① Gas Law $P\forall = \nu RT$ (N)

pressure · volume = #mols · R · Temperature (6.023×10^{23})

$P\forall = N \cdot \frac{M_{1mol}}{M_{molar}} \cdot RT$

number density $n \equiv \frac{N}{\forall} = \frac{M_{molar}}{M_{1mol}} \frac{P}{RT}$ (#mol)

mol1, mol2

free path l: $\frac{\text{Volume swept by mol}}{} \cdot n = 1$

自由间隔

$\frac{d^2}{4}\pi l \cdot n = 1$

$l = \frac{4}{\pi d^2 n} = \frac{4}{\pi d^2} \frac{M_{1mol} RT}{M_{molar} P}$

collision time: $T = \frac{l}{V_T}$ sound speed $\sqrt{\frac{\gamma RT}{M_{molar}}}$

$T = l\sqrt{\frac{M_{molar}}{\gamma RT}}$

★ Continuous?

Knudsen # $K_n = \frac{l}{\mathcal{L}}$ $\ll 1$ continuous; other NOT continuous

free path $\ll \mathcal{L}$

$P\forall = \nu RT$ #/$\forall$

$\nu = \frac{\#mol}{} = \frac{N - \#}{N_A - \#/mol}$ $N = n\forall$

$\nu = \frac{M}{M_{mol}}$ total mass / mass/mol $= \frac{N \cdot m}{}$

Continuum.

density: $\rho = \lim_{\Delta\forall \to 0} \frac{\sum m_i}{\Delta\forall}$ total mass / volume.

velocity: $\vec{u} = \lim_{\Delta\forall \to 0} \frac{\sum m_i \vec{v}_i}{\sum m_i}$ total momentum / total mass.

momentum: $\rho\vec{u} = \lim_{\Delta\forall \to 0} \frac{\sum m_i \vec{v}_i}{\Delta\forall^*}$ "momentum" / volume.

Velocity also $\vec{v}_i = \vec{u} + \vec{v}_i'$ every small pieces's own part movement

$\lim \frac{\sum m_i \vec{v}_i}{\sum m_i} = \lim \frac{\sum m_i(\vec{u} + \vec{v}_i')}{\sum m_i}$

$\cancel{\vec{u}} = \cancel{\vec{u}} + \frac{\sum m_i \vec{v}_i'}{\sum m_i} = 0$ total part movement is 0

Kinetic Energy: $\frac{\sum \frac{1}{2} m_i \vec{v}_i^2}{\sum m_i} = \frac{\sum \frac{1}{2} m_i (\vec{u} + \vec{v}_i')(\vec{u} + \vec{v}_i')}{\sum m_i}$

Energy / mass $= \frac{1}{2}\vec{u}\cdot\vec{u} + \frac{1}{2}\frac{\sum m_i \vec{v}_i' \cdot \vec{v}_i'}{\sum m_i}$

$\vec{\xi} = \alpha_1, \alpha_2, \alpha_3$

Langrange: Label each contious part.

$\vec{x}(\vec{\xi}, t)$ time, $\vec{x}(\vec{\xi}, 0) = \vec{\xi}$

velocity $\vec{u} = \frac{\partial \vec{x}}{\partial t}\Big|_{\xi}$ accelerate $\vec{a} = \frac{\partial^2 \vec{x}}{\partial t^2}\Big|_{\xi}$

Euler: Snap shot for field quantity $\vec{q}(\vec{x}, t)$

$\vec{q}(\vec{x}, t) = \vec{q}(\vec{x}_{old}, t - \Delta t) + \frac{\partial q}{\partial t}\Big|_{\vec{x}} \Delta t$

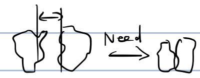
$\vec{q}(\vec{x} - \Delta x, t - \Delta t) = \vec{q}(\vec{x}, t) - \frac{\partial q}{\partial t}\Big|_{\vec{x}} \Delta t - \vec{u}\nabla q \Delta t$

?

$\frac{dq}{dt}\Big| = \frac{\partial q}{\partial t}\Big) + \vec{u}\nabla q\Big)$

$\sum u_i \frac{\partial}{\partial x_i}$

- Continuum vs. discrete description.
- Lagrangian vs. Eulerian description.
- Conservation Law.



Cont Assumption. large number molecules: free path $\ll \mathcal{L}$ (Length scale of incident / land scale interest?) ?

atom Radius 10^{-10} m. nucleus 10^{-15}. Volume occupied $\frac{1}{10^{15}}$. molecule # in micrometer

Gas. $P\forall = \nu RT$

pressure, Volume, # mols, constant, temperature

$P\forall = \frac{N \cdot M_{1mol}}{M_{molar}} \cdot RT$.

Standard cond. $T = 20°C$. $P = 101.3$ kPa.

free path l. (Mol 1, Mol 2)

number density $n \equiv \frac{N}{\forall} = \frac{M_{molar}}{m_{1mol}} \frac{P}{RT}$ (# mol)

① 1 mol. $\nu = 1$. $\forall = \frac{\nu RT}{P} = 0.024\ m^3$.

② # mol. $\forall_2 = (1\ \mu m)^3$ $\frac{N_1}{N_2} = \frac{\forall_1}{\forall_2}$ $N_2 = \frac{\forall_2}{\forall_1} \cdot N_1 = \frac{(1\times10^{-6})^3}{0.024} \times 1 \times 6.023 \times 10^{23}$

Volume swept · $n = 1$ (by mol)

$(\frac{1}{4}\pi d^2 l)(\frac{N}{\forall}) = 1$

$l_{\uparrow} = \frac{4}{\pi d^2 n_{\uparrow}}$ $n_{\uparrow} = \frac{M_{molar}}{m_{1mol}} \frac{P_{\uparrow}}{RT_{\downarrow}}$

ex. $T = 300$ k. $P = 1$ atm $= 101.3$ kPa.

$l \doteq 5\times10^{-8}$ m $= 50$ nm.

?

time between collision. $T = \frac{l}{\gamma}$ — speed of sound $\sqrt{\frac{\gamma RT}{M_{molar}}}$ $\gamma = \frac{C_p}{C_v}$

$T \doteq 10^{-10}$ sec

$T = l\sqrt{\frac{M_{molar}}{\gamma RT}}$

Knudson #

$K_n = \frac{l}{\mathcal{L}}$ → $\ll 1$. continuous. → otherwise. violate continuous.

Liquid.

$\rho_{H_2O} = 10^3\ kg/m^3$ $m_{1mol} = 18 \times 1.66\times10^{-27}$ kg.

$\simeq 30\times10^{-27}$ kg

$\frac{N}{\cancel{N_A}} = \frac{\text{mass of } (1\ \mu m)^3}{\text{mass of } 1\text{ mol}} = \frac{10^3 (10^{-6})^3}{3\times10^{-26}} = 3.33\times10^{10}$ mol. of water in $(1\ \mu m)^3$

free path: $\rho = \frac{m_{1mol}}{\pi l^3/6} = 10^3$ → 4×10^{-10} m

free path l: gas 5×10^{-8}, liquid 4×10^{-10}. more easy colli. in gas!

Continuum property

density $\rho = \lim_{\forall^* \to 0} \frac{\sum m_i}{\Delta\forall}$

velocity $\vec{u} = \lim_{\Delta\forall^* \to 0} \frac{\sum m_i \vec{v}_i}{\sum m_i}$ eq①

momentum. $\rho\vec{u} = \lim_{\Delta\forall^* \to 0} \frac{\sum m_i \vec{v}_i}{\Delta\forall^*}$?

$\vec{u}$ 为整体运动 $\vec{v}_i'$ 为分子自身的运动

tank stay. $\vec{u} = 0$

$\vec{v}_i = \vec{u} + \vec{v}_i'$

verify is constant with eqn.1

$\lim_{\Delta\forall^* \to 0} \frac{\sum m_i \vec{v}_i}{\sum m_i} = \frac{\sum m_i(\vec{u} + \vec{v}_i')}{\sum m_i}$ $\vec{u} \neq 0$.

$0\ \cancel{\vec{u}} = \cancel{\vec{u}} + \frac{\sum m_i \vec{v}_i'}{\sum m_i} \to 0$

Kinetic energy $\frac{\sum \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i}{\sum m_i} = \frac{\sum \frac{1}{2} m_i (\vec{u} + \vec{v}_i')(\vec{u} + \vec{v}_i')}{\sum m_i}$

$= \frac{\cancel{\sum} \frac{1}{2} \cancel{m_i}}{\cancel{\sum m_i}} \vec{u}\cdot\vec{u} + 2\vec{u} \frac{\cancel{\sum \frac{1}{2} m_i \vec{v}_i'}}{\cancel{\sum m_i}} + \frac{\sum \frac{1}{2} m_i \vec{v}_i' \cdot \vec{v}_i'}{\sum m_i}$ (0) → fluctuation kinetic energy. → T.

Lagrangium vs. Eulerium.

Largrongiaum: label each continuum parcle of fluid. ξ

describe its motion

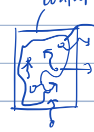
$\vec{x}(\vec{\xi}, t)$

its "label" natural label: initial position

$\vec{x} = (\vec{\xi}, t)$ where $\vec{x}(\vec{\xi}, t=0) = \vec{\xi}$

velocity $\vec{u} = \frac{\partial \vec{x}}{\partial t}\Big)_{\xi}$ $\vec{a} = \frac{\partial^2 \vec{x}}{\partial t^2}\Big)_{\xi}$

Euler control volume. Langrange parcel.



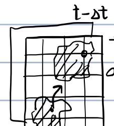
? field?

Eulerium description.

2 consecutive snap shots.

for fields. quantity $\vec{q}(\vec{x}, t)$

$\frac{D}{Dt}$



$\vec{q}(\vec{x}, t) = q(\vec{x}_{old}, t - \Delta t) + \frac{\partial q}{\partial t}\Big|_{\vec{x}} \Delta t$

$\vec{x}_{old} = \vec{x} - \vec{u}\Delta t$

$\vec{x}_{old} = \vec{x} - \delta\vec{x}$ $t_{old} = t - \delta t$ $\vec{q}(\vec{x} - \delta\vec{x}, t - \delta t)$

$q t$

The condition change of $\sum_{x,y,z} u_x \frac{\partial q}{\partial x}$

this article is its position's cond. change and the system's shape influence its cond.

if $q=\vec{v}$ $a_x = \frac{\partial u_x}{\partial t}\Big)_\xi + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y}$

$\vec{v}=(u_x, u_y)$

$\frac{\partial}{\partial t} + \nabla\cdot$

$\frac{d}{dt}\iiint = \iiint \frac{\partial}{\partial t}$

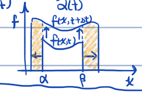
∇ nabla 算子 $\nabla = \mathrm{div}$

Δ laplace 算子 $\Delta = \nabla^2$

Math: • div Thm: $\int \nabla(*)\,d\forall = \oint (*)\vec{n}\,ds$

• Laibniz Thm: $\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} f\,dx = \int_{\alpha(t)}^{\beta(t)} \frac{\partial}{\partial t} f\,dx + \frac{d\beta}{dt}f - \frac{d\alpha}{dt}f$

boundary expand speed



Mass conservation:

Euler: $\Delta M = M_{in\,net}$

$\left[\frac{d}{dt}\int_{\forall_f} \rho\,d\forall = \oint_{S_f} (\rho\cdot\vec{u})\vec{n}\,ds\right]$ Integral of Mass conservation

div Thm

$\frac{d}{dt}\int_{\forall_f} \rho\,d\forall = -\int_{\forall_f} \nabla(\rho\vec{u})\,d\forall$

$\int_{\forall_f} \frac{\partial}{\partial t}\rho + \vec{u}\nabla\rho + \rho\nabla\vec{u}\,d\forall = 0$

$\left[\frac{D\rho}{Dt} + \rho\nabla\vec{u} = 0\right]$ Differential of Mass conservation

Langrangium: $\frac{dM_\ell}{dt} = 0$ Same object, mass not change

$\frac{d}{dt}\int_{\forall_\ell} \rho\,d\forall = 0$ Boundary velocity = flow $\vec{u}$

$\rho\to F$

Laibniz Thm $\to \frac{d}{dt}\int_{\forall_\ell} F\,d\forall = \int_{\forall_\ell} \frac{\partial}{\partial t}F\,d\forall + \oint_{S_\ell} F\cdot\vec{u}\cdot\vec{n}\,ds$ div Thm

$= \int_{\forall_\ell} \frac{\partial}{\partial t}F + \vec{u}\nabla F + F\nabla\vec{u}\,d\forall$

physical law $\frac{d}{dt}M_\ell = 0$ $= \int_{\forall_\ell} \frac{D}{Dt}F + F\nabla\vec{u}\,d\forall$

$F\to\rho$

$\to 0 = \frac{d}{dt}\int_{\forall_\ell} \rho\,d\forall = \int_{\forall_\ell} \frac{D}{Dt}\rho + \rho\nabla\vec{u}\,d\forall$

$\left[\frac{D}{Dt}\rho + \rho\nabla\vec{u} = 0\right]$ continuum eqn.

$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla\cdot\vec{u}$

change of density is the consumtion of flow

$F\to\rho F$, $\frac{d}{dt}\int_{\forall_\ell} \rho F\,d\forall = \int_{\forall_\ell} \rho\frac{D}{Dt}F + \left(F\frac{D}{Dt}\rho + \rho F\nabla\vec{u}\right)d\forall$

$\left[\frac{d}{dt}\int_{\forall_\ell} \rho F\,d\forall = \int_{\forall_\ell} \rho\frac{D}{Dt}F\,d\forall\right]$ 0

Laibniz Thm Boundary velocity $= \vec{v} \neq \vec{u}$

General: $\frac{d}{dt}\int_{\forall} F\,d\forall = \int_{\forall} \frac{\partial F}{\partial t}\,d\forall + \oint F\cdot\vec{v}\cdot\vec{n}\,ds$ div Thm

$F\to\rho F$ $= \int_{\forall} \frac{\partial F}{\partial t} + \nabla(F\cdot\vec{v})\,d\forall$

$\to \frac{d}{dt}\int_{\forall} \rho F\,d\forall = \int_{\forall} \frac{\partial\rho F}{\partial t} + \nabla(\rho F\vec{v})\,d\forall$

$= \int_{\forall} \rho\frac{\partial F}{\partial t} + F\left(\frac{\partial\rho}{\partial t} + \vec{u}\nabla\rho + \rho\nabla\vec{u}\right) - F\vec{u}\nabla\rho - F\rho\nabla\vec{u} + \nabla\rho F\vec{v}$

$\frac{D\rho}{Dt}$ $F\nabla(\rho\vec{u})$

continuum eqn $= 0$ $\rho\vec{v}F - \nabla F\rho\vec{u}$

$= \int_{\forall} \rho\left(\frac{\partial F}{\partial t} + \vec{u}\nabla F\right) + \nabla F\rho(\vec{v}-\vec{u})\,d\forall$

$\frac{DF}{Dt}$ div Thm

$\left[\frac{d}{dt}\int_{\forall} \rho F\,d\forall = \int_{\forall} \rho\frac{DF}{Dt}\,d\forall + \oint_{S} F\rho(\vec{v}-\vec{u})\vec{n}\,ds\right]$

$q(\vec{x}-\vec{u}\Delta t, t-\Delta t)$

$\frac{D\vec{x}}{Dt}$, $\frac{\partial\rho}{\partial t}$, $(\vec{u}\cdot\nabla)$

$\frac{\partial\rho\vec{u}}{\partial t} + \nabla\cdot(\rho\vec{u})$

$= \rho(\nabla\cdot\vec{u}) + \vec{u}\cdot\nabla\rho$

$\vec{u}\cdot\vec{n}\,dA$

$q(\vec{x}_{old}, t-\Delta t) = \vec{q}(\vec{x},t) - \frac{dq}{dt}\Big)_{\vec{x}}\Delta t - \vec{u}\nabla q\Delta t$

$\to \vec{q}(\vec{x},t) = \vec{q}(\vec{x},t) - \left[\frac{\partial q}{\partial t}\Big)_{\vec{x}} + \vec{u}\nabla q\right]\Delta t + \frac{dq}{dt}\Big|_\xi$

$\frac{dq}{dt}\Big|_\xi = \frac{\partial q}{\partial t}\Big)_\xi + \vec{u}\nabla q\Big)_t$

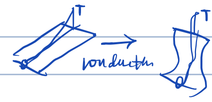
Langrange derivative

$\left.\begin{array}{l}\frac{dq}{dt}\\ \frac{Dq}{Dt}\end{array}\right| = \frac{\partial q}{\partial t} + \sum_{x,y,z} u_x\frac{\partial q}{\partial x}$

Conduction q change over the .

is the fix-point q change by the

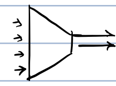
and control section change itself influence q ?



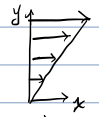
Ex. Acceleration.

$\vec{a} = \frac{d\vec{v}}{dt}$ $a_x = \underbrace{\frac{\partial u_x}{\partial t}}_{\text{unsteady}} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}$

Steady flow acceleration



$a_x = \underbrace{\frac{\partial u_x}{\partial t}}_{\text{unsteady}}^{0} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y}$



$\frac{\partial u_x}{\partial y} = c$

Langragian: Follow material particle. $q(\vec{\xi}, t)$ $\vec{x}(\vec{\xi}, t=0) = \vec{\xi}$ initial position.

