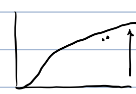
Another k_s same-grade roughness length

$k_s\approx30z_0$



★ Take log-log valid all the way to
- centerline $y=h$ for chamel
- cline $r=0$ for pipe
- $y=\delta_{99}$ for developing BL.

Assume



chamel $\frac{\bar{u}(h)}{u_*}=\frac{1}{k}\ln\frac{hu_*}{\nu}+B$

Given outer velocity $\bar{u}(h)$, what is u_*?

Solving for u_*

Transcendental equ. for u_*

$$C_f=\frac{1}{2}\left[\frac{u_*}{u_h}\right]^2$$

$$u_*^{n+1}=\frac{\bar{u}(h)}{\frac{1}{k}\ln\frac{hu_*^n}{\nu}+B}\to u_*^{n+1}=\frac{\bar{u}(h)}{\frac{1}{k}\ln\left(\frac{h}{\nu}\frac{\bar{u}}{\frac{1}{k}\ln\frac{hu_*^{n-1}}{\nu}+B}\right)+B}$$

C_f vs. Re_x formular log thm)

Statical Theory of Turbulene

homogenous: $f(\underset{\sim}{x}) = f(\underset{\sim}{x}')$ $\rightarrow \bar{u} = 0$, $u = u'$

Isotropic: $f(\vec{n}) = f(\vec{n}')$

Homogenous Isotropic Turbulene (HIT)

$B_{ij}(\underset{\sim}{x}, \underset{\sim}{x}') = \overline{u_i(\underset{\sim}{x}) u_j(\underset{\sim}{x}')}$

homogenous: $B_{ij}(\underset{\sim}{x}, \underset{\sim}{x}') = B_{ij}(\underset{\sim}{r})$

H + Isotropic: $B_{ij}(\underset{\sim}{r}) = B_{ij}(r)$

Schwartzian Inequality

$B_{ij}^2(\underset{\sim}{r}) \le \overline{u_i^2(\underset{\sim}{x})}\, \overline{u_j^2(\underset{\sim}{x}+\underset{\sim}{r})}$

$\underset{\sim}{x}' = \underset{\sim}{x} + \underset{\sim}{r}$

$\frac{\partial x'}{\partial x_j} = 1 + \frac{\partial r}{\partial x}$

$\partial \underset{\sim}{r} = -\partial \underset{\sim}{x}$

B_{ij} div free: $\nabla \cdot B_{ij} = 0$

$\frac{\partial}{\partial x_i} B_{ij} = \frac{\partial}{\partial r_i}$

$f_{ij}(\delta_{mq}) = A\delta_{ij} + B\delta_{mq} + C\delta_{me}\delta_{eq} \cdots$

$B_i(\underset{\sim}{r}) = A_i r_i$

$B_{ij}(\underset{\sim}{r}) = A_1 r_i r_j + B_1 \delta_{ij}$

$B_{ijk}(\underset{\sim}{r}) = A_2 r_i r_j r_k + B_2 r_i \delta_{jk} + C_2 r_j \delta_{ik} + D_2 r_k \delta_{ij} \cdots$

Statistical Theory of Turbulence

(?)

* homogenuous isotropic T. (HIT) $\bar{u} = 0$ $\overline{u_1'^2} = \overline{u_2'^2} = \overline{u_3'^2} = u'^2$ symm.

  $u' \neq 0$ $\overline{u_i' u_j'} = 0$ symm.

"Inner Structure of T."

Multipoint joint PDF $P(\underset{\sim}{v_1}, \underset{\sim}{x_1}; \underset{\sim}{v_2}, \underset{\sim}{x_2} \cdots \underset{\sim}{v_n}, \underset{\sim}{x_n}) \rightarrow$ Hopf. Eqn. ← N-S

↓ 2 pts. 2^{nd} moments

Linear! may work for Quanuntum Computing

Generalized 2-points Simpified moments for HIT.

$B_{ij}(\underset{\sim}{x}, \underset{\sim}{x}') = \overline{u_i(\underset{\sim}{x}) u_j(\underset{\sim}{x}')}$

new prime will denote 2^{nd} point

$\underset{\sim}{r} = \underset{\sim}{x}' - \underset{\sim}{x}$

value from fluctuation velocity? $\underset{\sim}{u}(\underset{\sim}{x})$ Func.

homog. T.

$B_{ij}(\underset{\sim}{x}, \underset{\sim}{x}') = B_{ij}(\underset{\sim}{r})$ iso-T. $\neq$ $B_{ij}(\underset{\sim}{r}) = B_{ij}(|\underset{\sim}{r}|)$

Correlation tensor. $i, j = 1, 2, 3 \rightarrow 9$ fns. for 3 components

$B_{22}(r_1, r_2, r_3)$

$B_{ij}(\underset{\sim}{r})$ continuous assume $\underset{\sim}{u}(\underset{\sim}{x})$ continuous.

$\lim_{h \to 0} \overline{u_i(\underset{\sim}{x})[u_j(\underset{\sim}{x} + \underset{\sim}{r} + \underset{\sim}{h}) - u_j(\underset{\sim}{x} + \underset{\sim}{r})]} = 0$

$\lim_{h \to 0} (B_{ij}(\underset{\sim}{r} + \underset{\sim}{h}) - B_{ij}(\underset{\sim}{r})) = 0 \rightarrow B_{ij}(\underset{\sim}{r})$ is continuous. $B_{ij}(\underset{\sim}{r}, t)$

omit notation for now

Schwartzian inequality $B_{ij}(\underset{\sim}{r})^2 \le \overline{u_i^2(\underset{\sim}{x})}\, \overline{u_j^2(\underset{\sim}{x} + \underset{\sim}{r})}$

no sum over i or j.

$B_{ij}(\underset{\sim}{r})$ for Div-free velocity fields

$\frac{\partial u_j(\underset{\sim}{x})}{\partial x_j} = 0$ $u_j(\underset{\sim}{x}) \frac{\partial u_i(\underset{\sim}{x})}{\partial x_i} = \frac{\partial}{\partial x_i} \overline{u_i(\underset{\sim}{x}) u_j(\underset{\sim}{x} + \underset{\sim}{r})} = 0$ $\underset{\sim}{x}' = \underset{\sim}{x} + \underset{\sim}{r}$, $\frac{\partial}{\partial x_i} = -\frac{\partial}{\partial r_i}$.

$-\frac{\partial}{\partial r_i} \overline{u_i(\underset{\sim}{x}) u_j(\underset{\sim}{x} + \underset{\sim}{r})} = \frac{\partial}{\partial r_i} B_{ij}(\underset{\sim}{r}) = 0$

Similarily. $\frac{\partial}{\partial r_j} B_{ij}(\underset{\sim}{r}) = 0$

Tensor fn. of vector in iso-tropic media

$f_{ij}(\delta_{mq}) = A\delta_{ij} + B\delta_{mq} + C\delta_{me}\delta_{eq} \cdots)$

Vector function $B_i(\underset{\sim}{r}) = A_i r_i$ (?)

2nd rank — * $B_{ij}(\underset{\sim}{r}) = A_1 r_i r_j + B_1 \delta_{ij}$

3rd rank — $B_{ijk}(\underset{\sim}{r}) = A_2 r_i r_j r_k + B_2 r_i \delta_{jk} + C_2 r_j \delta_{ik} + D_2 r_k \delta_{ij}$ + [$E_2 \varepsilon_{ijk} \cdots$ not use.

