PDE

PDE Partical Differential Eqn.

① # independent variables x, y, z.. ☆ Try
② # order $u_{xx} \to 2$ $u_{xxx} \to 3$.. $(u_t+2y, u_{t}+u_{x})$
③ Linearity: dependent variables linear u, u_{xx}..
- linear or $\boxed{?}u_x$ only independent variable Ex. xu_x
- Quasilinear: only highest derivative linear

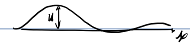
④ boundary condition (BC)
Initial condition (IC) $t=0$ $u(t=0)$ ...

⑤ Type of Eqn.
2nd PDE:
- Elliptic 椭圆 → Equilibrium. only BC No time
- Parabolic 抛物线 — time marching
- Hyperbolic 双曲线 — BC + IC

① # independent variables.
② order. #.
③ linearity 线性.
④ boundary condition
⑤ Type of eqn.

left line get shock wave first. slow down first.

Ex. $\frac{\partial u}{\partial t} + C\frac{\partial u}{\partial x} = 0$ wave eqn.
2 independent variables, 1 dependent variable, deformation



Ex. 1-D N-S. $\left[\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2}\right]$ velocity, pressure, $ma = F$, source, viscosity 粘性 Newton fluid $= C$. Not $\neq C$.

Ex. $[\nabla \cdot \vec{u} = 0]$ mass conservation — incompressible
$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$

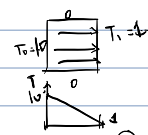
heat-transfer boundary: isotm., flux.

③ Linear:
$F(x.., u_x.., u_{xx}..) = 0$
u, u_x.. dependent variables $\xrightarrow{?}$ linear
NOT independent variable x...
(OR)
$\boxed{?}u_x u$.. only independent variables. x...

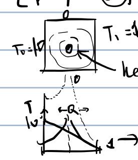
Ex. $u_t + Cu_x = 0$ linear, $=t^2$ linear
$u_t + uu_x = 0$ not linear

Quasilinear: only highest derivative linear (dependent variables) u_{xx}...
principle part.
Ex. $5u_{xx} + yu_{xy} = -uu_x - u^2 - x + 5$ Source term.
Q-linear. $\neq 0$ not homogeneous

Ex. $[\nabla^2 T = 0]$ Laplace eqn.

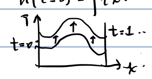


$[\nabla^2 T = Q]$ ← source



④ Boundary (BC)
① $u = f$ Dirchlet, isothermal
② $\frac{\partial u}{\partial n} = g$ flux.
③ $\alpha u + \beta\frac{\partial u}{\partial n} = h$. Mixed cond. Robin.
Ex. room

— Initial condition $t=0$.
(IC) $u(t=0) = f(x)$. $\frac{\partial}{\partial t}u(t=0) = f'(x)$.



$\frac{\partial}{\partial x}$

Well Posedness?

Problem Data = BC + IC + Source
small change → Solution — small change
Solution: exist, unique, depend continuously on data → well posedness

Well Posedness?
allow solution exists
Problem data: BC + IC + Source term.
Continuous dependence on data: small change. "自变量"连续" → 因变量"连续"; solution small change
→ Solution: exist. unique. depends continuously on data.
if yes.

— Types of Eqns. (classification)

2nd PDEs
- Elliptic → Equilibrium eqns. problems. not change by time (boundary problem) (no IC. only BC drives solu's.)
- parabolic (IC+BC) → contain time. Time-marching problem
- Hyperbolic (IC sometimes only) NEED IC.

$a\phi_{xx} + b\phi_{xy} + c\phi_{yy} + (d\phi_x + e\phi_y + f\phi) = g(x,y)$
linear eqn.
2nd eqn. classification: $a\phi_{xx} + b\phi_{xy} + c\phi_{yy} = \underbrace{g(x,y) - d\phi_x - e\phi_y - f\phi}_{H}$ (source part)

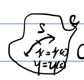
一般线性二阶偏微分方程
Linear 2nd Order PDE:
$a\phi_{xx} + b\phi_{xy} + c\phi_{yy} = g(x,y) - d\phi_x - e\phi_y - f\phi$
$a\phi_{xx} + b\phi_{xy} + c\phi_{yy} = H$ only 2nd order

① there is curve $\vec{s}$ for x, y satisfy eqn.
$\phi(s) = \phi(x,y)$

$a\phi_{xx} + b\phi_{xy} + c\phi_{yy} = H$ ①
$\frac{d}{ds}\phi_x = \frac{\phi_{xx}dx + \phi_{xy}dy}{ds}$ ② or $\phi_x = \phi_{xx}dx + \phi_{xy}dy$
$\frac{d}{ds}\phi_y = \frac{\phi_{yx}dx + \phi_{yy}dy}{ds}$ ③ $\phi_y = \phi_{yx}dx + \phi_{yy}dy$

$$\begin{bmatrix} a & b & c \\ \frac{dx}{ds} & \frac{dy}{ds} & \\ & \frac{dx}{ds} & \frac{dy}{ds} \end{bmatrix}\begin{bmatrix}\phi_{xx}\\ \phi_{xy}\\ \phi_{yy}\end{bmatrix} = \begin{bmatrix} H \\ \frac{d}{ds}\phi_x \\ \frac{d}{ds}\phi_y\end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ dx & dy & \\ & dx & dy \end{bmatrix}\begin{bmatrix}\phi_{xx}\\ \phi_{xy}\\ \phi_{yy}\end{bmatrix} = \begin{bmatrix} H \\ \phi_x \\ \phi_y\end{bmatrix}$$
$A \quad x = b$

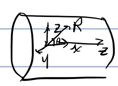
$a\phi_{xx} + b\phi_{xy} + c\phi_{yy} = g(x,y) - d\phi_x - e\phi_y - f\phi$
$u(s)$ $v(s)$ $w(s)$ $p(s)$ $q(s)$
curve. $\phi(x,y) \to \phi(s)$
ϕ, p, q are given.
C is boundary



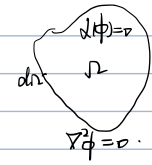
$au(s) + bv(s) + cw(s) = H$
$\frac{dp}{ds} = \frac{\partial p}{\partial x}\frac{dx}{ds} + \frac{\partial p}{\partial y}\frac{dy}{ds} = \frac{dx}{ds}\phi_{xx} + \frac{dy}{ds}\phi_{xy}$
$\frac{dq}{ds} = v\frac{dx}{ds} + w\frac{dy}{ds}$

$$\begin{bmatrix} a & b & c \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & \frac{dx}{ds} & \frac{dy}{ds} \end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix} = \begin{bmatrix} H \\ \frac{dp}{ds} \\ \frac{dq}{ds}\end{bmatrix}$$
$A \quad x = b$
for $\begin{bmatrix}u\\v\\w\end{bmatrix}$, $\det A \neq 0$. not singular

Ex. heat transfer
$\frac{\partial u}{\partial t} = \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]\alpha = \nabla^2 u\,\alpha$



(x, y, z) space coord.
(θ, R, z) cylinder coord. $= \left[\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial R^2} + \frac{\partial^2 u}{\partial z^2}\right]\beta$
Same.
① $\frac{\partial}{\partial \theta} = 0$ → 0 easier
② BC. $r = R$ easier.
$T = T_0$ much simplier.
$\sqrt{x^2+y^2} = R$
$T = T_0$



$d\Omega$, Ω, $d(\phi) = 0$, $\nabla^2\phi = 0$.

other term: $a\phi_{xx} + b\phi_{xy} + c\phi_{yy} = H$
$d\phi_x = \phi_{xx}dx + \phi_{xy}dy$
$d\phi_y = \phi_{yx}dx + \phi_{yy}dy$

$\det A = 0 \rightarrow$ singular $\rightarrow$ No soln.

$\det A = a\left(\frac{dy}{ds}\right)^2 - b\frac{dy}{ds}\frac{dx}{ds} + c\left(\frac{dx}{ds}\right)^2 = 0$

Characteristic equ. $a(dy)^2 - b(dy)(dx) + c(dx)^2 = 0$

$D = b^2 - 4ac$
- > 0 hyperbolic $\frac{dy}{dx} = (\lambda_1, \lambda_2)$ 2 family No soln
- $= 0$ parabolic $\frac{dy}{dx} = \lambda$ one family No soln
- < 0 Elliptic always soln.

$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2-4ac}}{2a}$

$y = \frac{b \pm \sqrt{b^2-4ac}}{2a}x + c$

在同曲线方程中

b^2-4ac 的值而划分

椭圆,双曲,抛物,线边界

相同偏分差方法

$\det A = a\left(\frac{dy}{ds}\right)^2 + c\left(\frac{dx}{ds}\right)^2 - b\left(\frac{dx}{ds}\right)\left(\frac{dy}{ds}\right) = 0$

$a(dy)^2 + c(dx)^2 - b\,dx\,dy \xrightarrow{/(dx)^2} a\left(\frac{dy}{dx}\right)^2 - b\frac{dy}{dx} + c$

$ah^2 - bh + c = 0$. characteristic equ.

$h = \frac{b \pm \sqrt{b^2-4ac}}{2a}$

$b^2 - 4ac < 0$. no soln h make $\det A = 0$. $\rightarrow$ always soln.

$b^2 - 4ac = 0$. one soln

$b^2 - 4ac = D$

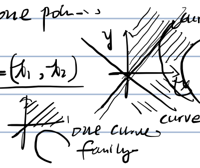
$D > 0$ two points $h \rightarrow \frac{dy}{dx} = (\lambda_1, \lambda_2)$

$D = 0$ one point $h \rightarrow \frac{dy}{dx} = \lambda$

$D < 0 \rightarrow$ Elliptic. always solvable.

$\begin{vmatrix} dx & dy \\ & dx & dy \end{vmatrix} \begin{vmatrix} \phi_{xy} \\ \phi_{yy} \end{vmatrix} = \begin{vmatrix} d\phi_x \\ d\phi_y \end{vmatrix}$

$\det A = a(dy)^2 - b\,dx\,dy + c(dx)^2$



hyperbolic

never teach

2 curve family no soln

to its asy.

Q1. $\phi(x,y) \rightarrow \phi(\xi,\eta)$

$L(u) = 0$.

$a\phi_{xx} + b\phi_{xy} + c\phi_{yy} = g(x,y) - d\phi_x - e\phi_y - f\phi$

$A\phi_{\xi\xi} + B\phi_{\xi\eta} + C\phi_{\eta\eta} = H$.

ξ_x.. "metrix" of transformation

$A = \xi_x^2 + 2\xi_x\xi_y + \xi_y^2$

$B =$

$C = \eta_x^2 + \eta_x\eta_y + \eta_y^2$

$B^2 - AC = (b^2 - ac)(\xi_x\eta_y - \xi_y\eta_x)^2$ > 0. $\neq 0$.

positive definition

"Jacobian" transformation

"onto" transformation ① unique 映射 both direction

Find ξ, η let. $A = C = 0$

$\boxed{\Delta}$ Need Calculation.

$\boxed{\Delta}$ $a\psi_x^2 + b\psi_x\psi_y + c\psi_y^2 = 0$

$\boxed{\Delta}$ $D > 0$ hyper $\rightarrow$ 2 solns. $\begin{cases} \xi = \psi_1 \\ \eta = \psi_2 \end{cases}$

parabolic $\rightarrow \begin{cases} \xi = \psi_1 \\ \eta = y \end{cases}$

elliptic. $\rightarrow \begin{cases} \psi = \psi_r + i\psi_i \\ \xi = \psi_r \\ \eta = \psi_i \end{cases}$

② $(x,y) \rightarrow (\xi,\eta)$

有对 a,b 偏导链式法则

$\phi_{ab} = \frac{\partial}{\partial b}\left(\frac{\partial\phi}{\partial\xi}\frac{\partial\xi}{\partial a} + \frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial a}\right)$ 前导项+后导项

$= \xi_a\left(\frac{\partial\phi_\xi}{\partial\xi}\frac{\partial\xi}{\partial b} + \frac{\partial\phi_\xi}{\partial\eta}\frac{\partial\eta}{\partial b}\right) + \xi_{ab}\phi_\xi + \eta_a\left(\frac{\partial\phi_\eta}{\partial\xi}\frac{\partial\xi}{\partial b} + \frac{\partial\phi_\eta}{\partial\eta}\frac{\partial\eta}{\partial b}\right) + \eta_{ab}\phi_\eta$

$= \xi_a\xi_b\phi_{\xi\xi} + (\xi_a\eta_b + \xi_b\eta_a)\phi_{\xi\eta} + \eta_a\eta_b\phi_{\eta\eta} + H'(\phi_\xi, \phi_\eta)$

$a=x, b=x$: $\phi_{xx} = \xi_x^2\phi_{\xi\xi} + 2\xi_x\eta_x\phi_{\xi\eta} + \eta_x^2\phi_{\eta\eta} + H''$

$a=x, b=y$: $\phi_{xy} = \xi_x\xi_y\phi_{\xi\xi} + (\xi_x\eta_y + \xi_y\eta_x)\phi_{\xi\eta} + \eta_x\eta_y\phi_{\eta\eta}$

$a=y, b=y$: $\phi_{yy} = \xi_y^2\phi_{\xi\xi} + 2\xi_y\eta_y\phi_{\xi\eta} + \eta_y^2\phi_{\eta\eta}$

代入 $a\phi_{xx} + b\phi_{xy} + c\phi_{yy} \rightarrow A\phi_{\xi\xi} + B\phi_{\xi\eta} + C\phi_{\eta\eta}$

$A = a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2$

$B = 2a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + 2c\xi_y\eta_y$

$C = a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2$

$A, C = a\psi_x^2 + b\psi_x\psi_y + c\psi_y^2$

let $A = C = 0 = a\psi_x^2 + b\psi_x\psi_y + c\psi_y^2 = 0$

ψ characteristic curve

$D = b^2 - 4ac$
- > 0 hyperbolic $\xi = \psi_1$, $\eta = \psi_2$
- $= 0$ parabolic $\xi = \psi_1$, $\eta = y \rightarrow$ could simplify transformed equ.
- < 0 Elliptic $\xi = \psi_r$, $\eta = \psi_i$; $\psi = \psi_r + i\psi_i$

$a^2u_{xx} - u_{tt} = 0$.

$w = au_x$, $v = u_t$

$\begin{cases} aw_x - v_t = 0 \\ w_t = av_x \end{cases}$ Cauchy Riemann. Equs.

$\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

large change

Ex. $m^2u_{xx} - u_{tt} = 0$

$b^2 - 4ac = 4m^2 > 0$. hyperbolic

$m\psi_x^2 - \psi_t^2 = 0 \rightarrow \frac{\psi_t}{\psi_x} = \pm m$

$\frac{d\psi}{dt} \pm \frac{d\psi}{dx}m = 0$.

$m^2u_{xx} - u_{tt} = 0$

$b^2 - 4ac = 4m^2 > 0$ hyperbolic

$m^2\psi_x^2 - \psi_t^2 = 0$

$\frac{\psi_t}{\psi_x} = \pm m \rightarrow \frac{dx}{dt} = \pm m$

$x = c \pm mt$

$x \pm mt = $ constant

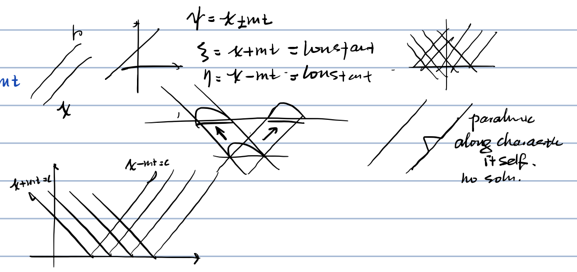
$\xi = x + mt$

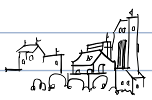
$\eta = x - mt$

$\psi = x \pm mt$

$\xi = x + mt = $ constant

$\eta = x - mt = $ constant





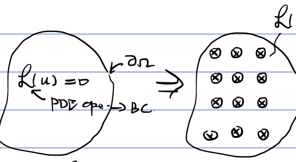
Basics of Finite Difference Methods

Discretization methods

give up on solving all $(\vec{x}, t)$

instead — Solution in discrete

number of points $\uparrow \Rightarrow$ accuracy $\uparrow$



error $\varepsilon \sim V - u \xrightarrow{\text{discrete}} 0$ (accurate)

$L(u) = 0$

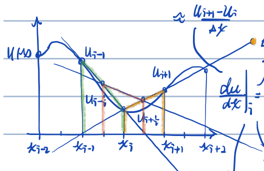
$\hat{L}(V)$ or $L(V) = 0$

Methods.
1) Finite Difference
2) Finite Volume
3) Compact difference — Padé scheme.
4) Finite-elements (FEM)
5) Molecular Dynamics.
6) LBM. 7) BEM. Boundary
8) Spectral Methods.

Finite Difference Method

obtain u_x: $\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0}\frac{u(x+\Delta x) - u(x)}{\Delta x}$

$\approx \frac{u_{i+1} - u_i}{\Delta x}$



$\frac{du}{dx}\Big|_i$

$\frac{\Delta u_i}{\Delta x} = \frac{u_{i+1} - u_i}{\Delta x}$ forward difference

$\frac{\delta u_i}{\Delta x} = \frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{\Delta x} = \frac{\frac{u_i + u_{i+1}}{2} - \frac{u_{i-1} + u_i}{2}}{\Delta x}$ Centeral difference

$\frac{\nabla u_i}{\Delta x} = \frac{u_i - u_{i-1}}{\Delta x}$ Backward difference

Taylor: $u_{i+1} = u_i + \left(\frac{\partial u_i}{\partial x}\right)\Delta x + \sum_{n=2}^{\infty} u_i^{(n)}\frac{\Delta x^n}{n!}$

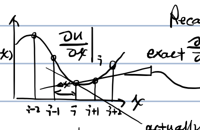
$\left(\frac{\partial u_i}{\partial x}\right) = \frac{u_{i+1} - u_i}{\Delta x} + \sum_{n=2}^{\infty} u_i^{(n)}\frac{(\Delta x)^{n-1}}{n!}$

forward difference $O(\Delta x)$

$u_t + c(u_x) = 0$

$(u_t) = \alpha u_{xx}$

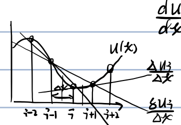
$u_{xx} + u_{yy} = 0$



exact $\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0}\frac{u(x+\Delta x) - u(x)}{\Delta x}$

$\approx \frac{u_{i+1} - u_i}{\Delta x}$

Recall.



$\frac{du}{dx}\Big|_i \approx \frac{\Delta u_i}{\Delta x}$ Forward difference $\Delta u_i = u_{i+1} - u_i$

$\approx \frac{\nabla u_i}{\Delta x}$ Backward difference $\nabla u_i = u_i - u_{i-1}$

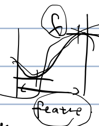
$\approx \frac{\delta u_i}{\Delta x}$ Centeral difference $\delta u_i = u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}$

$f(x)=o(\phi(x))$
$|f(x)|\le k|\phi(x)|$

$$\frac{d^2u}{dx^2}=\frac{\frac{du}{dx}(x+\Delta x)-\frac{du}{dx}(x)}{\Delta x}=\frac{\frac{du}{dx}\big|_{i+1}-\frac{du}{dx}\big|_i}{\Delta x}+\text{error}$$

? scale ?

measure feature part and Δx



$K_n=\frac{N_{f=o(\Delta x)}}{N_{f=o\,fall\,all}}$?

How to obtain { FD approximations? / the error of approximations?