Eulerian: Field properties is function space and time. $q(\vec{x}, t)$

$\frac{dq}{dt}\Big)_\xi = \frac{dq}{dt} = \frac{Dq}{Dt} = \frac{\partial q}{\partial t}\Big)_x + \vec{v}\nabla q$

$q(\vec{x}, t)$

$q(\vec{x}_{old}, t-\Delta t)$

1) Mass conservation

→ Euleriuan

→ Languglau

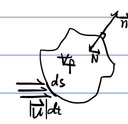
2) Reynold Transport therom. 雷诺输运定理

Mass conservation $\frac{d}{dt}$ $\int_{\forall_f} \rho\,d\forall$ ← fixed

Eulerimen the rate of change of mass in control volume

is equal to the net mass flux into the volume.

$\frac{1}{dt}\rho|\vec{u}|\,ds\,\cos\theta\,dt = \rho\vec{u}\cdot\vec{n}\,ds = -\oint_{S_f} \rho\vec{u}\cdot\vec{n}\,ds_f$



$\vec{n}$ leaving

$-\vec{n}$ coming

$\frac{d}{dt}\int_{\forall_f} \rho\,d\forall_f = -\oint_{S_f} \rho\vec{u}\cdot\vec{n}\,ds$

$\frac{d}{dt}$ move into. depend on space become partial $\frac{\partial}{\partial t}$

$\left[\int_{\forall_f} \frac{\partial\rho}{\partial t}\,d\forall_f = -\oint_{S_f} \rho\vec{u}\cdot\vec{n}\,ds\right]$ integral form of mass conservation

div Thm. $\int \nabla(*)\,d\forall = \oint (*)\vec{n}\,ds$

$\int \frac{\partial}{\partial x_j}(*)\,d\forall = \oint (*)n_j\,ds$

$\int_{\forall_f} \frac{\partial\rho}{\partial t} + \nabla(\rho\cdot\vec{u})\,d\forall_f = 0$

Involve divergence therom

$\frac{\partial\rho}{\partial t} + \nabla(\rho\cdot\vec{u}) = 0$ differencial form of mass conservation.

(The continuity equation.)

equal. $\underbrace{\frac{\partial\rho}{\partial t} + \vec{u}\cdot\nabla\rho}_{} + \rho\nabla\vec{u} = 0$

$\frac{D\rho}{Dt} = -\rho\nabla\vec{u}$

if velocity field divergeing, the $\rho\downarrow$

Incompressible fluids ρ = constant $\Rightarrow \frac{D\rho}{Dt} = 0 \Rightarrow \mathrm{div}\,\vec{u} = 0$

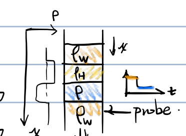
Incompressible flow $\frac{D\rho}{Dt} = 0$

Langrangiam form of mass conservation.

$\frac{d}{dt}M_\ell = 0$

$\frac{d}{dt}\int_{\forall_\ell(t)} \rho\,d\forall_\ell = 0$ the rate of change for mass = 0 the volume change

Ex.



$\frac{\partial\rho}{\partial x} \neq 0$

$\frac{\partial\rho}{\partial t} \neq 0$

$\frac{D\rho}{Dt} = 0$

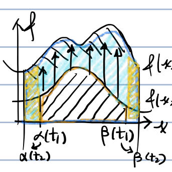
$\left(\frac{\partial\rho}{\partial t}\right) + u\left(\frac{\partial\rho}{\partial x}\right) = 0$

-ve (<0) +ve (>0)

probe. u = constant

Leibniz rule $\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} f(x,t)\,dx = \int_{\alpha(t)}^{\beta(t)} \frac{\partial f}{\partial t}\,dx + f(\beta,t)\frac{d\beta}{dt} - f(\alpha,t)\frac{d\alpha}{dt}$

Reynold Transport Rule



dropped. The boundary velocity

Reynold transport therom

Langrangeum Volume

$\frac{d}{dt}\int_{\forall_\ell} F\,d\forall_\ell = \int_{\forall_\ell} \frac{\partial F}{\partial t}\,d\forall_\ell + \oint_{S_\ell} F\cdot\vec{u}\cdot\vec{n}\,ds_\ell$

q in dimension

Boundary is (n-1) dimension

Langrange

$\vec{u}$ is velocity of flow

Boundary velocity is set to be fluid velocity because $\forall_\ell$ is Lagrange part.

div Thm

$\frac{d}{dt}\int F\,d\forall_\ell = \int \frac{\partial F}{\partial t}\,d\forall_\ell + \int \nabla\cdot(F\vec{u})\,d\forall_\ell$

Reynold Transport Thm.

Lagrange: $\vec{\nu}=\vec{u}$, $\forall=\forall_\ell$

fixed volume: $\vec{\nu}=0$, $\forall=\forall_f$

$$\frac{d}{dt}\int_{\forall_\ell} F\, d\forall_\ell = \int_{\forall_\ell}\left(\frac{\partial F}{\partial t}+\vec{u}\cdot\nabla F+F\nabla\cdot\vec{u}\right)d\forall_\ell$$

$$\frac{d}{dt}\int_{\forall_\ell} F\, d\forall_\ell = \int_{\forall_\ell}\left(\frac{DF}{Dt}+F\nabla\cdot\vec{u}\right)d\forall_\ell \longrightarrow \frac{d}{dt}\int_{\forall}\rho\, d\forall_\ell=\int_{\forall}\left(\frac{D\rho}{Dt}+\rho\nabla\cdot\vec{u}\right)d\forall_\ell$$

Set $F=\rho$

$$0=\frac{dM}{dt} \quad \bigstar \quad \boxed{\frac{D\rho}{Dt}+\rho\nabla\cdot\vec{u}=0} \quad \bigstar$$

continuity equ.

$$\frac{d}{dt}\int_a^{b(t)} f(t,x)\,dx=\int_{a(t)}^{b(t)}\frac{\partial f}{\partial t}\,dx+\dots$$

Other form of Reynold transport theorem

Imagine every $F\longrightarrow\rho F$

$$\frac{d}{dt}\int_{\forall_\ell}\rho F\, d\forall_\ell=\int_{\forall_\ell}\left(\frac{D\rho F}{Dt}+\rho F\nabla\cdot\vec{u}\right)d\forall_\ell$$

$$=\int_{\forall_\ell}\left[\underbrace{\left(F\frac{D\rho}{Dt}+\rho F\nabla\cdot\vec{u}\right)}_{=0}+\rho\frac{DF}{Dt}\right]d\forall_\ell$$

mass conservation / continuity

$$\left[\frac{d}{dt}\int_{\forall_\ell}\rho F\, d\forall_\ell=\int_{\forall_\ell}\rho\frac{DF}{Dt}\,d\forall_\ell\right]$$

How about general volume $\forall(t)$

Leibniz rule, surface velocity

$$\frac{d}{dt}\int_{\forall(t)}F\,d\forall=\int_{\forall(t)}\frac{\partial F}{\partial t}\,d\forall+\oint_{S(t)}F\,\vec{\nu}\cdot\vec{n}\,ds$$

↓ div Thm

$$\frac{d}{dt}\int_{\forall(t)}F\,d\forall=\int_{\forall(t)}\left[\frac{\partial F}{\partial t}+\nabla\cdot(F\vec{\nu})\right]d\forall$$

Imagine every $F\longrightarrow\rho F$

$$\frac{d}{dt}\int_{\forall}\rho F\,d\forall=\int_{\forall(t)}\frac{\partial\rho F}{\partial t}+\nabla\cdot(\rho F\vec{\nu})\,d\forall$$

$$\frac{\partial\rho}{\partial t}+\vec{u}\cdot\nabla\rho+\rho\nabla\cdot\vec{u}, \quad \underbrace{\frac{\partial\rho}{\partial t}+\vec{u}\cdot\nabla\rho}_{\frac{D\rho}{Dt}} \quad \text{continuity}$$

$$\frac{d}{dt}\int_{\forall}\rho F\,d\forall=\int_{\forall(t)}\rho\frac{\partial F}{\partial t}+F\left(\cancel{\frac{\partial\rho}{\partial t}+\nabla(\rho\vec{u})}\right)-F\cdot\nabla(\rho\vec{u})+\nabla(\rho F\vec{\nu})\;d\forall$$

0 continuity

$$=\int_{\forall(t)}\rho\frac{\partial F}{\partial t}-\nabla\cdot(F\rho\vec{u})+\rho\vec{u}\cdot\nabla F+\nabla(\rho F\vec{\nu})\;d\forall$$

$$\frac{d}{dt}\int_{\forall}\rho F\,d\forall=\int_{\forall}\rho\left(\frac{\partial F}{\partial t}+\vec{u}\cdot\nabla F\right)+\nabla\cdot\left(\rho F(\vec{\nu}-\vec{u})\right)d\forall$$

$$\frac{d}{dt}\int_{\forall}\rho F\,d\forall=\int_{\forall}\rho\frac{DF}{Dt}+\nabla\cdot\left(\rho F(\vec{\nu}-\vec{u})\right)d\forall$$

↓ div Thm.

$$\frac{d}{dt}\int_{\forall}\rho F\,d\forall=\int_{\forall}\rho\frac{DF}{Dt}+\oint_S\rho F(\vec{\nu}-\vec{u})\cdot\vec{r}\,ds$$

for Lagrange $\forall(t)=\forall_\ell(t)$, $\vec{\nu}=\vec{u}$

for fixed volume: $\forall(t)=\forall_f$, $\vec{\nu}=0$

Reynold's transport theorem

Lagrangian control volume

$$\frac{d}{dt}\int_{\forall_\ell}F\,d\forall=\int_{\forall_\ell}\frac{\partial F}{\partial t}+\nabla(F\vec{u})\,d\forall_\ell$$

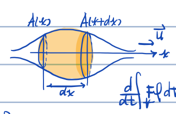
$$\frac{d}{dt}\int_{\forall_\ell}F\,d\forall=\int_{\forall_\ell}\frac{dF}{dt}+F\nabla\vec{u}\quad d\forall_\ell$$

$$\frac{d}{dt}\int_{\forall_\ell}\rho F\,d\forall=\int_{\forall_\ell}\rho\frac{dF}{dt}+F\nabla\vec{u}\quad d\forall_\ell$$

general: surface velocity

$$\frac{d}{dt}\int_{\forall}\rho F\,d\forall=\int_{\forall}\rho\frac{DF}{Dt}+\nabla\rho F(\vec{\nu}-\vec{u})\,d\forall$$

Ex. 对于流体,不知其边界固定/自由时,取一部分 (其体积) 用General 等式



General:

$$\frac{d}{dt}\int_{\forall}F\rho\,d\forall=\int_{\forall}\rho\frac{dF}{dt}+\nabla\rho F(\vec{\nu}-\vec{u})\,d\forall$$

↓ div Thm

Change of mass is equal to

$F=1$ $$\frac{d}{dt}\int_{\forall}\rho\,d\forall=\oint_S\rho(\vec{\nu}-\vec{u})\cdot\vec{n}\,ds$$

$\parallel$ ΔM ↓ Boundary

面可分为3部分, 其中侧面 当边界

固定时 $\vec{u}\cdot\vec{n}=0$, $\vec{\nu}=0$

自由时 $\vec{u}-\vec{\nu}=0$

则 $\int_{S(t)}\rho(\vec{\nu}-\vec{u})\vec{n}\,ds = 0$

if fixed: $\vec{u}\cdot\vec{n}=0$, $\vec{\nu}=0$

if free: $\vec{u}=\vec{\nu}$, $\vec{\nu}-\vec{u}=0$

$\vec{n}$与$\vec{u}$反向 $\vec{n}$与$\vec{u}$同向

$$\frac{d}{dt}\int_{\forall}\rho\,d\forall-\int_{A(x)}\rho\vec{u}\,dA+\int_{A(x+dx)}\rho\vec{u}\,dA=0$$

$d\forall = dA\,dx$

$$\boxed{\left(\frac{\partial}{\partial t}\int_A\rho\,dA\right)\cdot dx}$$

这里当把 dx 拿出时 $\frac{d}{dt}\to\frac{\partial}{\partial t}$ 因为拿出dx的动作造成其对t变为偏导

$$\frac{\partial}{\partial t}\int_A\rho\,dA+\frac{-\int_{A(x)}\rho\vec{u}\,dA+\int_{A(x+dx)}\rho\vec{u}\,dA}{dx}=0$$

$$\frac{\partial}{\partial t}\int_A\rho\,dA+\frac{\partial}{\partial x}\int_A(\rho\vec{u})\,dA=0$$

定义面积平均量 $\langle\circledcirc\rangle=\frac{1}{A}\int\circledcirc\,dA$

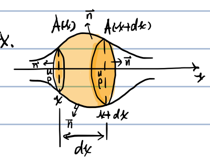
average Area value

则 $\langle\rho\rangle=\frac{1}{A}\int\rho\,dA$

$\langle\rho\vec{u}\rangle=\frac{1}{A}\int\rho\vec{u}\,dA$

$$\boxed{\frac{\partial}{\partial t}(A\langle\rho\rangle)+\frac{\partial}{\partial x}(\langle\rho\vec{u}\rangle A)=0}$$

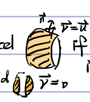
Ex.



F=1

$$\frac{d}{dt}\int_{\forall}\rho\,d\forall=\oint_S\rho(\vec{\nu}-\vec{u})\vec{n}\,ds$$

will cancel



If wall moving $\vec{\nu}=\vec{u}$

if wall fixed: $\vec{u}\perp\vec{n}$

$$\frac{d}{dt}\int_{\forall}\rho\,d\forall+\oint_S\rho\,\vec{u}\cdot\vec{n}\,ds=0$$

$$dx\cdot\underbrace{\frac{\partial}{\partial t}\left(\int_A\rho\cdot dA\right)}-\int_{A(x)}\rho\vec{u}\vec{n}\,dA+\int_{A(x+dx)}\rho\vec{u}\cdot\vec{n}\,dA=0$$

could influence by x

$$\frac{d}{dt}\int_{\forall}\circledcirc\,d\forall=\frac{d}{dt}\left[\left(\int\circledcirc\,dA\right)dx\right]=\frac{\partial}{\partial t}\int\circledcirc\,dA\cdot dx$$

(Integral form)

$$\frac{\partial}{\partial t}\left(\int_A\rho\cdot dA\right)+\frac{-\int_{A(x)}\rho\vec{u}\vec{n}\,dA+\int_{A(x+dx)}\rho\vec{u}\cdot\vec{n}\,dA}{dx}=0$$

$$\frac{\partial}{\partial t}\left(\int_A\rho\cdot dA\right)+\frac{\partial}{\partial x}\left(\int_{A(x)}\rho u\,dA\right)=0$$

面积均值

Introduce — area average volume $\langle\circ\rangle$

$\langle\rho\rangle=\frac{1}{A}\int_A\rho\,dA$ $\longrightarrow$ $\frac{\partial}{\partial t}(\langle\rho\rangle\cdot A)+\frac{\partial}{\partial x}(\langle\rho u\rangle A)=0$

$\langle\circ\rangle=\frac{1}{A}\int_A\circ\,dA$ if $\forall$ fixed. $A=A(x)$ $A\frac{\partial}{\partial t}\langle\rho\rangle+\frac{\partial}{\partial x}(\langle\rho u\rangle A)=0$

$\langle\rho u\rangle=\frac{1}{A}\int_A\rho u\,dA$ if $\rho\neq\rho(t)$ $\frac{\partial}{\partial x}(\langle\rho u\rangle A)=0$

$\langle\rho u\rangle A=$ constant

If $\rho=$ constant $\rho A\langle u\rangle=$ constant

Ex. Linearize problem

Basic state ρ_0 quiescent fluid $\longrightarrow$

(differential form)

Perturbed basic state

$\rho=\rho_0+\rho'$ $\rho'\ll\rho_0$ $\rho'\sim O(\varepsilon)$ order of magnitude

Ex. Linearlize 线性化

Continuity eqn. $\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{u}) = 0$

Base: $\begin{cases} \rho = \rho_0 \\ \vec{u} = 0 \end{cases} \to \begin{cases} \rho = \rho_0 + \rho' \\ \vec{u} = 0 + \vec{u} \end{cases}$ 在其态附近

$\frac{\partial(\rho_0+\rho')}{\partial t} + \nabla(\rho_0+\rho')\vec{u} = 0$ 只有 x 方向

$\cancel{\frac{\partial \rho_0}{\partial t}} + \frac{\partial \rho'}{\partial t} + \frac{\partial \rho_0 \vec{u}}{\partial x} + \frac{\partial \rho' \vec{u}}{\partial x} = 0$

ρ_0 constant; $O(\varepsilon)$, $O(\delta) \gg O(\varepsilon\delta)$ 2nd order; ρ' 可忽略 非线性化

线性化后 Linearized $\left[\frac{1}{\rho_0}\frac{\partial}{\partial t}(\rho') + \frac{\partial}{\partial x}(\vec{u}) = 0\right]$

$\rho' = \frac{\partial \rho}{\partial p}\Big|_s p' + \frac{\partial \rho}{\partial s}\Big|_p s'$ 压强, 熵; $=0$ isentropy

$\frac{\partial p}{\partial \rho}\Big|_s = c^2$ Incompressibility of Substance 不可压缩性

$\rho' = \frac{1}{c^2} p'$

$\left[\frac{1}{c^2}\frac{1}{\rho_0}\frac{\partial}{\partial t}(p') + \frac{\partial}{\partial x}(u) = 0\right]$

Euler Eqn. Newton 2nd law.