HIT: $B_{ij}(\underset{\sim}{r}) = A_1(|\underset{\sim}{r}|) r_i r_j + B_1(|\underset{\sim}{r}|) \delta_{ij}$ reduced description to only 2 scalar fn. from 9 single para. fn.

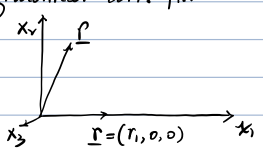
$\overline{u_1^2} = B_{11}(\underset{\sim}{0}) = B_1(0)$

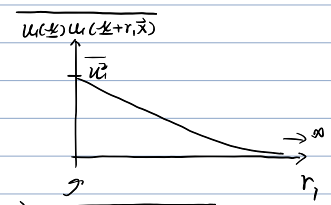
$\overline{u_2^2} = B_{22}(\underset{\sim}{0})$

$\overline{u_3^2} = B_{33}(\underset{\sim}{0})$

Pick 2 special "intuative" cases

Longitudinal corr. fn.





$i=1, j=1$ $B_{11}(\underset{\sim}{r} = (r_1, 0, 0)) = \overline{u_1(\underset{\sim}{x}) u_1(\underset{\sim}{x} + r_1\vec{x})}$

For $\underset{\sim}{r} = (r_1, 0, 0)$ $B_{11}(\underset{\sim}{r}) = A_{11} r_1^2 + B\delta_{11}$, 1

$i = j = 1$

$B_{ij}(r) = A_1 r_i r_j + B_1 \delta_{ij}$

B_{22}

Longitudinal $\quad B_{22}(r) = A_1 r_1^2 + B_1$

$u(x) \to \quad \to u(x+r_1)$

$i=j=1$ $\quad 0 \quad r_1 \quad r_1 \quad x$

B_{NN}

Transverse $\quad r_2 = 0 \quad B_{NN}(r) = B_1$

$u_2 \quad u_2(x+r_1)$

$$\left\{ A_1 = \frac{B_{22} - B_{NN}}{r_1^2}, \quad B_1 = B_{NN} \right.$$

$$B_{ij}(r) = \frac{1}{r^2}(B_{22} - B_{NN}) r_i r_j + B_{NN}\delta_{ij}$$

$$f(r,t) = \frac{B_{22}(r,t)}{u^2(t)} = \frac{B_{22}(r,t)}{B_{22}(0,t)}, \quad g(r,t) = \frac{B_{NN}(r,t)}{u^2(t)}$$

$$B_{ij} = u^2\left[\frac{1}{r^2}(f-g) r_i r_j + g\delta_{ij}\right]$$

$B_{22}(0,t) = B_{NN}(0,t) = u^2(t)$

$$\nabla\cdot B = 0 \to \frac{\partial B_{ij}}{\partial x_j} = \frac{\partial B_{ij}}{\partial r_j} = 0$$

$$B_{ij} = \frac{1}{r^2}(B_{22} - B_{NN}) r_i r_j + B_{NN}\delta_{ij} = u^2\left[\frac{1}{r^2}(f-g) r_i r_j + g\delta_{ij}\right]$$

$$\frac{\partial f}{\partial r_i} = f' \frac{\partial r}{\partial r_i} = f' \frac{r_i}{r}, \quad \frac{\partial r}{\partial r_i} = \frac{r_i}{r}$$

$\frac{\partial}{\partial r_j}$

$$\frac{\partial B_{ij}}{\partial r_j} = u^2\left[\left(\frac{\partial f}{\partial r_j} - \frac{\partial g}{\partial r_j}\right)\frac{r_i r_j}{r^2} + (f-g)\left(\frac{\partial r_i}{\partial r_j}\cdot\frac{r_j}{r^2} + \frac{\partial r_j}{\partial r_j}\frac{r_i}{r^2} + r_i r_j (-2) r^{-3}\frac{\partial r}{\partial r_j}\right) + \frac{\partial g}{\partial r_j}\delta_{ij}\right]$$

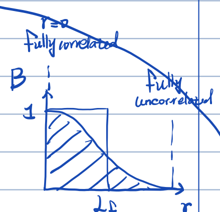
$$= u^2\left[(f'-g')\frac{\cancel{r_j}}{r}\frac{\cancel{r_i r_j}}{r^2} + (f-g)\left(\delta_{ij}\frac{r_j}{r^2} + \delta_{jj}\frac{r_i}{r^2} - 2\frac{r_i r_j}{r^3}\frac{r_j}{r}\right) + g'\frac{r_j}{r}\delta_{ij}\right]$$

$$0 = u^2\left[(f'-\cancel{g'})\frac{r_i}{r} + (f-g)\left(\frac{r_i}{r^2} + \frac{3 r_i}{r^2} - 2\frac{r_i}{r^2}\right) + \cancel{g'}\frac{r_i}{r}\right]$$

$$0 = f' + 2(f-g)\frac{1}{r}$$

$$g = \frac{r}{2}f' + f$$

Integral Scale



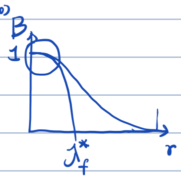
$$\left\{ L_f = \int_0^\infty f(r,t)\,dr, \quad L_g = \int_0^\infty g(r,t)\,dr = \frac{1}{2}L_f \right.$$

Taylor Series Expan.

$$f(r) = f(0) + r\cancel{f'(0)} + \frac{1}{2}r^2 f''(0)$$

$$f(\lambda_f^*) = 0$$

$$\lambda_f^* = \frac{-2f(0)}{f''(0)} \quad ?$$



Viscous dissipation rate.

$$\varepsilon = \nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j}$$

$$D = \frac{\partial u_i(x)}{\partial x_j}\left(\frac{\partial u_i(x')}{\partial x_j'} + \frac{\partial u_j(x')}{\partial x_i'}\right)$$

$$\bar{D} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j'}\left(\overline{u_i(x) u_i(x')}\right) + \frac{\partial^2}{\partial x_i \partial x_j'}\left(\overline{u_i(x) u_j(x')}\right)$$

$\underbrace{}_{x+r}$

$$\bar{D} = -\frac{\partial^2}{\partial r_j \partial r_j} B_{ii}(r) - \frac{\partial^2}{\partial r_i \partial r_j} B_{ij}(r)$$

3D Laplacian $\quad 0 \quad \nabla\cdot B = 0$

$$\bar{D} = -\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) B_{ii}(r)$$

$$B_{ii} = u^2\left((f-g)\frac{\cancel{r_i r_i}}{\cancel{r^2}} + 3g\right) = u^2(f+2g)$$

$$u^2(3f + rf') \quad g = 2f + \frac{r}{2}f'$$

$$B_{ij}(r) = A_1 r_i r_j + B_1 \delta_{ij} \quad A, B \text{ fns of } |r| \text{ only}$$

Longitudinal corr. fn: $\quad u(x) \quad u(x+r_1) \quad i=j=1 \quad x = r_1, 0, 0$