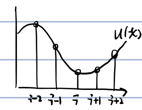
① Taylor Series
② Polynomial Fitting
③ Integral Method → Finite Volume, Finite Element

① Taylor Series

(1) choose stencil points $(i,i+1)$
(2) expand all points not at i in terms T series not at i

$$u_{i+1}=u_i+\frac{du}{dx}\Big|_i\Delta x+\frac{d^2u}{dx^2}\Big|_i\frac{\Delta x^2}{2!}+\frac{d^3u}{dx^3}\Big|_i\frac{\Delta x^3}{3!}\cdots$$

$$\frac{du}{dx}\Big|_i=\frac{u_{i+1}-u_i}{\Delta x}-\underbrace{\frac{d^2u}{dx^2}\Big|_i\frac{\Delta x}{2!}-\frac{d^3u}{dx^3}\Big|_i\frac{\Delta x^2}{3!}\cdots}_{\text{neglected}}$$



$O(\Delta x)$ Truncation Error

"order" of Δx → dominate term

"order" if $f(x)=O(\phi(x))$
then Exist a Positive Constant K independent of x
$|f(x)|\le k|\phi(x)|$

$\varepsilon=O(\Delta x)$
2 operator

VS FLOP Floating points operator

5 operator
$\varepsilon=O(\Delta x^2)$

$i,\ i+1,\ i+2$: $\frac{2\Delta x^2}{2}\frac{\partial^2u}{\partial x^2}$

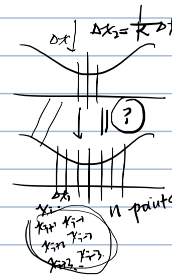
$$u_{i+2}=u_i+2\Delta x\frac{\partial u}{\partial x}\Big|_i+\frac{(2\Delta x)^2}{2}\frac{\partial^2u}{\partial x^2}\Big|_i+\frac{(2\Delta x)^3}{3!}\frac{\partial^3u}{\partial x^3}\Big|_i$$

$$-4\times\ u_{i+1}=\left(u_i+\frac{du}{dx}\Big|_i\Delta x+\frac{d^2u}{dx^2}\Big|_i\frac{\Delta x^2}{2!}+\frac{d^3u}{dx^3}\Big|_i\frac{\Delta x^3}{3!}\cdots\right)\times 4$$

$$\frac{\partial u}{\partial x}\Big|_i=\frac{4u_{i+1}-u_{i+2}-3u_i}{\Delta x}+\frac{2(\Delta x)^2}{3!}u_{3x}\ldots$$

$O(\Delta x^2)$

$\Delta x_2=k\Delta x_1$



B1:
B2:

Taylor series $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_i)}{n!}(x-x_i)^n$

① $u_{i+1}=u_i+\Delta x\left(\frac{\partial u}{\partial x}\right)_i+\frac{\Delta x^2}{2!}\left(\frac{\partial^2u}{\partial x^2}\right)_i+\frac{\Delta x^3}{3!}\left(\frac{\partial^3u}{\partial x^3}\right)_i\cdots\longrightarrow\left(\frac{\partial u}{\partial x}\right)_i=\frac{u_{i+1}-u_i}{\Delta x}+O(\Delta x)$

② $u_{i+2}=u_i+2\Delta x\left(\frac{\partial u}{\partial x}\right)_i+\frac{(2\Delta x)^2}{2!}\left(\frac{\partial^2u}{\partial x^2}\right)_i+\frac{(2\Delta x)^3}{3!}\left(\frac{\partial^3u}{\partial x^3}\right)_i\cdots$

$4\times① - ② = 4u_{i+1}-u_{i+2}=3u_i+2\Delta x\left(\frac{\partial u}{\partial x}\right)_i-4\frac{\Delta x^3}{6}\left(\frac{\partial^3u}{\partial x^3}\right)_i$

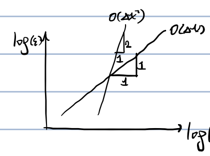
$\Rightarrow\left(\frac{\partial u}{\partial x}\right)_i=\frac{4u_{i+1}-u_{i+2}-3u_i}{\Delta x}+O(x^2)$

$x_{i-1}\quad x_i\quad x_{i+1}$

FD: $\frac{\partial u}{\partial x}=\frac{u_{i+1}-u_i}{\Delta x}$

BD: $\frac{\partial u}{\partial x}=\frac{u_i-u_{i-1}}{\Delta x}$

central difference CD: $\frac{\partial u}{\partial x}=\frac{u_{i+1}-u_{i-1}}{2(\Delta x)}+O(\Delta x^2)$



for several

2D: $\left(\frac{\partial^2u}{\partial x^2}\right)_{ij}=\frac{u_{i+2,j}-2u_{i+1,j}+u_{i+1,j}}{\Delta x^2}+O(\Delta x)$

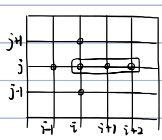
$\frac{\partial^n u}{\partial x^n}\Big|_{ij}\approx\frac{\Delta_x^n u_{ij}}{\Delta x}$ Forward difference

$=\frac{\Delta_x\Delta_x^{n-1}u_{ij}}{\Delta x}=\frac{\Delta_x^{n-1}\Delta_x u_{ij}}{\Delta x}$

2D. $\frac{\partial^2u}{\partial x^2}\Big|_{ij}$

② $-2\times$① $\longrightarrow u_{i+2,j}-2u_{i+1,j}=-u_{ij}+0+(\Delta x)^2\left(\frac{\partial^2u}{\partial x^2}\right)_{i,j}+(\Delta x)^3\left(\frac{\partial^3u}{\partial x^3}\right)_{ij}+\cdots$

$\left(\frac{\partial^2u}{\partial x^2}\right)_{ij}=\frac{u_{i+2,j}-2u_{i+1,j}+u_{i,j}}{\Delta x^2}+O(\Delta x)$



$\frac{\partial^n u}{\partial x^n}\Big|_{i,j}\approx\frac{\Delta_x^n u_{ij}}{\Delta x^n}$ or $\frac{\nabla_x^n u_{ij}}{\Delta x^n}$ or $\frac{\delta_x^n u_{ij}}{(\Delta x)^n}$

Forward difference, Backward difference, Central difference

$\Delta_x^n u_{ij}=\Delta_x\left(\Delta_x^{n-1}u_{ij}\right)=\frac{\left(\Delta_x^{n-1}u_{ij}\right)_{k+1}-\left(\Delta_x^{n-1}u_{ij}\right)_k}{\Delta x}$

$$\frac{\partial^n u}{\partial x^n}\Big|_{i,j}\approx\frac{\Delta_x^n u_{ij}}{\Delta x^n}=\frac{\Delta_x\Delta_x u_{ij}}{\Delta x^n}=\frac{\Delta_x(u_{i+1,j}-u_{i,j})}{\Delta x^n}=\frac{u_{i+2}-u_{i+1}-(u_{i+1}-u_i)}{\Delta x^2},\ \frac{u_{i+2}-2u_{i+1}+u_i}{\Delta x^2}$$

$(\odot+O(\Delta x))(\odot+O(\Delta y))$
$=\odot\Delta x+\odot\Delta y+\odot\Delta x\Delta y$

$$\frac{\partial^2u}{\partial x\partial y}\Big|_{ij}=\frac{\Delta_x}{\Delta x}\left(\frac{\Delta_y}{\Delta y}u_{ij}\right)=\frac{\Delta_x}{\Delta x}\left(\frac{u_{i,j+1}-u_{ij}}{\Delta y}\right)=\frac{(u_{i+1,j+1}-u_{i+1,j})-(u_{i,j+1}-u_{i,j})}{\Delta x\Delta y}=\frac{u_{i+1,j+1}-u_{i+1,j}-u_{i,j+1}+u_{ij}}{\Delta x\Delta y}+O(\Delta x,\Delta y)$$

$u(x)=\sum_{i=0}^{n}C_i x^i$

$u^{(k)}(x)=\sum_{i=k}^{n}C_i\frac{i!}{(i-k)!}x^{i-k}$

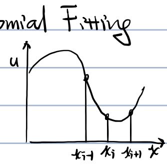
$u^{(n)}(x)=(n!)C_n\longrightarrow C_n=\frac{u^{(n)}}{n!}$

$u^{(n-1)}=\frac{n!}{1}C_n x+(n-1)!C_{n-1}\longrightarrow C_{n-1}=\frac{u^{(n-1)}-u^{(n)}x}{(n-1)!}$

$u^{(n-2)}=\frac{n!}{2}C_n x^2+\frac{(n-1)!}{1}C_{n-1}x+(n-2)!C_{n-2}\longrightarrow C_{n-2}=\frac{u^{(n-2)}-\frac{u^{(n)}}{2}x^2-\frac{u^{(n-1)}-u^{(n)}x}{1}x}{(n-2)!}$

$C_{n-\alpha}=\frac{u^{(n-\alpha)}-\sum_{i=1}^{i=\alpha}\frac{u^{i}}{i!}x^i}{(n-\alpha)!}$

Polynomial Fitting



$\frac{\partial^2u}{\partial x^2}$ $\quad x_{i-1},\ x_i,\ x_{i+1}$
$\quad u_{i-1},\ u_i,\ u_{i+1}$

Ex. $\frac{du}{dx}=2ax+b$

$u(x)=ax^2+bx+c$

$\frac{\partial u}{\partial x}=2ax+b$

$\frac{\partial^2u}{\partial x^2}=2a$

$x_i=0\longrightarrow u(x_i)=a(0)^2+b(0)+c=C$ or $u_i=C$

$u_{i-1}=ax_{i-1}^2+bx_{i-1}+C$
$=a(-\Delta x)^2+b(-\Delta x)+u_i$

$u_{i+1}=a(\Delta x)^2+b(\Delta x)+u_i$

$a=\frac{u_{i+1}+u_{i-1}-2u_i}{\Delta x^2}\qquad b=\frac{u_{i+1}-u_{i-1}}{2\Delta x}$

$u(x)=\frac{u_{i+1}+u_{i-1}-2u_i}{\Delta x^2}x^2+\frac{u_{i+1}-u_{i-1}}{2\Delta x}x+u_i\longrightarrow$

$\frac{\partial u}{\partial x}=b=\frac{u_{i+1}-u_{i-1}}{2\Delta x}+O(\Delta x^2)$

$\frac{\partial^2u}{\partial x^2}=a=\frac{u_{i+1}+u_{i-1}-2u_i}{\Delta x^2}+O(\Delta x^2)$

? Is that means for every function f can be poly-fitted to $\sum_{i=0}^{\infty}C_i x^i$ and the steps to find C_i is same to find $f^{(n)}$

$i+1$ points

Taylor Table

$f_j' = \sum_{k=0}^{2} a_k f_{j+k} + O(?)$

$f_j' + \sum_{k=0}^{2} a_k f_{j+k} = O(?)$

$f_j' + a_0 f_j + a_1 f_{j+1} + a_2 f_{j+2} = O(?)$

Compressible

	f_j	f_j'	f_j''	f_j'''
f_j'	0	1	0	0
$a_0 f_j$	a_0	0	0	0
$a_1 f_{j+1}$	a_1	$a_1 h$	$a_1 \frac{h^2}{2}$	$a_1 \frac{h^3}{3!}$
$a_2 f_{j+2}$	a_2	$a_2 2h$	$a_2 \frac{4h^2}{2}$	$a_2 \frac{8h^3}{3!}$

$O(?) = (a_0+a_1+a_2) f_j + (1+a_1 h + 2h a_2) f_j' + (a_1 \frac{h^2}{2} + 2 a_2 h^2) f_j'' + (a_1 \frac{h^3}{6} + a_2 \frac{4}{3} h^3) f_j'''$

① $a_0 + a_1 + a_2 = 0 \rightarrow a_0 = \frac{3}{2h}$

② $1 + a_1 h + 2a_2 h = 0 \rightarrow a_1 + 2a_2 = -\frac{1}{h}$, $a_1 = -\frac{2}{h}$, $a_2 = \frac{1}{2h}$

③ $\frac{a_1 h^2}{2} + 2a_2 h^2 = 0 \rightarrow a_1 = -4a_2$

leading Truncation Error:

$f_j'''(\frac{a_1 h^3}{6} + a_2 \frac{4}{3} h^3)$

$= f_j'''(-\frac{h^2}{3} + \frac{2}{3}h^2) = \frac{1}{3}h^2 f_j^{(3)}$

Taylor Table.

$\sum_{k=0}^{K} \sum_{\Delta=0}^{\Delta} a_i f_{j+\Delta}^{(k)} = 0$

	f_j	f_j'	f_j''
$a_i f_{j+\Delta}$	a_i	Δa_i	$\frac{\Delta^2}{2} a_i$
$a_i f_{j+\Delta}^{(k)}$		0	

Simulate function

Find Truncation Error

$\sum_{k=0}^{K} \sum_{\Delta=0}^{\Delta} a_i f_{j+\Delta}^{(k)} = a_i f_j + a_i f_{j+1}$ ... $k = 0, 1, \ldots$ mea... $\Delta = \pm\alpha\Delta x$ ...

Taylor Table

for $T_i^{(n)}$, rth order accuracy $O(\Delta x^r)$

Taylor table: → # derivatives $T_i^{(n+r+k-1)}$; $n+r$ points; First k terms of T error;

$[\Delta_{c(r)}^{n}$: $n+r-1$ points (Ti 不包括)$]$

	$T_i^{(0)}$ .. $T_i^{(n)}$ .. $T_i^{(n+k)}$	$T_i^{(n+r+k-1)}$
$n+r$ points T_{i+1}	factors	factors
$T_i^{(n)}$	0 ... 1 ... 0	$O(\Delta x^{(n+r)})$

to get the simulation function; to get accuracy terms

<顶前移>

central difference Δ_c 本身不需要 T_i, 因为计算时约去

故在用点相同时 Δ_c 比 Δ_F / Δ_B 高一阶精度

同一阶精度时 Δ_c 比 Δ_F / Δ_B 少用一个点 (T_i)

$\Delta_{(2)}^{c} T_i = \frac{T_{i+1} - T_{i-1}}{2\Delta x} = \frac{T_{i+1} - T_i}{2\Delta x} \ldots$

Pade's approximation.

Compact scheme. $f_j' = a_0 f_j + a_1 f_{j+1} + a_2 f_{j-1}$

$a_3 f_{j+1}' + a_4 f_{j-1}' + f_j' + a_0 f_j + a_1 f_{j+1} + a_2 f_{j-1} = O(?)$

	f_j	f_j'	f_j''	f_j'''	f_j^{iv}	f_j^{v}
f_j'	0	1	0	0	0	0
$a_0 f_j$	a_0	0	0	0	0	0
$a_1 f_{j+1}$	a_1	$a_1 h$	$a_1 \frac{h^2}{2}$	$a_1 \frac{h^3}{6}$	$a_1 \frac{h^4}{24}$	$a_1 \frac{h^5}{120}$
$a_2 f_{j-1}$	a_2	$-a_2 h$	$a_2 \frac{h^2}{2}$	$-a_2 \frac{h^3}{6}$	$a_2 \frac{h^4}{24}$	$-a_2 \frac{h^5}{120}$
$a_3 f_{j+1}'$	0	a_3	$a_3 h$	$a_3 \frac{h^2}{2}$	$a_3 \frac{h^3}{6}$	$a_3 \frac{h^4}{24}$
$a_4 f_{j-1}'$	0	a_4	$-a_4 h$	$a_4 \frac{h^2}{2}$	$-a_4 \frac{h^3}{6}$	$a_4 \frac{h^4}{24}$
	①	②	③	④	⑤	⑥

① $(a_0 + a_1 + a_2) = 0$

② $1 + a_1 h - a_2 h + a_3 + a_4 = 0$

③ $a_1 \frac{h^2}{2} + a_2 \frac{h^2}{2} + a_3 h - a_4 h = 0$

④ $a_1 \frac{h^3}{6} - a_2 \frac{h^3}{6} + a_3 \frac{h^2}{2} + a_4 \frac{h^2}{2} = 0$

⑤ $a_1 \frac{h^4}{24} + a_2 \frac{h^4}{24} + a_3 \frac{h^3}{6} - a_4 \frac{h^3}{6} = 0$

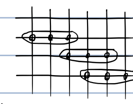
$\Rightarrow$ $a_0 = 0$, $a_1 = -\frac{3}{4h}$, $a_2 = \frac{3}{4h}$, $a_3 = a_4 = \frac{1}{4}$

$f_{j-1}' + 4f_j' + f_{j+1}' - \frac{3}{h}(f_{j+1} - f_{j-1}) + \frac{h^4}{30} f_j^{v}$ ⑥ $O(h^4)$

$Ax = b$ $j = 1, 2, 3 \ldots$

B1: $f_0' + 2f_1' = \frac{1}{h}(-\frac{5}{2}f_0 + 2f_1 + \frac{1}{2}f_2)$

B2: $f_N' + 2f_{N-1}' = \frac{1}{h}(\frac{5}{2}f_N - 2f_{N-1} - \frac{1}{2}f_{N-2}) + O(h^3)$



$$\begin{bmatrix} 1 & 2 & & & 0 \\ 1 & 4 & 1 & & \\ & 1 & 4 & 1 & \\ & & \ddots & \ddots & \ddots \\ 0 & & & 1\ 4 & 1 \\ & & & 2 & 1 \end{bmatrix} \begin{bmatrix} f_0' \\ \vdots \\ f_N' \end{bmatrix} = \frac{1}{h} \begin{bmatrix} -\frac{5}{2}f_0 + 2f_1 + \frac{1}{2}f_2 \\ 3(f_2 - f_0) \\ \vdots \\ 3(f_N - f_{N-2}) \\ \frac{5}{2}f_N - 2f_{N-1} - \frac{1}{2}f_{N-2} \end{bmatrix}$$

$f_{j-1}' + 4f_j' + f_{j+1}' = \frac{3}{h}(f_{j+1} - f_{j-1}) + \frac{h^4}{30} f_n^{(v)}$

$[A]_{N\times N} \{f'\} = \{R(f)\}$

$\{f'\} = [A]^{-1} \{R(f)\}$

$f' = A^{-1}$

$\begin{bmatrix} 1 & 1 \\ & \ddots \end{bmatrix} [f] = [f']$

Thomas Algorithm. (TDMA)

Compact scheme.

use $f_{i+\Delta}$, $f_{i+\Delta}^{(k)}$ (highest derivative), find f_i^{k}

$\sum_i a_i f_i + \sum_i b_i f_i^{k} = 0$

	f_j	f_j'	f_j'' ..	$f_j^{(k)}$ Taylor Ser
$a_i f_{j+\Delta}$	a_i	Δa_i	$\frac{\Delta^2}{2} a_i$	.. $\frac{\Delta^k}{k!}$
$a_i f_{j+\Delta}^{(k)}$		0		a_i Δa_i

Simulate function

Approximation Error.

$f(x) = e^{ikx}$

$k = \frac{2\pi}{L} n$

$f'(x) = ik e^{ikx} = ik f$

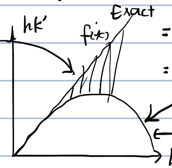
$x_j = \frac{L}{N} j$

$\frac{\partial f}{\partial x}\Big|_j = \frac{f_{j+1} - f_{j-1}}{2h} = \frac{e^{i2\pi n \frac{L}{N}(j+1)} - e^{i2\pi n \frac{L}{N}(j-1)}}{2h}$

$= \frac{e^{i\frac{2\pi}{N}n} - e^{-i\frac{2\pi}{N}n}}{2h} f_j$

$= \frac{\sin(\frac{2\pi n}{N})}{h} f_j$

hk' $f(x)$ Exact $k = \frac{2\pi}{L}n$ k' hk



$\begin{bmatrix} f_i'' \end{bmatrix} = \begin{bmatrix} \frac{2}{\Delta x^2} & \frac{1}{\Delta x^2} \\ \frac{1}{\Delta x^2} & \ddots \end{bmatrix} \begin{bmatrix} f_i \end{bmatrix}$

Perade's scheme.