$\Delta M v = \frac{d}{dt}\int_V \rho\vec{u}\,dV = -\int_S p\cdot\vec{n}\,ds + \int_{V} \rho\vec{g}\,dV$

Momentum change = ⊖ Pressure effect (to other) ⊕ gravity effect

$\frac{d}{dt}\int_V \rho\vec{u}\,dV = \int_V (-\nabla p + \rho\vec{g})\,dV$ (div Thm)

Lagrange $\frac{d}{dt}\int_{V} F\,dV = \int_{V} \frac{\partial F}{\partial t}\,dV + \oint_S F\cdot\vec{u}\cdot\vec{n}\,ds$ (div Thm)

① $\frac{d}{dt}\int F\,dV = \int \left(\frac{\partial F}{\partial t} + \nabla(F\vec{u})\right)dV$

OR. $\frac{d}{dt}\int \rho F\,dV = \int \frac{\partial \rho F}{\partial t}\,dV + \oint_S \rho F\cdot\vec{u}\cdot\vec{n}\,ds$

$= \int \rho\frac{\partial F}{\partial t} + F\left(\frac{\partial \rho}{\partial t} + \nabla(\rho\vec{u})\right) + \rho\vec{u}\nabla F\,dV$

② $\frac{d}{dt}\int \rho F\,dV = \int \rho\frac{DF}{Dt}\,dV$ (Conservation of mass $=0$)

(1) Reynold Transport ①($F\to\vec{u}$)

$\int \rho\frac{D\vec{u}}{Dt}\,dV = \int -\nabla p + \rho g\,dV$

$\left[\rho\frac{D\vec{u}}{Dt} + \nabla p - \rho g = 0\right]$

(2) Raplace ①($F \to \rho\vec{u}$)

$\frac{\partial \rho\vec{u}}{\partial t} + \nabla(\rho\vec{u}^2) + \nabla p - \rho g = 0$

$(\textcircled{1}=\textcircled{2})$ $\frac{\partial \rho\vec{u}}{\partial t} + \nabla\rho\vec{u}^2 = \rho\frac{\partial\vec{u}}{\partial t} + \vec{u}\left(\frac{\partial \rho}{\partial t} + \nabla\rho\vec{u}\right)$ mass conservation $- \vec{u}\nabla\rho - \vec{u}\rho\nabla\vec{u} + \vec{u}\nabla\rho + \vec{u}\rho\nabla\vec{u} + \rho\vec{u}\nabla\vec{u} = \rho\frac{D\vec{u}}{Dt}$

Bernoulli eqn: $\rho\frac{D\vec{u}}{Dt} + \nabla p - \rho\vec{g} = 0$

$\frac{\partial\vec{u}}{\partial t} + \vec{u}\nabla\vec{u} = -\frac{1}{\rho}\nabla p + \vec{g}$

$\frac{\partial\vec{u}}{\partial t} + \nabla\frac{1}{2}\vec{u}^2 + \vec{\omega}\times\vec{u} = -\frac{1}{\rho}\nabla p + \vec{g}$

$\left[\frac{\partial\vec{u}}{\partial t} + \vec{\omega}\times\vec{u} + \nabla\left(\frac{1}{2}\vec{u}^2 + h + \nabla\vec{u}\right) = 0\right]$

not steady, rotation 值场剩项 B

$\vec{u} = 0 + \vec{u}$ $\quad \vec{u} \sim O(\delta)$ ($O(\varepsilon)$ less than)

Continuity $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$ Base state $\rho = \rho_0$ constant, $u = 0$ ✓ satisfied.

But must be purturbed state $\rho = \rho_0 + \rho'$, $\vec{u} = 0 + \vec{u}$

$\frac{\partial(\rho_0+\rho')}{\partial t} + \frac{\partial}{\partial x}(\rho_0+\rho')u = 0$

ρ', u unknown — $\frac{\partial \rho'}{\partial t} + \frac{\partial \rho_0 u}{\partial x} + \frac{\partial \rho' u}{\partial x} = 0$

$O(\frac{\varepsilon}{\partial t})$ $\quad O(\frac{\delta}{\partial x}) \gg O(\frac{\varepsilon\delta}{\partial x})$

Linearlize — Drop nonlinear term

$\left[\frac{1}{\rho_0}\frac{\partial \rho'}{\partial t} + \frac{\partial u}{\partial x} = 0\right]$

re-write in term of pressure $\rho' = \frac{\partial \rho}{\partial p}\Big|_s p' + \frac{\partial \rho}{\partial s}\Big|_p s'$ $\quad P$ pressure $\quad S$ Entropy

isentropic: $s' = 0$

$\rho' = \frac{\partial \rho}{\partial p}\Big|_s p' = \frac{1}{c^2}p'$ $\qquad \frac{\partial p}{\partial \rho}\Big|_s = c^2$ incompressibility of substance, speed of sound

$\left[\frac{1}{\rho_0 c^2}\frac{\partial p'}{\partial t} + \frac{\partial u}{\partial x} = 0\right]$

Euler eqn. (Newton's 2nd law)

$\frac{d}{dt}\int_V \rho\vec{u}\,dV = -\int_S p\cdot\vec{n}\,ds + \int_V \rho g\,dV$

$\frac{d}{dt}\int_V \rho\vec{u}\,dV = \int_V -\nabla p + \rho g\,dV$

Lagrange

$\frac{d}{dt}\int_{V} F\,dV = \int_{V}\frac{\partial F}{\partial t} + \nabla(F\vec{u})\,dV$

$\frac{d}{dt}\int F\,dV = \int \frac{dF}{dt} + F\nabla\vec{u}\,dV$

$\frac{d}{dt}\int \rho F\,dV = \int \rho\frac{dF}{dt} + \cancel{F\nabla\vec{u}}\,dV$

$\rho\frac{\partial F}{\partial t} + \rho\vec{u}\nabla F = \rho\frac{DF}{Dt}$

① Reynold Transport ($F \to \vec{u}$)

$\int_V \rho\frac{D\vec{u}}{Dt} + \nabla p - \rho g\,dV = 0$

$\rho\frac{D\vec{u}}{Dt} + \nabla p - \rho g = 0$

$\rho\frac{\partial\vec{u}}{\partial t} + \rho\cdot\vec{u}\nabla\vec{u}$

② Raplace $F \to \rho\vec{u}$

$\int_V \frac{\partial \rho\vec{u}}{\partial t} + \nabla(\rho\vec{u}^2) + \nabla p - \rho g\,dV = 0$

$\frac{\partial \rho\vec{u}}{\partial t} + \nabla\rho\vec{u}^2 + \nabla p - \rho g = 0$

$\rho\frac{\partial\vec{u}}{\partial t} + \underbrace{\vec{u}\frac{\partial \rho}{\partial t} + \rho\vec{u}\nabla\vec{u} + \vec{u}\nabla\rho\vec{u}}_{0}$

Ex. 1

Bernoulli equation

$\frac{\partial\vec{u}}{\partial t} + \vec{u}\nabla\vec{u} = -\frac{1}{\rho}\nabla p + \vec{g}$

$\frac{\partial\vec{u}}{\partial t} + (\nabla\cdot\frac{1}{2}|\vec{u}|^2 - \vec{u}\times(\nabla\times\vec{u})) = -\frac{1}{\rho}\nabla p + \vec{g}$

$\frac{\partial\vec{u}}{\partial t} + \nabla\frac{1}{2}|\vec{u}|^2 + \vec{\omega}\times\vec{u} = -\frac{1}{\rho}\nabla p + \vec{g}$ $\quad \frac{1}{\rho}\nabla p = h$ $\quad \vec{g} = -\nabla^2 u$

$\left[\frac{\partial\vec{u}}{\partial t} + \vec{\omega}\times\vec{u} + \nabla\left(\frac{1}{2}\vec{u}^2 + h + \nabla u\right) = 0\right]$ (B)

Assume steady $\frac{\partial\vec{u}}{\partial t} = 0$, no retusing $\vec{\omega} = 0$

$\nabla B = 0$

Ex. 2 $\qquad B = \frac{1}{2}\vec{u}^2 + h + \nabla\vec{u} = \text{constant}$

Stream Line. Assume steady. $\frac{\partial\vec{u}}{\partial t} = 0$

project along streamline. $\vec{u}\cdot\left[\frac{\partial\vec{u}}{\partial t} + \vec{\omega}\times\vec{u} + \nabla\left(\frac{1}{2}\vec{u}^2 + h + \nabla u\right) = 0\right]$ (B)

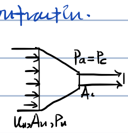
$\vec{u}\cdot\vec{\omega}\times\vec{u} + \vec{u}\cdot\nabla B = 0$ ($\vec{\omega}\times\vec{u}$, $\vec{u}$, $\vec{\omega}$ orthogonal, 0)

$\vec{u}\cdot\nabla B = 0$

$\vec{u}\cdot\left(\frac{1}{2}\vec{u}^2 + h + \nabla u\right) = 0$

B constant along stream line.

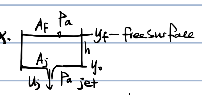
Contraction.

Ex. 

B: $\frac{1}{2}u_u^2 + \frac{P_u}{\rho} = \frac{1}{2}u_c^2 + \frac{P_c}{\rho}$

Continuity $Q = \rho u_u A_u = \rho u_c A_c$

$Q = \sqrt{\left(\frac{1}{A_c} - \frac{1}{A_u}\right)^{-1}\frac{(P_u - P_c)}{\rho}}$

Ex. 

B: $P_a + \frac{1}{2}\rho u_f^2 + \rho g y_f = P_a + \frac{1}{2}\rho u_j^2 + \rho g f_0$

$\frac{1}{2}\rho u_f^2 + \rho g h = \frac{1}{2}\rho u_j^2$

can be ignored if $A_f \ggg A_j$

$u_j = \sqrt{2gh}$

Continuity $\rho u_f \cdot A_f = \rho u_j A_j$

$u_j = \frac{(u_f A_f)}{A_j}$

$\frac{\partial}{\partial t}\int \rho\,dV = \rho u_f A_f - \rho u_j A_j$

Linearlized #2

1-dim flow.

Continuous (diff-form)

$\frac{D\rho}{Dt} + \rho\nabla\vec{u} = 0$

$\frac{\partial \rho}{\partial t} + \nabla(\rho\vec{u}) = 0$ (1x dir.)

$\rho = \rho_0 + \rho'$ $\quad \rho' \sim O(\varepsilon)$; $u = 0 + u'$ $\quad u' \sim O(\delta)$; $\frac{\partial p}{\partial \rho}\Big|_s = c^2$

$\frac{\partial \rho_0}{\partial t} + \frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x}(\rho_0 + \rho')u' = 0$

$\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x}(\rho_0 u') + \frac{\partial}{\partial x}\rho' u' = 0$

$O(\frac{\varepsilon}{\partial t})$ $\quad O(\frac{\delta}{\partial x})$ $\quad O(\frac{\varepsilon\delta}{\partial x})$

$\left[\frac{\partial \rho'}{\partial t} + \rho_0\frac{\partial u'}{\partial x} = 0\right]$ Linearized continuity eqn.

$\left[\frac{1}{c^2 \rho_0}\frac{\partial p'}{\partial t} + \frac{\partial u'}{\partial x} = 0\right]$ ①

Euler eqn. x-dir

$\rho\frac{D\vec{u}}{Dt} = -\nabla p + \rho\vec{g}$

$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + \frac{\partial p}{\partial x} = 0$ 1-dim

$(\rho_0 + \rho')\left(\frac{\partial u'}{\partial t} + u'\frac{\partial u'}{\partial x}\right) + \frac{\partial p'}{\partial x} = 0$

$\rho_0\frac{\partial u'}{\partial t} + \rho_0 u'\frac{\partial u'}{\partial x} + \rho'\frac{\partial u'}{\partial t} + \rho' u'\frac{\partial u'}{\partial x} + \frac{\partial p'}{\partial x} = 0$

$\rho_0\frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0$

$\left[\frac{1}{\rho_0}\frac{\partial p'}{\partial x} + \frac{\partial u'}{\partial t} = 0\right]$ ②

$\frac{\partial^2 p'}{\partial x \partial y}$ equal $\quad c^2\cdot\frac{\partial}{\partial x}① - \frac{\partial}{\partial t}② = c^2\frac{\partial^2 u'}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$

$\frac{\partial^2 u'}{\partial x \partial t}$ equal $\quad \frac{\partial}{\partial t}① - \frac{\partial}{\partial x}② = \frac{1}{c^2 \rho_0}\frac{\partial^2 p'}{\partial t^2} - \frac{1}{\rho_0}\frac{\partial^2 p'}{\partial x^2} = 0$

$c^2\frac{\partial^2 p'}{\partial x^2} - \frac{\partial^2 p'}{\partial t^2} = 0$

$p' = f(x-ct) + g(x+ct)$ $\qquad$ wave eqn!

$u' = F(x-ct) + G(x+ct)$ $\qquad \frac{\partial^2 w}{\partial t^2} - c^2\frac{\partial^2 w}{\partial x^2} = 0$

$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)w = 0$

$① \to \frac{1}{c^2 \rho_0}(-cf' + cg') + (F' + G') = 0$

$w = H_1(x-ct) + H_2(x+ct)$

$② \to \frac{1}{\rho_0}(f' + g') + (-cF' + cG') = 0$

$\begin{cases} F' + G' = \frac{1}{c\rho_0}(f' - g') \\ F' - G' = \frac{1}{c\rho_0}(f' + g') \end{cases} \quad \begin{cases} F' = \frac{f'}{\rho_0 c} \\ G' = -\frac{1}{\rho_0 c}g' \end{cases} \quad u' = \frac{1}{\rho_0 c}\left(f(x-ct) + g(x+ct)\right)$

∘ Euler vs. Lagrange

∘ Mass Conservation

∘ Reynold transport

∘ Euler eqns.

— ∘ Contraction

— ∘ Empty tank.

— Linilarized

∘ Potential flow.

— uniform.

1 dimensional acoustic eqn.

∘ Continuity eqn. $\frac{D\rho}{Dt} + \rho\nabla\vec{u} = 0$ $\qquad \rho = \rho_0 + \rho'$ $\quad \rho' \sim O(\varepsilon)$; $u = 0 + u'$ $\quad u' \sim O(\delta)$; $p = p_0 + p'$

$\frac{\partial \rho}{\partial t} + \nabla(\rho\vec{u}) = 0$ $\qquad \frac{\partial p}{\partial \rho}\Big|_s = c^2 \Rightarrow \rho' = \frac{1}{c^2}p'$

$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$

$(\rho_0 + \rho', u', p')$ $\quad \cancel{\frac{\partial \rho_0}{\partial t}} + \frac{\partial \rho'}{\partial t} + \cancel{u'\frac{\partial \rho_0}{\partial x}} + u'\frac{\partial \rho'}{\partial x} + \rho_0\frac{\partial u'}{\partial x} + \rho'\frac{\partial u'}{\partial x} = 0$

$O(\frac{\varepsilon}{\partial t})$ $\quad O(\frac{\varepsilon\delta}{\partial x})$ $\quad O(\frac{\delta\varepsilon}{\partial x})$ $\quad O(\frac{\delta}{\partial x}) = 0$

$\left[\frac{1}{\rho_0}\frac{\partial \rho'}{\partial t} + \frac{\partial u'}{\partial x} = 0\right]$ Linilarized continuity eqn.

$\left[\frac{1}{\rho_0 c^2}\frac{\partial p'}{\partial t} + \frac{\partial u'}{\partial x} = 0\right]$ ①

Euler eqn. x direction $\quad \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial p}{\partial x} + \cancel{g_x}$

$(\rho_0 + \rho')\left(\frac{\partial u'}{\partial t} + u'\frac{\partial u'}{\partial x}\right) = -\frac{\partial p}{\partial x}$

$\rho_0\frac{\partial u'}{\partial t} + \cancel{\rho_0 u'\frac{\partial u'}{\partial x}} + \cancel{\rho'\frac{\partial u'}{\partial t}} + \cancel{\rho' u'\frac{\partial u'}{\partial x}} = -\frac{\partial p}{\partial x}$

$\left[\rho_0\frac{\partial u'}{\partial t} = -\frac{\partial p}{\partial x}\right]$ ②

$\rho_0\frac{\partial ①}{\partial t} - \frac{\partial ②}{\partial x} \Rightarrow \frac{\partial^2 p'}{\partial t^2} - c^2\frac{\partial^2 p'}{\partial x^2} = 0$

$\rho_0 c^2\frac{\partial ①}{\partial x} - \frac{\partial ②}{\partial t} \Rightarrow \frac{\partial^2 u'}{\partial t^2} - c^2\frac{\partial^2 u'}{\partial x^2} = 0$

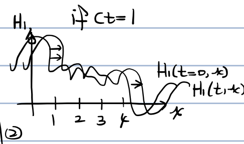
wave eqn.