$0 \quad r_1 \quad r_1 \quad x$

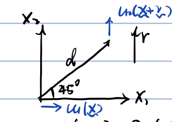
$$B_{22}(r) = A_1 r_1^2 + B_1$$

Transverse corr. $\quad u_2 \quad u_2 \quad i=j=2 \quad r_2 = 0$

$0 \quad r_1 \quad r_1 \quad x$

$$B_{NN}(r) = B_1$$

$$A_1 = \frac{B_{22} - B_{NN}}{r^2}, \quad B_1 = B_{NN}$$



$$u_1(x) u_2(x+r) = B_{12}(r) \quad r_1 = \frac{d}{\sqrt{2}}, \; r_2 = \frac{d}{\sqrt{2}}, \; r_3 = 0$$

$$B_{12}(r) = \frac{1}{d^2}(B_{22} - B_{NN})\cdot\frac{d}{\sqrt{2}}\cdot\frac{d}{\sqrt{2}} + 0 = \frac{1}{2}(B_{22} - B_{NN})$$

$$B_{ij}(r) = \frac{1}{r^2}(B_{22} - B_{NN}) r_i r_j + B_{NN}\delta_{ij}$$

$$f(r,t) = \frac{B_{22}(r,t)}{u^2(t)} = \frac{B_{22}(r,t)}{B_{22}(0,t)}, \quad g(r,t) = \frac{B_{NN}(r,t)}{u^2(t)}$$

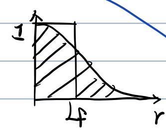
$$B_{22}(0) = B_{NN}(0) = u^2$$

Div-free cond.: $\frac{\partial B_{ij}}{\partial r_j} = 0$ $\quad$ $B_{ij}(r)$ only depends on single scale fn. of determination

$$g(r) = f(r) + \frac{1}{2}r f'(r)$$

Integral Scale:

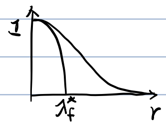
$$L_f = \int_0^\infty f(r,t)\,dr$$



$$L_g = \int_0^\infty g(r,t)\,dr = \frac{1}{2}L_f$$

$$g = f + \frac{1}{2}rf'$$

another length scale:



Taylor Series Expan at $r=0$

$$f(r) = f(0) + rf' + \frac{1}{2}r^2 f'' \cdots$$

$$= f(0) + \frac{1}{2}r^2 f''(0) \longrightarrow f(r=\lambda^*) = 0 = 1 + \frac{1}{2}\lambda^{*2} f''(0)$$

$$\lambda^{*2} = -\frac{1}{2f''(0)} \quad ?$$

$$f(r) = 1 - \left(\frac{r}{\lambda^*}\right)^2 \cdots$$

Viscous dissipation rate

$$\varepsilon = \nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j} \quad \text{viscous force } \frac{F^\nu}{\rho} = \frac{\nabla\cdot\tau}{\rho}$$

x' not relate on x

Two-point: $\quad x' = x + r$ $\quad \partial x' = \partial r$ $\quad \partial x = -\partial r$

$$\cancel{\frac{\partial x'}{\partial x}} = 1 + \frac{\partial r}{\partial x}, \quad \partial r = -\partial x$$

Consider $$D = \frac{\partial u_i(x)}{\partial x_j}\left(\frac{\partial u_i(x')}{\partial x_j'} + \frac{\partial u_j(x')}{\partial x_i'}\right) \quad ?$$

$$\bar{D} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j'}\left(\overline{u_i(x) u_i(x')}\right) + \frac{\partial^2}{\partial x_i \partial x_j'}\overline{u_i(x) u_j(x')}$$

$\underbrace{}_{x+r}$ $\quad$?

$$\bar{D} = -\frac{\partial^2}{\partial r_j \partial r_j} B_{ii}(r) - \frac{\partial^2}{\partial r_i \partial r_j} B_{ij}(r)$$

3D Laplace $\quad 0 \text{ as } \frac{\partial}{\partial r_j}B_{ij} = 0 \; (\nabla\cdot B = 0)$

$$\bar{D} = -\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) B_{ii}(r)$$

$$B_{ii}(r) = u^2(f+2g)$$

$$\frac{1}{u^2}B_{ij}(r) = \frac{1}{r^2}(f-g) r_i r_j + g\delta_{ij} \quad \delta_{ii} = 3$$

$$i=j \quad \frac{1}{u^2}B_{ii}(r) = (f-g) + 3g = f + 2g$$

$$\bar{D} = -u^2\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)(f+2g) \quad g = f + \frac{r}{2}f'$$

$$(3f + rf')$$

$$= -u^2\left(3f'' + \frac{6}{r}f' + \frac{\partial}{\partial r}(f' + rf'') + \frac{2}{r}(f' + rf'')\right)$$

$$f'' + f'' + rf''' \quad \frac{2}{r}f' \quad 2f''$$

$$\bar{D} = -u^2\left(7f'' + \frac{8}{r}f' + rf'''\right)$$

L'Hopital

$$\bar{D} = -u'^2\left(3f'' + \frac{6}{r}f' + \frac{\partial}{\partial r}(f' + r f'') + \frac{2}{r}f' + 2f''\right)$$

$$\bar{D} = -u'^2\left(7f'' + \frac{8}{r}f' + r f'''\right)$$

$r \to 0$, $\frac{d}{dr}$, L'Hopital

$$\bar{D}(r\to 0) = -u'^2(7f'' + 8f'' + 0) = -u'^2\,15 f''(0)$$

$$\epsilon = \nu(-u'^2)15 f''(0)$$

$$\left[\epsilon = 15\nu\left(\frac{u'^2}{\lambda_x^2}\right)\right] \quad \lambda_x^2 = -\frac{1}{f''(0)} \quad \text{Taylor Microscale}$$

$$\epsilon = 15\nu\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2} \sim \frac{u'^2}{\lambda^2}$$

Assumption of Local "Isotropic": Small scale Statically isotropic

$$\frac{\partial r_i}{\partial r_i} = \delta_{ii} = 3$$

Velocity-pressure correlation

$B_i(r) = \overline{p(x)u_i(x+r)}$ should finite

iso → $B_i(r) = A_0(r) r_i = C\frac{r_i}{r^3} \to C = 0,\ B_i = 0$

$$\nabla\cdot u = 0 \to \frac{\partial B_i}{\partial r_i} = 0 \to \frac{r_i}{r}A_0' r_i + A_0\underbrace{\frac{\partial r_i}{\partial r_i}}_{3} = 0$$

$B_i = \overline{p u_i}$

$$\frac{\partial B_i}{\partial r_i} = \overline{p\frac{\partial u_i}{\partial r_i}} + \underbrace{\overline{u_i\frac{\partial p}{\partial r_i}}}_{0}$$

$$\frac{dA_0}{A_0} = -3\frac{dr}{r}, \quad A_0 = C r^{-3}$$

Triple components

$$B_{ij,k}(r) = \overline{u_i(x)u_j(x)u_k(x+r)}$$

$r \to 0$: $D(r\to 0) = -u'^2(7f''(0) + 8f'' + 0) = -u'^2\,15 f''(0)$

$$\epsilon = \nu(-u'^2)15 f''(0)$$

$$\left[\epsilon = 15\nu\frac{u'^2}{\lambda_x^2}\right] \quad \lambda_x^2 = -\frac{1}{f''(0)}$$

Taylor microscale

$$\epsilon = 15\nu\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2} \sim \frac{u'^2}{\lambda^2}$$

$$f(r) = 1 - \left(\frac{r}{\sqrt{2}\lambda}\right)^2$$

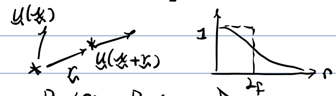
should complete $\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2} = \overline{\left(\frac{\partial u_1(x)}{\partial x_1} \cdot \frac{\partial u_1(x+r_1)}{\partial x'}\right)} = \frac{\partial}{\partial r_1}B_{11}(r_1)$

Assumption of "Local isotropy" (Kolmogorov)

Small eddy stat. isotropic ($\checkmark$ in practice)

$B_{ij}(r)$ may NOT iso.