$[\ B\ ] [f_i''] = [\ A\ ] [f_i] + [\ C\ ] [f_i']$ if use f_i'

$[B]\{f''\} = [A]\{f\}$

$\{f''\} = [B]^{-1}[A]\{f\}$

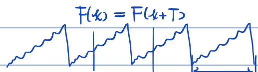
$[C][f_i'] = [D]\{f\}$

$\{f_i'\} = [C]^{-1}[D]\{f\}$

$\{f''\} = [B]^{-1}[A + C^{-1}D]\{f\}$

Periodic Fn. 周期函数

$F(x) = F(x+T)$



x $x+T$ T

Fourier Transform

$\boxed{F(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx\frac{2\pi}{T}) + b_n \sin(nx\frac{2\pi}{T})\, dx}$ even   odd

$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} F(x)\cos(nx\frac{2\pi}{T})\, dx$

$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} F(x)\sin(nx\frac{2\pi}{T})\, dx$

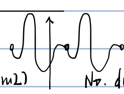
Euler

$\frac{T}{2} = l$ $\frac{1}{2}$ period.

$\boxed{F(x) = \sum_{-\infty}^{\infty} A_n e^{inx(\frac{2\pi}{T})}} = A_0 + \sum_{1}^{\infty}(i\sin + \cos)$

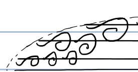
$A_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} F(x) e^{-inx(\frac{2\pi}{T})}\, dx$

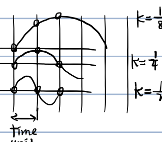
Periodic Fn.



$F(x) = F(x + mL)$ No. discontinuity

m integer





$k = \frac{1}{8}$ $k = \frac{1}{4}$ $k = \frac{1}{2}$ Time unit

can we only use bigger sequence

$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} f_{i-1}'' \\ f_i'' \\ f_{i+1}'' \end{bmatrix} = \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} f_{i-1} \\ f_i \\ f_{i+1} \end{bmatrix}$

$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} f_i'' \end{bmatrix} = \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} f_{i-1} \\ f_i \\ f_{i+1} \end{bmatrix}$

k — wave number $(\frac{1}{T})$ (period)$^{-1}$

number of wave of unit length

注: 不需确定起始点, 因为同期函数处处是周期

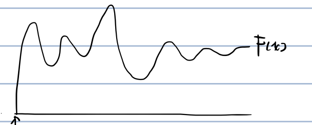
Sampleing

$k_{max} = \frac{N}{2}$ Need sample at least 2 per wave.

$N = \frac{2\pi}{\Delta x} \rightarrow (k_{max}\Delta x = \pi)$

Sample in 2π period

$a_n = \frac{1}{l}\int_{-l}^{l} F(x)\cos\frac{n\pi x}{l}\, dx$ $b_n = \frac{1}{l}\int_{-l}^{l} F(x)\sin\frac{n\pi x}{l}\, dx$ wave number $\frac{1}{2}$ period

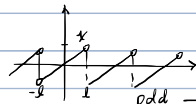


F(x)

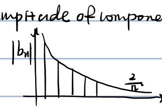
$\frac{1}{2}a_0$

Fourier $F(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{l}) + (b_n\sin\frac{n\pi x}{l})$

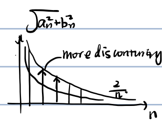
average value of function.   even   odd   amplitude of component



$f(x) = x$ odd $\rightarrow$ $\sum(-1)^{n+1}\frac{2}{n}(\sin nx) \cdot \frac{1}{x}$ odd $\rightarrow a_n = 0$



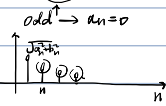
$|b_n|$ $\frac{2}{n}$



$\sqrt{a_n^2 + b_n^2}$ more discontinuity. $\frac{2}{n}$ n

any discontinuity → infinite fourier series.

$\sqrt{a^2 + b^2}$ special term.



$\sqrt{a_n^2 + b_n^2}$ n

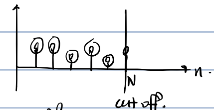
any function: $F(x)=\sum_{n=0}^{\infty} C_n \sin(nx+\varphi)$ 任何函数都为 $\sin x$ 的和 (周期为2π)

$T=2\pi$ | 即. $F(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx + b_n \sin nx$ 为傅里叶级数

$a_n: \int_{-\pi}^{\pi} F(x)\cos kx\,dx = \int_{-\pi}^{\pi}\left(\frac{a_0}{2}+\sum a_n\cos + b_n\sin\right)\cos kx\,dx$ 代入 $F(x)$

$=\frac{a_0}{2}\cos$

$$F(x)=\frac{1}{2}a_0+\sum_{n=1}^{N}\left(a_n\cos\frac{n\pi x}{\ell}\right)+\left(b_n\sin\frac{n\pi x}{\ell}\right)$$



Euler $e^{i\theta}=\cos\theta+i\sin\theta$ — wave number

$$F(x)=\sum A_n e^{\frac{i\pi n x}{\ell}} \qquad A_n=\frac{1}{2\ell}\int_{-\ell}^{\ell}F(x)e^{-\frac{i\pi n x}{\ell}}dx$$

Fourier Transform.

$$F[-\ell,\ell]\to(-\infty,\infty): \hat{f}(\xi)=\int f(x)e^{-i2\pi\xi x}dx$$

$$f(x)=\int \hat{f}(\xi)e^{i2\pi\xi x}d\xi$$

Discrete Fourier Transform. Parseval Identity.

$$\hat{f}(k)=\sum f(x)e^{-\frac{i2\pi k x}{L}}$$

$$f(x)=\sum_{k=-\frac{N}{2}}^{k=\frac{N}{2}}\hat{f}(k)e^{\frac{i2\pi k x}{L}}$$

$$h=\frac{L}{N}$$

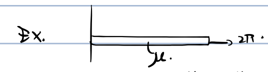
$$f(s)=\sum_{k=0}^{\pi}\hat{f}_k e^{iks} \qquad k=\frac{2\pi kh}{L} \qquad s=\frac{x}{h}$$

if is linear transf.

$$f(x)\sim\hat{f}_k e^{iks}$$

if $f(x)$ real, $\hat{f}_k=\hat{f}_{-k}$

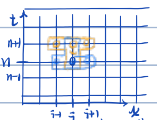
$$T(\Delta x^2)=T(s^2)$$



$u_t=\mu u_{xx}$ if μ discont. → discont..

1-dimension heat transf.

For $\frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2}$ — central diff.



FE Forward Euler (n)(i): $\frac{u_i^{n+1}-u_i^n}{\Delta t}=\alpha\frac{u_{i+1}^n-2u_i^n+2u_{i-1}^n}{\Delta x^2}$ ✓ Explicit ⊗ Δt small May Not Stable — Explicit

BE Backward Euler (n+1)(i): $\frac{u_i^{n+1}-u_i^n}{\Delta t}=\alpha\frac{u_{i+1}^{n+1}-2u_i^{n+1}+2u_{i-1}^{n+1}}{\Delta x^2}$ ✓ Stability Δt big ⊗ Implicit — Inplicit

CN central (n+½)(i): $\frac{u_i^{n+1}-u_i^n}{\Delta t}=\frac{\alpha}{2}\left(\frac{u_{i+1}^n-2u_i^n+2u_{i-1}^n}{\Delta x^2}+\frac{u_{i+1}^{n+1}-2u_i^{n+1}+2u_{i-1}^{n+1}}{\Delta x^2}\right)$

Crank Nicolson scheme. ✓ 2nd order ⊗ implicit stable

DF Dufort-Franked scheme. (central) (n)(i): $\frac{u_i^{n+1}-u_i^{n-1}}{\Delta t}=\alpha\frac{u_{i+1}^n-(u_i^{n+1}+u_i^{n-1})+2u_{i-1}^n}{\Delta x^2}$ (u_i^n)

✓ Explicit ($u_i^{n+1}=\dots$)

✓ un-conditionally stable

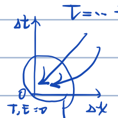
⊗ $t\neq-1$ IC.

⊗ inconsistancy 不一致.

$T=\dots+\alpha\left(\frac{\Delta t}{\Delta x}\right)^2$

$\beta=\frac{\Delta t}{\Delta x}$

$\Delta t=\Delta x^{\mu},\ \mu>1.$



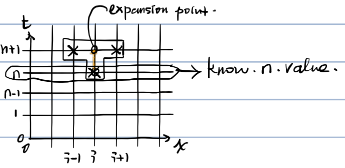
Not same as using different β

Parabolic Equs.

$$\frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2}$$

0 — L

BC. → Isothermal.. → flux..



$\frac{\partial u}{\partial t}$ → Forward / Backward / Central. — 2 points

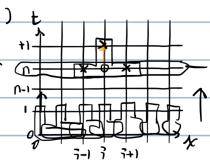
$$\frac{\partial u}{\partial t}=\frac{u_i^{n+1}-u_i^n}{\Delta t} \quad \text{(from n+1)}$$

as heat conduct to all directions

$\frac{\partial^2 u}{\partial x^2}$ central difference. $\frac{\partial^2 u}{\partial x^2}=\frac{u_{i-1}^{n+1}-2u_i^{n+1}+u_{i+1}^{n+1}}{\Delta x}$ (from n+1 point)

BE $\frac{u_i^{n+1}-u_i^n}{\Delta t}=\alpha\frac{u_{i-1}^{n+1}-2u_i^{n+1}+u_{i+1}^{n+1}}{\Delta x^2}$ Backward Euler Scheme. (for point n+1)

Fully. Implicit Bigger Δt. small stability.



FE $\frac{u_i^{n+1}-u_i^n}{\Delta t}=\alpha\frac{u_{i-1}^n-2u_i^n+u_{i+1}^n}{\Delta x^2}$ Forward in time. (for point n)

Forward Euler. Explicit

smaller Δt. more stability.

central to $n+\frac{1}{2}$ — $\frac{u_i^{n+1}-u_i^n}{\Delta t}=\alpha\frac{u_{i-1}^{n+\frac{1}{2}}-2u_i^{n+\frac{1}{2}}+u_{i+1}^{n+\frac{1}{2}}}{\Delta x^2}$ Centeral

if $\frac{\partial u}{\partial t}=f\to\frac{u^{n+1}-u^n}{\Delta t}=f^{n+\frac{1}{2}}$

CN $\frac{u_i^{n+1}-u_i^n}{\Delta t}=\frac{\alpha}{2}\left[\frac{u_{i-1}^{n+1}-2u_i^{n+1}+u_{i+1}^{n+1}}{\Delta x^2}+\frac{u_{i-1}^n-2u_i^n+u_{i+1}^n}{\Delta x^2}\right]$

central in time, central in space — Crank Nicolson Scheme

T.E. $\sim O(\Delta x^2,\Delta t^2)$

PDE $\checkmark\ [u_t-\alpha u_{xx}=0]$

FDE $\checkmark\ \left[\frac{u_i^{n+1}-u_i^n}{\Delta t}-\alpha\frac{u_{i-1}^n-2u_i^n+u_{i+1}^n}{\Delta x^2}=0\right]$

Expansion point (i, n)

FE: $\left[\frac{u_i^{n+1}-u_i^n}{\Delta t}-\alpha\frac{u_{i-1}^n-2u_i^n+u_{i+1}^n}{\Delta x^2}=0\right]$

consistency — $\lim_{\Delta t,\Delta x\to0}$ FDE → PDE

$\lim_{\Delta t,\Delta x\to0}$ T.E. → 0

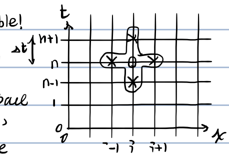
EP. (i, n+1)

$$\text{BE: }\underbrace{\frac{u_i^{n+1}-u_i^n}{\Delta t}-\alpha\frac{u_{i-1}^{n+1}-2u_i^{n+1}+u_{i+1}^{n+1}}{\Delta x^2}}_{\text{FDE}}=\underbrace{\frac{\partial u}{\partial t}-\alpha\frac{\partial^2 u}{\partial x^2}}_{\text{PDE}}+\underbrace{\frac{\Delta t}{2}u_{tt}\Big|_i^{n+1}-\alpha\frac{\Delta x^2}{12}u_x^{(4)}\Big|_i^{n+1}+\dots}_{\text{T.E.}}$$

$\lim_{\Delta t,\Delta x\to0}$ T.E $=0$ ✓ consistent!

DuFort-Franked Scheme. u_i^n stable!

$$\frac{u_i^{n+1}-u_i^{n-1}}{2\Delta t}=\alpha\frac{u_{i+1}^n-(u_i^{n+1}+u_i^{n-1})+u_{i-1}^n}{\Delta x^2}$$



central in time, central in space

advantage: → Explicit! $u_i^{n+1}=f(\dots)$

→ Un-conditional stable

problem: ✱ initialization: $t=-1$

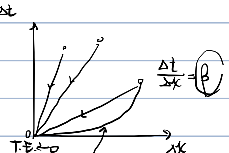
✱ T.E. $=\alpha\frac{\Delta x^2}{12}u_x^{(4)}\Big|_i^n-\alpha\left(\frac{\Delta t}{\Delta x}\right)^2 u_t^{(2)}\Big|_i^n-\alpha\frac{\Delta t^2}{6}u_t^{(3)}\Big|_i^n$

$$\lim_{\Delta t,\Delta x\to0}\text{T.E.}=-\alpha\beta^2u_t^{(2)}\Big|_i^n$$

$$[u_t-\alpha u_{xx}-\alpha\beta^2u_{tt}=0]$$

X. inconsistant!

But. if $\Delta t\to\Delta x^{\mu}$



$\frac{\Delta t}{\Delta x}=\beta$

$\mu=2$. recover 2nd order accuracy

others:

Adams-Bashforth — 2,3 order

Runge-Kutta — 3,4th order

$$[\text{FDE}]\xrightarrow[\Delta t,\Delta x\to0]{\text{consistancy}}[\text{PDE}]$$

$$U\xrightarrow[\text{convergence}]{\Delta t,\Delta x\to0}U_{exact}$$

Lax Equivalence Thm.

Consistency + stability → Convergence

consistant (T.E → 0)

$[FDE] \xrightarrow{\Delta x, \Delta t \to 0} [PDE]$; stable

$(U) \xrightarrow[\text{convergence}]{\Delta x, \Delta t \to 0} (u\ \text{exact})$ $(u \to V)$

$V + \varepsilon$ round-off error.

Round off error: floating points $\boxed{a.bcdX|E}$ ...

Ex. $\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$

Finite-Diff-eqn: $f(u_n, u_{n+1}, \cdots u_{n+k}) = 0$

$u_{n+k} = g(u_n, u_{n+1}, \cdots u_{n+k-1})$

① $u_{n+k} + \sum_{i=1}^{k} a_i u_{n+k-i} = 0$ Linear Homogenous

⊙ Exact Soln. No Round-off Err.

Taylor: $u_{n+k} = \left[\sum_{h=0}^{\infty} \frac{(k\Delta x)^h}{h!} \frac{d^h}{dx^h}\right] u_n$

$e^x = \left[\sum_{h=0}^{\infty} \frac{x^h}{h!} \frac{d^h e^0}{dx^h} \cdot 1\right]$

$u_{n+k} = (\underbrace{e^{\Delta x \frac{d}{dx}}}_{\lambda})^k \cdot u_n$

$u_{n+k} = \lambda^k u_n$

$u_n \left(\boxed{\sum_{i=0}^{k} a_i \lambda^{k-i}}\right) = 0 \to \lambda_i$

general soln: $u_n = \sum_{i=1}^{k} C_i \lambda_i^n$

Repeated roots: $u_n = \left(\sum_{i=1}^{k} C_i n^i\right) \lambda^n$

② Not homogenous → Round off error.

⊗ No Exact Soln.

冯一诺伊曼稳定性分析

Von-Neumann Stability Analysis:

① ignore BC.

② Periodic Problem.

$f(u_n, u_{n+1}, \cdots u_{n+k}) = 0$

$u = \underset{\text{Exact}}{V} + \underset{\text{Error}}{\varepsilon}$

$\underbrace{f(V_n \cdots V_{n+k})}_{0} + f(\varepsilon_n, \cdots \varepsilon_{n+k}) = 0$

$f(\varepsilon_n, \cdots \varepsilon_{n+k}) = 0$

Assume $\varepsilon(x,t) = \sum_k b_k(t) e^{ikx(\frac{2\pi}{2L})}$, $b \sim e^{at}$

$\varepsilon(x, t_n) = \sum_k (e^{a\Delta t})^n e^{ikx(\frac{2\pi}{2L})}$

$\varepsilon_{i+1}^n = \varepsilon(n\Delta t, x+\Delta x) = (e^{a\Delta t})^n e^{ik(x+\Delta x)(\frac{\pi}{L})}$

$\boxed{\text{Stability: } e^{a\Delta t} \le 1}$

$|k(e^{a\Delta t}) = k(\Delta t, \Delta x)| \le 1$

Von-Neumann for FE:

$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$

$\cancel{\frac{V_i^{n+1} - V_i^n}{\Delta t}} + \frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = \cancel{\alpha \frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{\Delta x^2}} + \alpha \frac{\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n}{\Delta x^2}$

$\frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = \alpha \frac{\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n}{\Delta x^2}$

Assume Periodic: $\varepsilon(x,t) = \sum_k b_k(t) e^{ikx(\frac{2\pi}{2L})}$

$\varepsilon_k \sim b_k(t) e^{ikx(\frac{2\pi}{2L})}$

$b_k(t) = e^{at} = a^{(n\Delta t)}$

$n+1 \to (n+1)\Delta t$

$i+1 \to x + \Delta x$

$[\varepsilon_k \sim (e^{a\Delta t})^n e^{ikx(\frac{2\pi}{2L})}]$

$[e^{a\Delta t} \le 1 \text{ stable}]$

$\frac{e^{a\Delta t} - 1}{\Delta t} = \alpha \frac{e^{ik\Delta x(\frac{2\pi}{2L})} + e^{-ik\Delta x(\frac{2\pi}{2L})} - 2}{\Delta x^2}$ ↓ Euler

$e^{a\Delta t} = 1 + \frac{\alpha \Delta t}{\Delta x^2}\left(2\cos k\Delta x(\tfrac{2\pi}{2L}) - 2\right)$

$\left(\left|e^{a\Delta t} = 1 - 4r\sin^2\left(\frac{k\Delta x}{2} \cdot \frac{2\pi}{2L}\right)\right| \le 1\right)$

$V + \varepsilon$ — round off error (floating point error) 截断误差

Ex. 4 digit $1 \to 0.9999$, 0.1001; last digit not accurate

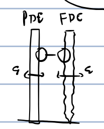
Stability of numerical soln. to round off error?

Lax Equivalence Theorem.

Convergence ← Consistency + Stability → only challenge! expl( ) in error?

properly posed.

Given IVP (initial value Prob.) and FDE satisfied Consistency, Stability is necessary & Sufficient condition to convergence. Time dependent!



Round-off error

Heat eqn. $\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$ $0 < x \le 2L$

Soln: $u = V + \varepsilon$

① How does error introduce at n ↑ or ↓ at $n+1$?

② How does the SHAPE of error effect soln. V?

Machine ε < relative error

rounding in floating point a ε

Floats $m \times 2^E$ IEEE 754 binary 32 2^{-24}

$1 \le m < 2$ binary 64 $2^{-53} \sim 10^{-16}$

$\varepsilon = \frac{b^{-(p-1)}}{2}$

Exact soln. to FDE

$f(u_n, u_{n+1}, \cdots u_{n+k}) = 0$

$u_{n+k} = g(u_n, u_{n+1}, \ldots, u_{n+k-1})$

Linear Homogenous eqn.