$\frac{\partial^2 w}{\partial t^2} - c^2 \frac{\partial^2 w}{\partial x^2} = 0$

$(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}) w = 0$

$w = \underbrace{H_1(x-ct)}_{\text{Right}} + \underbrace{H_2(x+ct)}_{\text{Left}}$

if $ct = 1$



$p' = f(x-ct) + g(x+ct)$

$u' = F(x-ct) + G(x+ct)$

rotate (f & F) on (g & G) substitute ①|②

$\dot F = \frac{\dot f}{\rho_0 c}$ $\quad \dot G = \frac{\dot F}{\rho_0 c}$ $\quad \dot F = \frac{dF}{d(x-ct)}$ $\quad u' = \frac{1}{\rho_0 c}[f(x-ct) - g(x+ct)]$

$F = \frac{f}{\rho_0 c}$ $\quad G = \frac{F}{\rho_0 c}$ $\quad \dot G = \frac{dG}{d(x-ct)}$

Potential Flow. 势流

Euler $\rho\frac{D\vec u}{Dt} = -\nabla p + \rho\vec g$

$\rho\frac{\partial \vec u}{\partial t} + \rho u \nabla u + \nabla p - \rho g = 0$

$\frac{\partial \vec u}{\partial t} + \nabla\frac{1}{2}|\vec u|^2 + \vec\omega\times\vec u + \frac{\nabla p}{\rho} + \nabla u = 0$

$\nabla\times\vec u = \vec\omega = 0 \rightarrow \vec u = \nabla\phi$ means no votex

$\frac{\partial\nabla\phi}{\partial t} + \nabla\frac{1}{2}(\nabla\phi)^2 + \frac{\nabla p}{\rho} + \nabla^2\phi = 0$

$\nabla(\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} + \nabla^2\phi) = 0$

$\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} + \nabla^2\phi = F(t)\right]$ non-linear

Continuous $\cancel{\frac{D\rho}{Dt}} + \rho\nabla\vec u = 0$

0 Assume ρ constant

$\nabla\vec u = 0$

$[\nabla^2\phi = \Delta\phi = 0]$ Laplace

Boundary condition: ① Free stream

$\lim_{|\vec x|\to\infty} \nabla\phi = \vec U_\infty(\vec x, t)$

② Relatively normal velosity

$\vec n \cdot (\nabla\phi - \vec\nu) = 0$

fluid velocity

relatively speed to object

No crossing into

polar coordinate:

$V_r = \frac{\partial\phi}{\partial r} = V(1-\frac{R^2}{r^2})\cos\theta$

$V_\theta = \frac{\partial\phi}{\partial\theta} = -V(1+\frac{R^2}{r^2})\sin\theta$

$P = \frac{1}{2}\rho V^2(2\frac{R^2}{r^2}\cos(2\theta) - \frac{R^4}{r^4}) + P_\infty$

(cylinder) polar coordinate: $\frac{1}{r}\frac{\partial r V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} = 0$

$\nabla^2\phi = 0$ spherical: $\frac{1}{r^2}\partial_r(r^2\partial_r\phi) +$

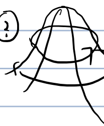
Laplace eqn. in different coordinate

Potential Flow.

Start from Euler $\qquad \vec u\nabla\vec u = \nabla\frac{1}{2}|\vec u|^2 + \vec u\times\nabla\times\vec u$

$\frac{\partial\vec u}{\partial t} + \vec u\nabla\vec u = -\frac{1}{\rho}\nabla P + \vec g \qquad = \nabla\frac{1}{2}u^2 - \vec u\times\vec\omega$

$\frac{\partial\vec u}{\partial t} + \nabla\frac{1}{2}|\vec u|^2 + \vec\omega\times\vec u = -\nabla h - \nabla^2 \mathcal{U} \qquad = \nabla\frac{1}{2}u^2 + \vec\omega\times\vec u$

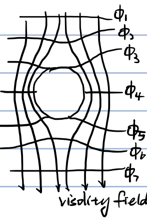
①  same no gradient.

Assume $\nabla\times\vec u \equiv \vec\omega = 0 \rightarrow \vec u = \nabla\phi \qquad \nabla\times\nabla\phi = 0$ garantee $\nabla\times\vec u (\equiv\vec\omega) = 0$

$\frac{\partial}{\partial t}(\nabla\phi) + \nabla\frac{1}{2}(\nabla\phi)^2 + \nabla h + \nabla u = 0$

$\nabla(\frac{\partial}{\partial t}\phi + \frac{1}{2}(\nabla\phi)^2 + h + u) = 0$) as the gradient to space = 0, it won't be influenced by space coordinate then it will only be function of time

$\left[\frac{\partial}{\partial t}\phi + \underbrace{\frac{1}{2}(\nabla\phi)^2}_{\text{nonlinear}} + \frac{P - P_0}{\rho} + u = F(t)\right]$



ϕ: gradient

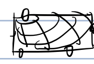
Continuity eqn. $\frac{\partial\rho}{\partial t} + \vec u\nabla\rho + \rho\nabla\vec u = 0$

Assume. ρ = constant.

$\nabla\vec u = 0$

$[\nabla\cdot\nabla\phi = 0]$ linear

Laplace $\Delta\phi = 0$

②  

Solu: $(\phi_1, P_1)\ (\phi_2, P_2)$

$(\phi_1+\phi_2, P_?)$ solu.

$\neq P_1 + P_2$

Some Boundary condition

* Free stream condition $\lim_{|\vec x|\to\infty} \nabla\phi = \vec u_\infty(\vec x, t)$

* Relative Normal velosity $\vec n \cdot (\nabla\phi - \vec\nu) = 0$

fluid velosity

relative to object

Compability condition.

$\nabla^2\phi = f \qquad \nabla^2\phi = f$

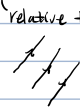
Subject to. $((\nabla\phi)\cdot\vec n) = g$

$\partial\Omega \quad \phi\to T$

$\int \nabla^2 T\, d\Omega = \int \nabla\cdot\nabla T\, d\Omega = \int \nabla T\cdot\vec n\, ds = \int g\, ds$

$\int f\, d\Omega = \int g\, ds$

Ex. ① uniform flow. $\vec U$



$\phi = \vec U\cdot\vec x = \Sigma U_i x_i$

$\nabla\phi = (\cdots U_i \cdots) = \vec U$

② Conal flow

Cylinder coordinate.

$\vec u = \nabla\phi$

$\left[u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta}\Big|_{\theta=\{0,\alpha\}} = 0\right]$ zero flux to the wall

only work on boundary

Continuity $\nabla^2\phi = 0$

① $\left[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0\right]$

$\phi = R(r)\Theta(\theta)$ Sepration variables

Substitute ① & equate to $(m+1)^2$

$\frac{r}{R}\frac{d}{dr}(r\frac{dR}{dr}) = (m+1)^2 \qquad \frac{d^2\Theta}{d\theta^2} = -(m+1)^2\Theta$ only sin/cos 's 2nd $\frac{d}{dt}$ is itself inversed.

$R = r^p \qquad \Theta = C_1\cos((m+1)\theta) + C_2\sin((m+1)\theta)$

$R = Pr^{m+1} + Qr^{-(m+1)}$

$\phi = Pr^{m+1}(C_1\cos(m+1)\theta + C_2\sin(m+1)\theta)$

To find constants. use Boundary condition $u_\theta\Big| = \frac{1}{r}\frac{\partial\phi}{\partial\theta}\Big|_{\theta=\{0,\alpha\}} = 0 \longrightarrow = (m+1)Pr^{m+1}[C_1\cos(m+1)\theta + C_2\sin(m+1)\theta]$

$\theta = 0 \quad u_\theta = 0 \quad C_1 = 0$

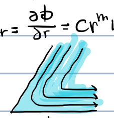
$\theta = \alpha \Rightarrow (m+1)\alpha = k\pi \quad k = 1, 2\ldots$

C_1 already to 0.

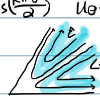
try $C_2 \neq 0$.

$\left[\phi = \frac{C}{m+1} r^{m+1}\cos((m+1)\theta)\right]$

$u_r = \frac{\partial\phi}{\partial r} = Cr^m\cos(\frac{k\pi\theta}{\alpha}) \qquad u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -Cr^m\sin(\frac{k\pi\theta}{\alpha})$



k=1



k=2

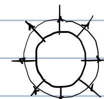
③ Growing on collapsing bubbles

$r > R$ Spherical Coordinate

Continuity $\nabla^2\phi = 0$

$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\phi}{\partial r}) = 0$

$\frac{\partial}{\partial r}(r^2\frac{\partial\phi}{\partial r}) = 0$



$\left[\phi = \frac{a(t)}{r} + b(t)\right]$ Can still be depended of time. solu.

$\left[\frac{\partial\phi}{\partial r}\Big|_{r=R} = \dot R\right]$ B.C.

$-\frac{a(t)}{R^2} = \dot R \rightarrow a(t) = -\dot R R^2$

$\phi = -\frac{\dot R R^2}{r} + b(t)$

$u_r = \frac{\partial\phi}{\partial r} = \frac{\dot R R^2}{r^2}$ velocity drop $\frac{1}{r^2}$

$u_r(4\pi r^2) = \dot R(4\pi R^2)$

Mass conservation

Pressure from Euler

$\frac{\partial\phi}{\partial t} + \frac{1}{2}u_r^2 + \frac{P}{\rho} = F(t)$

$(-\frac{\ddot R R^2}{r} - \frac{2R\dot R^2}{r} + \dot b) + \frac{1}{2}\frac{\dot R^2 R^4}{r^4} + \frac{P}{\rho} = F(t)$

far away $r\to\infty$

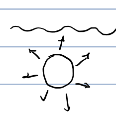
$0 \qquad 0 \quad + \dot b + \frac{P_\infty}{\rho} = F(t)$

Pressure only drop $\frac{1}{r}$

$-\frac{\ddot R R^2}{r} - \frac{2R\dot R^2}{r} + \frac{1}{2}\frac{\dot R^2 R^4}{r^4} + \frac{P}{\rho} = \frac{P_\infty}{\rho}$

on surface. $r = R$

$-\ddot R R - 2\dot R^2 + \frac{1}{2}\dot R^2 = \frac{P_\infty}{\rho} - \frac{P}{\rho}$



$\left[P_s = P_\infty + \rho(R\ddot R + \frac{3}{2}\dot R^2)\right]$

Rayleigh-Plesset eqn.

○ Potential flow

e.g. uniform.

✓ vortex

✓ Sphere

○ flow over sphere

○ Index notation

potential flow eqn.

Euler $\frac{\partial \vec{u}}{\partial t} + \nabla \frac{1}{2}u^2 + h + \psi + \vec{\omega} \times \vec{u} = 0$

assume. $\vec{\omega}(=\nabla \times \vec{u}) = 0$

$\Rightarrow \vec{u} = \nabla\phi$

$\therefore \nabla\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}u^2 + \frac{P}{\rho} + \chi\right) = 0$

Non-Linear $\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}u^2 + \frac{P}{\rho} + \chi = F(t)\right]$

Continuity $\cancel{\frac{d\rho}{dt}} + \rho \nabla \cdot \vec{u} = 0$ assume. $\rho = c.$

Linear. $[\nabla^2 \phi = 0.]$

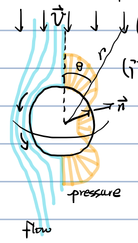
B.C. e.g. $\lim_{|\vec{x}| \to \infty} \nabla\phi = \vec{u}$ farfield

$\vec{n} \cdot (\underbrace{\nabla\phi - \vec{v}}_{\text{fluid body}}) = 0$ relatively velocity normal to surface.

normal

Ex.

e.g. Potential flow past a sphere.



(i) continuity $\nabla^2\phi = 0$ subject to $\vec{n} \cdot \nabla\phi = 0$ at $r = R$ (a)

$\nabla\phi \to \vec{u}$ as $r \to \infty$ (b)

(ii) $\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\phi\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\phi\right) + \underset{\text{symmetric.}}{0} = 0\right]$ — ①

(a) $\left.\frac{\partial\phi}{\partial r}\right|_{r=R} = 0.$

&

(b) $\frac{\partial\phi}{\partial r} \to -V\cos\theta$ & $\frac{1}{r}\frac{\partial\phi}{\partial\theta} \to V\sin\theta$ as $r \to \infty$

$\phi \to -Vr\cos\theta + C_1(\theta)$ $\phi = -Vr\cos\theta + C_2(r)$

$\therefore \phi_\infty = -Vr\cos\theta + \cancel{C}.$

Substitute in ① $(r^2 f')' - 2f = 0$ $\Leftarrow$ $[\phi = V f(r)\cos\theta]$ — ②

subject to $f'(R) = 0$

$f(\infty) = -r$

assume. $f = r^\alpha$

$f = A_1 r + A_2 r^{-2} \longrightarrow A_1 = -1$

$f' = A_1 - 2A_2 r^{-3} \longrightarrow -\frac{1}{2}R^3 = A_2$

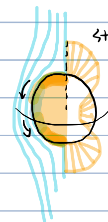
$f = -r - \frac{R^3}{2r^2}$

$\phi = -V\left(1 + \frac{R^3}{2r^3}\right) r\cos\theta$

$V_r = \frac{\partial\phi}{\partial r} = V\left(-1 + \frac{R^3}{r^3}\right)\cos\theta$

$V_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = V\left(1 + \frac{R^3}{2r^3}\right)\sin\theta$ $\left.V_\theta\right|_{r=R} = \frac{3}{2}V\sin\theta$

Pressure from Bernoulli. eqn.



Steady $\cancel{\frac{\partial\phi}{\partial t}} + \frac{1}{2}|\vec{u}|^2 + \frac{P}{\rho} = F(t) = \text{constant}.$

Steady.

$\frac{1}{2}(u_r^2 + u_\theta^2) + \frac{P}{\rho} = \frac{1}{2}V_\infty^2 + \frac{P_\infty}{\rho}$

on surface. of sphere.

P on surface $P_s = P_\infty + \frac{1}{2}\rho V_\infty^2 \left(1 - \frac{9}{4}\sin^2\theta\right)$

Index notation

$\vec{u} = (u_x, u_y, u_z) = u_x\vec{i} + u_y\vec{j} + u_z\vec{z} = u_x\vec{e}_x + u_y\vec{e}_y + u_z\vec{e}_z$

$= \sum_{j=1}^{3} u_j\vec{e}_j = u_j\vec{e}_j$

$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3 = \sum_{j=1}^{3} u_jv_j = u_jv_j$

Dummy indices

① (a) repeated index implies summation over all possible values. (所有值求和) n values in nD

(b) repeated index can only be appear twice $u_j v_j$ (appear 2.) $u_m \cancel{v_m} x_m$

② repeated indices are dummy indices can be replaced by any dummy index $u_jv_j = u_mv_m = u_kv_k = \sum_{i}^{\text{all } i} u_iv_i$

Free indices

$\vec{u} = u_i$

① Appear once And must appear 1 in every term of eqn.

Ex. $\cancel{u_j = v_j x_i}$

Ex. $u_i = v_i$ $\begin{cases} u_1 = v_1 \\ u_2 = v_2 \\ u_3 = v_3 \end{cases}$ $\frac{\partial u_i}{\partial t} = \frac{\partial u_i}{\partial x}\begin{cases}\frac{\partial u_1}{\partial t} = \frac{\partial u_1}{\partial x} \\ \vdots\end{cases}$

$\cancel{\frac{\partial u_i}{\partial t} = \frac{\partial u_i}{\partial x_i}}$ dummy

$\frac{\partial u_i}{\partial t} = \frac{\partial u_i}{\partial x_i} v_i\ V.$

$\cancel{u_i = v_j}$ both i, j free.

means $u_1 = v_1, v_2, v_3 \ldots$

Gradient

$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}$ $\begin{cases} \nabla\phi = \frac{\partial\phi}{\partial x_i} = \left(\frac{\partial\phi}{\partial x_1} + \frac{\partial\phi}{\partial x_2} + \frac{\partial\phi}{\partial x_3}\right) \\ \nabla \cdot \vec{u} = \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) = \frac{\partial u_k}{\partial x_k} \\ \vec{u} \cdot \nabla\phi = u_i\frac{\partial\phi}{\partial x_i} = u_1\frac{\partial\phi}{\partial x_1} + u_2\frac{\partial\phi}{\partial x_2} + u_3\frac{\partial\phi}{\partial x_3} \end{cases}$

Euler. $\frac{\partial\vec{u}}{\partial t} + \vec{u} \cdot \nabla\vec{u} = -\frac{\nabla P}{\rho} + \vec{g}$

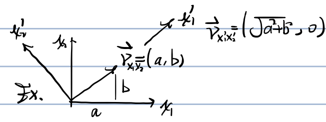
$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + g_i$ $\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}$

$u_i = (u_1, u_2, u_3)$ Vector

$A_{ij} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ Tensor

B_{ijk} 3rd order Tensor

Ex.



Kronecker delta independent to coord.

$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ isotropic Tensor

$\delta_{ij}u_j = u_i$ j as dummy index. i free index $\to$ vector

$\delta_{ij}u_iv_j = u_iv_i = u_jv_j$

$\delta_{ij}\delta_{jk} = \delta_{ik}$

$\delta_{kk} = n$ in n dimension.

Ex $\delta_{ij}A_{jm} = A_{im}$ j dummy, i, m free. & replace $j \to i$

Look note on Thursday

Index Notation. 张量计算\指标计算

○ $u_i v_i = \sum_i u_i v_i$ 哑指标 Dummy index means: SUM appear 2. sum all $\vec{u} \cdot \vec{v} = u_iv_i$

○ $v_i = \vec{v}$ 自由指标 Free index means: Vector appear 1. every term. $v_i = (v_1, v_2, v_3)$

○ Tensor 张量 $A_{ij} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ 2nd order Tensor

○ Kronecker 符号.

delta $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ isotropic tensor, 各向同性张量.

$\begin{cases} \delta_{ij}u_j = u_i \text{ (dummy) } \sim \text{vector} \\ \delta_{ij}u_iv_j = u_iv_i = u_jv_j \quad \text{means } i = j \\ \delta_{ij}\delta_{jk} = \delta_{ik} \quad \text{means } i = j = k \\ * \ \delta_{kk} = N \text{ in } N \text{ dim.} \end{cases}$

○ Alternating Tensor.

$\epsilon_{ijk} = \begin{cases} 1 & i,j,k = 1,2,3 \backslash 3,1,2 \backslash 2,3,1 \\ -1 & i,j,k = 2,1,3 \backslash 1,3,2 \backslash 3,2,1 \\ 0 & i = j \backslash j = k \backslash i = k \end{cases}$

$\vec{u} \times \vec{v} = \epsilon_{ijk}\underbrace{u_jv_k}_{\text{dummy}}$

$= \epsilon_{1jk}u_jv_k + \epsilon_{2jk}u_jv_k + \epsilon_{3jk}u_jv_k$

$\underset{1}{\epsilon_{123}}u_2v_3$ $\underset{-1}{\epsilon_{132}}u_3v_2$ $\underset{-1}{\epsilon_{213}}$ $\underset{1}{\epsilon_{231}}$ ...