But $\nabla\nabla_r B$ at $r \to 0$ iso.



$B_{ij}(r) \to B_{11}(r)$ or $f(r)$

Velocity pressure correlation.

$$B_i(r) = \overline{p(x)\,u_i(x+r)}$$

isotropy: $B_i(r) = A_0(r) r_i$

$C\frac{1}{r^3}r_i = C\frac{\hat{r}_i}{r^2}$

$A_0 = 0$

$$\nabla\cdot u = 0 \to \frac{\partial}{\partial r_i}B_i(r) = 0 \longrightarrow \frac{\partial}{\partial r_i}(A_0(r) r_i) = 0$$

$p(x)$ not on r, only u

$$r_i\frac{\partial r}{\partial r}\cdot\frac{dA}{dr} + A_0\underbrace{\frac{\partial r_i}{\partial r_i}}_{\delta_{ii}=3} = 0$$

$$r\frac{dA_0}{dr} + 3A_0 = 0$$

$$\frac{dA}{A} = -3\frac{dr}{r} \to A_0 = C\cdot r^{-3}$$

$C = 0$ since $\overline{u(x)p(x)}$ should be finite

Triple moments.

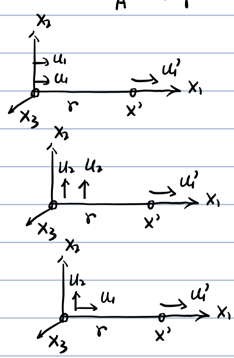
$$B_{ij,k}(r) = \overline{u_i(x)u_j(x)u_k(\underbrace{x+r}_{x'})}$$

Low-dim.

Scalar fn. $B_{11,1}(r) = \overline{u_1^2(x)\,u_1(x+re_1)}$

$$B_{NN1}(r) = \overline{u_2^2(x)\,u_1(x+re_1)}$$

$$B_{1NN}(r) = \overline{u_1(x)\,u_2^2(x+re_1)}$$



P & u uncorrelated at any 2 pts.

But NOT stat. indept.

$$Q(r) = \overline{p(x)\,u_i^2(x+r)} \neq 0$$

Isotropy

$C_2 r_i\delta_{jk} + D_2 r_j\delta_{ik}$ $(C_2 = D_2)$

$$B_{ijk}(r) = A_2(r)\,r_i r_j r_k + B_2 r_k\delta_{ij} + C_2(r_i\delta_{jk} + r_j\delta_{ik})$$

$$\begin{cases} B_{111} = A_2 r^3 + B_2 r + 2C_2 r \\ B_{NN1} = r B_2 \\ B_{1NN} = r C_2 \end{cases} \quad r_1 = r \quad r_N = 0 \quad \begin{aligned} A_2(r) &= \frac{1}{r^3}(B_{111} - B_{NN1} - 2B_{1NN}) \\ B_2(r) &= \frac{1}{r}B_{NN1} \\ C_2(r) &= \frac{1}{r}B_{1NN} \end{aligned}$$

HW:

Prove incompressibility

$B_{NN1}, B_{1NN} \to f(B_{111})$

$$\frac{\partial}{\partial r_k}B_{ijk}(r) = 0$$

$$B_{jk,i} = \frac{1}{2r^3}(B_{111} - rB_{111}')r_i r_j r_k - \frac{1}{2r}B_{111}r_i\delta_{jk} + \frac{1}{4r}(B_{111} + \frac{r}{2}B_{111}')(r_j\delta_{ik} + r_k\delta_{ij})$$

B_{111} contains all relevant info!

Def. $k(r) = \frac{B_{111}(r)}{B_{11}(0)^{\frac{3}{2}}} = \frac{1}{u'^3}B_{111}(r)$

B_{111}, k odd: $B_{111}(0) = 0$, $B_{jk,i}(r) = B_{jk,i}(-r) \to \overline{u_j(x)u_k(x)u_i(x+r)} = -\overline{u_j(x+r)u_k(x+r)u_i(x)}$

Dynamical Eqn. $B_{11}(r,t)$ and $B_{111}(r,t)$

or $f(r,t)$ and $k(r,t)$

N-S at x and x'

$$\left[\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_k}(u_i u_k) = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\nabla_x^2 u_i\right]$$

$$\left[\frac{\partial u_j'}{\partial t} + \frac{\partial}{\partial x_k'}(u_j' u_k') = -\frac{1}{\rho}\frac{\partial p'}{\partial x_j'} + \nu\nabla_{x'}^2 u_j'\right] \times u_j'$$

$u_j \times$ $\oplus$

$[N\text{-}S]$

$u_i \quad u_j'$

$(B_{ij}) = (B_{ik,j}, B_{jk,i})$

$\downarrow i=j$

$(\ )(B_{22}, B_{222}) = 0$

$Q = 0$

$(B_{22}) = (B_{222})$ Von-Karman H.

$D =$

$(B_{22}(0), D_{22}) = (D_{222})$

$$\overline{\frac{\partial}{\partial t}[u_i u_j'] + \frac{\partial}{\partial x_k}[u_i u_k u_j'] + \frac{\partial}{\partial x_k'}(u_i u_k' u_j')} = -\frac{1}{\rho}\left(\frac{\partial}{\partial x_i}(p u_j') + \frac{\partial}{\partial x_j'}(p u_i)\right) + \nu(\nabla_x^2 + \nabla_{x'}^2)(u_i u_j')$$

avg

$\overline{u_i u_j'} = B_{ij}(r,t)$

$\frac{\partial}{\partial x_k'} = \frac{\partial}{\partial r_k}$, $\frac{\partial}{\partial x_k} = -\frac{\partial}{\partial r_k}$, $\overline{p u_j'} = \overline{p u_i'} = 0$

$$\frac{\partial}{\partial t} B_{ij}(r,t) - 2\nu \nabla_r^2 B_{ij}(r,t) = \frac{\partial}{\partial r_k}(\overline{u_i u_k u_j'} - \overline{u_i u_j' u_k'})$$

$\overline{u_i u_k' u_j'} = -\overline{u_k u_j u_i'} = -3B_{kj,i}(r,t)$

$$\left(\frac{\partial}{\partial t} - 2\nu\nabla_x^2\right) B_{ij}(r,t) = \frac{\partial}{\partial r_k}(B_{ik,j} + B_{jk,i})$$

if $i=j$: $\nabla_x^2 B_{ii}(r) = 2\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)(3B_{22} + rB_{22}')$

$$\frac{\partial}{\partial r_k}(B_{ik,i} + B_{ik,i}) = \left(r\frac{\partial}{\partial r} + 3\right)\left(\frac{\partial}{\partial r} + \frac{4}{r}\right) B_{222}(r,t)$$

$\Downarrow$

$$\left(r\frac{\partial}{\partial r}+3\right)\underbrace{\left[\frac{\partial}{\partial t} - 2\nu\left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right) B_{22}(r,t) - \left(\frac{\partial}{\partial r} + \frac{4}{r}\right) B_{222}\right]}_{Q(r,t)} = 0$$

$$\left(r\frac{\partial}{\partial r} + 3\right) Q = 0$$

$$\frac{dQ}{Q} = -3\frac{dr}{r} \rightarrow Q = cr^{-3}$$

$$\left[\frac{\partial}{\partial t} - 2\nu\left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\right)\right] B_{22}(r,t) = \left(\frac{\partial}{\partial r} + \frac{4}{r}\right) B_{222}(r,t)$$

Von-Karmann-H. Eqn.