$u_{n+k} + \sum_{i=1}^{k} a_i u_{n+k-i} = 0$

$u_{n+k} = \left[1 + k\Delta x \frac{d}{dx} + \frac{(k\Delta x)^2}{2} \frac{d^2}{dx^2} + \ldots\right] u_n$

$= e^{k\Delta x \frac{d}{dx}} \sim e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

$u_{n+k} = \lambda^k \cdot u_n$

? u_n λ

$a_n = u_0 \frac{1-\lambda^n}{1-\lambda}$ $\frac{q}{1-q}$

$u_{n+k} = e^{k\Delta x \frac{d}{dx}} u_n$

$u_{n+k} = [e^{\Delta x \frac{d}{dx}}]^k u_n$; λ

$u_{n+k} = \lambda^k u_n$

$u_n \left(\sum_{i=0}^{k} a_i \lambda^{k-i}\right) = 0$ characteristic eqn. k^{th} order polynomial.

$\sum_i a_i \lambda_k^{k-i}$

General Soln:

distinct roots: $u_n = \sum_{i=1}^{k} C_i \lambda_i^n$

If repeated roots: $u_n = \left(\sum_{i=0}^{k} C_i n^i\right) \lambda^n$

Second order: $u_n = a\lambda^n + bn\lambda^n$

$= C_1 2^n + C_2 n 2^n$

$u_0 = 0$, $u_1 = 1$, $u_2 = 4$

$u_{n+2} - 4u_{n+1} + 4u_n = 0$

$(\lambda - 2)^2 = 0$

$\lambda_1 = \lambda_2 = 2$

$u_n = \sum a_i 2^{k-1}$

Ex. $u_{n+2} - 5u_{n+1} + 6u_n = 0$

IC $u_0 = 0$ $u_1 = 1$

Chareqn. $(\lambda^2 - 5\lambda + 6) = 0$

$\lambda_1 = 3$ $\lambda_2 = 2$

$u_n = C_1 3^n + C_2 2^n$

$n=0$ $u_0 = 0 = C_1 + C_2$

$n=1$ $u_1 = 1 = 3C_1 + 2C_2$

$C_1 = 1$, $C_2 = -1$

$u_n = 3^n - 2^n$

if homogenue. exact soln.

if Inhom → No soln.

all can use numerical. roundoff error. analysis.

Not-homogenous.

No exact soln → have round-off error.

Von-Neumann Stability analysis:

↳ Ignore BCs

↳ Periodic problem.

Explicit $u = V + \varepsilon \to$ FDE $\overbrace{\frac{V_i^{n+1} - V_i^n}{\Delta t}}^{\text{exact}} + \frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = \alpha \frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{\Delta x^2} + \alpha \frac{\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n}{\Delta x^2}$

$\therefore \frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = \alpha \frac{\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n}{\Delta x^2}$

Assume $\varepsilon(x,t) = \sum_k b_k(t) \exp(ikx(\frac{2\pi}{2L}))$ $k\frac{2\pi}{2L} \sim$ wave number (normalized) $2L = T$

$\varepsilon \sim b_k(t) e^{ikx(\frac{2\pi}{2L})}$

$b_k(t) \sim e^{at} \Rightarrow \varepsilon(x, t_n) \sim e^{an\Delta t} e^{ikx(\frac{2\pi}{2L})}$; $e^{an\Delta t} = (e^{a\Delta t})^n$

$b_k(t_n) = e^{a(n\Delta t)}$

$|e^{a\Delta t}| < 1$ stable

$|e^{a\Delta t}| > 1$ unstable.

(need $\left|\frac{\varepsilon^{n+1}}{\varepsilon^n}\right| < 1$)

? means unstable → no soln exactly get. NOT use BE?

$\frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = \alpha \frac{\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n}{\Delta x^2}$

$e^{ikx(\frac{2\pi}{2L})} \frac{e^{a(n+1)\Delta t} - e^{an\Delta t}}{\Delta t} = e^{an\Delta t} \cdot \alpha \frac{e^{ik(x+\Delta x)(\frac{2\pi}{2L})} - 2e^{ikx(\frac{2\pi}{2L})} + e^{ik(x-\Delta x)(\frac{2\pi}{2L})}}{\Delta x^2}$

$\cancel{e^{ikx(\frac{2\pi}{2L})}} \cancel{a^{n\Delta t}} \frac{e^{a\Delta t} - 1}{\Delta t} = \cancel{e^{an\Delta t}} \cancel{e^{ikx(\frac{2\pi}{2L})}} \alpha \frac{e^{ik\Delta x(\frac{2\pi}{2L})} - 2 + e^{-ik\Delta x(\frac{2\pi}{2L})}}{\Delta x^2}$

$\frac{e^{a\Delta t} - 1}{\Delta t} = \alpha \frac{e^{ik\Delta x(\frac{2\pi}{2L})} + e^{-ik\Delta x(\frac{2\pi}{2L})} - 2}{\Delta x^2}$

$e^{a\Delta t} - 1 = \alpha \frac{\Delta t}{\Delta x^2}\left(e^{ik\Delta x(\frac{2\pi}{2L})} + e^{-ik\Delta x(\frac{2\pi}{2L})} - 2\right)$

$e^{a\Delta t} - 1 = \alpha \frac{\Delta t}{\Delta x^2}\left(2\cos(k\Delta x(\tfrac{2\pi}{2L})) - 2\right)$

$r = \frac{\alpha \Delta t}{\Delta x^2}$ $e^{a\Delta t} = 1 + 2r[\cos k\Delta x \frac{2\pi}{2L} - 1]$

$e^{a\Delta t} = 1 - 4r\sin^2\left(\frac{k\Delta x}{2} \cdot \frac{2\pi}{2L}\right)$

$|e^{a\Delta t}| < 1 \to -1 \le e^{a\Delta t} \le 1$

$-1 \le 1-4r\sin^2\frac{\beta_k}{2} \le 1$

$r\sin^2\frac{\beta_k}{2} \ge \frac{1}{2}$ $\quad r \ge 0,\ a \ge 0$

$\Delta t \le \frac{(\Delta x)^2}{2}$

FDM → $f(\cdot\cdot) \to U_{n+k} = g(\cdot\cdot)$:

FDM 将微分方程转化为差分方程，差分方程有存在式
(Finite difference Method)
可整理为一阶非齐次差分递推值 U_i 的方程

$f(U_{n}+\dots+U_{n+k})=0$ 为求解高值 U_{n+k}，化为

$U_{n+k}=g(U_n,\dots U_{n+k-1})$,

Exact Soln:

此时，若 $U_{n+k}=g(U_n,\dots U_{n+k-1})$ 线性，且齐次，

可化为 $U_{n+k}+\sum_{i=1}^{k}a_i U_{n+k-i}=0$，则可直接算出

精确解 (Exact Soln)

Non-Exact Soln:

若 $U_{n+k}=g(U_n,\dots U_{n+k-1})$ 不为齐次方程，则求近似解，存在 Round-off error，为证随迭代误差逐渐变小，需要用 Von-Neumann Stability analysis.

注：即使可算精确解，也可不算，用迭代求近似解，则也要用 Von-Neumann Stability analysis

Round off Error
→ Von-Neumann stability analysis ②
→ Matrix Stability ∼ ✓ Better ①

因为 Von-Neumann Stability analysis 不考虑边界条件

假设 Error 是 periodic 故需用 Matrix Stability 可考虑边界

Incorporate BC.

③ Matrix Stability

Iteration: $u^{n+1} = L(u_{j-1}^n, u_j^n, \dots)$ — linear Fn.

BC: $u_j^0 = 0 \quad u_j^{N-1} = 0$

$$\begin{bmatrix} u_1 \\ u_j \\ u_{N-1}\end{bmatrix}^{n+1} = \begin{bmatrix} A \end{bmatrix}\begin{bmatrix} u_1 \\ u_j \\ u_{N-1}\end{bmatrix}^{n}$$

$\{u^{n+1}\} = [A]\{u^n\}$

Iteration: Numerical soln. $\left[\{u^{n+1}\} = [A]^n\{u^0\}\right]$

exact soln $\{\hat{u}^{n+1}\} = [A]\{\hat{u}^n\}$

$\{\epsilon^{n+1}\} = \{\hat{u}^{n+1}\} - \{u^{n+1}\} = [A]\{\epsilon^n\}$

Iteration of Error: $\left[\{\epsilon^{n+1}\} = [A]^n\{\epsilon^0\}\right]$

$[A]\{V_m\} = \lambda_m\{V_m\}$

let $\{\epsilon^0\} = \sum_{m=1}^{N-1} C_m\{V_m\}$

$\{\epsilon^1\} = \sum_{m=1}^{N-1} C_m[A]\{V_m\} = \sum_{m=1}^{N-1} C_m\lambda_m V_m$

$\{\epsilon^n\} = \sum_{m=1}^{N-1} C_m \lambda_m^n \{V_m\}$

$|\lambda_m| \le 1$ stable

$[A] = f([K]r) \to |\lambda(A) = f(\lambda(K)r)| \le 1$

For Backward / C-N. $\quad Bu^{n+1} = Au^n.$ $\quad u^{n+1} = B^{-1}Au^n$

$\lambda(B^{-1}A) = \frac{1}{\lambda_{min}(B)} \cdot \lambda_m(A)_{at}$

Ex. Heat Transfer $\partial_t T = \partial_x^2 T$ $\quad$ BC: $\begin{cases} u(0)=u(1)=0 \\ u(x,0)=f(x)\end{cases}$

Forward Euler (Explicit): $u_j^{n+1} - u_j^n = r(u_{j-1}^n - 2u_j^n + u_{j+1}^n)$

$u_j^{n+1} = ru_{j-1}^n + (1-2r)u_j^n + ru_{j+1}^n$

$B\begin{cases} u_1^{n+1} = r\cancel{u_0^n} + (1-2r)u_1^n + ru_2^n & (BC \to 0) \\ u_{N-1}^{n+1} = ru_{N-2}^n + (1-2r)u_{N-1}^n + r\cancel{u_N^n} & (BC \to 0)\end{cases}$

$$\begin{bmatrix} \uparrow \\ u_j \\ \downarrow \end{bmatrix}^{n+1} = \begin{bmatrix} 1-2r & r & & 0 \\ r & \ddots & \ddots & \\ & \ddots & \ddots & \\ 0 & & & \end{bmatrix}\begin{bmatrix} u_1 \\ u_j \\ u_{N-1}\end{bmatrix}^{n}$$

$\{u^{n+1}\} = [A]\{u^n\}$

$\{u^{n+1}\} = [A]^n\{u^0\}$, $\quad u^0 = f$

Value $\{\hat{u}^{n+1}\} = [A]^n\{\hat{u}^0\}$

Introduce Error $\{\epsilon^{n+1}\} = \{\hat{u}^{n+1}\} - \{u^{n+1}\}$

$= [A]^n(\{\hat{u}^0\} - \{u^0\})$

$\beta_k = k\Delta x\left(\frac{2\pi}{2L}\right)$ $\quad -1 \le 1-4r\sin^2\left(\frac{\beta_k}{2}\right) \le 1$

$r\sin^2\frac{\beta_k}{2} \ge \frac{1}{2}$ $\qquad r \ge 0,\ \alpha \ge 0$

$r = \frac{\alpha\Delta t}{\Delta x^2} \le \frac{1}{2}$

$\Delta t \le \frac{(\Delta x)^2}{2}$ Stable condition

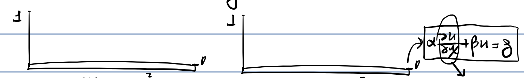
Implicit scheme in time. $\quad \frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}$

$e^{a\Delta t}[1+2r(1-\cos\beta_k)] = 1$

$e^{a\Delta t}[1+4r\sin^2\frac{\beta_k}{2}] = 1$

$e^{a\Delta t} = \frac{1}{1+4r\sin^2\frac{\beta_k}{2}}$ $\quad r>0.$ Stable for Δt (any)

Von-Neumann Stability.



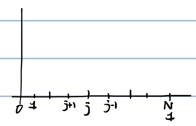
$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}$ $\qquad \frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}$ $\qquad \boxed{\alpha\frac{\partial u}{\partial x} + \beta u = g}$

Matrix Stability.

$u_t = u_{xx}$

$u(0) = u(1) = 0$

$u(x,0) = f(x)$



Simple Explicit. $\quad r = \frac{\alpha\Delta t}{\Delta x^2}$

$u_j^{n+1} - u_j^n = r(u_{j-1}^n - 2u_j^n + u_{j+1}^n)$

$j=1$: $u_1^{n+1} = r\cancel{u_0^n} + (1-2r)u_1^n + ru_2^n$ (BC. → 0)

$j=2$: $u_2^{n+1} = ru_1^n + (1-2r)u_2^n + ru_3^n$

$\vdots$

$j=N-1$: $u_{N-1}^{n+1} = ru_{N-2}^n + (1-2r)u_{N-1}^n + r\cancel{u_N^n}$ (BC.)

$$\Rightarrow \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ u_{N-1}^{n+1}\end{bmatrix} = \overbrace{\begin{bmatrix} (1-2r) & r & & 0 \\ r & (1-2r) & r & \\ & \ddots & \ddots & \ddots \\ 0 & & r & (1-2r)\end{bmatrix}}^{A}\begin{bmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{N-1}^n\end{bmatrix} \to \{u^{n+1}\} = [A]\{u^n\}$$

Tri-diagonal matrix

$[A]\{u^n\} = A\{u^{n-1}\}$ ↓ $\{u^{n+1}\} = [A]^n\{u^0\}$, $\{u^0\} = \{f\}$

$\left[\{u^{n+1}\} = [A]^n\{u^0\}\right]$

Introduce error in $\{u^0\}$ ↓

$\{\hat{u}^0\}$ erroneous value. $\qquad \{\hat{u}^{n+1}\} = [A]^n\{\hat{u}^n\}$

$\{\epsilon^{n+1}\} = \{\hat{u}^{n+1}\} - \{u^{n+1}\}$

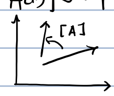
$= [A]^n(\{\hat{u}^0\} - \{u^0\})$

$\left[\{\epsilon^{n+1}\} = [A]^n\{\epsilon^0\}\right]$

Does Error grow with n? ∼ Depend on $[A] = [A(r)]$ ∼ r $\qquad \frac{|\{\epsilon^{n+1}\}|}{|\{\epsilon^n\}|} > 1$

$[A]\{V_m\} = \lambda_m\{V_m\}$

m^{th} eigenvalue $\qquad$ eigenvector



eigenvectors are vectors CANNOT been rotate by $[A]$

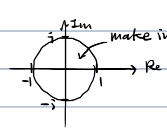
$\{\epsilon^0\} = \sum_{m=1}^{N-1} C_m\{V_m\}$

$\{\epsilon^1\} = [A]\{\epsilon^0\} = [A]\sum_{m=1}^{N-1} C_m\{V_m\} = \sum_{m=1}^{N-1} C_m\lambda_m\{V_m\}$

$\{\epsilon^n\} = \sum_{m=1}^{N-1} C_m\lambda_m^n\{V_m\}$

$|\lambda_m| \le 1$ for all m, want it not going

↓ r ?



make inside.

λ_m: $\quad [A] = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1\end{bmatrix} + r\underbrace{\begin{bmatrix} -2 & 1 & \\ 1 & \ddots & \ddots \\ & \ddots & \ddots\end{bmatrix}}_{T} = I + r[T]$ $\qquad \begin{bmatrix} a & b & \\ c & \ddots & \ddots \\ & \ddots & \ddots\end{bmatrix}$

$\lambda_m(T) = a + 2\sqrt{bc}\cos\left(\frac{m\pi}{N+1}\right)$

m: λ_m

n: size of $[T]$

$\lambda_m(A) = 1 + r\lambda_m(T) = 1 + r(-2 - 2\cos(\frac{m\pi}{N}))$

$= 1 - 2r[1 - \cos(\frac{m\pi}{N})]$

$= 1 - 4r\sin^2\left(\frac{m\pi}{2N}\right)$ $\qquad r > 0$, $r \le \frac{1}{2}$.

Problem: NOT linear eqn. for N. $\lambda_{m,n_k} > 1$?

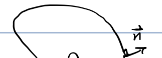
Conservation Form & Conservation Form

PDE = FDE + T.E. → Stability

Any other error?

Conservation Law
- conservation of Energy
- conservation of momentum
- conservation of mass
- conservation of species. (chemical)

- mass
- momentum
- Energy



Can be anything carried with Material inside

Iteration of Error

$\{\epsilon^{n+1}\}=[A]^{n+1}\{\epsilon^0\}$

For A: $[A]\{V_m\}=\lambda_m\{V_m\}$ eigenvector, eigenvalue

$\{\epsilon^0\}=\sum_{m=1}^{N-1}C_m\{V_m\}$

$\{\epsilon^1\}=[A]\sum_{m=1}^{N-1}C_m\{V_m\}=\sum_{m=1}^{N-1}C_m\lambda_m\{V_m\}$

$\{\epsilon^n\}=\sum_{m=1}^{N-1}C_m(\lambda_m^n)\{V_m\}$

want $|\lambda_m|\le 1$

$\lambda(A) \leftarrow \lambda_m(r)$

$[A]=\begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}+\begin{bmatrix}-2 & 1 & \\ 1 & \ddots & 1\\ & 1 & -2\end{bmatrix}r = I+r[T]$

$\lambda_m(T)=-2+2\cos(\frac{m\pi}{N})$

$\lambda_m(A)=1+r\lambda_m(T)$
$=1-2r(1-\cos\frac{m\pi}{N})$

$|\lambda_m(A)|=|1-4r\sin^2(\frac{m\pi}{2N})|\le 1$

$\begin{bmatrix}a & b & \\ c & \ddots & \\ & & \end{bmatrix}:\ \lambda_m=a+2\sqrt{bc}\cos(\frac{m\pi}{N+1})$

Rate of Change of Total Amount "Substance" contained in domain Ω

Is EQUAL to "Flux" accross boundary (NO SOURCE)

$q=\frac{\text{Substance}}{\text{Volume}}$ Intensive Property: density → Mass; $\rho\vec{u}$ → Momentum; E → Energy

$$\frac{\partial}{\partial t}\int_\Omega q(x)\,d\forall=-\int_{\partial\Omega}\vec{F}\cdot\hat{n}\,ds$$

$\vec{F}$ Flux Vector: Fick's Law of diffusion; Fourier Law of conduction

div Thm

$$\int_\Omega\left(\partial_t q(\vec{x})+\nabla\cdot\vec{F}\right)d\forall=0$$

$\boxed{q_t+\vec{\nabla}\cdot\vec{F}=0}$ ~ PDE. Every point of domain

Mass Conservation $\frac{\partial\rho}{\partial t}+\vec{\nabla}(\rho\vec{v})=0$

1D: $\boxed{\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0}$

Heat Conduction. $\vec{F}=-k\vec{\nabla}T$ high to low.

$q\to$ heat ~ ρcT specific

$\rho c\frac{\partial T}{\partial t}=\vec{\nabla}(k\vec{\nabla}T)$

k,ρ,c constant

$\frac{\partial T}{\partial t}=(\frac{k}{\rho c})\nabla^2T$

$\frac{\partial T}{\partial t}=\alpha\nabla^2T$

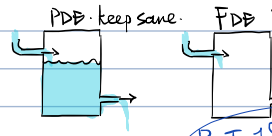
1D: $\boxed{\frac{\partial T}{\partial t}=\alpha\frac{\partial^2T}{\partial x^2}}$ PDE → FDE

T $\quad T'$

$T'-T=TE$ ~ $O(\Delta x^k,\Delta t^m)$

Numerical Soln. Exact Soln.

PDE keep same. FDE ? $T\to n$.



But is Final Matter than T(t)?

守恒 Conservation form ~ Conservative Method

Keep Product Inside $\nabla(\rho u)$ → Error Conservative ε

Conservative Method Let Error Finally → 0.