$\nabla \times \vec{u} = \epsilon_{ijk}\frac{\partial}{\partial x_j}u_k$

$\epsilon_{ijk}\epsilon_{iab} = \delta_{ja}\delta_{kb} - \delta_{jb}\delta_{ka}$

Gradient $\nabla\phi = \frac{\partial\phi}{\partial x_i}$ (vector)

$\nabla \cdot \vec{u} = \frac{\partial u_i}{\partial x_i}$ (Σ number)

$\vec{u} \cdot \nabla\phi = u_i\frac{\partial\phi}{\partial x_i}$ (Σ number)

Euler: $\frac{\partial\vec{u}}{\partial t} + \vec{u}\nabla\cdot\vec{u} = -\frac{\nabla P}{\rho} + \vec{g}$

$\frac{\partial u_i}{\partial t} + u_i\frac{\partial u_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + g_i$ i: free j: dummy

$\vec\nabla\cdot\vec x = \delta_{ij}\frac{\partial}{\partial x_i}x_j$

$\vec A\times\vec B\times\vec C = \epsilon_{abc}A_b(\epsilon_{cij}B_iC_j)$ same index

$\epsilon_{abc}=\epsilon_{cab}$

$= \epsilon_{cab}\epsilon_{cij}A_bB_iC_j$

$=(\delta_{ai}\delta_{bj}-\delta_{aj}\delta_{bi})A_bB_iC_j$ ($a=i$, $b=j$; $a=j$, $b=i$)

$=A_bB_aC_b - A_bB_bC_a$

As multi-cross product

$\vec A\times\vec B\times\vec C = \epsilon_{abc}A_b(\vec B\times\vec C)_c$ means $\vec B\times\vec C$, free index c

$=\epsilon_{abc}A_b(\epsilon_{cij}B_iC_j)$

$\Delta=\nabla^2=\vec\nabla\cdot\vec\nabla=\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}$ dummy

$\nabla\cdot\vec x=\frac{\partial}{\partial x_i}x_j=\delta_{ij}$

$\vec a\vec b = a_ib_j$

$\neq$

$\vec a\cdot\vec b = a_ib_i$

$\vec u\cdot\nabla\vec u=(\vec u\cdot\nabla)\vec u = u_k\nabla_k u_j$

$\vec u\cdot(\nabla\vec u)=u_k\nabla_j u_k$

Alternating Tensor

$$\epsilon_{ijk}=\begin{cases}1 & i,j,k=1,2,3 \text{ or } 3,1,2 \text{ or } 2,3,1\\ -1 & i,j,k=2,1,3 \text{ or } 1,3,2 \text{ or } 3,2,1\\ 0 & i=j,\ j=k, \text{ or } k=i\end{cases}$$

$\epsilon_{123}=-\epsilon_{213}$

$\epsilon_{ijk}=\epsilon_{kij}=\epsilon_{jki}=-\epsilon_{jik}=-\epsilon_{ikj}=-\epsilon_{kji}$

$\epsilon_{iik}=0=\epsilon_{kii}=\epsilon_{kjj}$

$\vec u\times\vec v=\epsilon_{ijk}u_jv_k=\epsilon_{1jk}u_jv_k+\epsilon_{2jk}u_jv_k+\epsilon_{3jk}u_jv_k$

$=\underset{1}{\epsilon_{123}}u_2v_3+\underset{-1}{\epsilon_{132}}u_3v_2+\cdots$

$=u_2v_3-u_3v_2+\cdots$

$\nabla\times\vec u=\epsilon_{ijk}\frac{\partial}{\partial x_j}u_k$

prove: $\vec u\times\vec v=-\vec v\times\vec u$

$\epsilon_{ijk}u_jv_k=-\epsilon_{ikj}v_ku_j=-\vec v\times\vec u$

$\nabla\cdot\nabla\vec u=\nabla^2\vec u=\nabla\nabla\cdot\vec u-\nabla\times\nabla\times\vec u$

$\frac{\partial(u_i)}{\partial x_j\partial x_j}=\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right)-\epsilon_{ipk}\frac{\partial}{\partial x_p}\left(\epsilon_{kem}\frac{\partial u_m}{\partial x_e}\right)$

$=\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right)-(\delta_{ie}\delta_{pm}-\delta_{im}\delta_{pe})\frac{\partial}{\partial x_p}\frac{\partial u_m}{\partial x_e}$ (ϵ_{kip}; $e\to i$, $p\to m$; $m\to i$, $p\to e$)

$\frac{\partial^2u_i}{\partial x_j\partial x_j}=\frac{\partial^2}{\partial x_i\partial x_j}\frac{\partial u_j}{\partial x_j}-\frac{\partial}{\partial x_m}\frac{\partial u_m}{\partial x_i}+\frac{\partial}{\partial x_e}\frac{\partial u_i}{\partial x_e}$

free.

$=$

$\frac{\partial}{\partial x_m}\frac{\partial u_m}{\partial x_i}=\frac{\partial}{\partial x_i}\frac{\partial u_m}{\partial x_m}$

$\vec w\cdot(\vec u\times\vec v)=\vec v\cdot(\vec w\times\vec u)=\vec u\cdot(\vec v\times\vec w)$

$w_i\epsilon_{ijk}u_jv_k=v_k\epsilon_{kij}w_iu_j=u_j\epsilon_{jki}v_kw_i$

$\epsilon_{ijk}\epsilon_{iab}=\delta_{ja}\delta_{kb}-\delta_{jb}\delta_{ka}$ $\quad$ $\epsilon_{ijk}\epsilon_{kab}=\delta_{ia}\delta_{jb}-\delta_{ib}\delta_{ja}$ $\quad$ ϵ

if $j=a$ $\quad =3\delta_{kb}-\delta_{jb}\delta_{kj}$

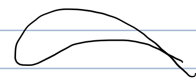
$=3\delta_{kb}-\delta_{kb}$

if $b=k$ $\quad =9-3=6$

$\frac{\partial\vec u}{\partial t}+\frac12\nabla|\vec u|^2+\vec\omega\times\vec u=-\frac{\nabla p}{\rho}+\vec g+\nu\nabla^2\vec u$ $\quad \nabla\cdot\vec u=0$

$=-\nu\nabla\times(\nabla\times\vec u)$

$=-\nu\nabla\times\vec\omega$



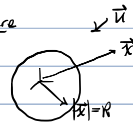
- Potential flow over sphere
- Index notation

$\oint\vec n\,ds=0$

Potential Flow around sphere

$\nabla\phi\to\vec u$ at infinity

$\vec n\cdot\nabla\phi=0$ at $|\vec x|=R$



How can we construct ϕ?

① ϕ scalar

② ϕ must be linear in $\vec U$

$\phi=\vec U\cdot\vec x\,g(r)$ linear; need another vector to create scalar.

$\vec U\cdot\nabla f=\vec U\cdot\frac{\vec x}{r}\hat f$ $\quad g(r)$

$\phi=\vec U\cdot\nabla f(r)$

Substitute $\nabla^2\phi=0$ $\quad \vec U\cdot\vec a$

$\vec U\cdot\nabla(\nabla^2f(r))=0$

$\nabla\nabla^2f(r)=0$

$\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}f(r)\right)=0$

$\left(f''+\frac2rf'\right)'\frac{x_i}{r}=0$

$f''+\frac2rf'=\text{constant}$

$f'=C_1r+\frac{C_3}{r^2}$

$\phi=\vec U\cdot\nabla f=\vec U\cdot\frac{\vec x}{r}f'$

$\nabla f(r)$

$=\frac{\partial}{\partial x_i}f(r)$

$=f'(r)\frac{\partial r}{\partial x_i}$

$=f'(r)\frac{\partial}{\partial x_i}(x_jx_j)^{\frac12}$

$=f'(r)\frac12(x_mx_m)^{-\frac12}\cdot2x_j\delta_{ij}$

$=f'\frac{x_i}{r}$

$\oint\vec n\vec n\,ds=\oint n_in_j\,ds=C\delta_{ij}=\frac43\pi R^2\delta_{ij}$ (tensor; sphere) $\quad i=j$ $\delta_{ij}=3$

$i=j$ $\quad \oint_{sphere}ds=4\pi R^2=3C$

Boundary conditions infinity

$u_i\xrightarrow{r\to\infty}V_jC_2\delta_{ij}=C_2V_i\longrightarrow C_2=1$

on surface

$\vec u\cdot\vec x=0$ at $r=R$ $\quad |\vec x|=R\longrightarrow C_3=\frac{R^3}{2}$

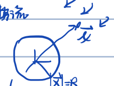
$$\boxed{\begin{aligned}\phi&=\vec V\cdot\frac{\vec x}{r}\left(C_2r+\frac{C_3}{r^2}\right)\\ \vec u=\nabla\phi&=V_j\left[-3\frac{C_3}{r^3}\frac{x_ix_j}{r^2}+\left(C_2+\frac{C_3}{r^3}\right)\delta_{ij}\right]=u_i\end{aligned}}$$

$$\boxed{\begin{aligned}\phi&=\vec V\cdot\vec x\left(1+\frac{R^3}{2r^3}\right)\\ u_i&=V_j\left[-3\frac{R^3}{2r^3}\frac{x_ix_j}{r^2}+\left(1+\frac{R^3}{2r^3}\right)\delta_{ij}\right]\end{aligned}}$$

Sphere Potential Flow 绕球势流

BC: $\nabla\phi\xrightarrow{\infty}\vec U$ infinite $\vec U$

Boundary condition: $\vec n\cdot\nabla\phi\big|_{R=a}=0$ No-across Boundary



ϕ: ① scalar; ② linear in $\vec U$ (what we know about ϕ?) $\to\phi=\vec U\cdot\vec x\,g(r)$ must relate to r; make it scalar

$\phi=\vec U\cdot\nabla f(r)$

$=V_j\frac{\partial}{\partial x_j}f(r)=V_jf'\frac{x_j}{r}=\vec V\cdot\vec x\frac{f'}{r}$

Mass conservation $\nabla^2\phi=0$ $\quad \nabla^2[\vec U\cdot\nabla f(r)]$ $\quad \vec U$ constant vector

$=\vec U\cdot\nabla(\nabla^2f(r))$ can change

$\vec U\cdot\nabla(\nabla^2f(r))=0$

$\nabla\nabla^2f(r)=0$

$\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}f(r)\right)=0$

$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\left[\frac{f'}{r}x_j\right]=0$

$\frac{\partial}{\partial x_i}\left[\frac{x_j}{r}\frac{\partial f'}{\partial x_j}-\frac{f'}{r^2}\frac{\partial r}{\partial x_j}x_j+\frac{f'}{r}\delta_{jj}\right]=0$

$\frac{\partial}{\partial x_i}\left(f''+\frac{2f'}{r}\right)=0$

$f''+\frac2rf'=\text{constant}$ $\quad i^2+i=0$

$f=\sum C_ir^i$ $\quad i(i-1)+2i=0$

$f=C_1r+\frac{C_2}{r}$

$\phi=\vec U\cdot\nabla f=\vec U\cdot\frac{\vec x}{r}f'$

$\phi=\vec U\cdot\frac{\vec x}{r}\left(C_1r+\frac{C_2}{r^2}\right)$

$\vec u=\nabla\phi=V_j\left[-3\frac{C_3}{r^3}\frac{x_ix_j}{r^2}+\left(C_2+\frac{C_3}{r^3}\right)\delta_{ij}\right]$

BC: $u\xrightarrow{\infty}\vec U$: $u_i=V_jC_2\delta_{ij}=C_2V_i\to C_2=1$

$\vec u\cdot\vec x\big|_{r=R}=0$: $C_3=\frac{R^3}{2}$

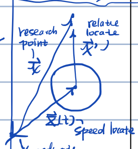
$$\boxed{\begin{aligned}\phi&=\vec U\cdot\vec x\left(1+\frac{R^3}{2r^3}\right)\\ \vec u=\nabla\phi&=V_j\left[-3\frac{R^3}{2r^3}\frac{x_ix_j}{r^2}+\left(1+\frac{R^3}{2r^3}\right)\delta_{ij}\right]\end{aligned}}$$

Moving Sphere 动球绕流

① Attach on sphere, see ∞ flow

$\vec V=-\vec V$ (1 come, like it close) 相对运动

"∞ flow" $\quad$ sphere velocity



② $\vec x\longrightarrow\vec x'=\vec x-\vec X(t)$ relative distance from center

$\phi=\underbrace{-\vec V\cdot(\vec x-\vec X)\left(1+\frac{R^3}{2r^3}\right)}_{\text{flow potential on sphere coord.}}+\underbrace{\vec V\cdot(\vec x-\vec X)}_{\text{sphere coord. moving (speed}\cdot\vec x\text{) potential}}$

total flow potential

$$\to\left[\begin{aligned}\phi&=-\vec V\cdot(\vec x-\vec X)\frac{R^3}{2r^3}\\ \vec u&=-\frac{R^3}{2r^3}\vec V+\frac{3R^3}{2r^3}\frac{[\vec V\cdot(\vec x-\vec X)](\vec x-\vec X)}{r^2}\end{aligned}\right.$$ ②

Moving Sphere:

- If we attach ourself to sphere

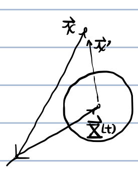
We see flow from ∞ with $\vec V=-\vec V$

(where $\vec V$ is sphere velocity)

- Replace $\vec x=\vec x-\vec X(t)$

$\therefore\phi=-\vec V\cdot(\vec x-\vec X(t))\left(\frac{R^3}{2r^3}+1\right)+\vec V\cdot(\vec x-\vec X)$

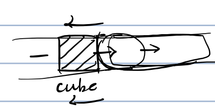
$$\boxed{\begin{aligned}\phi&=-\vec V\cdot(\vec x-\vec X)\frac{R^2}{2r^3}\\ \vec u&=-\frac{R^3}{2r^3}\vec V+\frac{3R^3}{2r^3}\left[\frac{(\vec V\cdot(\vec x-\vec X))(\vec x-\vec X)}{r^2}\right]\end{aligned}}$$



$u_r=\frac{u_ix_i}{r}=\vec U\cdot\frac{\vec x}{r}\left(1-\frac{R^3}{r^3}\right)$

$u_\perp=\vec u-u_r\frac{\vec x}{r}$

radio velocity point to normal direction



cube

Bernulli's Equation

moving sphere = moving coordinate

$\frac{\partial\phi}{\partial t}+\frac12u^2+\frac{p}{\rho}=F(t)$ $\quad \frac{\partial\phi}{\partial\vec x}=-\frac{\partial\phi}{\partial\vec x}$

$\frac{\partial\phi}{\partial t}\Big|_{\vec x}=\frac{\partial\phi}{\partial t}\Big|_{\vec x,\vec X}+\dot{\vec X}\cdot\nabla_{\vec X}\phi=\frac{\partial\phi}{\partial t}\Big|+\vec V\cdot(-\nabla_{\vec x}\phi)=\frac{\partial\phi}{\partial t}\Big|_{\vec x,\vec X}-\vec V\cdot\vec u$

fixed $\vec x$ location; change because $\vec x$, $\vec X$ motion of sphere

see class notes

$\frac{\partial\phi}{\partial t}+\frac12u^2+\frac{p}{\rho}=F(t)$

To find $F(t)$, evaluate ∞ where $\frac{\partial\phi}{\partial t}=0$, $u=0$, $p=p_0$

$\frac{\partial\phi}{\partial t}+\frac12u^2+\frac{p}{\rho}=\frac{p_0}{\rho}$

evaluate on sphere surfaces

$P=P_a+\frac{\rho}{2}\left[R\dot{\vec V}\cdot\vec n-\frac14V^2\,\frac94(\vec V\cdot\vec n)^2\right]$

Bernulli's Eqn.