$u' = u - \frac{u}{0}$

unclosed: $B_{222} \sim B_{22}$

- $B_{222}(r,t) = \alpha B_{22}(r,t)^{\frac{3}{2}}$ $\times$ $B_{222}(0,t) = 0$ But $B_{22}(0,t) = u'^2 \neq 0$
- $B_{222}(r,t) = \alpha[B_{22}(r,t)^{\frac{3}{2}} - B_{22}(0,t)^{\frac{3}{2}}]$ might work, not good one
- $\sim\sim$

$$\frac{\partial}{\partial t}[u'^2(t) f(r,t)] = [u'^2(t)]^{\frac{3}{2}}\left(\frac{\partial}{\partial r} + \frac{4}{r}\right) k + 2\nu u'^2(t)\left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right) f(r,t)$$

At $r=0$

$k(0,t) = 0$, $k'(0,t) = 0$, $k''(0,t) = 0$, $B_{222} \sim r^3$

$$\frac{\partial}{\partial t}[u'^2(t)] = 2\nu u'^2(t)\left(f''(0,t) + \frac{4}{r} f'(0,t)\right)$$

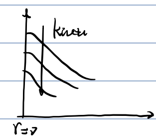
L'Hopital: $5f''(0,t)$

$$\frac{d}{dt}u'^2(t) = 2\nu \cdot 5 \cdot u'^2(t) f''(0,t) = -10\nu u'^2(t)\frac{1}{\lambda^2}$$

$$\frac{d}{dt}\underbrace{\frac{3}{2}u'^2(t)}_{K(t)} = -15\nu\frac{u'^2}{\lambda^2} = -\epsilon$$

Kinetic energy

$$\left[\frac{dK_{TKE}}{dt} = -\epsilon\right]$$



At $r=0$: $\frac{\partial}{\partial t}[u'^2(t)] = 2\nu u'^2(t)\left(f''(0,t) + \frac{4}{r}f'(0,t)\right) k(0,t) = 0$

$f(0,t) = 1$

L'Hopital

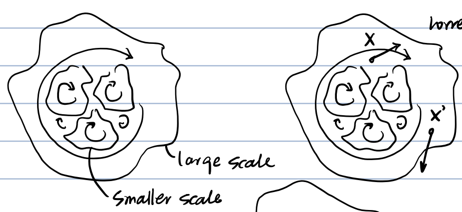
$$\frac{d}{dt}u'^2(t) = 2\nu \cdot 5u'^2(t) f''(0,t) \Leftarrow 5f''(0,t) \cdot k''(0,t)$$

$$= -10\nu\frac{u'^2}{\lambda^2}$$

$$\frac{d}{dt}\underbrace{\frac{3}{2}u'^2(t)}_{k(t)} = -15\nu\frac{u'^2}{\lambda^2} = -\epsilon$$

$B_{ijk}(r)$

Structure Fn.



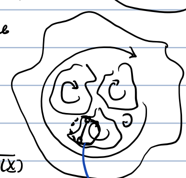
Cannot capture Small scale

Scalar field

$\theta(x)$

$$D(r) = \overline{(\theta(x+r) - \theta(x))^2}$$
$$= \overline{\theta^2(x+r)} - 2\overline{\theta(x+r)\theta(x)} + \overline{\theta^2(x)}$$
$$= 2[B_0(0) - B(r)]$$

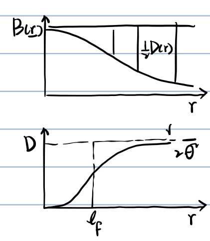
$B_0(r) = \overline{\theta(x+r)\theta(x)}$



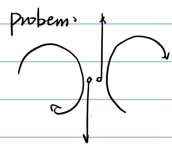
- scale energy

cancel large scale eddy effect.
capture Small scale character.

1D: $D(r) = 2(B(0) - B(r))$



problem:



only works for stat. (averaged).

Vector fields

$$D_{ij}(r) = \overline{[u_i(x+r) - u_i(x)][u_j(x+r) - u_j(x)]}$$
$$D_{ij}(r) = 2[B_{ij}(0) - B_{ij}(r)]$$

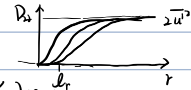
incompressibility $\frac{\partial D_{ij}}{\partial r_i} = 0$

isotropic div. free. $D_{ij}(r) = -\frac{1}{2}\frac{1}{r}D_{22}'(r) r_i r_j + D_{22}(r) + \frac{r}{2}D_{22}'\delta_{ij}$

$B_0 - B_r$

$$\left[\begin{array}{l} D_{22}(r) = \overline{[u_i(x+r) - u_i(x)]^2} \\ \text{Longitudinal structure Fn. } 2^{nd} \text{ order} \end{array}\right]$$

Third-Order Structure Tensor

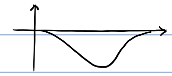


$$D_{ijk}(\underline{r}) = \overline{(u_i(\underline{x}+\underline{r}) - u_i(\underline{x}))(u_j(\underline{x}+\underline{r}) - u_j(\underline{x}))(u_k(\underline{x}+\underline{r}) - u_k(\underline{x}))}$$

$$D_{ijk}(\underline{r}) = \frac{1}{6r}(D_{LLL} + rD'_{LLL})(r_k\delta_{ij} + r_i\delta_{jk} + r_j\delta_{ik}) + \frac{1}{2r^3}(D_{LLL} - rD'_{LLL})r_i r_j r_k$$

$D_{LLL}(r) = \overline{(u_1(\underline{x}+r\underline{e}_1) - u_1(\underline{x}))^3}$ 3rd Order Longitudinal Structure Fn.

data shows $D_{LLL}(r) < 0$



$(\ )^3 = \|(\ )^3\|(\ )$ + S P? avg. → 0 Tur. ≠0 difference

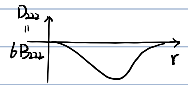
$D_{LLL}(r) = \overline{(u(x+r) - u(x))^3}$

$= \cancel{\overline{u^3(x+r)}} - 3\overline{u^2(x+r)u(x)} + 3\overline{u(x+r)u^2(x)} - \cancel{\overline{u^3(x)}}$ homogeneity

$+3$ odd.