$\rho c\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x})$ ✓ $\equiv$ $\rho c\frac{\partial T}{\partial t}=\frac{\partial k}{\partial x}\frac{\partial T}{\partial x}+k\frac{\partial^2T}{\partial x^2}$

Conservation Law form ~ divergence in Flux term Identifible

NON-conservative form.

Ex. Steady-State continuity

$\vec{\nabla}(\rho\vec{v})=0$

$\int_0^L\nabla(\rho\vec{v})\,dl=\rho(L)v(L)-\rho(0)v(0)=0$ outflux influx

$\frac{\partial}{\partial x}(\rho u)$ or $\frac{\partial\rho}{\partial x}u+\rho\frac{\partial u}{\partial x}$?

$\left[\frac{\rho_2v_2-\rho_1v_1}{\Delta x}+\frac{\rho_3v_3-\rho_2v_2}{\Delta x}+\frac{\rho_4v_4-\rho_3v_3}{\Delta x}+\frac{\rho_5v_5-\rho_4v_4}{\Delta x}\right]\Delta x$

~ $O(\Delta x)$ $=\rho_5v_5-\rho_1v_1$ ~ Exact

Errors Cancelled out! Telescope property

$\rho u_x+\rho_x u$

$\left[\rho_2\frac{(u_2-u_1)}{\Delta x}+u_2\frac{(\rho_2-\rho_1)}{\Delta x}+(\ )+(\ )\cdots\right]\Delta x$?

$\neq\rho_5u_5-\rho_1u_1$ global conservation Error

"Numeric Method" Is Integral.

Can we (a.b)

For Eqns. Ex. $\frac{\partial u}{\partial t}=\alpha\frac{\partial^2u}{\partial x^2}$ BC fixed.., IC fixed..

Solns $u(x,t)$ ~ $e^{-\alpha k^2t}e^{ikx}$

① Smoothing property: larger wavenumber decay faster

② Min-Max Principle: Max, BC, Min, IC. Min\Max Soln only on BC\IC → $A,B,C\ge0$, $A+B+L\le1$ Explicit

③ Speed propagation-∞

$\frac{\partial u}{\partial t}=\alpha\frac{\partial^2u}{\partial x^2}$ $u(x,0)=u^0(x)$ IC $u^0(x)\to0$ at $\pm\infty$

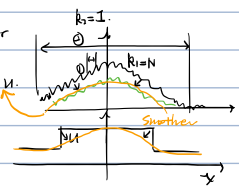
sep variables

Sdn. $u(x,t)$ ~ $e^{-\alpha k^2t}e^{ikx}$ sin/cos

k: wave number; high — decay faster; low

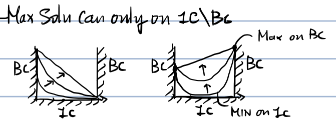
① Smoothing property ← high wavenumber ① k_1 decay faster

Ex. Sdn.



② Min-Max Principle:

Min-Max Sdn Can only on IC\BC



? Are These Principles Work for Most Eqns For ~?

③ Speed propagation ~ ∞ infinite speed

Energy ~ Eq. $u_t=\alpha u_{xx}$

$E(t)=\int_{-\infty}^{+\infty}u^2dx$

$\frac{d}{dt}E=\frac{d}{dt}\int_{-\infty}^{+\infty}u^2dx$
$=\int_{-\infty}^{+\infty}2uu_t\,dx=\int_{-\infty}^{\infty}2\alpha uu_{xx}\,dx$
$=\int_{-\infty}^{\infty}2\alpha\,d(uu_x)-\int_{-\infty}^{\infty}2\alpha(u_x)^2dx$
$=-2\alpha\int_{-\infty}^{\infty}(u_x)^2dx<0$

Energy decrease

Energy Ex. $u_t=\alpha u_{xx}$ u temperature

? why Introduce

$E(t)=\int_{-\infty}^{\infty}u^2dx$

$\frac{d}{dt}E=\int_{-\infty}^{\infty}2uu_tdx=\int_{-\infty}^{+\infty}2\alpha uu_{xx}dx=\int_{-\infty}^{+\infty}2\alpha(uu_x)_xdx-\int_{-\infty}^{\infty}2\alpha(u_x)^2dx=uu_x\Big|_{-\infty}^{+\infty}-2\alpha\int_{-\infty}^{\infty}(u_x)^2dx<0$

Energy Continuous Decrease

Explicit Scheme: $A+B+C\le1$, $A,B,C\ge0$ → Stable

Analysis: $r=\alpha\frac{\Delta t}{\Delta x^2}$

Forward Euler — Explicit $u_j^{n+1} = \overset{A}{r u_{j-1}^n} + \overset{B}{(1-2r)u_j^n} + \overset{C}{r u_{j+1}^n}$
- Consistant
- Von-Neumann \ Matrix Stability $\sim r \le \frac{1}{2}$
- ② Min-Max $\begin{cases} A+B+C \le 1 \\ A,B,C \ge 0 \end{cases} \to \begin{cases} A\ge0 & r\ge0 \\ B\ge0 & r\le\frac{1}{2} \end{cases}$
- ③ Propogation Speed $\frac{\Delta x}{\Delta t} \ne \infty$

Backward Euler — Implicit $-r u_{j-1}^{n+1} + (1+2r)u_j^{n+1} - r u_{j+1}^{n+1} = u_j^n$
- Stable
- ② Min-Max? NOT obvious
- ③ Speed of proporgation ∞ all depended

Spatial Deriatue ODD/Even → Imaginary on t

n: $u_t = \alpha\, u_{(nx)}$ + nth order of spacial derative

$u = \hat{u}\, e^{(ik)^n t} e^{ikx}$

n odd: $u = \hat{u}\, e^{ik(x-kt)}$ NOT amplify, but different wave number shift different ϕ — dispersive

n even: $u = (\hat{u}e^{-kt})e^{ikx}$ u decay among t — dissipative

$\frac{\partial u}{\partial t} = \alpha\frac{\partial^n u}{\partial x^n}$ Get Exact Soln.

seperation of variable: $u = A(t)B(x)$

$A'B = \alpha A B^{(n)}$

$\frac{A'}{A} = \alpha\frac{B^{(n)}}{B} = R$

$A = A e^{Rt}$, $\frac{B^{(n)}}{B} = \frac{R}{\alpha}$

$B = e^{\lambda t}$, $\lambda^n = \frac{R}{\alpha}$

$B = B_0 \exp\left(\left(\frac{R}{\alpha}\right)^{\frac{1}{n}} x\right)$

$\Rightarrow u = \hat{u}\, e^{Rt} e^{\left(\frac{R}{\alpha}\right)^{\frac{1}{n}} x}$

let $\frac{R}{\alpha} = ik$

$u = \hat{u}\, e^{(ik)^n t} e^{ikx}$

FDE Simulation: PDE = FDE + TE

↓ solving FDE = 0

PDE = TE "Modified Eqn"

$\frac{\partial u}{\partial t} - \alpha\frac{\partial^2 u}{\partial x^2} = -\frac{(\Delta t)}{2!}\frac{\partial^2 u}{\partial t^2} + \alpha\frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\partial x^4} + o(\Delta x^4, \Delta t^2)$

? $\frac{\partial}{\partial t} = \alpha\frac{\partial^2}{\partial x^2} - \frac{(\Delta t)}{2!}\frac{\partial^2}{\partial t^2} + \alpha\frac{(\Delta x)^2}{12}\frac{\partial^4}{\partial x^4}$

$\frac{\partial^2}{\partial t^2} = \alpha\frac{\partial^4}{\partial x^4} - \frac{(\Delta t)}{2!}\frac{\partial^4}{\partial t^4} + \alpha\frac{(\Delta x)^2}{12}\frac{\partial^8}{\partial x^8}$

$\frac{\partial u}{\partial t} - \alpha\frac{\partial^2 u}{\partial x^2} = \left[-\alpha\frac{\Delta t}{2!} + \alpha\frac{\Delta x^2}{12}\right]\frac{\partial^4 u}{\partial x^4} + \frac{1}{3}o(\Delta t^2, \Delta x^4)\frac{\partial^6 u}{\partial x^6}$

Low order term — High order Term

$\alpha\frac{\Delta x^2}{2}\left[-r + \frac{1}{6}\right]$

let $r = \frac{1}{6}$ → 0

stable $r = \frac{1}{2}$, $r = \frac{1}{6}$ accurate

T.E $\sim o(\Delta t^2, \Delta x^4)$ twice accuracy

Soln. $(e^{-k^2\alpha t} + e^{k^4\beta t})e^{ikx}$

? $r = \frac{1}{6}$ FE 2nd accuracy

$N = \frac{2\pi}{\Delta x}$ $k_{max} = \frac{N}{2} \to k_{max} = \frac{2\pi}{2\Delta x}$ $\beta_{k_{max}} = k_{max}\Delta x = \pi$

$r = \alpha\frac{\Delta t}{\Delta x^2}$ $k\Delta x$

Smoothing Property $\quad u \sim e^{-\alpha k^2 t} = e^{-r\beta_k^2}$

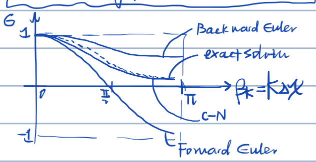
Amplification number G

$G = \frac{u^{n+1}}{u^n} = \frac{\epsilon^{n+1}}{\epsilon^n}$ $|G| \le 1$

Smoothing: $|G|$ decrease

$|G| \le 1$ means solution decrease, error decrease to get accurate error

$|G|$ decrease as β_k growing, means higher wave number (β_k) decay faster



Simple Explicit Scheme — FE.

$u_j^{n+1} = r u_{j+1}^n + (1-2r)u_j^n + r u_{j-1}^n$ $\quad r = \alpha\frac{\Delta t}{\Delta x^2}$

$\frac{u^{n+1}|_k}{u^n|_k} = 1 - 4r\sin^2\left(\frac{\beta_k}{2}\right) \to$ Smoothing property?
- Consistant
- Von-Neumann & Matrix $\to r \le \frac{1}{2}$ Conditionally Stable. (same)
- Min-Max principle: $A,B,C \ge 0$, $A+B+C \le 1$ $\to r \le \frac{1}{2}$ $r \ge 0$ $\sim$ heuristic Stability
- Speed proporgation $\frac{\Delta x}{\Delta t}$ — NOT ∞, only to one time-step

Is that means Physical World (Realworld) the proporgation have some speed limit? (already known Simulate need limit meet)

(It is ∞, but Not need that much) ?

Backward Euler: Implicit

$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \alpha\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2}$

$\to -r u_{j-1}^{n+1} + (1+2r)u_j^{n+1} - r u_{j+1}^{n+1} = u_j^n$

- Min-Max? Not easy to say
- Speed Propogation: ∞ all depend on others

$i=1$ $\quad -r u_0^{n+1} + (1+2r)u_1^{n+1} - u_2^{n+1} = u_1^n$

$$\to \begin{bmatrix} 1+2r & -r & & 0 \\ -r & \ddots & \ddots & \\ & \ddots & \ddots & \\ 0 & & & \end{bmatrix}\begin{bmatrix} u_1 \\ u_j \\ \vdots \\ u_{N-1}\end{bmatrix}^{n+1} = \begin{bmatrix} ru_1 \\ ru_j \\ \vdots \\ ru_{N-1}\end{bmatrix}^n + \begin{bmatrix} ru_0 \\ 0 \\ \vdots \\ 0 \\ ru_N\end{bmatrix}^n$$

TDMA:

$③' = ② - \frac{a_{21}}{a_{11}}① \to ③' = ② + \frac{①}{1+2r}r$

$③' = ③ - \frac{a_{32}}{a_{22}}②\ldots$

$[A]\{u^{n+1}\} = \{u^n\} + \{BC\}$

Neumann Bc $\quad \frac{u_1^{n+1} - u_0^{n+1}}{\Delta x} = \beta \to u_0^{n+1} = u_1^{n+1} - \beta\Delta x$ $\quad r u_0^n = r\beta\Delta x$

$(1+r)u_1^{n+1} - r u_2^{n+1} = u_1^n + r\beta\Delta x$

$[A]\{u^{n+1}\} = \{u^n\} + \{BC\}$
- Gauss Elimination $[A] \to [\,]$

$[A]$ NOT Tridiagonal $[A]_{N\times N} \to O(N^3)$ operations

$[A]$ Tridiagonal. $\to O(N)$ operations

PDE = FDE + TE

Solving FDE = 0 $\quad$ FDE $= \frac{u^{n+1} - u^n}{\Delta t} - \alpha\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} = 0$

ME $\quad$ PDE = TE "Modified Equation"

$\frac{\partial u}{\partial t} - \alpha\frac{\partial^2 u}{\partial x^2} = -\frac{(\Delta t)}{2!}\frac{\partial^2 u}{\partial t^2} + \alpha\frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\partial x^4} + o(\Delta t^2, \Delta x^4) \sim$ PDE

$\frac{\partial}{\partial t} = \alpha\frac{\partial^2}{\partial x^2} - \frac{(\Delta t)}{2!}\frac{\partial^2}{\partial t^2} + \alpha\frac{(\Delta x)^2}{12}\frac{\partial^4}{\partial x^4}$

$\frac{\partial^2}{\partial t^2} = \alpha\frac{\partial^4}{\partial x^4} - \frac{(\Delta t)}{2!}\frac{\partial^4}{\partial t^4} + \alpha\frac{(\Delta x)^2}{12}\frac{\partial^8}{\partial x^8}$

only $\frac{\partial}{\partial x}$ — not worried

$\frac{\partial u}{\partial t} - \alpha\frac{\partial^2 u}{\partial x^2} = \left[-\alpha^2\frac{(\Delta t)}{2!} + \alpha\frac{(\Delta x)^2}{12}\right]\frac{\partial^4 u}{\partial x^4} + \frac{1}{3}\left[\frac{1}{3}\alpha^3(\Delta t)^2 - \frac{1}{12}\alpha^2\Delta t(\Delta x)^2 + \frac{1}{360}\alpha(\Delta x)^4\right]\frac{\partial^6 u}{\partial x^6}$

$r = \frac{\alpha\Delta t}{(\Delta x)^2} \le \frac{1}{2}$ $\quad$ LOT leading order Term $\quad$ high order term

$\alpha\frac{\Delta x^2}{2}\left[-r + \frac{1}{6}\right]$ NOT 1 ?

If $r = \frac{1}{6}$ get twice accuracy

T.E $\sim o(\Delta t^2, \Delta x^4)$

$\left[u_t = \alpha u_{xx} + \beta u_{4x}\right] + \gamma u_{6x} \cdots$ (LOT, HOT)

Soln: $u = \left[e^{-k^2\alpha t} + e^{k^4\beta t}\right]e^{ikx}$

① Smoothly growing $k^4\beta$

If also doing for this, will that possible to find $r = ?$ $\quad r = \alpha\frac{\Delta t}{(\Delta x)^2}$

$\frac{1}{3}\alpha(\Delta x)^6$

u_{xx}: $\frac{\partial \hat{u}}{\partial t} = \alpha\hat{u}_{xx}$ $\quad \hat{u} = e^{-\alpha k^2 t}$

$\frac{\partial \hat{u}}{\partial t} = \alpha(-k)^2\hat{u}$ $\quad u(x,t) = e^{-\alpha k^2 t}e^{ikx}$

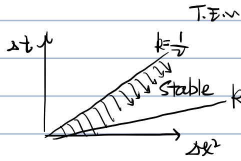
u_{3x}: $\frac{\partial \hat{u}}{\partial t} = \alpha\hat{u}_{xxx}$

$\frac{\partial \hat{u}}{\partial t} = \alpha(ik)^3\hat{u} \sim \hat{u} = e^{-ik^3\alpha t}$

u_{4x}: $\hat{u} = e^{-ik^4\alpha t}$

only t derivatives

$\frac{\partial u}{\partial t} - \alpha\frac{\partial^2 u}{\partial x^2}$



FDE Smoothing property $\quad u_t = \alpha u_{xx}$ $\quad u \sim e^{-\alpha k^2 t}e^{ikx}$

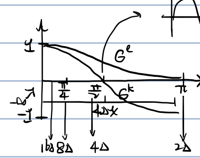
Amplification factor $G = \frac{u(t+\Delta t)}{u(t)}$ $\quad G_k^e = e^{-\alpha k^2\Delta t} = e^{-r\beta_k^2}$ Exact

$G^k = 1 + 2r(\cos\beta_k - 1)$ FDE

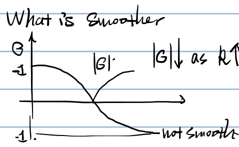
Recall $\sim$ Von-Neumann $\left|\frac{\epsilon_k^{n+1}}{\epsilon_k^n}\right| \le 1$

$\beta_k \sim k\Delta x$

$0 \sim \pi \leftarrow$ not loss information?



What is Smoother



$|G| \downarrow$ as $k \uparrow$

close Book

EXAM NOV. 2 1h 50min

Von-Neumann / Matrix Analysis. $\quad$ BE vs CE $\quad$ Even odd derivative

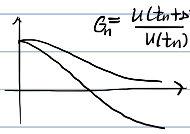
Short answer qu. $\quad$ Von Vs Matrix advantage dead. $\quad$ concise [Von | Mat] differences

FE CN NOT Smoother $r = \frac{1}{2}$

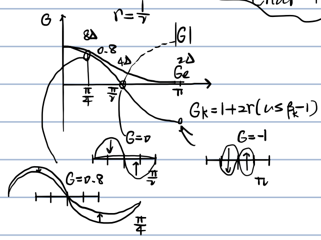
Amp factor. G_k derivative

$|G|$ NOT keep decrease

$G_n = \frac{u(t_n + \Delta t)}{u(t_n)}$



Char PDE.



Spectral Method — $T.E. \sim e^{-N}$

only simple geometry — $T.E. \sim (N^{-1})^b$

10.17.

Boundary conditions

$U_t=\alpha U_{xx}$ $\quad 0\le x\le 1$

I.C. $u(x,0)=f(x)$

B.C. $a_1\frac{\partial u}{\partial x}+b_1u=c$ at $x=0$, $a_2\frac{\partial u}{\partial x}+b_2u=c$ at $x=1$

$O[\Delta t,(\Delta x)^2]$

- Dirichlet $a_1=a_2=0$
- Neumann $b_1=b_2=0$
- Robin $a_1\frac{\partial u}{\partial x}+b_1u=c$

Boundary condition (BC)

BC: $a\frac{\partial u}{\partial x}+bu=c$

BC
- Dirichlet $a=0$ $u=c$
- Neumann $b=0$ $\frac{\partial u}{\partial x}=c$
- Robin $a\frac{\partial u}{\partial x}+bu=c$ (mixed)

How to evaluate $\frac{\partial u}{\partial x}$ at boundary?