$\frac{\partial \phi}{\partial t} + \frac{1}{2}u^2 + \frac{p}{\rho} = F(t)$

$d\phi = \frac{\partial \phi}{\partial t} dt + dx \frac{\partial \phi}{\partial x}$

Cartesian Tensors

A_{ij} : Second Rank Tensor

$a_i b_j$: dyadic tensor $u_i u_j = \begin{pmatrix} u_1u_1 & u_1u_2 & u_1u_3 \\ u_2u_1 & u_2u_2 & u_2u_3 \\ u_3u_1 & u_3u_2 & u_3u_3 \end{pmatrix}$

New Frame $\vec{u}$ = Old Frame $\vec{u}$

$u_j' \vec{e}_j' = u_k \vec{e}_k = \vec{u}$ (new basis, old basis)

$\cdot \vec{e}_j'$: $u_j' \vec{e}_j' \cdot \vec{e}_j' = u_k \vec{e}_k \cdot \vec{e}_j'$ (rotation tensor)

$[u_j' = u_k q_{jk}] \to [\vec{u}' = Q\vec{u}]$ new $\vec{u}$ = old $\vec{u}$ · rotation matrix Q

For Tensor $u_i v_j$ Transform: $u_i' v_j' = (u_k q_{ik})(v_l q_{jl})$ $u_i v_j = u_i \cdot v_j$ (transform transform transform)

$\sigma' = Q \sigma Q^T$ $[u_i'][v_j']^T = Q[u_i](Q[v_j])^T$

$T' = Q T Q^T$

$T_{ij}' = q_{ik} T_{kl} q_{lj}$

For Kron tensor δ: $\delta_{ij}' = q_{ik}\delta_{kl} q_{jl}$ ($l=j$)

$= q_{ik} q_{jl}$

$= QQ^T = \delta_{ij}$ ORThogonal $q_{ai} q_{aj} = \delta_{ij}$

Product Tensor $\underline{a} \cdot \vec{u} = a_{ij} u_j$

$\vec{u} \cdot \underline{a} = u_i a_{ij}$

$(\underline{a} \cdot \underline{b})_{ij} = a_{ik} b_{kj}$

$(\underline{b} \cdot \underline{a})_{ij} = b_{ik} a_{kj}$

$A:b = A_{ij} B_{ij}$

~~$\underline{a} : \underline{b} = a_{ik} b_{ki} = \underline{b} : \underline{a}$~~

Symmetric Tensor $S_{ij} = S_{ji}$

Anti-Symmetric Tensor $A_{ij} = -A_{ji}$

Kinetic Energy

$K = \frac{1}{2}(\sum m_i v_i^2) = \frac{1}{2}\int_V \rho |\nabla\phi|^2 dV = \frac{1}{2}\int_V \rho \nabla \cdot (\phi \nabla \phi) dV$

$= \frac{1}{2}\oint_S \rho \phi \nabla \phi \cdot \vec{n}\, ds$

Flow on Moving Sphere:

$\phi = -\vec{v}(\vec{x} - \vec{X}) \frac{R^3}{2r^3}$ (control volume ∞)

$K = -\frac{1}{2}\oint_S \rho \phi \nabla\phi \cdot \vec{n}\, ds$

$= -\frac{\rho}{2}\left[\int_{sphere} \phi \nabla\phi \cdot \vec{n}\, ds + \int_{S\infty} \phi \nabla\phi \cdot \vec{n}\, ds\right]$ ($\phi = \phi_R$, $\nabla\phi = 0$, only sphere move)

$= -\frac{\rho}{2}\int_{sphere} -\frac{1}{2}\vec{v}\vec{n}R(\vec{v}\cdot\vec{n}) dS_{sphere}$

$= \frac{\rho}{4}\int_S v_i n_i R\, v_k n_k\, dS_{sphere}$

$= \frac{\rho}{4} v_i v_k \int_S n_i n_k R\, dS_{sphere}$ only $i=k$, $n_i n_k = 1$

$= \frac{\rho}{4} v_i v_k \frac{4}{3}\pi R^3 \delta_{ik}$

$= \frac{\rho}{4}\left(\frac{4}{3}\pi R^3\right)\vec{v}^2$

$\frac{1}{2} m_{added} \vec{v}^2$

hydrodynamics Force $\vec{F}_{hd} = -\oint_{sphere} p \cdot \vec{n}\, ds = \oint_{sphere} \left[p_\infty + \frac{\rho}{2}\left[R\dot{\vec{v}}\cdot\vec{n} - \frac{1}{4}\dot{\vec{x}}^2 - \frac{9}{4}(\dot{\vec{x}}\cdot\vec{n})^2\right]\right]\vec{n}\, ds$

$= -\oint_{sphere} \frac{\rho R}{2}\dot{\vec{v}}\cdot\vec{n}\,\vec{n}\, ds = -\oint_{sphere} \frac{\rho}{2} R \dot{v}_j n_j n_i\, ds$

$= -\frac{\rho}{2}\left(R\dot{v}_j \delta_{ij} \frac{4}{3}\pi R^2\right)$

$$\left[\vec{F}_{hd} = -\underbrace{\frac{1}{2}\frac{4}{3}\pi R^3 \rho}_{\text{added mass}}\dot{\vec{v}}_i\right]$$ hydrodynamic field of [Flow]

If accelerating → hydrodynamic Force

also need to accelerate fluid around

Eqn. of motion of sphere

$\vec{F} + \vec{F}_{hd} = m_s \dot{\vec{v}}$

$\vec{F} - m_{added}\dot{\vec{v}} = m_s \dot{\vec{v}}$

$\vec{F} = (m_s + m_{added})\dot{\vec{v}}$

Effect moving (also depend on shape)

因为球体对称，不拘于形状

故此时 Madded 为一个常数

若为 cube / 其他，则取决于物体形状

moving a solid sphere in air: $m_s = \frac{4}{3}\pi R^3 \rho_{solid}$, $m_a = \frac{4}{3}\pi R^3 \rho_{air}$ NOT important

moving a bubble in water: ~~$m_s = \frac{4}{3}\pi R^3 \rho_{air}$~~, $m_a = \frac{4}{3}\pi R^3 \rho_{water}$

Cartesian tensors

A_{ij}: Second rank tensor Ex. $u_i u_j = \begin{pmatrix} u_1u_1 & u_1u_2 & u_1u_3 \\ u_2u_1 & u_2u_2 & u_2u_3 \\ u_3u_1 & u_3u_2 & u_3u_3 \end{pmatrix}$ (momentum, velocity)

$a_i b_j$: dyadic tensor

how to transform a vector?

$u_j' \vec{e}_j' = u_k \vec{e}_k$ ✓

Take dot product with $\vec{e}_j'$

$u_j' = u_k \vec{e}_j' \cdot \vec{e}_k$

$u_j' = q_{jk} u_k$

$\vec{u}' = Q\vec{u}$ Q rotation matrix $q_{ik}(q_{kj})^T = \delta_{ik} q_j$

$Q^T Q = Q Q^T = I$ Orthogonal matrix

$u_i' v_j' = q_{ik} u_k q_{jl} v_l$

$\sigma' = Q \sigma Q^T$

Kronecker δ_{ij} in new coord.

$\delta_{ij}' = q_{ik}\delta_{kl} q_{lj}^T = q_{ik}\delta_{kl} q_{jl}$

$\delta_{ij}' = q_{ik} q_{kj}^T = QQ^T = \delta_{ij}$

Product tensor & vector

$\underline{a} \cdot \vec{u} = a_{ij} u_j$

$\vec{u} \cdot \underline{a} = u_i a_{ij}$ ≠

Product of 2 tensors $\underline{a}$ & $\underline{b}$

$(a \cdot b)_{ij} = a_{ik} b_{kj}$

$(b \cdot a)_{ij} = b_{ik} a_{kj}$ ≠

$\underline{a} : \underline{b} = a_{ij} b_{ji} = Tr(a \cdot b)$

$\underline{a} : \underline{b} = \underline{b} : \underline{a}$ ✓

QAQ^T

$= q_{ai} A_{ij} q_{bj}$

Trace$(Q \cdot A \cdot Q^T)$

~~$= q_{ij} A_{ij} q_{jk}$~~

Symmetric Tensor $S_{ij} = S_{ji}$

Anti-Symmetric $A_{ij} = -A_{ji}$

$\sigma_{ij} = \underbrace{\frac{1}{2}(\sigma_{ij} + \sigma_{ji})}_{\text{Symmetric}} + \underbrace{\frac{1}{2}(\sigma_{ij} - \sigma_{ji})}_{\text{anti-Symmetric}}$

$A_{ij} = u_i v_j$

$A\vec{x} = \lambda\vec{x}$

$A_{ij} x_j = \lambda x_i$

$u_i v_j x_j = \lambda x_i$

$\lambda = \vec{u} \cdot \vec{v}$

$x = A_i$

- Kinetic energy in potential flow
- Mathematical foundation

Infinity ∞, out normal to fluid

$K = \frac{1}{2}\int_V \rho |\nabla\phi|^2 dV = \frac{\rho}{2}\int_V \nabla \cdot (\phi\nabla\phi) dV = \frac{\rho}{2}\oint_S \phi\nabla\phi \cdot \vec{n}\, ds$

$\nabla \cdot (\phi\nabla\phi) = \nabla\phi \cdot \nabla\phi + \phi \nabla^2\phi$ (0 for potential flow)

(often $\to$ 0 as $r \to \infty$)

$\frac{\partial}{\partial x_j}\left(\phi \frac{\partial\phi}{\partial x_j}\right) = \frac{\partial\phi}{\partial x_j}\cdot\frac{\partial\phi}{\partial x_j} + \phi \cdot \frac{\partial^2\phi}{\partial x_j \partial x_j}$

Ex. Moving Sphere

$K = -\frac{\rho}{2}\oint_S \phi\nabla\phi \cdot \vec{n}\, ds$

$\phi = -\vec{v}(\vec{x} - \vec{X})\frac{R^3}{r^3}$

$K = -\frac{\rho}{2}\left[\int_S \phi(\nabla\phi\cdot\vec{n}) dS_{sphere} + \int_{S\infty} \phi\nabla\phi \cdot \vec{n}\, ds\right]$ ($\to 0$)

$= -\frac{\rho}{2}\int_{sphere} -\frac{1}{2}\vec{v}\vec{n}R(\vec{v}\cdot\vec{n})\, dS_{sphere}$ on sphere $\vec{x} - \vec{X} = \vec{x}'$, $\vec{x}' = \vec{n}R$

$= \frac{\rho}{4}\int_S v_i n_i v_j n_j R\, dS_{sphere}$

$= \frac{\rho}{4} v_i v_j \left[\int_{sphere} \frac{R}{r} n_i n_j\, dS_{sphere}\right]$, $\frac{4}{3}\pi R^3 \delta_{ij}$

$= \frac{\rho}{4} v_i v_j \frac{4}{3}\pi R^3 \delta_{ij}$

$= \frac{\rho}{2}\left(\frac{4}{3}\pi R^3\right)\frac{1}{2}|\vec{v}|^2$

$= \frac{1}{2} m_{added} |\vec{v}|^2$

Madded $= \frac{1}{2}\frac{4}{3}\pi R^3\rho$ Madded $= 3\left(\frac{4}{3}\pi R^3\right)\rho$

Ex. Expanding Bubble

$\phi = -\frac{R^2}{r}\dot{R}$ on the sphere $\phi = -R\dot{R}$

$\nabla\phi \cdot \vec{n} = \dot{R}$

$K = +\frac{\rho}{2}\int_S \dot{R}\cdot R\dot{R}^2\, dS_{sphere} - \frac{\rho}{2}\int_{S\infty} \phi(\nabla\phi\cdot\vec{n}) dS = \frac{\rho}{2}(4\pi R^3)\dot{R}^2$

$= \frac{1}{2}\rho\left(3 \cdot \frac{4}{3}\pi R^3\right)\dot{R}^2$ (Madded)

① + Practice

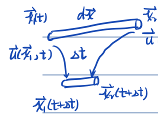
MIDTERM 30th. 3h close book NEXT 2 Assignment

Tensors. $M_{ij} = a_i b_j$

$M_{ij} = \frac{1}{2}(a_i b_j + a_j b_i) + \frac{1}{2}(a_i b_j - a_j b_i)$

$M_{ij} = S_{ij} + A_{ij}$

Local kinematic.



$\vec{x}_1(t)$ known

$\vec{x}_2(t) = \vec{x}_1(t) + d\vec{x}\cdot\partial_{\vec{x}}\vec{x}_1(t)$

$\vec{u}(\vec{x}_2,t) = \vec{u}(\vec{x}_1,t) + d\vec{x}\cdot\partial_{\vec{x}}\vec{u}(\vec{x}_1,t)$

Δt after $\vec{x}_1(t+\Delta t) = \vec{x}_1(t) + \vec{u}(\vec{x}_1,t)\,dt$

$\vec{x}_2(t+\Delta t) = \vec{x}_2(t) + \vec{u}(\vec{x}_2,t)\,dt$

$\vec{x}_2(t+\Delta t) = \vec{x}_2(t) + \vec{u}(\vec{x}_1,t)\,dt + d\vec{x}\,\partial_{\vec{x}}\vec{u}(\vec{x}_1,t)\,dt$

$\ominus$

$d\vec{x}(t+\Delta t) = d\vec{x}(t) + d\vec{x}\,\nabla\vec{u}\,dt$

$\frac{d}{dt}(d\vec{x}) = d\vec{x}\cdot\nabla\vec{u}$

Tensors $M_{ij} = a_i b_j$

$M_{ij} = \frac{1}{2}(a_ib_j + a_jb_i) + \frac{1}{2}(a_ib_j - a_jb_i)$

$M_{ij} = S_{ij}$ symmetric matrix $+ A_{ij}$ Anti-Symmetric matrix

$A_{ij}\nu_j = \frac{1}{2}(a_ib_j - a_jb_i)\nu_j$

$= \frac{1}{2}(a_ib_j\nu_j - \nu_j a_j b_i)$

$= \frac{1}{2}[\vec{a}(\vec{b}\cdot\vec{\nu}) - (\vec{\nu}\cdot\vec{a})\vec{b}]$

$= \vec{\nu}\times(\frac{1}{2}\vec{a}\times\vec{b})$?

Velocity Gradient Tensor

$$\nabla\vec{u} = \frac{\partial u_j}{\partial x_i}\hat{e}_i\hat{e}_j = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

Stretch

$$\nabla\vec{u} = \frac{\partial u_j}{\partial x_i} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) + \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right) = e_{ij} + \Omega_{ij}$$

e_{ij} deformation, e_{ii} Stretch, Ω_{ij} rotation

Volume change $\frac{1}{3}e_{kk}\delta_{ij}$

Rotation $\nabla_i u_j - \nabla_j u_i$? $e = n_i S_{ij} n_j$

Stretch $\boxed{e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}}$ e_{ij}

angle change $\frac{d}{dt}\cos\theta_{ij} = 2e_{ij} = \nabla_i u_j + \nabla_j u_i$ (i, j are orthogonal)

If NOT orthogonal: $\frac{d}{dt}\cos\theta_{12} = n_1\cdot e\cdot n_2 + (n_1\cdot e\cdot n_1 + n_2\cdot e\cdot n_2)\,n_1\cdot n_2$

Eigen Value problem

$\lambda^3 - \mathrm{I}\lambda^2 + \mathrm{II}\lambda - \mathrm{III}\,\mathbb{1} = 0$

$\mathrm{Tr}(\sigma)$, $\frac{1}{2}(\mathrm{Tr}^2 - \mathrm{Tr}(\sigma^2))$, $\det(\sigma)$

$\lambda_1+\lambda_2+\lambda_3$ length; $\frac{1}{2}(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)$ Area; $\lambda_1\lambda_2\lambda_3$ Volume

Cayley-Hamilton Thm.

$\sigma^3 - \mathrm{I}\sigma^2 + \mathrm{II}\sigma - \mathrm{III}\,\mathbb{1} = 0$

Cauchy Momentum Eqn.

Momentum change rate $\frac{d}{dt}\langle M\vec{v}\rangle = \vec{F} = \vec{F}_{volume} + \vec{F}_{surface}$

Newton 2nd law. $\frac{d}{dt}\int_{V_2}\rho\vec{u}\,dV_2 = \int_{V_2}\rho\vec{g}\,dV_2 + \oint\vec{t}(\vec{n})\,ds$

$\frac{d}{dt}\int_{V_2}\rho\vec{u}\,dV + \oint_S \rho\vec{u}(\vec{u}\cdot\vec{n})$ div Thm $\tau = \sigma_{ij}r_j$

a) $\int_{V_2}\rho\left(\frac{D\vec{u}}{Dt}\right)dV_2 = \int_{V_2}\rho\vec{g} + \nabla\cdot\underline{\underline{\tau}}\,dV_2$

$= \frac{D\rho u}{Dt} - u\frac{D\rho}{Dt}$

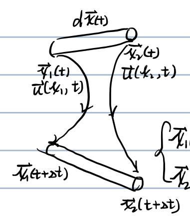
b) $\int_{V_2}\frac{\partial\rho u}{\partial t} + \nabla(\rho\vec{u}\vec{u})\,dV_2 = \int_{V_2}\rho\vec{g} + \nabla\cdot\tau\,dV_2$

get. $\begin{cases}\rho\frac{D\vec{u}}{Dt} = \rho\vec{g} + \nabla\cdot\underline{\underline{\tau}} \\ \frac{\partial\rho u}{\partial t} + \nabla\cdot(\rho u u) = \rho\vec{g} + \nabla\cdot\tau\end{cases}$

Anti Symmetric tensor has an associated dual vector $\vec{d}$

$A_{ij}\nu_j = \frac{1}{2}(a_ib_j - a_jb_i)\nu_j$ $= \vec{d}$

$= \frac{1}{2}((\vec{b}\cdot\vec{\nu})\vec{a} - (\vec{a}\cdot\vec{\nu})\vec{b}) = (\vec{\nu}\times(\frac{1}{2}\vec{a}\times\vec{b}))_i$



$\vec{u}(\vec{x}_2,t) = \vec{u}(\vec{x}_1,t) + d\vec{x}\cdot\nabla\vec{u}(\vec{x}_1,t)$

$u_j(\vec{x}_2,t) = u_j(\vec{x}_1,t) + dx_j\frac{\partial u_i(\vec{x}_1,t)}{\partial x_j}$

Velocity gradient tensor. $\nabla\vec{u} = \frac{\partial u_j}{\partial x_i} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$ Shear, stretch

$\begin{cases}\vec{x}_1(t+\Delta t) = \vec{x}_1(t) + \vec{u}(\vec{x}_1,t)\,dt \\ \vec{x}_2(t+\Delta t) = \vec{x}_2(t) + \vec{u}(\vec{x}_2,t)\,dt\end{cases}$

$= \vec{x}_2(t) + [\vec{u}(\vec{x}_1,t) + d\vec{x}\cdot\nabla\vec{u}(\vec{x}_1,t)]\,dt$

$\ominus$ $d\vec{x}(t+\Delta t) = d\vec{x}(t) + d\vec{x}\cdot\nabla\vec{u}(\vec{x},t)\,dt$

$\frac{d\vec{x}(t+\Delta t) - d\vec{x}(t)}{dt} = d\vec{x}\cdot\nabla\vec{u}(\vec{x}_1,t)$