$= 6\,\overline{u(x+r)u^2(x)} = 6B_{LLL}(r)$

$0 = 0$, $= 0$



returning to Von-Karman-Howard Eqn.

$$\left[\frac{\partial}{\partial t} - 2\nu\left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right)\right]\underbrace{B_{LL}(r,t)} = \left(\frac{\partial}{\partial r} + \frac{4}{r}\right)\underbrace{B_{LLL}(r,t)}_{\frac{1}{6}D_{LLL}}$$

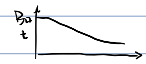
$D_{LL}(r,t) = 2[B_{LL}(0,t) - B_{LL}(r,t)]$

$B_{LL}(r,t) = B_{LL}(0,t) - \frac{1}{2}D_{LL}(r,t)$

$$\frac{\partial}{\partial t}\left(B_{LL}(0,t) - \frac{1}{2}D_{LL}(r,t)\right) - 2\nu\left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right)\left(-\frac{1}{2}\right)D_{LL}(r,t) = \frac{1}{6}\left(\frac{\partial}{\partial r} + \frac{4}{r}\right)D_{LLL}(r,t)$$

$\frac{\partial}{\partial t}B_{LL}(0,t) = \frac{\partial}{\partial t}(\overline{u^2}) = \frac{d}{dt}\left(\frac{2}{3}k\right) = \frac{2}{3}(-\varepsilon)$

TKE



for decaying isotropic turbulence.

$-\frac{2}{3}\epsilon - \frac{1}{2}\frac{\partial}{\partial t}D_{LL}(r,t)$

rate of total energy ≫ rate of shape changing

what about $\frac{\partial}{\partial t}D_{LL}(r,t) \ll \epsilon$

Same profile. (not for higher r)

neglect

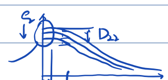
$\left|-\frac{2}{3}\epsilon - \frac{1}{2}\frac{\partial}{\partial t}D_{LL}(r,t)\right|$

$\frac{\partial}{\partial t}D_{LL}$ compare to ϵ

at VERY SMALL r

works

$\frac{\partial}{\partial t}D_{LL} \ll \epsilon$



$$\frac{\partial}{\partial t}\left(B_{LL}(0,t) - \frac{1}{2}D_{LL}(r,t)\right) - 2\nu\left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right)\left(-\frac{1}{2}\right)D_{LL}(r,t) = \frac{1}{6}\left(\frac{\partial}{\partial r} + \frac{4}{r}\right)D_{LLL}(r,t)$$

$\frac{2}{3}\times 6 = 4$

$\left(\frac{\partial}{\partial r} + \frac{4}{r}\right)(D_{LLL} - 6\nu D'_{LL}) = -4\epsilon$ $\quad\left(\frac{\partial}{\partial r} + \frac{4}{r}\right)(\epsilon r) = \epsilon + \cancel{\epsilon r} + 4\epsilon = 5\epsilon$

?

$\left(\frac{\partial}{\partial r} + \frac{4}{r}\right)(D_{LLL} - 6\nu D'_{LL} + \frac{4}{5}\epsilon r) = 0$

$\left(\frac{\partial}{\partial r} + \frac{4}{r}\right)Q = 0 \longrightarrow Q \sim Ar^{-4}$

no singularity at $r = 0$ $\quad A = 0$

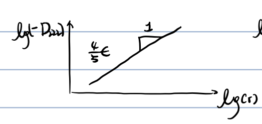
$$\boxed{D_{LLL} - 6\nu D'_{LL} = -\frac{4}{5}\epsilon r} \longleftarrow Q = 0 \qquad \text{Data.}$$

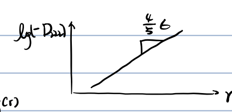
Kolmogorov Eqn. or $\left(\frac{4}{5}\right)^{th}$ Eq.

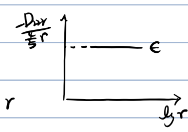
Behaviour in inertial range of T.: $\eta_k \ll r \ll l_f$

$6\nu D'_{LL} \ll |\varepsilon \cdot r|$

in Inertial range. $D_{LLL}(r) = -\frac{4}{5}\varepsilon \cdot r$







Behavior $D_{LLL}(r)$ at $r \to 0$:

$r \sim \eta_k$ or smaller, u is smooth & Taylor Series expans.

$D_{LLL} = \overline{(u(x+r) - u(x))^3} \triangleq \overline{\left[u(x) + \frac{\partial u}{\partial x}r - u(x)\right]^3} = \overline{\left(\frac{\partial u}{\partial x}\right)^3}\cdot r^3$

also, $D_{LL}(r) = \overline{\left(\frac{\partial u}{\partial x}\right)^2}\cdot r^2$

$r \to 0$ $(r \ll \eta_k)$ $\quad \overline{\left(\frac{\partial u}{\partial x}\right)^3}\cdot r^3 - 6\nu\cdot 2r\,\overline{\left(\frac{\partial u}{\partial x}\right)^2} = -\frac{4}{5}\varepsilon r$

$\overline{\left(\frac{\partial u}{\partial x}\right)^3}r^3 - 15r\nu\overline{\left(\frac{\partial u}{\partial x}\right)^2} = -\varepsilon r$

constant. $\varepsilon = 15\nu\overline{\left(\frac{\partial u}{\partial x}\right)^2}$

$D_{LLL}(r,t) - 6\nu D'_{LL}(r) = -\frac{4}{5}r\epsilon$

$r \gg \eta \rightarrow$ neglect $\nu \longrightarrow \frac{4}{5}$th law $D_{LLL} = -\frac{4}{5}\epsilon r$

$\overline{(\delta u)^3} \sim r \longrightarrow \overline{\delta u^2} \sim \varepsilon^{\frac{2}{3}} r^{\frac{2}{3}}$

$\delta u = u(x+r) - u(x)$ $\quad$ 2 th order Structural Fn.

Tur.

$r \ll \eta$ $\quad D_{22} = \left(\frac{\partial u_1}{\partial x_1}\right)^3 r^3$ $\quad$ recall $\varepsilon = 15\nu\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2} \rightarrow \overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2} = \frac{\varepsilon}{15\nu}$ $\quad \varepsilon \sim \frac{u^3}{l}$

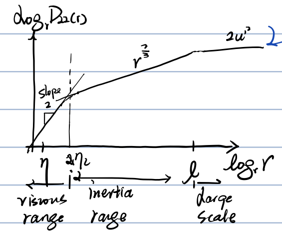
$D_{22} = \overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2} r^2 \longrightarrow D_{22}(r) = \frac{\varepsilon}{15\nu} r^2$ Small r

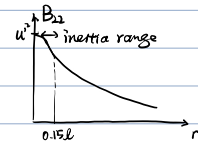
Inertial range $\eta \ll r \ll l$

if is indeed observed

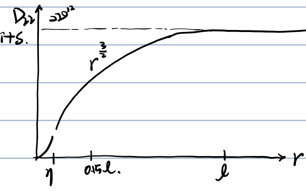
$$D_{22}(r) = C_2 \varepsilon^{\frac{2}{3}} r^{\frac{2}{3}}$$

measurment $C_2 \hat{=} 2.1$





in linear units.



Cross-over scale

$$\frac{\varepsilon}{15\nu} r^2 = C_2 \varepsilon^{\frac{2}{3}} r^{\frac{2}{3}}$$

$$\eta_c^2 \qquad \eta_c^{\frac{2}{3}}$$

$$\eta_c^{\frac{4}{3}} = C_2\, 15 \frac{\nu}{\varepsilon^{\frac{1}{3}}} = 15 C_2 \left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{3}}$$

$$\eta_c = (C_2 \cdot 15)^{\frac{3}{4}} \underbrace{\left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}}}_{\eta_k}$$

Kolomorgov length scale.