$\left.\frac{\partial u}{\partial x}\right|_0^n=\frac{u_1^n-u_0^n}{\Delta x}+O(\Delta x)$ Forward. But not accurate

$\left.\frac{\partial u}{\partial x}\right|_0^n=\frac{-3u_0^n+4u_1^n-u_2^n}{2\Delta x}+O(\Delta x^2)$ Forward 2nd Order accuracy

But NOT triagonal

Consider $a_1\frac{\partial u}{\partial x}+b_1u=c_1$ at $x=0$ put BC. in equ.

get $a_1\frac{-3u_0^n+4u_1^n-u_2^n}{2\Delta x}+b_1u_0^n=r_1$

$u_0^n(b_1-\frac{3a_1}{2\Delta x})=c_1-\frac{a_1}{2\Delta x}(4u_1^n-u_2^n)$

$u_0^n=q_1u_1^n+q_2u_2^n+p$ — constant

$\frac{\partial u}{\partial x}$ at boundary
- ① FD NOT accurate $O(\Delta x)$
- ② FD 2nd accuracy $O(\Delta x)^2$ NOT Triagonal
- ③ Ghost point (cell)

ghost points (cell)

$\left.\frac{\partial u}{\partial x}\right|_0^n=\frac{u_1^n-u_{-1}^n}{2\Delta x}$

BC. $a_1\frac{u_1^n-u_{-1}^n}{2\Delta x}+b_1u_0^n=c_1$

governing Eqn. $\frac{u_0^{n+1}-u_0^n}{\Delta t}=\alpha\frac{u_{-1}^n-2u_0^n+u_1^n}{\Delta x^2}$

$\hookrightarrow u_0^{n+1}=r\left[u_{-1}^n-2u_0^n+u_1^n\right]+u_0^n$

$\hookrightarrow u_0^{n+1}=ru_{-1}^n+(1-2r)u_0^n+ru_1^n$

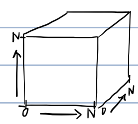
$u_0^{n+1}=2ru_1^n+(1-2r)u_0^n+2r\Delta x\frac{b_1}{a_1}u_0^n-2r\Delta x\frac{c_1}{a_1}$

Put ghost point outside Boundary $\to FD(u,u_g)=0$
use boundary condition (BC) $\to BC(u,u_g)=0$ u_g
Set ghost point out

Ex: FD for $f(x=0)$: $\frac{u_0^{n+1}-u_0^n}{\Delta t}=\alpha\frac{u_{-1}^n-2u_0^n+u_1^n}{\Delta x^2}$ $\} u_0^{n+1}, u_{-1}^n$

BC for $f(x=0)$: $a\frac{u_0^n-u_{-1}^n}{\Delta t}+bu_0=c$

Local / Global Error



interior $O(\Delta^m)$ — local error?

boundary $O(\Delta^P)$

total error $\mathcal{E}_{tot}\sim\alpha N^3\Delta^m+\beta N^2\Delta^P$

Average $\frac{\mathcal{E}_{tot}}{N^3}=\alpha\Delta^m+\frac{\beta}{N}\Delta^P$ — Is that mean with larger N, $\mathcal{E}_{Ave}\downarrow$?

$N\sim\Delta^{-1}$, $\bar{\mathcal{E}}=\alpha\Delta^m+\beta\Delta^{P+1}$ — means boundary can drop 1th order accuracy than interior

As long as $P+1=m$, global error will stay $O(\Delta^m)$

$P\ge m-1$

Global Error $=\sum$ local Error

$\mathcal{E}_{total}=\alpha N^3 O(\Delta^m)+\beta N^2 O(\Delta^P)$ (interior $O(\Delta^m)$, Boundary $O(\Delta^P)$)

$\bar{\mathcal{E}}_{average}=\frac{\mathcal{E}_{tot}}{N^3}=\alpha O(\Delta^m)+\beta\frac{O(\Delta^P)}{N}$

Boundary can accept 1 lower accuracy

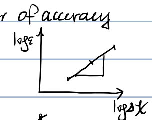
$P+1\ge m$

$P\ge m-1$

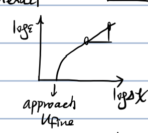
Numerically calculate order of accuracy

If exact soln. available

$\mathcal{E}=u-u_{exact}$



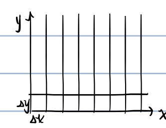
$\mathcal{E}=u-u_{fine}$



u fine? tend?

Multi-Dimensional problems

2D: $U_t=\alpha(U_{xx}+U_{yy})$



Simple explicit

$u_{j,k}^{n+1}=u_{j,k}^n+r_x(u_{j-1,k}^n-2u_{j,k}^n+u_{j+1,k}^n)+r_y(u_{j,k-1}^n-2u_{j,k}^n+u_{j,k+1}^n)$

$r_x=\alpha\frac{\Delta t}{\Delta x^2}$ $\quad r_y=\alpha\frac{\Delta t}{\Delta y^2}$ $\quad T.E.\sim O[\Delta t,(\Delta x)^2,(\Delta y)^2]$

Stability Analysis

$u_{j,k}^n=e^{at}e^{ik_xx}e^{ik_yy}$

$G_{k_x,k_y}=1+2r_x(\cos\beta_x-1)+2r_y(\cos\beta_y-1)\longrightarrow G=1-4r_x\sin^2\frac{\beta_x}{2}-4r_y\sin^2\frac{\beta_y}{2}$

$\beta_x=k_x\Delta x$ $\quad \beta_y=k_y\Delta y$

Stability $|G|\le 1$

$\Rightarrow 4r_x\sin^2\frac{\beta_x}{2}+4r_y\sin^2\frac{\beta_y}{2}\le 2$

$r_x+r_y\le\frac{1}{2}$

$\Rightarrow\alpha\Delta t\left[\frac{1}{(\Delta x)^2}+\frac{1}{(\Delta y)^2}\right]\le\frac{1}{2}$

$r_x=r_y=r\to r\le\frac{1}{4}$

1D: $r\le\frac{1}{2}$

2D: $r\le\frac{1}{4}$

3D: $r\le\frac{1}{6}$ $\quad(r_x(\ )+r_y(\ )+r_z(\ ))$

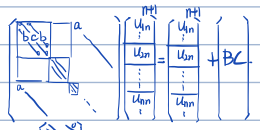
$\delta(\ )=(\ )_{r-1}-2(\ )_r+(\ )_{r+1}$

10.24.

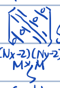
2D Heat Transfer

central operator for x, y

CN: $\frac{u_{ij}^{n+1}-u_{ij}^{n}}{\Delta t}=\alpha\left[\frac{\delta_x^2}{\Delta x^2}+\frac{\delta_y^2}{\Delta y^2}\right]\frac{u_{ij}^{n+1}+u_{ij}^{n}}{2}$



+BC



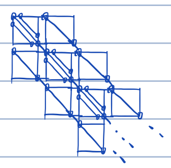
Sparse Matrix 稀疏矩阵

$(N_x-2)(N_y-2)$

$M\times M$

Can't use TDMA

Solve: ① Matrix Inversion ② Gauss Elimination } $O(M^3)$ ③ Carmer's Rule ~



(ADI) Alternative Direction Implicit

$u^n \xrightarrow{\frac{\Delta t}{2}} u^{n+\frac12} \xrightarrow{\frac{\Delta t}{2}} u^{n+1}$

$\frac{u_{ij}^{n+\frac12}-u_{ij}^{n}}{(\frac{\Delta t}{2})}=\alpha\left[\frac{\delta_x^2}{(\Delta x)^2}u_{ij}^{n+\frac12}+\frac{\delta_y^2}{(\Delta y)^2}u_{ij}^{n}\right]$

$\frac{u_{ij}^{n+1}-u_{ij}^{n+\frac12}}{(\frac{\Delta t}{2})}=\alpha\left[\frac{\delta_x^2}{(\Delta x)^2}u_{ij}^{n+\frac12}+\frac{\delta_y^2}{(\Delta y)^2}u_{ij}^{n+1}\right]$

✓ 2D: unconditionally stable

! 3D: conditionally stable

Approximate Factorization

$\Delta u_{ij}=u_{ij}^{n+1}-u_{ij}^{n}$

$u_{ij}^{n+1}=u_{ij}^{n}+\Delta u_{ij}$

CN

$\frac{u_{ij}^{n+1}-u_{ij}^{n}}{\Delta t}=\alpha\left[\frac{\delta_x^2}{(\Delta x)^2}+\frac{\delta_y^2}{(\Delta y)^2}\right]\frac{u_{ij}^{n+1}+u_{ij}^{n}}{2} \Rightarrow \Delta u_{ij}=\frac{\alpha\Delta t}{2}\left[\frac{\delta_x^2}{(\Delta x)^2}+\frac{\delta_y^2}{(\Delta y)^2}\right](2u_{ij}^{n}+\Delta u_{ij})$ ✓CN

ij: $r_x=\frac{\alpha\Delta t}{\Delta x^2}$ $r_y=\frac{\alpha\Delta t}{\Delta y^2}$ $\Delta u=\frac12[r_x\delta_x^2+r_y\delta_y^2](2u+\Delta u)$

$[1-\frac{r_x}{2}\delta_x^2-\frac{r_y}{2}\delta_y^2]\Delta u=[r_x\delta_x^2+r_y\delta_y^2]u^n$

$\frac{r_x r_y}{4}\delta_x^2\delta_y^2+(1-\frac{r_x}{2}\delta_x^2)(1-\frac{r_y}{2}\delta_y^2)\Delta u$ Approximate Factorization

$(1-\frac{r_y}{2}\delta_y^2)\Delta u=\Delta u^*$

$(1-\frac{r_x}{2}\delta_x^2)\Delta u_{ij}^*=[r_x\delta_x^2+r_y\delta_y^2]u_{ij}^n$

Laplace

Elliptic Eqns — Steady (without time)

① Min-Max principle → Homogenous ONLY

② Eqn. require 1 only condition on BC

③ Dirichelet BC $u=c$ (BC) → Sdn Exist & Unique

2D Heat Transfer

δ Central Difference Operator

$\delta^2 u_p=u_{p-1}-2u_p+u_{p+1}$

$\frac{\partial u}{\partial t}=\alpha(u_{xx}+u_{yy})$

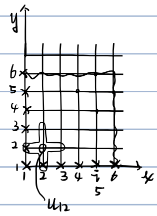
CN: $\frac{u_{ij}^{n+1}-u_{ij}^{n}}{\Delta t}=\alpha\left[\frac{\delta_x^2}{\Delta x^2}+\frac{\delta_y^2}{\Delta y^2}\right]\frac{u_{ij}^{n+1}+u_{ij}^{n}}{2}$

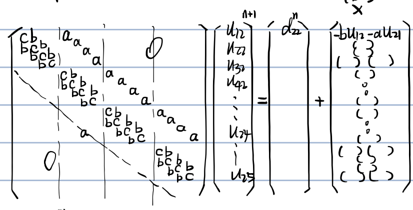
$a\,u_{i,j-1}^{n+1}+b\,u_{i-1,j}^{n+1}+c\,u_{i,j}^{n+1}+b\,u_{i+1,j}^{n+1}+a\,u_{i,j+1}^{n+1}=d_{ij}^n$

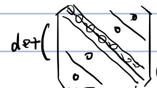
where, $a=-\frac{\alpha\Delta t}{2\Delta y^2}=-\frac{r_x}{2}$ $\quad d_{ij}^n=u_{ij}^n+\frac{\alpha\Delta t}{2}\left[\frac{\delta_x^2}{\Delta x^2}+\frac{\delta_y^2}{\Delta y^2}\right]u_{ij}^n$

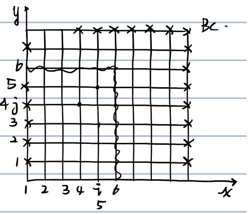
$b=-\frac{\alpha\Delta t}{2\Delta x^2}=-\frac{r_y}{2}$

$C=1+r_x+r_y$









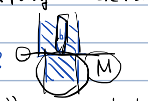
NOT Tri-diagonal

BUT Sparse Matrix (only connected on same line)

$[A]\{u^{n+1}\}=\{u^n\}+\{bc^n\}$

$M\times M$ · 10×10

By the way is $|A|=C^n$?



Matrix Inversion, Gauss Elimination — $O(M^3)$

Cramer's Rule — $O(M+1)!$

② (TDMA ~ $O(M)$) Clock Rate 5GHz.

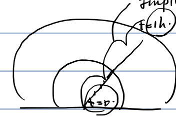
Add/subtract 10

Multi. 20

Divide ~ Most Expensive. 5r.

1f/1000.

Implicit can Jump Bigger Δt.



New Method.

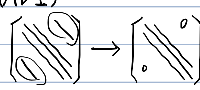
Alternative Direction Implicit (ADI)

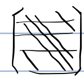
Peaceman & Rankford 1955

Douglas. 1955

1955

$u^n \xrightarrow{\frac{\Delta t}{2}} u^{n+\frac12} \xrightarrow{\frac{\Delta t}{2}} u^{n+1}$





$\frac{u^{n+1}-u^n}{\Delta t}$ $\quad n\to n+\frac12$

$\frac{u_{ij}^{n+\frac12}-u_{ij}^{n}}{\frac{\Delta t}{2}}=\alpha\left[\frac{\delta_x^2}{(\Delta x)^2}u_{ij}^{n+\frac12}+\frac{\delta_y^2}{(\Delta y)^2}u_{ij}^{n}\right]$ (N_x-2)TDMA

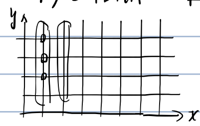
Implicit, Explicit

Ny-2 TDMA

$n+\frac12\to n+1$ explicit

$\frac{u_{ij}^{n+1}-u_{ij}^{n+\frac12}}{\frac{\Delta t}{2}}=\alpha\left[\frac{\delta_x^2}{(\Delta x)^2}u_{ij}^{n+\frac12}+\frac{\delta_y^2}{(\Delta y)^2}u_{ij}^{n+1}\right]$ (N_y-2)TDMA

Ex, Im



other	$(N_x N_y)^3$
ADI	$(N_x N_y)^1$

✓ 2D: unconditionally stable $\sim O(N_x N_y)^1$.

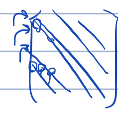
! 3D: $u_t=\alpha(u_{xx}+u_{yy}+u_{zz})$ conditionally stable only

0 ~ 1/3 ~ 2/3 ~ 1: Ex, Im, Ex. 2 Explicit, 1 Implicit.

0 ~ 1/3: $\frac{\Delta u_t}{\Delta t/3}=\alpha(u_{xx}^{n+\frac13}+u_{yy}^{n}+u_{zz}^{n})$

1/3 ~ 2/3: $\frac{\Delta u_t}{\Delta t/3}=\alpha(u_{xx}^{n}+u_{yy}^{n+\frac23}+u_{zz}^{n+\frac23})$

2/3 ~ 1: $\frac{\Delta u_t}{\Delta t/3}=\alpha(u_{xx}^{n+1}+u_{yy}^{n+1}+u_{zz}^{n})$



1964 — Douglas & Gunn.

① fix conditionally stable

② fix $(\Delta x)^2$

Approximate Factorization

$\Delta u_{ij}=(u_{ij}^{n+1}-u_{ij}^{n})$

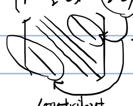
replace $u_{ij}^{n+1}=\Delta u_{ij}+u_{ij}$

$\Delta u_{ij}=\frac{\alpha\Delta t}{2}\left[\frac{\delta_x^2}{(\Delta x)^2}\Delta u_{ij}+\frac{\delta_y^2}{(\Delta y)^2}\Delta u_{ij}+\frac{2\delta_x^2}{(\Delta x)^2}u_{ij}^n+\frac{2\delta_y^2}{(\Delta y)^2}u_{ij}^n\right]$ Crank-Nicolson.

$r=\frac{\alpha\Delta t}{\Delta x^2}$

$[1-\frac{r_x}{2}\delta_x^2-\frac{r_y}{2}\delta_y^2]\{\Delta u_{ij}\}=[r_x\delta_x^2+r_y\delta_y^2]\{u_{ij}\}$

Cross-product term $O(\Delta t)^3$



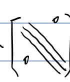
contribute to

$[1-\frac{r_x}{2}\delta_x^2][1-\frac{r_y}{2}\delta_y^2]\{\Delta u_{ij}\}=[r_x\delta_x^2+r_y\delta_y^2]\{u_{ij}^n\}+\left[\cancel{\frac{r_x r_y}{4}\delta_x^2\delta_y^2}\right]\{\Delta u_{ij}\}$ $O(\Delta t)$

Approximate Factorization

let $[1-\frac{r_y}{2}\delta_y^2]\{\Delta u_{ij}\}=\{\Delta u_{ij}^*\}$

$[1-\frac{r_x}{2}\delta_x^2]\{\Delta u_{ij}^*\}=[r_x\delta_x^2+r_y\delta_y^2]\{u_{ij}^n\}$ ~



$\frac{\partial u}{\partial t}+\vec u\cdot\nabla\vec u=-\nabla p+\frac{1}{Re}\nabla^2\vec u$

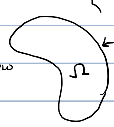
Source Terms

Fractional-Step Method.

Elliptic Equations.

Steady-state heat cond.

potential flow



Laplace Eqn. $\nabla^2 u=0$

Possion Eqn. $\nabla^2 u=f(x)$

Helmhotz Eqn. $\nabla^2 u+\alpha u=g(x)$

$[\nabla^2(\nabla^2 u)=0]$ — Equiblium Stress in Solids.

① Min/Max property for Homogenous. Only.

② Eqn. require 1 only condition every point of boundary close boundary

③ Dirichlet BC → Sdn Exist & Unique

$\nabla\cdot\left[\cancel{\frac{\partial u}{\partial t}}+\vec u\cdot\nabla\vec u=-\nabla p+\cancel{\frac{1}{Re}\nabla^2\vec u}\right]$ $\nabla\cdot\vec u=0$

$\nabla\cdot(\vec u\cdot\nabla\vec u)=-\nabla^2 p$

Pressure Possion Eqn.

10.31 week 9. Neumann BC. Exist → Non-unique $\frac{\partial P}{\partial n} = ?$

Ex. Possion Eqn

$\nabla^2 u = f(x)$ in Ω — source

flux $\frac{\partial u}{\partial n} = g(x)$ on $\partial\Omega$

$\int_\Omega \nabla\cdot(\nabla u)\,dV = \int_\Omega f(x)\,dx$

div therom

$\int_{\partial\Omega} \vec{n}\cdot\nabla u\, d\sigma = \left[\int_\Omega f(x)\,dx = \int_{\partial\Omega} g(x)\,dx\right]$ compatibility condition → Soln exist $u(x) + C$. NOT unique.

BC

total generated inside = flux out → Elliptic (without time)

$\nabla^2\phi = 0$

$n\cdot\nabla\phi = 0$. No generation. (net flux)

not time but pseudo time

$\frac{\partial u}{\partial \tau} = \nabla^2 u - f(x)$

solve till steady state

@ steady st. $\nabla^2 u = f(x)$

Consider 2D Laplace Eqn.

$u_{xx} + u_{yy} = 0$ with BC

2nd Order CD Scheme

$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta y^2} = 0$

T.E. $\frac{\Delta x^2}{12} u_{xx} + \frac{\Delta y^2}{12} u_{yy} + \text{HoT}$.

$[A]\{u\} = \{b\}$

Neumann. ⟷ Dirichlet

Singular Matrix ! Unique. Not singular

on $J = 0$

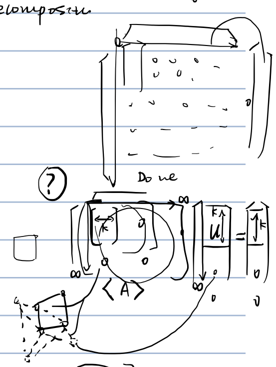
SVD Singular value decomposition

$[A \times A^{-1} = I] + \varepsilon$

?

Do we

$\langle A \rangle$



If Neumann BC

$\frac{\partial u}{\partial n} = g$

$\frac{\partial u}{\partial x} = g_{1,2}$

$\frac{u_{2,2} - u_{1,2}}{\Delta x} = g_{12} \rightarrow u_{1,2} = u_{2,2} - g_{12}\Delta x$

$u_{1,2} = u_{22} - g_{12}\Delta x$

for $i=2, j=2$: $\frac{\cancel{u_{i-1,j}} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta y^2} = 0$

Dirchlet

$[A]\{u\} = \{b\}$ $[A]^{n\times n} \rightarrow |\lambda_1| < \dots < |\lambda_n|$

Condition number $K = \frac{|\lambda_1|}{|\lambda_n|} \sim 10^{\alpha} \sim 10^{32}$? physically.