$\frac{d}{dt}(d\vec{x}) = d\vec{x}\cdot\nabla\vec{u}$ Project Velocity Gradient $\nabla\vec{u}$ to element direction $d\vec{x}$

$$\nabla\vec{u} = \frac{\partial u_j}{\partial x_i} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) + \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right)$$

$$\frac{\partial u_j}{\partial x_i} = e_{ij} + \Omega_{ij}$$

$$= \frac{1}{2}\begin{pmatrix} & \frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1} & \frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1} \\ & & \frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2} \\ & & \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & \frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1} & \frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1} \\ & 0 & \frac{\partial u_2}{\partial x_3}-\frac{\partial u_3}{\partial x_2} \\ & & 0 \end{pmatrix}$$

e_{ij}, e_{ii}, Ω_{ij}

Interpretation

$e_{11} = \frac{\partial u_1}{\partial x_1}$ $\frac{\partial u_1}{\partial x_1} > 0$ Stretch

$e_{22} = \frac{\partial u_2}{\partial x_2}$ $\frac{\partial u_2}{\partial x_2} > 0$ Stretch

$e_{kk} = \nabla\cdot\vec{u} = \frac{1}{V}\frac{dV}{dt}$ volume change e_{ii} Diagonal elements of e_{ij} Volume change

$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$ deformation e_{ij} off-Diagonal $\rightarrow$ volume pussing deformation

$\frac{d}{dt}\cos\theta_{12} = 2\cdot e_{12}$

Ω_{ij} related to rotation

$\frac{\partial u_j}{\partial x_i} = e_{ij} + \Omega_{ij}$ $\tilde{e}_{ij} = e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}$ Remove trace de

$= (\tilde{e}_{ij} + \frac{1}{3}e_{kk}\delta_{ij}) + \Omega_{ij}$ $\underbrace{\tilde{e}_{mm}}_{\text{trace}} = e_{mm} - \frac{1}{3}e_{kk}\delta_{mm} = 0$

$\frac{\partial}{\partial x_j}$ $\vec{b}$ $\frac{\partial u_j}{\partial x_i}a_j = (\nabla u)_{ij}a_j$

$(\nabla u)\cdot\vec{a}$ Dual of N-S

Examples. ① $u_i = \frac{1}{3}\dot{\epsilon}x_i = (\frac{\dot{\epsilon}}{3}x_1, \frac{\dot{\epsilon}}{3}x_2, \frac{\dot{\epsilon}}{3}x_3)$

$\nabla u = \begin{pmatrix}\frac{\dot{\epsilon}}{3} & 0 & 0 \\ 0 & \frac{\dot{\epsilon}}{3} & 0 \\ 0 & 0 & \frac{\dot{\epsilon}}{3}\end{pmatrix}$ $a_j\frac{\partial u_i}{\partial x_j}$

$\vec{\nabla}\cdot\vec{u} = \frac{\dot{\epsilon}}{3} + \frac{\dot{\epsilon}}{3} + \frac{\dot{\epsilon}}{3} = \dot{\epsilon}$ (trace) uniform dilatation

? $\vec{a}\cdot\nabla\vec{u} = a_j(\nabla u)_{ij}$ $a_j(\nabla u)_{ij}$

$(\vec{a}\cdot\vec{\nabla})\vec{u} = a_i\nabla_i u_j$

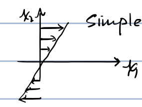
② $\nabla\vec{u} = \begin{pmatrix}\dot{\epsilon} & 0 & 0 \\ 0 & \dot{\epsilon} & 0 \\ 0 & 0 & \dot{\epsilon}\end{pmatrix}$ $\left(\frac{\partial}{\partial x_j}\right)(\leftarrow u_j \rightarrow)$

Incompressible $\nabla\cdot\vec{u} = 0$ $\dot{\epsilon}_1 + \dot{\epsilon}_2 + \dot{\epsilon}_3 = 0$

③ Simple Shear $u_1 = \dot{\gamma}x_2$ Simple shear flow. $\vec{b}\cdot\underline{A} \neq \underline{A}\cdot\vec{b}$

$$\frac{\partial u_j}{\partial x_i} = \begin{pmatrix}0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & \frac{\dot{\gamma}}{2} & 0 \\ \frac{\dot{\gamma}}{2} & 0 & 0 \\ 0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & \frac{\dot{\gamma}}{2} & 0 \\ -\frac{\dot{\gamma}}{2} & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}$$

rotation



$\frac{\partial\phi}{\partial t}$

Laudue Paintain white   think about problem itself

$\frac{\partial u_j}{\partial x_i} = \underbrace{e_{ij}}_{\text{Symmetric}} + \underbrace{\Omega_{ij}}_{\text{Anti-symmetric}}$

$= \underbrace{\tilde{e}_{ij}}_{\text{deviatoric}} + \underbrace{\frac{1}{3}e_{kk}\delta_{ij}}_{\text{isotropic}} + \Omega_{ij}$ $\mathrm{Tr}(\tilde{e}_{ij}) = 0$

Ex. $\begin{pmatrix}2a & b & c \\ b & 2a & d \\ c & d & -a\end{pmatrix} = e = \begin{pmatrix}a & b & c \\ b & a & d \\ c & d & -2a\end{pmatrix} + \frac{1}{3}\begin{pmatrix}3a & 0 & 0 \\ 0 & 3a & 0 \\ 0 & 0 & 3a\end{pmatrix}$

$\mathrm{Tr} = 3a$ $\tilde{e}$ Stretch $+\frac{1}{3}e_{kk}\delta_{ij}$ Volume change

Ex. $u = (\dot{\gamma}y, 0, 0)$ $y = h$, $y = 0$

$\nabla u = \begin{pmatrix}0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & \frac{\dot{\gamma}}{2} & 0 \\ \frac{\dot{\gamma}}{2} & 0 & 0 \\ 0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & \frac{\dot{\gamma}}{2} & 0 \\ -\frac{\dot{\gamma}}{2} & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}$

$\vec{u} = (\dot{\gamma}y, \dot{\gamma}x, 0)$

$\nabla u = \begin{pmatrix}0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0\end{pmatrix} + 0$ NOT rotate

$\frac{d\theta_{12}}{dt} = 2e_{12}$

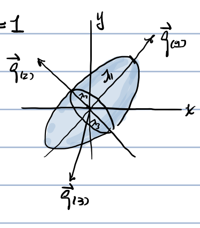
Diagonlize Symmetric Tensor

Symmetric tensor σ

$\sigma = Q\underbrace{\Lambda}_{\text{diagonal}}Q^T$ $QQ^T = \mathbb{1}$

$\sigma\vec{q}_{(i)} = \lambda_{(i)}\vec{q}_{(i)}$

$Q^T\sigma Q = \Lambda$



Eigen Value problem

$\lambda^3 - \mathrm{I}\lambda^2 + \mathrm{II}\lambda - \mathrm{III}\,\mathbb{1} = 0$

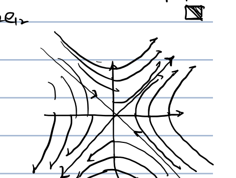
$\mathrm{Tr}(\sigma)$ $\frac{1}{2}(\mathrm{Tr}^2 - \mathrm{Tr}(\sigma^2))$ $\det(\sigma)$

The three invariants

$\mathrm{I} = \mathrm{Tr}(\sigma) = \lambda_1 + \lambda_2 + \lambda_3$ length

$\mathrm{II} = \frac{1}{2}[\mathrm{Tr}^2(\sigma) - \mathrm{Tr}(\sigma^2)] = \frac{1}{2}[\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1]$ Area

$\mathrm{III} = \lambda_1\lambda_2\lambda_3$ Volume



Cauchy Momentum Eqn.

$ma = \Sigma F$ Momentum change speed $=$ Volume force $+$ surface force

$\frac{d}{dt}\int_{V_2}\rho\vec{u}\,dV_2 = \int_{V_2}\underbrace{\rho\vec{g}\,dV_2}_{\text{Volume}} + \oint\underbrace{\vec{t}(\vec{n})\,ds}_{\text{Surface}}$

$t_i = \sigma_{ij}r_j$?

$\frac{d}{dt}\int_{V_2}\rho\vec{u}\,dV_2 = \int_{V_2}\rho\vec{g}\,dV_2 + \oint\underline{\underline{\sigma}}\cdot\vec{n}\,ds$ div Thm

$\frac{d}{dt}\int_{V_2}\rho\vec{u}\,dV_2 = \int_{V_2}\rho\vec{g} + \nabla\cdot\underline{\underline{\tau}}\,dV_2$

Mass conservation

a) $\int_{V_2}\rho\frac{D\vec{u}}{Dt}\,dV_2 = \int_{V_2}\rho\vec{g} + \nabla\cdot\tau\,dV_2$ $\longrightarrow$ $\begin{cases}\rho\frac{D\vec{u}}{Dt} = \rho\vec{g} + \nabla\cdot\sigma \\ \frac{\partial\rho u}{\partial t} + \nabla(\rho\vec{u}\vec{u}) = \rho\vec{g} + \nabla\cdot\tau\end{cases}$

$\rho\vec{u} \rightarrow \rho\vec{u}$ b) $\int_{V_2}\frac{\partial\rho u}{\partial t} + \nabla(\rho\vec{u}\vec{u})\,dV_2 = \int_{V_2}\rho\vec{g} + \nabla\cdot\tau\,dV_2$

Must model σ in terms of $\frac{\partial u_i}{\partial x_j}$

Cayley-Hamiltion Thm.

σ satisfy its char-eqn.

$\sigma^3 - \mathrm{I}\sigma^2 + \mathrm{II}\sigma - \mathrm{III}\,\mathbb{1}_d = 0$

Can write any $\sigma^{n\geq 3}$ to $\sigma^0, \sigma^1, \sigma^3$

In general, $\sigma_{ij} = \tilde{\sigma}_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij}$, isotropic

$\sigma_{ij} = \tilde{\sigma}_{ij} - P_m\delta_{ij}$, P_m Mechanical pressure

Model σ_{ij}

$$\left[\sigma_{ij} = \tau_{ij} - P\delta_{ij}\right]$$

$$= \tau_{ij} - \left(-\tfrac{1}{3}\tau_{kk} + P\right)\delta_{ij}$$

$$\tau_{ij} = f_{ij}(e_{kl}, P, T, \cancel{\nabla_{ij}})$$

Stokeram Flow.

$$\tau_{ij} = 2\mu e_{ij} + \mu' e_{kk}\delta_{ij} + \cancel{\mu'' e_{ik}e_{kj}}$$

τ_{ij} linear

$$\tau_{ij} = 2\mu e_{ij} + \mu' e_{kk}\delta_{ij}$$

$$\tau_{ij} = 2\mu e_{ij} + \underbrace{\left(\tfrac{2}{3}\mu + \mu'\right)}_{\xi}\cancel{e_{kk}\delta_{ij}}$$

Incompressible Flow: $e_{kk} = \frac{\partial u_k}{\partial x_k} = \nabla\cdot\vec{u} = 0$

$$\tau_{ij} = 2\mu e_{ij} = 2\mu\left(\nabla\underset{\sim}{u} + \nabla\underset{\sim}{u}^T\right)$$

relate P_m, P:

$$\sigma_{ij} = \sigma_{ij}$$

$$\tilde{\sigma}_{ij} - P_m\delta_{ij} = \tau_{ij} - P\delta_{ij}$$

$$\tilde{\sigma}_{ij} - P_m\delta_{ij} = 2\mu e_{ij} + \xi e_{kk}\delta_{ij} - P\delta_{ij}$$

$$P_m = P - \xi e_{kk}$$

or $P_m = P - \xi\nabla\cdot\vec{u}$

Navier-Stokes

$$\rho\left(\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla p + \mu\left(\nabla^2\vec{u} + \rho\vec{g}\right)$$

$$\rho\left(\frac{\partial u_j}{\partial t} + (u_k\nabla_k)u_j\right) = -\nabla_j p + \mu\left(\nabla^2 u_j + \rho g_j\right)$$

uni-direction flow. 单维流动 $\vec{u} = (u, \overset{0}{v}, \overset{0}{w})$

continuity $\rho\nabla\cdot\vec{u} = 0 \longrightarrow \frac{\partial u}{\partial x} = 0$

fully developed / Streamwise invariant

Momentum.

$$\rho\left(\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla p + \mu\left(\nabla^2\vec{u}\right) + \rho\vec{g}$$

x: $\frac{\partial u}{\partial t} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} + \cancel{w\frac{\partial u}{\partial z}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\left[\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla_\perp^2 u\right]$$

Let's model σ_{ij}

① Objectivity: Stress should not depend on solid body motion

express σ in terms of e_{ij}

② Seperate σ_{ij} into viscous & pressure terms

$$\sigma_{ij} = \underbrace{\tau_{ij}}_{\text{viscous}} - P\delta_{ij}$$

$$= \tau_{ij} - \left(-\tfrac{1}{3}\tau_{kk} + P\right)\delta_{ij}$$

$$\tau_{ij} = f_{ij}(e_{kl}, P, T, \cancel{\nabla_{ij}})$$

$$\tau_{ij} = A_{ijab}e_{ab} + B_{ijabcd}e_{ab}e_{cd} + \dots$$

$$\tau_{ij} = Ae_{ij} + Be_{ik}e_{kj} + Ce_{ik}\cancel{e_{kl}}e_{lj} + \dots$$

Cayley-Hamilton

$$+\left[M\nabla\cdot\vec{u} + N(\nabla\cdot\vec{u})^2 + \dots + P(\mathrm{II}) + Q(\mathrm{III}) + \dots\right]\delta_{ij}$$

Stokeram flow

$$\tau_{ij} = 2\mu e_{ij} + \mu' e_{kk}\delta_{ij} + \cancel{\mu'' e_{ik}e_{kj}}$$

Newtonian fluid τ_{ij} linear 0

$$\left[\tau_{ij} = 2\mu e_{ij} + \mu' e_{kk}\delta_{ij}\right]$$

$$\left[\tau_{ij} = 2\underbrace{\mu}_{\text{shear viscosity}} e_{ij} + \underbrace{\left(\tfrac{2}{3}\mu + \mu'\right)}_{\xi}\underbrace{e_{kk}}_{\text{divergence}}\delta_{ij}\right]$$

For Incompressible flow $e_{kk} = \frac{\partial u_k}{\partial x_k} = \nabla\cdot\vec{u} = 0$

$$\tau_{ij} = 2\mu e_{ij} = 2\mu e_{ij} = 2\mu\left(\nabla\underset{\sim}{u} + \nabla\underset{\sim}{u}^T\right)$$

Let's relate P_m, P

$$\sigma_{ij} = \sigma_{ij}$$

$$\tilde{\sigma}_{ij} - P_m\delta_{ij} = \tau_{ij} - P\delta_{ij}$$

$$\tilde{\sigma}_{ij} - P_m\delta_{ij} = 2\mu e_{ij} + \xi e_{kk}\delta_{ij} - P\delta_{ij}$$

$$\left[P_m = P - \xi e_{kk}\right]$$

or $\left[P_m = P - \xi\nabla\cdot\vec{u}\right]$

Now, substitute ($\sigma_{ij} = \tau_{ij} - P\delta_{ij}$) in Cauchy-Momentum to obtain N-S

$$\rho\frac{D\vec{u}}{Dt} = \nabla\cdot\underline{\Sigma} + \rho\vec{g}$$

$$\rho\frac{D\vec{u}}{Dt} = -\nabla p + \nabla\cdot\left[2\mu\underline{e} + \xi\nabla\cdot\vec{u}\,\underline{1}_d\right] + \rho\vec{g}$$

Continuity $\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{u} = 0$

Simpify to incompressible

$$\frac{1}{\rho}\frac{D\rho}{Dt} = 0 \Rightarrow \nabla\cdot\vec{u} = 0, \quad e_{kk} = 0$$

$\Rightarrow$ continuity $\nabla\cdot\vec{u} = 0$

Navier-Stokes $\rho\left(\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla p + \mu\left(\nabla^2\vec{u} + \rho\vec{g}\right)$

$\frac{\partial u_j}{\partial x_j} = 0$ $\quad \rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu\left(\frac{\partial^2 u_i}{\partial x_j\partial x_j} + \rho g_i\right)$

Navier Stokes
↳ Incompressible
↳ unidirectional

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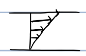
1) N-S Eqn

→ Incompressible fluid

2) Uni-directional flow

→ Couette flow.



Unidirectional flow. $\vec{u} = (u, v=0, w=0)$

Continuity $\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}} + \cancel{\frac{\partial w}{\partial z}} = 0$

$\frac{\partial u}{\partial x} = 0$ fully developed / stream-wise invariant

$u = u(t, y, z)$

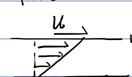
Momentum x-direction $\underbrace{\frac{\partial u}{\partial t}}_{\text{unsteady}} + \underbrace{\cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} + w\frac{\partial u}{\partial z}}_{=0} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$\left[\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla_\perp^2 u\right]$ Laplace orthogonal to $\vec{x}$

Ex. unidirectional flows

1) Couette



$u(y=h) = U$

$u(t=0) = 0$

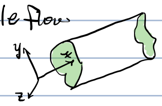
$\frac{\partial u}{\partial x} = 0$

$\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\text{steady}} = -\cancel{\frac{1}{\rho}\frac{\partial p}{\partial x}}_{0} + \nu\left(\frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}_{\text{2D}}\right)$

$$u = U\frac{y}{h}$$

Poiseuille flow

2) Duct flow.



$\frac{\partial u}{\partial x} = 0$

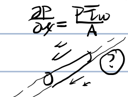
$\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\text{steady}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla_\perp^2 u$

$\frac{\partial p}{\partial x} = \mu\nabla_\perp^2 u \longleftarrow$ Integrate over cross flow plane.

$$\frac{A}{\rho}\frac{\partial p}{\partial x} = \frac{\mu}{\rho}\oint\underbrace{\nabla u\cdot\vec{n}\,dl}_{\tau_w}$$

Impulsively started plate (Rayleigh's problem)
(Stoke's 1st problem)

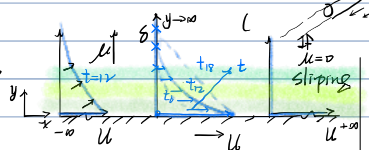
$\frac{\partial P}{\partial x} = \frac{F_{\tau w}}{A}$



Continuity $\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}}^{0} = 0$

$\frac{\partial u}{\partial x} = 0$

$u = u(y,t)$



x-momentum.