$(21.15)^{\frac{3}{4}} \doteq 13$

$$\left[\ \eta_c = 13\eta_k\ \right]$$

$\hat{u}(k,t)$ Spectual Method Isotropic?

1D Scalar field $u(x)$

$B(r) = \overline{u(x)u(x+r)}$

Spectual density

$$E(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overset{\text{even}}{B(r)}\, e^{-ikr} dr = \frac{1}{\pi}\int_0^{\infty} B(r)\cos(kr)\, dr$$

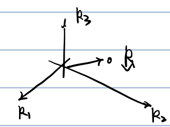
cos even + sin; $\int_{-\infty}^{\infty} = 0$

inverse: $B(r) = \int_{-\infty}^{\infty} E(k) e^{ikr} dk = 2\int_0^{\infty} E(k)\cos(kr)\, dk$

at $r=0$: $B(0) = \overline{u^2} = 2\int_0^{\infty} E(k)\, dk \rightarrow$ "area under $E(k)$ is Kinetic Energy ($\frac{1}{2}$ variance)"

$$\int_0^{\infty} E(k)\, dk = \frac{\overline{u^2}}{2}$$

3D vector fields. $B_{ij}(\underline{r}) = \iiint \Phi_{ij}(\underline{k}) e^{i\underline{k}\cdot\underline{r}} d\underline{k}$



inverse $\Phi_{ij}(\underline{k}) = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} B_{ij}(\underline{r}) e^{-i\underline{k}\cdot\underline{r}} d^3\underline{r}$

Spectual Tensor

Isotropic $\Phi_{ij}(\underline{k}) = A(k) k_i k_j + B(k)\delta_{ij}$

recall $\frac{\partial B_{ij}}{\partial r_i} = 0 \rightarrow k_i \Phi_{ij}(\underline{k}) = 0$ $\quad$ or recall every Fourier $\underline{k}$

$k_j \Phi_{ij}(\underline{k}) = 0 \quad \Longleftrightarrow \quad \hat{u}_i(\underline{k})$ Fourier Mode $\perp$ $\underline{k}$

$$k_i(A k_i k_j + B\delta_{ij}) = 0$$

$$A k^2 k_j + B k_j = 0$$

$$B = -Ak^2 \leadsto A = -\frac{B}{k^2}$$

$$\Phi_{ij}(\underline{k}) = A(k) k_i k_j + B(k)\delta_{ij} = B(k)\underbrace{\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)}_{P(\underline{k})}$$

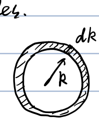
Single scaler fn. of $|k|$

(different $E(k)$)

Def. $E(k)$: radial 3-D Energy spectual density of

$E(k)\,dk$ is kinetic Energy in modes.

with $\underline{k}$ $\quad k < |k| < k+dk$



energy density

$$E(k) = \iint_{k \le |k| \le k+dk} \frac{1}{2}\Phi_{ii}(\underline{k})\, d\underline{k}^2$$

or

$$E(k) = \underbrace{\left\langle \frac{1}{2}\Phi_{ii} \right\rangle}_{\text{spectial avg.}}^{B} \cdot \underbrace{4\pi k^2}_{\text{avg of sph.}} = 2\pi k^2 \langle \Phi_{ii}(\underline{k}) \rangle$$

$$\Phi_{ii}(\underline{k}) = B(k)\left(\underset{3}{\delta_{ii}} - \underset{1}{\frac{k_i k_i}{k^2}}\right) = 2B(k) \longrightarrow B = \frac{1}{2}\Phi_{ii}(\underline{k})$$

or $\Phi_{ii}(\underline{k}) = 2B$

$$B(\underline{k}) = \frac{E(k)}{4\pi k^2}, \quad \left[\ \Phi_{ij}(\underline{k}) = \frac{E(k)}{4\pi k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\ \right]$$

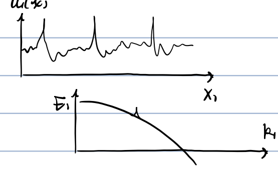
Energy spectrum

$$E_1(k) = \frac{1}{\pi}\int_0^\infty B_{11}(r)e^{-ikr}\,dr = \frac{1}{\pi}\int_0^\infty B_{11}(r)\cos(kr)\,dr$$

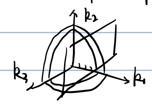
NOT Same

Can show. $E(k) = \frac{1}{2}\left(k^2\frac{d^2}{dk^2} - k\frac{d}{dk}\right)E_1(k)$

radial



To compute spectura in various directions.



$\Phi_{ij}(k)$

Ex. $E_1(k_1) = \iint_{k_2\,k_3} \Phi_{ij}(k)\ldots \cos dk_2 dk_3$

Return to Von-Karman-Howard Eqn.

$$\left(\frac{\partial}{\partial t} - 2\nu\nabla_r^2\right)B_{ij}(r) = \frac{\partial}{\partial r_k}\left(\overline{u_i u_k u_j'} - \overline{u_i u_k' u_j'}\right) \quad [B_{ik,j} + B_{jk,i}]$$

2D Fourier Transform

$$\frac{\partial}{\partial t}\Phi_{ij}(k,t) + 2\nu k^2\Phi_{ij}(k,t) = \Gamma_{ij}(k) \qquad \Gamma_{ij}(k) = ik_p\left[\Phi_{ik,j} + \Phi_{jk,i}\right]$$

non-linear interactions

$$\left[\,T(k,t) = 2\pi k^2\langle\Gamma_{jj}(k)\rangle\,\right]$$

Set $i=j$ $\left[\frac{\partial E(k,t)}{\partial t} = T(k,t) - 2\nu k^2E(k,t)\right]$ Spectral verison of Von-Karman-Howard Eqn.

Some connections to Structural Fn. ~ 1D Verison

$$D_{22}(r) = 2(B_{22}(0) - B_{22}(r))$$
$$= 2\left(\int_0^\infty E_1(k)\,dk - \int_0^\infty E_1(k)\cos(kr)\,dk\right)$$
$$= 2\int_0^\infty E_1(k)(1-\cos(kr))\,dk$$

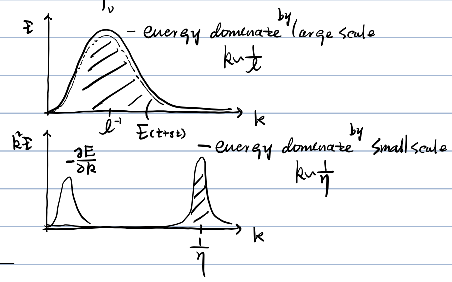
$$D_{22}''(r) = 2\int_0^\infty k^2E_1(k)\cos(kr)\,dk$$

$$D_{22}''(0) = 2\int_0^\infty k^2E_1(k)\,dk = -2B_{22}''(0) = 2\left(\frac{\varepsilon}{15\nu}\right)$$

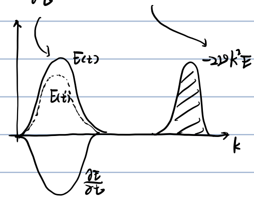
$$\left[\,\varepsilon = 15\nu\underbrace{\int_0^\infty k^2E_1(k)\,dk}_{\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2}}\,\right]$$

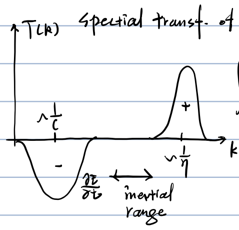
Also.