λ_n largest wave number

λ_1 smallest wave number

① consistancy. approach to exact. ⓥ $K(N)$ Number of grid points. $N\uparrow \Rightarrow K\uparrow \Rightarrow K\uparrow\uparrow$

ill-condition number

②stability ? — Round-off error

③ convergeny

11.7. week 10

$N \begin{bmatrix} \diagdown \end{bmatrix}_N [u] = [\text{RHS}] + [\text{BC}]$

Direct Methods: Cramer's Rule — O(N+1)!

Gauss Elimination — $O(N^3)$ $[A][u] = [b]$ row/column opn. → $[\,][u] = [b']$ → Back substitution

But if $\nabla^2 u = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ \dots \end{bmatrix}$ Multiple Right hand side?

Matrix Factorization

LU Factorization $[A]\{x\} = \{b\}$ Incomplete LU

$O(N^3) \sim [L][U] = [A]$

$O(N^2)$ store

$[L][U]\{x\} = \{b\}$

$[L]\{b^*\} = \{b\}$

No need to know Right hand Side ☆ «Numerical Recipes»

$O(N^3) \longrightarrow O(M^9)$

$N \sim M_x M_y M_z$

Ex.

$\nu_a = 10^{-5}$

$\nu_w = 10^{-6}$

$Re = \frac{UD}{\nu} \sim 1000$.

? $[\diagdown] \rightarrow [\diagdown] + [\diagdown] + [\diagdown]$

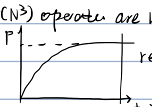
? $O(N)$. Best we can do.

Iterative Method

Vast number of $O(N^3)$ operation are multiplied by "zero"

$\nabla^2 P = S$ reach elliptic

add time derivative → $\frac{\partial P}{\partial t} = \nabla^2 P - S$



Forward Euler, central difference (STCS) in time in space

$\frac{P_{ij}^{n+1} - P_{ij}^n}{\Delta t} = \frac{1}{\Delta^2}\left[P_{i+1,j}^n - 2P_{i,j}^n + P_{i-1,j}^n + P_{i,j-1}^n - 2P_{ij}^n + P_{i,j+1}^n\right] - S_{ij}$ — Source term.

$P_{ij}^{n+1} = P_{ij}^n + \frac{\Delta t}{\Delta^2}\left[P_{i-1,j}^n + P_{i+1,j}^n + P_{i,j-1}^n + P_{i,j+1}^n - 4P_{ij}^n\right] - S_{ij}\Delta t$

temporal accuracy not revelant

Try to move with large Δt 2D: $r \le \frac{1}{4}$

It is aready observed. as $t\uparrow$ Error↑. how to prove

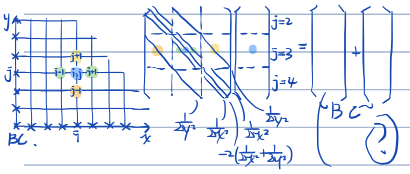
soln. as exact when t very large to reach steady.

α?

2D Laplace Eqn. $u_{xx} + u_{yy} = 0$

$\frac{\delta_x^2}{\Delta x^2} u_{ij} + \frac{\delta_y^2}{\Delta y^2} u_{ij} = 0$

$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta y^2} = 0$



Source Term

Dirchlet BC.

$[A]\{u\} = \{b\}$

$[A] \rightarrow \lambda_1 < \dots < \lambda_n$ (λ_{min}, λ_{max})

condition number $K = \left|\frac{\lambda_{max}}{\lambda_{min}}\right| \sim 10^{\alpha}$

? $K > \alpha$. λ_1 or λ_n

Matrix Factorization (source)

$N\begin{bmatrix}\diagdown\end{bmatrix}_N [u] = [\text{RHS}] + [\text{BC}]$

$[A]\{x\} = \{b\}$ $[A] = [L][U]$ $O(N^3)$

$[L][U]\{x\} = \{b\}$

$[L]\{b^*\} = \{b\}$

still $O(N^3)$

O(N) Iterative Method

$\nabla^2 P = S$ Elliptic Eqn. ← Steady

add time. → $\frac{\partial P}{\partial t} = \nabla^2 P - S$

FE CD

$\frac{P_{ij}^{n+1} - P_{ij}^n}{\Delta t} = \frac{1}{\Delta^2}\left[P_{i+1,j}^n - 2P_{ij}^n + P_{i-1,j}^n + P_{i,j-1}^n - 2P_{ij}^n + P_{i,j+1}^n\right] - S_{ij}$

NOT need temporal accuracy

Δt larger.

$\frac{\Delta t}{\Delta^2} = r \le \frac{1}{4}$

$\Delta t_{max} = \frac{\Delta^2}{4}$

$P_{ij}^{n+1} - \cancel{P_{ij}^n} = \frac{1}{4}\left[P_{i+1,j}^n - \cancel{2P_{ij}^n} + P_{i-1,j}^n + P_{i,j-1}^n - \cancel{2P_{ij}^n} + P_{i,j+1}^n\right] - S_{ij}\Delta t$

$P_{ij}^{n+1} = \frac{1}{4}\left[P_{i-1,j}^n + P_{i+1,j}^n + P_{i,j-1}^n + P_{i,j+1}^n\right] - S_{ij}\Delta t$

Jacobi Method

$\frac{1}{4} \times \rightarrow ④$

$S_{ij}\times\Delta t \rightarrow ①$ $O(6N)$

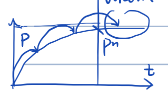
$-S_{ij}\Delta t \rightarrow ①$

when?

R^n residual

$[A]\{P\} = \{S\}$

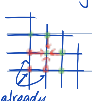
$[A]\{P^n\} = \{S^n\}$ $\|S - S^n\| \le \epsilon$



Two times faster

Gauss-Sedial

$P_{ij}^{n+1} = \frac{1}{4}[P_{i-1,j}^{n+1} + P_{i+1,j}^{n} + P_{i,j-1}^{n+1} + P_{i,j+1}^{n}] - S_{ij}\Delta t$



Generalized Iteration Method

$[A] = [L] + [D] + [U]$

Lower Triangle, Diagonal, Upper Triangle

$[A]\{P\} = \{S\}$

$[L+U+D]\{P\} = \{S\}$

Jacobi $[L+U]\{P^k\} + [D]\{P^{k+1}\} = \{S\}$

$[D]\{P^{k+1}\} = \{S\} - [L+U]\{P^k\}$

Gauss-Sedial

$[L+D]\{P^{k+1}\} + [U]\{P^k\} = \{S\}$

$[D]\{P^{k+1}\} = \{S\} - \underbrace{[L]\{P^{k+1}\}}_{\text{already known}} - [U]\{P^k\}$

General Splitting

$([A] = [M] - [N])\{P\} = \{S\}$

$[M]\{P^{k+1}\} = [N]\{P^k\} + \{S\}$

$\{P^{k+1}\} = \underbrace{[M]^{-1}[N]}\{P^k\} + [M]^{-1}\{S\}$

$= [Q]\{P^k\} + [M]^{-1}\{S\}$

Iteration Matrix

Convergence ($P(FDE) \xrightarrow{\Delta\to 0} P(PDE)$?)

FDE: $[M]\{P^{k+1}\} = [N]\{P^k\} + \{S\}$

PDE: $[M]\{P_e\} = [N]\{P_e\} + \{S\}$

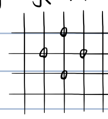
$[M]\{\epsilon^{k+1}\} = [N]\{\epsilon^k\}$

Convergence $\left|\frac{\{\epsilon^{k+1}\}}{\{\epsilon^k\}} = \underbrace{[M]^{-1}[N]}_{[Q]}\right| < 1$

Max $\Delta t = \frac{\Delta^2}{4}$

$P_{ij}^{n+1} = \cancel{P_{ij}^n} + \frac{1}{4}[P_{i-1,j}^n + P_{i+1,j}^n + P_{i,j-1}^n + P_{i,j+1}^n - \cancel{4P_{ij}^n}] - S_{ij}\frac{\Delta^2}{4}$ Jacobi Method

Iterative Method



$m \times m = N$

$O(bN)$ bN operations $P_{ij}^{n+1} = \cancel{P_{ij}^n} + \frac{1}{4}[P_{i-1,j}^n + P_{i+1,j}^n + P_{i,j-1}^n + P_{i,j+1}^n - \cancel{4P_{ij}^n}] - S_{ij}\frac{\Delta^2}{4}$

$P \to P_{steady} + \epsilon$

Initial condition P^0 ?

$\nabla^2 P - S = R_{exact}$

how many $K(=n)$ do we need? To converge to steady state

$\|P^k - P_{steady}\| \le \epsilon$ then converged

$\|P^{k+1} - P^k\| \le \epsilon$

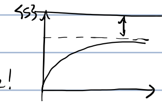
or Check Residual $\nabla^2 P = S$

$[A]\{P\} = \{S\}$

$[A]\{P^k\} = \{S^k\}$ $\to \{S\} - \{S^k\} = \{R^k\}$ Residual

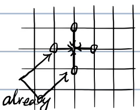
$\|R^k\| \le \epsilon$

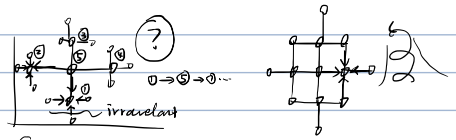
Make sure not reach wrong steady state!



Gauss-Sedial

$P_{ij}^{n+1} = \cancel{P_{ij}^n} + \frac{1}{4}[P_{i-1,j}^{n+1} + P_{i+1,j}^n + P_{i,j-1}^{n+1} + P_{i,j+1}^n - \cancel{4P_{ij}^n}] - S_{ij}\frac{\Delta^2}{4}$ two times faster than Jacobi





11.9

General Iterative Method

(Splitting)

$A = \begin{bmatrix} \diagdown & U \\ L & D\ \diagdown \end{bmatrix} = [L] + [D] + [U]$

$[A]\{P\} = \{S\}$

$([L] + [U] + [D])\{P\} = \{S\}$

Jacobi Method

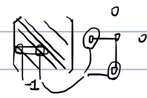
$[L + D + U]\{P^{k+1}\}$

$[L+U]\{P^k\} + [D]\{P^{k+1}\} = \{S\}$ approximation

$[D]\{P^{k+1}\} = \{S\} - [L+U]\{P^k\}$

D^{-1}

Gauss-Sedel



$[L+D]\{P^{k+1}\} = \{S\} - [U]\{P^k\}$

$[D]\{P^{k+1}\} = \{S\} - \underbrace{[L]\{P^{k+1}\}}_{\text{already known}} - [U]\{P^k\}$

$\{P^{k+1}\} = -[D]^{-1}[L]\{P^{k+1}\} - [D]^{-1}[U]\{P^k\} + [D]^{-1}\{S\}$

General Splitting

$[A] = [M] - [N]$

Iterative scheme.

$[M]\{P^{k+1}\} = [N]\{P^k\} + \{S\}$

$\{P^{k+1}\} = [M]^{-1}[N]\{P^k\} + [M]^{-1}\{S\}$

$= [Q]\{P^k\} + [M]^{-1}\{S\}$

$[Q]$ Iteration Matrix

Jacobi $\to [Q] = [D]^{-1}[L+U]$

G-S $\to [Q] = [L+D]^{-1}[U]$

$ax^{k} + bx + c = d$

x^{k+1} x^k

x^k x^{k+1} ?

$[\nabla^2 P - S] = \{Re\}$

$\|Re\| \le \epsilon$

Convergence $[A]\{P\} = \{S\}$

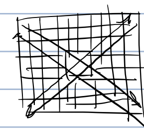
let $\{Pe\} \to$ exact soln.

$[M]\{P^{k+1}\} = [N]\{P^k\} + \{S\}$

$[M]\{P_e\} = [N]\{P_e\} + \{S\}$ $\ominus \to [M]\{\epsilon^{k+1}\} = [N]\{\epsilon^k\}$

$\{\epsilon^{k+1}\} = [M]^{-1}[N]\{\epsilon^k\}$

$\{\epsilon^{k+1}\} = [Q]\{\epsilon^k\}$



Convergence of Iterative Method $\frac{\|\epsilon^{k+1}\|}{\|\epsilon^k\|} < 1$ $|\lambda(Q)| < 1$

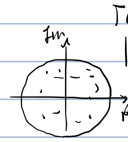
$[Q] \to \lambda_1 \dots \lambda_n$

$|\lambda| < 1$ for all $i = 1, N$

$|\lambda| \to$ Spectral Radius

$\frac{\{\epsilon^{k+1}\}}{\{\epsilon^k\}} = [Q]$

? What is imaginary part meaning

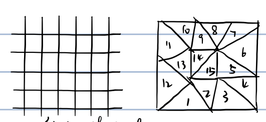


$\rho(Q) < 1$ $N \cdot T = 1$

for convergence

also tell how fast convergence

? smallest $\rho(Q)$ pick $[M]$ $[N]$?



$[A]=[M]-[N]$

1) $[M]\{z\}=\{g\}$ → easy to solve (easier $[A]\{x\}=\{b\}$)

2) $[M]^{-1}[N]=[Q]$, $\rho(Q)<1$

If $[\,\cdot\,]=\underbrace{[\,\cdot\,]}_{[M]}+\underbrace{[\,\cdot\,]}_{[N]}$

Young (1950)

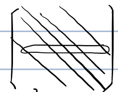
Jacobi $[Q_J]=[D]^{-1}[L+U]$

$\rho(Q_J)<1$

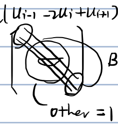
$[A]_{n\times n}\to a_{ij}$

one sufficient condition: diagonal dominance → $\underset{1\le i\le n}{\text{Max}}\ \frac{1}{|a_{ii}|}\sum_{\substack{j=1\\ j\ne i}}^{N}|a_{ij}|\le 1$ diagonal part dominate Non-diag-part

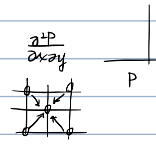
and strictly <1 for at least one i



$\frac{\partial^2 P}{\partial x^2}=\frac{1}{\Delta x^2}(u_{i-1}-2u_i+u_{i+1})$



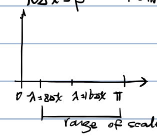
$\frac{\partial^2 P}{\partial x\partial y}$



$[A]\{x\}=\{b\}$

$\left\|\frac{\lambda_1}{\lambda_N}\right\|=\frac{\beta_{max}}{\beta_{min}}$

$k\Delta x=\beta$



11.14

Jacobi → Diagonal Dominance → converge

Gau

Jacobi → Sufficient

Condition is diagonal dominance

$P^{n+1}=([Q]=[D]^{-1}[L+U])P^n$

Gauss-Seidel

$[A]\{P\}=\{b\}$

$[L+D]\{P^{k+1}\}=-[U]\{P^k\}+\{b\}$

G-S convergence for all 正定矩阵 ~~positive definite~~ $[A]$ sufficient Proof G.D. Smith Numerical Solns for PDEs Finite difference methods

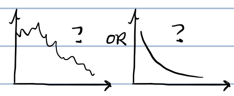
$\{x\}^T[A]\{x\}>0$ for all $\{x\}$

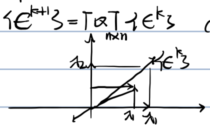
Diagonal Dominance

Sufficient for G-S converge (More easier to converge than Jacobi)

Asymptotic Rate to Converge

$\{\epsilon^{k+1}\}=[Q]_{n\times n}\{\epsilon^k\}$ can $[Q]\{\epsilon\}=\lambda_i v_i$?





$[Q]$

order of eig-values

$|\lambda_1|\ge|\lambda_2|\ldots\ge|\lambda_n|$

$\{\epsilon^0\}=\sum_{m=1}^{n}C_m\{v_m\}$

$\{\epsilon^1\}=[Q]\{\epsilon^0\}=\lambda\sum_{m=1}^{n}C_m\{v_m\}$

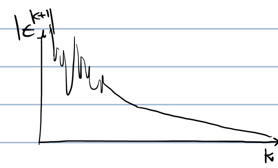
$\{\epsilon^{k+1}\}=\lambda_1^{k+1}C_1\{v_1\}\ldots=\sum C_m\lambda_m^{k+1}\{v_m\}$

$=\lambda_1^{k+1}[C_1\{v_1\}+C_2(\frac{\lambda_2}{\lambda_1})^{k+1}C_2\{v_2\}\ldots]$

let $k\gg1$

$\{\epsilon^{k+1}\}=\lambda_1^{k+1}C_1\{v_1\}$ Asymptotic converge

Spectral Radius $\rho(Q)$



$\|\{\epsilon^{k+1}\}\|=\lambda_1\|\{\epsilon^k\}\|$ → asymptotic

$=\rho(Q)\|\{\epsilon^k\}\|$

number of decimal places of error reduction per iteration $=\log_{10}(\|\epsilon^k\|)-\log_{10}(\|\epsilon^{k+1}\|)$

$=-\log_{10}[\rho(Q)]$

Suppose we want to reduce error by 10^{-9}

then number of iteration p require $P\ge\frac{9}{-\log_{10}[\rho(Q)]}$

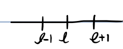
Matrix or Von-Neumann

$P^n_{\ell m}\sim e^{an}e^{ik_x x_\ell}e^{ik_m y_m}$

G^n

Amp factor.

Ex. $\frac{\partial^2 P}{\partial x^2}=0$ $0\le x\le 2\pi$



Jacobi

$P_\ell^{n+1}=\frac{P^n_{\ell-1}+P^n_{\ell+1}}{2}$

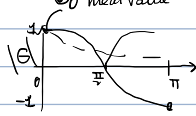
$\epsilon_\ell^n=G^n e^{ik x_\ell}$

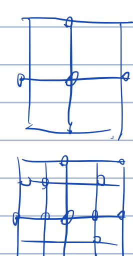
$G^{n+1}e^{ik\ell\Delta x}=G^n\frac{e^{ik(\ell+1)\Delta x}+e^{ik(\ell-1)\Delta x}}{2}$

$G=\cos k\Delta x=\cos\beta$

$|G|<1$

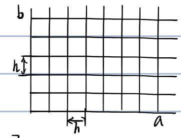
$\mathcal{O}$ mean value.





11.16

$\nabla^2 p = 0$ 2^{nd} order central Difference



Jacobi: (Matrix stability)

$|\lambda|_{max} = \frac{1}{2}[\cos\frac{\pi h}{a} + \cos\frac{\pi h}{b}]$ how long can it converge?

Small h: $|\lambda|_{max} \simeq 1 - \frac{1}{4}[\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}]h^2 = 1 - \alpha h^2$

$-\log_{10}[\rho(Q)] = -\log_{10}[1-\alpha h^2] \simeq \alpha h^2 - O(h^4)$

$= \frac{1}{4}[\frac{1}{a^2} + \frac{1}{b^2}]\pi^2 h^2 = \frac{1}{4}[\frac{1}{N_x^2} + \frac{1}{N_y^2}]\pi^2$

$\frac{a}{h} = N_x$, $\frac{b}{h} = N_y$

If $N_x = N_y = N$. $= \frac{1}{2}\frac{\pi^2}{N^2}$

$\frac{1}{b}$

$P \simeq \frac{+q}{-\log_{10}[\rho(Q)]} = \frac{2N^2}{\pi^2} q$ $\sim N^2$: Total # grid points ($M = N^2$)

4 pt. average for each pt.

FLOP/Iteration $4N^2$.

Total FLOP M^2

Gauss-Seidel: $|\lambda|_{max} = \frac{1}{4}[\cos\frac{\pi h}{a} + \cos\frac{\pi h}{b}]^2$

$P \simeq \frac{N^2}{\pi^2} q$ — $2\times$ for Jacobi

Gauss-Seidel 2 times faster

$O(M^3)$ direct.

$O(M^2)$ Iteration Method.

$O(M)$ – Multigrid.

Successive Overelaxation Method.