$$2\nu\int_0^\infty k^2E(k)\,dk = \varepsilon$$



$$\frac{\partial E}{\partial t} = T - 2\nu k^2E$$



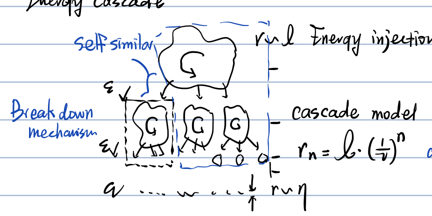


$T(k)$ spectial transf. of E.

$$\int_0^\infty T(k)\,dk = 0 \longleftrightarrow B_{22}(0) = 0$$

$$\int_0^\infty E(k)\,dk = \frac{1}{2}\overline{u^2} = \frac{1}{2}B_{22}(0)$$

Energy cascade

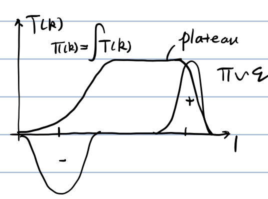


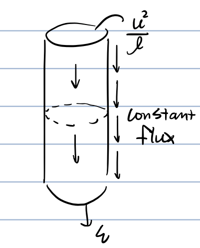
$T(k)$ spectial transf.

$\Pi(k)$ spectial flux

$$\Pi(k) = \int_k^\infty T(k)\,dk \;\text{ or }\; \int_k^\infty -T(k)\,dk$$

$$\varepsilon = \int_0^\infty 2\nu k^2E(k)\,dk$$

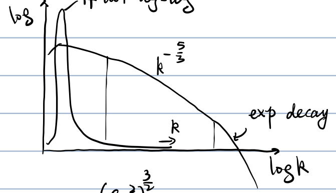




$$D_{22}(r) = C_2\varepsilon^{\frac{2}{3}}r^{\frac{2}{3}}$$

if not [illegible]

$\int (1-n s k r) E\, dk.$



$E_1(k) = C_1 \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$

1D radial. $E(k) = C_k \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$

$C_k \sim 1.6$

$\varepsilon \sim \frac{(\delta u_r)^{\frac{3}{2}}}{r}$

Kolom. Eqn.

$D_{LL} - 6\nu \frac{dD_{LL}}{dr} = -\frac{4}{5}\varepsilon r$

neglected $\frac{\partial D_{LL}}{\partial t}$ check for constancy $\varepsilon = \frac{u'^3}{\ell}$

$6\nu \frac{dD_{LL}}{dr}$ $\quad \nu \frac{\partial}{\partial r}(\varepsilon^{\frac{2}{3}} r^{\frac{2}{3}})$

$\sim \nu \varepsilon^{\frac{2}{3}} r^{-\frac{1}{3}}$

$\sim \frac{\nu}{u'\ell} u'\ell \left(\frac{u'^3}{\ell}\right)^{\frac{2}{3}} r^{-\frac{1}{3}}$

$\sim \frac{1}{Re_\ell} \varepsilon \left(\frac{r}{\ell}\right)^{-\frac{1}{3}}$

$Re_\ell \sim \frac{u'\ell}{\nu}$

$Re_\ell \to \infty,\ r \gg \ell$

viscous neglected

to $-\frac{4}{5}$ flow.

$\frac{\partial}{\partial t}(C_2 \varepsilon^{\frac{2}{3}} r^{\frac{2}{3}})$

$\sim r^{\frac{2}{3}} \varepsilon^{-\frac{1}{3}} \frac{\partial \varepsilon}{\partial t}$

$\sim (r\varepsilon)^{\frac{2}{3}} \underbrace{\frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial t}}_{\frac{u'}{\ell}}$ inverse time scale large eddy turnover time

$\sim r\left(\frac{u'^3}{\ell}\right)^{\frac{2}{3}}$

$\sim \left(\frac{r}{\ell}\right)^{\frac{2}{3}} \frac{u'^3}{\ell}$

$\frac{\partial D_{LL}}{\partial t} \sim \left(\frac{r}{\ell}\right)^{\frac{2}{3}} \varepsilon$

$\frac{r}{\ell} \to 0,\ \frac{\partial D_{LL}}{\partial t} \ll \varepsilon$

In inertial range

given ε, r $\quad (\varepsilon \sim \frac{u'^3}{\ell})$

velocity scale $\delta u_r \sim \varepsilon^{\frac{1}{3}} r^{\frac{1}{3}}$

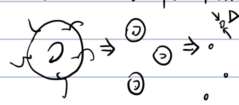
time scale $\frac{r}{\delta u_r} \sim \tau_r \sim \frac{r}{\varepsilon^{\frac{1}{3}} r^{\frac{1}{3}}} \sim \varepsilon^{-\frac{1}{3}} r^{\frac{2}{3}}$

shear scale /shear rate $\gamma_r \approx \frac{1}{\tau_r} \sim \frac{\delta u_r}{r} \sim \varepsilon^{\frac{1}{3}} r^{-\frac{2}{3}}$

→ Flamelets of thickness r

$\eta \ll r \ll \ell$ $\quad \frac{1}{A}\frac{dA}{dt} \sim \gamma_r \sim \varepsilon^{\frac{1}{3}} r^{-\frac{2}{3}}$

→ Drops or bubbles. surface tension σ



D small enough for surface tension prevent breakup.

D_H Hinze scale

$We = \frac{\rho U^2 L}{\sigma}$

Break up if $We > We_{cr}$

$We_{cr} = \frac{\rho D_H (\varepsilon D_H)^{\frac{2}{3}}}{\sigma}$ $\quad L = D_H$

$U = \delta u_{D_H}$ relative velocity across scale D_H

$D_H = \left(\frac{We_{cr}}{\rho}\sigma\, \varepsilon^{-\frac{2}{3}}\right)^{\frac{3}{5}}$

$\left[D_H = \left(\frac{We_{cr}}{\rho}\sigma\right)^{\frac{3}{5}} \varepsilon^{-\frac{2}{5}} \right]$

Local iso: $r \in$ I.R.

T close to isotropic

Turbulence Diffusion (Relative Diff.)

Fluid particles

$R(t)$ distance between plt.

$\frac{d}{dt}\overline{R^2(t)}$

$D = \frac{1}{2}\frac{d}{dt}\overline{R^2(t)}$ Diffusion coef.

For R in Inertia range. (I.R.)

$\frac{dR}{dt} \sim (\varepsilon \cdot R)^{\frac{1}{3}}$ relative V across R

$\underbrace{\frac{1}{2}\frac{dR^2}{dt} \sim R\frac{dR}{dt}} \sim \varepsilon^{\frac{1}{3}} R^{\frac{4}{3}} \sim \varepsilon^{\frac{1}{3}} (R^2)^{\frac{2}{3}}$

$\frac{d\overline{R^2}}{(\overline{R^2})^{\frac{2}{3}}} = \varepsilon^{\frac{1}{3}}$ at $(\overline{R^2})^{\frac{1}{3}} \sim (\overline{R^2(0)})^{\frac{1}{3}} = \frac{1}{3}\varepsilon^{\frac{1}{3}}(t)$

at large time I.C. $\left[\overline{R^2(t)} \sim \varepsilon t^3 \right]$

Richardson diffusion

much faster diffusion

$D = cR^{\frac{4}{3}} \cdot \varepsilon^{\frac{1}{3}}$