SOR Method Successive Overelaxation

$P^{k+1} = P^k + \omega(P^* - P^k)$

$P^*_{i,j} = [P_{i-1,j} + P_{i+1,j} + P_{i,j-1} + P_{i,j+1}]\frac{1}{4} - \frac{\Delta^2}{4}S_{ij}$

$\omega = \begin{cases} (0,1) & \text{under relaxation} \\ (1,2) & \text{over relaxation} \end{cases}$

Jacobi SOR

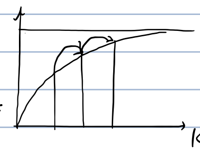
2D Jacobi: $P^*_{ij} = \frac{1}{4}[P^k_{i+1,j} + P^k_{i-1,j} + P^k_{i,j-1} + P^k_{i,j+1}]$

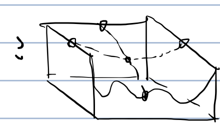
$(P^{n+1}_{ij} - P^n_{ij})$ step size

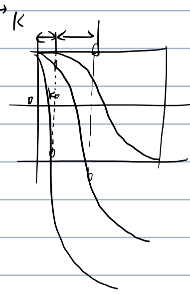
SOR: $P^{n+1}_{ij} = P^k_{ij} + \omega(P^*_{ij} - P^k_{ij}) = \omega P^* + (1-\omega)P^k$

$\{$ $\omega > 1$. Overrelaxation

$\omega < 1$ underrelaxation

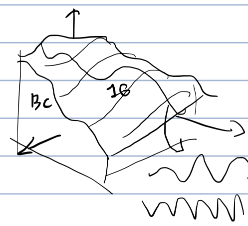






not multiply
But plus ②

Fourier → k.



Jacobi SOR

$P^{k+1} = (1-\omega)P^k + \omega P^*$

$P^* = \frac{1}{4}[P^k_{i,j-1} + P^k_{i,j+1} + P^k_{i-1,j} + P^k_{i+1,j}] - \frac{\Delta^2}{4}S_{ij}$

Gauss-Seidel SOR

$P^{k+1}_{ij} = (1-\omega)P^k_{ij} + \omega P^*$

$P^* = \frac{1}{4}[P^{k+1}_{i,j-1} + P^k_{i,j+1} + P^{k+1}_{i-1,j} + P^k_{i+1,j}] - \frac{\Delta^2}{4}S_{ij}$

Gauss-Seidel SOR

$P^*_\ell = \frac{P^k_{\ell+1} + P^k_{\ell-1}}{2}$

$P^{k+1}_\ell = P^k_\ell + \omega(P^*_\ell - P^k_\ell)$

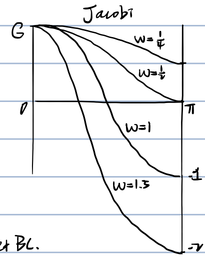
$P^{k+1}_\ell = (1-\omega)P^k_\ell + \omega\frac{P^k_{\ell+1} + P^k_{\ell-1}}{2}$ $\xrightarrow{G^k e^{ikx}}$ $\left| \begin{array}{l} G = 1-\omega + \omega\cos\beta \\ = 1 - 2\omega\sin^2(\frac{\beta}{2}) \end{array} \right| < 1$

$[\omega \leq 1]$ Cannot Over relax Jacobi

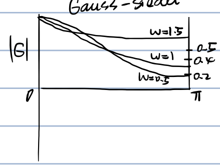
$P^*_\ell = \frac{P^{k+1}_{\ell+1} + P^k_{\ell-1}}{2}$

$P^*_\ell = (1-\omega)P^k_\ell + \omega P^*_\ell$

Jacobi



Gauss-Seidel



Gauss-Seidel

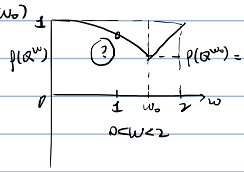
Young (1954)

$\lambda[Q^\omega] = 1 - \omega[1 - \lambda(Q)]$

Square domain, $\Delta x = \Delta y$, dirichlet BC.

Optimal value of ω (ω_0)

$\omega_0 = \frac{2}{1 + [1 - \rho(Q)]^{\frac{1}{2}}}$



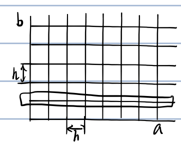
$\rho(Q^{\omega_0}) = \omega_0 - 1 = \frac{2}{1 + [1 - \rho(Q)]^{\frac{1}{2}}} - 1$

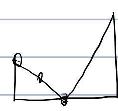
$\rho(Q_{GS}) \simeq \cos^2\pi h$

$\rho[Q^{\omega_0}_{G\text{-}S}] = \frac{1 - \sin\pi h}{1 + \sin\pi h} \simeq 1 - 2\pi h$

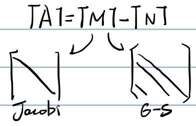
$-\log_{10}[\rho(Q^{\omega_0}_{GS})] \simeq 0.43(2\pi h)$

$P \simeq \frac{q}{-\log_{10}(\cdot)} \propto N \sim M^{\frac{1}{2}}$ ⟶ Total Flop $\sim M^{\frac{3}{2}}$

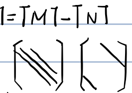




$[A] = [M] - [N]$



If $[A] = [M] - [N]$

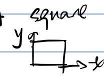


Line-Iterative Method

$\frac{\delta_x^2}{\Delta^2}P^{k+1} + \frac{\delta_y^2}{\Delta^2}P^k = 0$

Line.

〰 ③ If not square

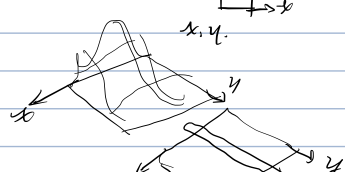


x, y.

Line Jacobi: $2\times$ point Jacobi

Line Gauss-Seidel: $2\times$ point G-S

Line SOR: $\sqrt{2}$ point SOR



11.28.

Multi-grid. Hyperbolic Equ.

Hyperbolic PDE

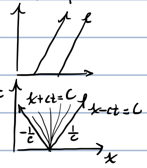
Hyperbolic PDE

allow discontinuity. Initial value problem.

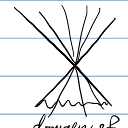
1st. order Hyperbolic PDE. $u_t + cu_x = 0$ IC: $u(x,0) = f(x)$.

soln: $u(x,t) = f(x-ct)$

2nd order $u_{tt} - c^2 u_{xx} = 0$ IC: $\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$



$$u(x,t) = \frac{1}{2}\underbrace{[f(x+ct) + f(x-ct)]}_{\text{Boundary of propagation}} + \underbrace{\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}_{\text{derivative part}}$$



CD:
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\,\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0$$

FE.
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\,\frac{u_{i+1}^n - u_i^n}{\Delta x} = 0$$

VN-Stability analysis.

$$u_i^{n+1} = u_i^n - c\frac{\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)$$

ν

$u_i^n = G^n e^{ikx}$

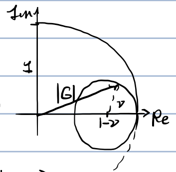
BE.
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\,\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

...

$|G| > 1$. unconditionally unstable.

$G = (1-\nu) + \nu e^{-ik\Delta x}$

$= [(1-\nu) + \nu\cos(k\Delta x)] - i\nu\sin(k\Delta x)$



$|G| > 1$ if $(1-\nu)$ cycle > unit cycle.

$|G| \le 1$ where $\nu = c\frac{\Delta t}{\Delta x} \le 1$ stability requirement

CFL Condition.

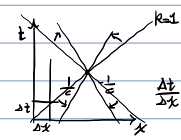
Cauchy Fred. Lowe.

$E(xy - ue)$

11.30

CFL Condition.

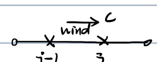
$u_{tt} - c^2 u_{xx} = 0$



$\frac{\Delta t}{\Delta x} < \frac{1}{c}$

up-wind scheme.

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\,\frac{u_{i+1}^n - u_i^n}{\Delta x} = 0 \quad c<0$$



"go upwind to grab more information"

$$u_t + cu_x = -\frac{\Delta t}{2}u_{tt} + c\frac{\Delta x}{2}u_{xx} - \frac{\Delta t^2}{6}u_{ttt} - c\frac{\Delta x^2}{6}u_{xxx} \ldots$$

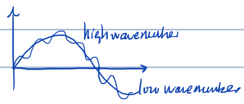
$\nu = \frac{c\Delta t}{\Delta x}$

$$= \frac{c\Delta x}{2}(1-\nu)u_{xx} - \frac{c(\Delta x)^2}{6}(2\nu^2 - 3\nu + 1)u_{xxx} \ldots$$

dissipation dispersion.

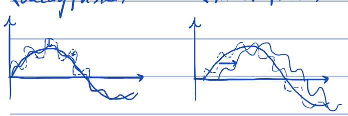
$\nu = 1$

all error vanishes.



Dissipation — high wavenumber <decay faster>

Dispersion — high wave number <travel faster>



$\hat{u}' = -ik\hat{u}$

$\hat{u} = e^{ikt}\hat{u}(0)$

k im → $e^{ikt} = e^{kt}$ → Amp change

k Re. → $e^{ikt} = \cos + i\sin$. → Frequency change

Dissipation error

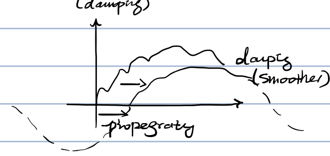
$$u_{tt} + cu_{xx} = \frac{c\Delta x}{2}(1-\nu)u_{xx} = \mu u_{xx}$$

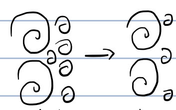
where $u = \sum \hat{u}_k e^{ikx}$

$$\frac{d\hat{u}}{dt} + (ick + \mu k^2)\hat{u} = 0$$

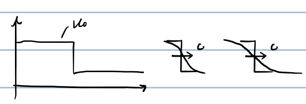
$$\hat{u} = \underset{\text{Initial.}}{\hat{u}_0}\, e^{-ickt}\, e^{-\mu k^2 t}$$

$$u = \sum \underbrace{(\hat{u}_0 e^{-\mu k^2 t})}_{\substack{\text{Amp}\\ \text{decay}\\ \text{(damping)}}} \overbrace{e^{ik(x-ct)}}^{\text{propagating}}$$





high wave number vortex disappear



Dispersion Error

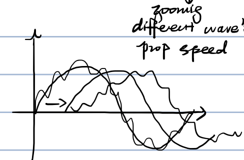
$$u_t + cu_{xx} = \beta u_{xxx}$$

$$\frac{d\hat{u}}{dt} + ik(c + \beta k^2)\hat{u} = 0$$

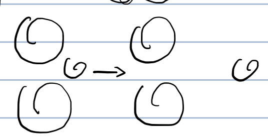
$u = \hat{u}_0 e^{-ik(c+\beta k^2)t}$ $c' = c + \beta k^2$

$u = \sum \hat{u}_0 e^{-ik(x-c't)}$

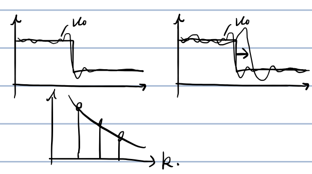
zooming different wave's prop speed

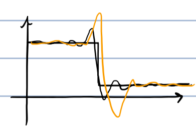


"decomposing"



vortex seperate.





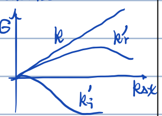
12.5

Modified Wavenumber

$u_j = \hat{u} e^{ikx}$ k each wavenumber

$$\frac{\partial u_j}{\partial x} = i(k) u_j \quad \text{PDE}$$

$$\left.\frac{\partial u}{\partial x}\right|_j = i(k') u_j \quad \text{FDE}$$



modified wavenumber $\left\{ \begin{array}{l} k_r' \\ k_i' \end{array} \right.$

$k' = k'(k, \Delta x)$

Modified wave number

each wave number $u_j = \hat{u} e^{ikx}$ $k = [0, \frac{\pi}{\Delta x}]$

Exact: $\frac{\partial u}{\partial x} = ik u$ true wavenumber k

FDE: (BE) $\left.\frac{\partial u}{\partial x}\right|_j \approx \frac{u_j - u_{j-1}}{\Delta x} = \frac{\hat{u}e^{ikx} - \hat{u}e^{ik(x-\Delta x)}}{\Delta x} = \frac{1-e^{-ik\Delta x}}{\Delta x}\hat{u}e^{ikx} = i\underbrace{\left(-i\frac{1-e^{-ik\Delta x}}{\Delta x}\right)}_{k'} u_j$

modified wavenumber k'

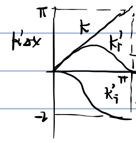
$\Delta x \downarrow \quad \lim_{\Delta x \to 0} -i\frac{1-e^{-ik\Delta x}}{\Delta x} =$

$$k' = \left(-i\frac{1-e^{-ik\Delta x}}{\Delta x}\right) = \underbrace{\frac{\sin k\Delta x}{\Delta x}}_{k_r'} + i\underbrace{\frac{\cos k\Delta x - 1}{\Delta x}}_{k_i'}$$

real part   imaginary part

$\Delta x \downarrow \quad \lim_{\Delta x \to 0} \sqrt{|k_r'|^2 + |k_i'|^2} = \lim \sqrt{1 + \frac{1}{\Delta x^2}}$

If match $\Delta x \uparrow$ ~



Semi-discrete analysis.

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

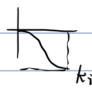
$u = \sum \hat{u}_k e^{ik(x-ct)}$ $\frac{\partial \hat{u}}{\partial t} + ik'c\hat{u} = 0$

im → Re. $c' = \frac{k_r'}{k}c$ phase error

$$u = \sum \hat{u}_0 e^{(ikx - ik'ct)} = \sum \hat{u}_0 e^{\left(k_i'ct + ik\left(x - \frac{k_r'}{k}ct\right)\right)} = \sum\left(\hat{u}_0 e^{k_i'ct}\right) e^{ik(x-c't)}$$

amperror   low wave number less damping

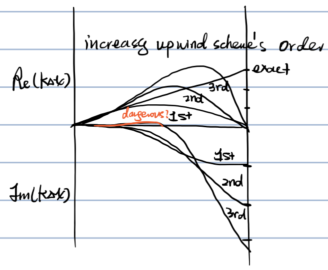
$\hat{u}_0 e^{\frac{\cos k\Delta x - 1}{\Delta x}ct}$

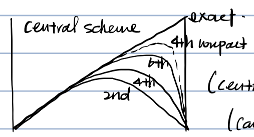


1 iterative error ν

$\hat{u}_0 e^{\frac{\cos k\Delta x - 1}{\Delta x}c\Delta t}$

$= \hat{u}_0 e^{\nu(\cos k\Delta x - 1)}$





(central difference don't have Im(kΔx))

(cancelled out)

only spatial method can get exact line.

Multigrid

- Direct Inversion ~ $O(N^3)$
- J, GS $\Big\rangle\ O(N^2)$ ~ $N\uparrow$, wavenumber → 0, G → 1
- Optimal SOR ~ $O(N^{1.5})$
- $O(N)$ Multigrid

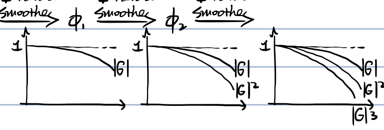
Initial guess (IG)

$$\mathcal{L}\,\phi_0 = R_0 \longrightarrow \mathcal{L}(\phi - \phi_0) = R - R_0$$

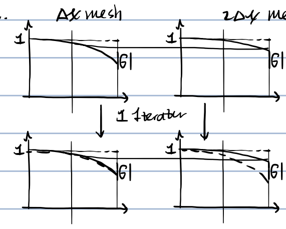
$$\mathcal{L}\,\epsilon_0 = r_0$$

IG error   residual

$\phi_0 \xrightarrow[\text{smoother}]{\text{1 iterate}} \phi_1 \xrightarrow[\text{smoother}]{\text{1 iterate}} \phi_2 \xrightarrow[\text{smoother}]{\text{1 iterate}}$



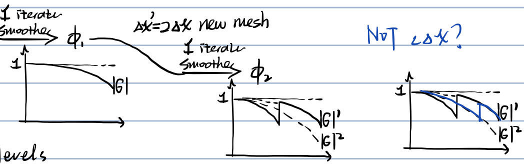
Different Grid.   Δx mesh   2Δx mesh



Multigrid

$\phi_0 \xrightarrow[\text{smoother}]{\text{1 iterate}} \phi_1$ $\Delta x' = 2\Delta x$ new mesh $\xrightarrow[\text{smoother}]{\text{1 iterate}} \phi_2$

NOT $2\Delta x$?



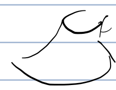
> 2 levels

32×32 100 10^{-3}

96×96 2000.

$\frac{\partial^2 u}{\partial r^2} + (\ )\frac{\partial^2 u}{\partial \theta^2} = 0$

D



12.7

Conjugate Gradient

$[A]\vec{x} = \vec{b}$ $[A]$ sparse

$\underbrace{\left(f(x) = \frac{1}{2}\vec{x}^T A\vec{x} + \vec{b}^T\vec{x} + c\right)'}$ quadratic form.

scalar function

$[A]\vec{x} = \vec{b}$ critical point

$\nabla f(\vec{x}) = \frac{1}{2}[A]^T\vec{x} + \frac{1}{2}[A]^T\vec{x} - \vec{b}$

if $[A]$ symmetric $f'(\vec{x}) = 0$ $[A]\vec{x} = b$

Multigrid tutorial-

Painless Conjugated gradient

Robert

Steepest Descent

$f(\vec{x}) = \frac{3}{2}x_1^2 + 3x_2^2 + 2x_1x_2 - 2x_1 + 8x_2$

$-f'(\vec{x}) = b - [A]\vec{x} = \vec{r}$ residual

$\vec{x} = \vec{x}_0 + \alpha\vec{r}_0$

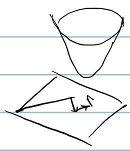
$\frac{d}{d\alpha}f(\vec{x}_1) = f'(x_1)\frac{dx_1}{d\alpha} = f'(\vec{x})^T\vec{r}_0$

$\vec{r}_1 \cdot \vec{r}_0 = 0$

$(b - [A]\vec{x}_1)^T\vec{r}_0 = 0$

$\vec{r}_0^T\vec{r}_0 = \alpha\vec{r}_0[A]\vec{r}_0$

$\alpha = \frac{\vec{r}_0^T\vec{r}_0}{\vec{r}_0[A]\vec{r}_0}$



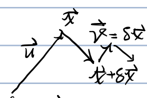
$\vec{x}_{i+1} = \vec{x}_i + \alpha\vec{r}_0$

Conjugate Gradient (CG)

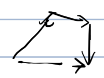
$f'(\vec{x}) = [A]\vec{x} - b$

$\vec{x} = \vec{x} + \delta\vec{x}$

$\delta f'(\vec{x}) = f'(\vec{x} + \delta\vec{x}) - f'(\vec{x}) = [A]\delta\vec{x}$

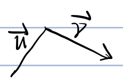


the change in gradient orthogonal to $\vec{u}$



$\vec{u} \cdot \delta\vec{f}' = 0$

$\vec{u}[A]\vec{v} = 0$ conjugate vectors



n-dimensional $\longrightarrow$ N direction

$\vec{x} = \sum_{k=1}^{N}\alpha_k\vec{P}_k$

Preconditioning (reduce condition number)

$[M]^{-1}[A]\vec{x} = [M]^{-1}[A]\vec{b}$ $[M]$ easily invertable

$\kappa([M]^{-1}[A]) \ll \kappa([A])$

ex. $[M] = [A]$

$\kappa([M]^{-1}[A]) = 1$

1) Jacobi $M_{ij} = \begin{cases} A_{ij} & i=j \\ 0 & \text{otherwise} \end{cases}$

BiCGSTAB.