$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right)$

$\frac{\partial u}{\partial t} = -\cancel{\frac{1}{\rho}\frac{\partial P}{\partial x}} + \nu\frac{\partial^2 u}{\partial y^2}$ $\quad \mathcal{U} = \mathcal{U}$

No pressure gradient

$\delta] =]L$, $\nu] =]\frac{L^2}{T}$, $t] =]T$ $\Big\} \; \delta \propto \sqrt{\nu t}$ $\quad \eta = \frac{y}{a\sqrt{\nu t}}$

$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}$ s.t. $u(y=0,t) = \mathcal{U}$, $u(y\to\infty, t) = 0$) same as heat equ!

Change of varibles

$u = f(\mathcal{U}, y, t, \nu)$

$u = \mathcal{U}\, F(y,t,\nu)$

$u = \mathcal{U}\, g\left(\frac{y}{\sqrt{\nu t}}\right) = \mathcal{U} f(\eta)$ $\quad \eta = \frac{y}{\sqrt{\nu t}}$

$\frac{\partial}{\partial t} = \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t} = \frac{\partial}{\partial \eta}\left(-\frac{1}{2}\frac{y}{t\sqrt{\nu t}}\right) = \frac{\partial}{\partial t}\left(-\frac{1}{2}\frac{\eta}{t}\right)$

$\frac{\partial}{\partial y} = \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial}{\partial \eta}\frac{1}{\sqrt{\nu t}}$

Rewrite eqn.

$\left(-\frac{1}{2}\frac{\eta}{\cancel{t}}\right)\frac{\partial u}{\partial \eta} = \cancel{\nu}\frac{\partial^2 u}{\partial \eta^2}\left(\frac{1}{\cancel{4\nu t}}\right)$

$\frac{\partial^2 u}{\partial \eta^2} + 2\eta\frac{\partial u}{\partial \eta} = 0$ (subject to)

$\frac{d^2 u}{d\eta^2} + 2\eta\frac{du}{d\eta} = 0$ s.t. $u(\eta = 0) = \mathcal{U}$, $u(\eta\to\infty) = 0$

$u = \mathcal{U} F(\eta)$

$\therefore \frac{d^2F}{d\eta^2} + 2\eta\frac{dF}{d\eta} = 0$ s.t. $F(\eta=0) = 1$, $F(\eta\to\infty) = 0$

$F = A_0 + A_1\, \mathrm{erf}(\eta)$

$F(\eta\to\infty) = 0 \Rightarrow 0 = A_0 + A_1$

$F(\eta = 0) = 1 \Rightarrow A_0 = 1$

$F = 1 - \mathrm{erf}(\eta) = \mathrm{erfc}(\eta)$

$u = \mathcal{U} F$

$\left[\frac{u}{\mathcal{U}} = 1 - \mathrm{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right]$

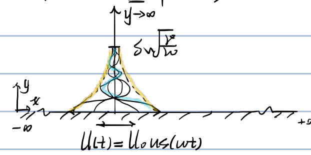
$\left[\tau_{yx}\Big|_{y=0} = -\mu\frac{\partial u}{\partial y}\Big|_0 = -\mu\frac{\mathcal{U}}{\sqrt{\pi\nu t}}\right]$

Stokes' oscillating plate (Stokes' 2nd problem)

$u = U_0 f(y, \nu, t, \omega)$

$u = U_0 F\left(\sqrt{\frac{\omega}{\nu}}\, y, \omega t\right)$

$= U_0 F(\eta, \tau)$



$U(t) = U_0 \cos(\omega t)$

$U(t) = U_0 \mathrm{Re}(e^{i\omega t})$

Continuity

$\frac{\partial u}{\partial x} = 0$

x-Momentum.

$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}$ $\quad$ s.t. $u(y=0,t) = U_0\mathrm{Re}(e^{i\omega t})$, $u(y\to\infty, t) = 0$

$u = U(y)e^{i\omega t}$ assumed form of soln.

$u(y,t) = U_0\exp\left(-\sqrt{\frac{i\omega}{\nu}}\, y + i\omega t\right)$

take real part

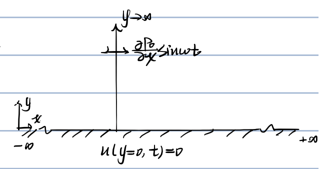
$\left[u(y,t) = \underbrace{U_0\exp\left(-\sqrt{\frac{\omega}{2\nu}}\, y\right)}_{\text{exponential decay}}\underbrace{\cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}\, y\right)}_{\text{ossilation}}\right]$

Shear wave propagating in y
with speed $\frac{\omega}{\sqrt{\frac{\omega}{2\nu}}} = \sqrt{2\nu\omega}$

Related Problem: oscillating pressure gradient

$\frac{\partial u}{\partial x} = 0$

$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial P_0}{\partial x}\sin\omega t + \nu\frac{\partial^2 u}{\partial y^2}$



change of varibles

$x' = x - X_0\sin\omega t$

$u' = u - \omega X_0\cos\omega t$

$t' = t$

Caution: new x' is a non-inertial frame

$\Rightarrow$ Must add a force to RHS of Momentum eqn

equal to minus the acceleraty of frame

$\frac{\partial u'}{\partial t} = -\frac{1}{\rho}\frac{\partial P_0}{\partial x}\sin(\omega t) + \underbrace{\omega^2 X_0\sin(\omega t)}_{\text{acceleratn term}} + \nu\frac{\partial^2 u'}{\partial y^2}$

at $y\to\infty$, $u' = 0$

$+\frac{1}{\rho}\frac{\partial P_0}{\partial x}\cancel{\sin(\omega t)} = \omega^2 X_0\cancel{\sin(\omega t)}$

$X_0 = \frac{1}{\rho\omega^2}\frac{\partial P_0}{\partial x}$

$V_0 = \omega X_0 = \frac{1}{\rho\omega}\frac{\partial P_0}{\partial x}$

at the wall $y=0$ $\quad u'(y=0) = -V_0\cos(\omega t)$

Replace $\mathcal{U} = -V_0$ in previous soln.

$\left[u = U_0\left[-\cos(\omega t) + \exp\left(-\sqrt{\frac{\omega}{2\nu}}\, y\right)\cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}\, y\right)\right]\right]$

<Repeat>

Navier-Stokes. (compressible)

$\left[\begin{array}{l}\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{u} = 0 \\ \rho\frac{D\vec{u}}{Dt} = -\nabla p + \nabla\cdot(\mu\check{\underline{e}}\,2 + \xi\nabla\cdot\vec{u}\,\underline{1}) + \rho\vec{g}\end{array}\right.$

$\check{e} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

$\check{e}_{ij} = e_{ij} - \frac{1}{3}\check{e}_{kk}\delta_{ij}$

$\longrightarrow$ Incompressible

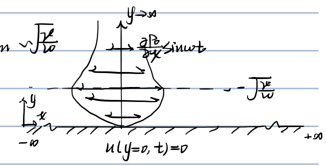
$\frac{1}{\rho}\frac{D\rho}{Dt} \to 0$

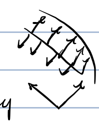
① $\nabla\cdot\vec{u} = 0$

② $\rho\frac{D\vec{u}}{Dt} = -\nabla p + \mu\nabla^2\vec{u} + \rho\vec{g}$

Momentum eqn.

Velocity maxium $\sim\sqrt{\frac{\nu}{\omega}}$





uni-direction flow.

$\vec{u} = (u, v=0, w=0)$ only one direction have velocity

① $\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}}^{0} + \cancel{\frac{\partial w}{\partial z}}^{0} = 0$

$u = u(y,z,t)$ cannot depend on x, but y, z and t

② x-momentum.

$\frac{\partial u}{\partial t} + \underbrace{u\cancel{\frac{\partial u}{\partial x}}}_{\text{contiuty}} + \cancel{v}^{0}\frac{\partial u}{\partial y} + \cancel{w}^{0}\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$

$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$

$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\nabla_1^2 u$

y-momentum

$\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}} + \cancel{w\frac{\partial v}{\partial z}} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}} + \cancel{\frac{\partial^2 v}{\partial z^2}}\right)$

(same as z)

$\frac{\partial P}{\partial y} = 0$

$p \neq f(y,z)$ $\quad p \neq f(y) \neq f(z)$

Uni-Direction flow $\quad \nu = \frac{\mu}{\rho}$

$\left[\begin{array}{l}\text{Mass conservation } \frac{\partial u}{\partial x} = 0 \\ \text{Momentum. } \frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\frac{1}{\rho}\nabla p + \nu(\nabla^2\vec{u} + \rho\vec{g})\end{array}\right.$

x-direction: $\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\nabla_1^2 u$

If μ is NOT constant: $\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) = 0$

Ⓐ Cauette flow

eqn. $\left[\begin{array}{l}\frac{\partial u}{\partial x}=0\\ \frac{\partial \vec{u}}{\partial t}+\vec{u}\cdot\nabla\vec{u}=-\frac{1}{\rho}\nabla P+\nu(\nabla^2\vec{u}+\rho\vec{g})\end{array}\right.$

$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+\rho g_x\right]$

Steady flow, $\frac{\partial u}{\partial x}=0$, $v=0$, $w=0$, No P radient, $\frac{\partial u}{\partial x}=0$, 2D, No gravity

$\nu\frac{\partial^2 u}{\partial y^2}=0 \longrightarrow u=ky$

BC $\left[\begin{array}{l}u(y=0)=0\\ u(y=h)=\mathcal{U}\end{array}\right. \longrightarrow u=\frac{y}{h}\mathcal{U}$

$\tau=\mu\frac{du}{dy}\Big|_h=\mu\frac{\mathcal{U}}{h}$

Ⓑ Laminar duct flow.

eqn. $\left[\begin{array}{l}\frac{\partial u}{\partial x}=0\\ \frac{\partial \vec{u}}{\partial t}+\vec{u}\cdot\nabla\vec{u}=-\frac{1}{\rho}\nabla P+\nu(\nabla^2\vec{u}+\rho\vec{g})\end{array}\right.$

$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho g_x$

Steady flow, $\frac{\partial u}{\partial x}=0$, $v=0$, $w=0$, $\frac{\partial u}{\partial x}=0$, No g

$\int_A\left(\frac{dp}{dx}=\nu\nabla_\perp^2 u\right)$

$A\frac{dP}{dx}=\mathcal{P}\tau_w$

$\frac{dP}{dx}=\frac{1}{A}\mathcal{P}\tau_w$

Implusively started plate

$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}$

$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}$

let $\delta=\sqrt{\nu t}$, then let $\eta=\frac{1}{2}\frac{y}{\delta}=\frac{y}{2\sqrt{\nu t}}$

$u=U_0(1-\mathrm{erf}(\frac{y}{2\sqrt{\nu t}}))$

$\delta\sim\sqrt{\nu t}$

Oscillating plate:

$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}$

$\delta=\sqrt{\frac{\nu}{\omega}}$, $\eta=\frac{y}{\sqrt{2}\delta}=\sqrt{\frac{\omega}{2\nu}}y$

$u=U_0\exp(\sqrt{\frac{\omega}{2\nu}}y)\cdot\cos(\omega t-\sqrt{\frac{\omega}{2\nu}}y)$

$\delta\sim\sqrt{\frac{2\nu}{\omega}}$

$=\frac{\partial b}{\partial t}$

Oscillating Pressure gradiene $\frac{\partial P}{\partial x}=\frac{\partial P_0}{\partial x}\sin\omega t$

as $\frac{\partial P}{\partial x}=\frac{\partial P_0}{\partial x}\sin\omega t$

let $x'=x-X_0\sin\omega t$

$u'=u-X_0\omega\cos\omega t$

then $u=U_0(-\cos\omega t+\exp(\sqrt{\frac{\omega}{2\nu}}y)\cos(\omega t-\sqrt{\frac{\omega}{2\nu}}y))$

Ex.

ⓐ Cauette flow.

$\frac{\partial u}{\partial x}=0$ $u=f(y,z,t)$

$\cancel{\frac{\partial u}{\partial t}}=-\frac{1}{\rho}\cancel{\frac{\partial P}{\partial x}}+\nu\frac{\partial^2 u}{\partial y^2}$ subject to. $u(y=0)=0$, $u(y=h)=\mathcal{U}$

steady

$u=\mathcal{U}\frac{y}{h}$

$\tau=\mu\frac{du}{dy}\Big|_h=\mu\frac{\mathcal{U}}{h}$

ⓑ Laminar duct flow

$\frac{\partial u}{\partial x}=0$ $u=f(y,z)$ steady

$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\nabla_\perp^2 u$

$\frac{dP}{dx}=\mu\nabla_\perp^2 u$

$\int_A\left(\frac{dP}{dx}=\mu\nabla_\perp^2 u\right)$

$A\frac{dP}{dx}=\mathcal{P}\tau_w \rightarrow \frac{dP}{dx}=\frac{\mathcal{P}\tau_w}{A}$

ⓒ $\frac{\partial u}{\partial x}=0$ $u=f(y,\cancel{z},t)$ fluid.

$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\cancel{\frac{\partial P}{\partial x}}+\nu\frac{\partial^2 u}{\partial y^2}$

$u=\mathcal{U}f(y,t,\nu)$

$\eta=\frac{y}{2\sqrt{\nu t}}$

$\frac{\partial}{\partial t}=\frac{\partial}{\partial\eta}\frac{\partial\eta}{\partial t}$

$\frac{\partial}{\partial y}=\frac{\partial}{\partial\eta}\frac{\partial\eta}{\partial y}$

Similarity

$\delta\sim\sqrt{\nu t}$

$u(y=0)=\mathcal{U}$

$u(y\to\infty)=0$

$\frac{u}{\mathcal{U}}=f(\eta=\frac{y}{2\sqrt{\nu t}})$

$\frac{u}{\mathcal{U}}=1-\mathrm{erf}(\frac{y}{2\sqrt{\nu t}})$

η_4 — y_4, η_3 — y_3, η_2 — y_2, η_1 — y_1 $\Rightarrow$ t_0 y_3, t_0 y_2, t_0 y_1, $2\sqrt{\nu t_0}$

ⓓ Oscillating plate (Stokes' 2nd problem)

Brian Kantum Symmetric

$\frac{\partial u}{\partial x}=0$ $u=f(y,t)$

$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\cancel{\frac{\partial P}{\partial x}}+\nu\frac{\partial^2 u}{\partial y^2}$

$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}$

$u=U_0f(y,t,\nu,\omega)$

$\frac{u}{U_0}=f(\frac{y}{\sqrt{2\nu/\omega}},\omega t)$

$u(y,t)=U(y)e^{i\omega t}$ take the real part

$u(y,t)=U_0\exp(-\sqrt{\frac{\omega}{2\nu}}y+i\omega t)$

take real part

$[u(y,t)=U_0\exp(-\sqrt{\frac{\omega}{2\nu}}y)\cos(\omega t-\sqrt{\frac{\omega}{2\nu}}y)]$

$\frac{y}{\sqrt{\frac{2\nu}{\omega}}}$ moving cosine.

$\delta\sim\sqrt{\frac{\nu}{f}}$ fluid. $\mathcal{U}=U_0\cos(\omega t)$

$\delta\sim\sqrt{\frac{\nu}{f}}$ fluid. $\mathcal{U}=U_0\cos(\omega t)$

ⓔ Oscillating pressure gradient $\frac{dP_0}{dx}\sin\omega t$

$\frac{\partial u}{\partial x}=0$

$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$

$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{dP_0}{dx}\sin\omega t+\nu\frac{\partial^2 u}{\partial y^2}$ s.t. $u(y=0)=0$

change of coordinate (moving with fluid)

$x'=x-X_0\sin\omega t$

$u'=u-X_0\omega\cos\omega t$

Caution: New x-coordinate Non inertial

$\Rightarrow$ Must add a force on R.H.S of momentum eqn.

equals to minus acceleration of frame

$\frac{\partial u'}{\partial t}=\underbrace{-\frac{1}{\rho}\frac{\partial P_0}{\partial x}\sin(\omega t)+\omega^2X_0\sin(\omega t)}+\nu\frac{\partial^2 u'}{\partial y^2}$

want these canceled

$y\to\infty$ $u'(\infty)=0$, $\frac{\partial u'}{\partial t}(\infty)=0$, $\frac{\partial^2 u'}{\partial y^2}(\infty)=0$ $\rightarrow$ $-\frac{1}{\rho}\frac{\partial P_0}{\partial x}\sin(\omega t)+\omega^2X_0\sin(\omega t)=0$

$X_0=\frac{1}{\rho\omega^2}\frac{dP_0}{dx}$

$V_0=\frac{1}{\rho\omega}\frac{dP_0}{dx}$

At wall, $y=0$ $u'=0-V_0\cos\omega t$

Replace $\mathcal{U}$ in pervious problem with $-V_0$

$u'=-V_0\left[\exp(\frac{-y}{\sqrt{\frac{2\nu}{\omega}}})\cos(\omega t-\sqrt{\frac{\omega}{2\nu}}y)\right]$

$u=-V_0\left[\underbrace{-\cos(\omega t)}_{\text{oscillate with no wall}}+\underbrace{\exp(\frac{-y}{\sqrt{\frac{2\nu}{\omega}}})\cos(\omega t-\sqrt{\frac{\omega}{2\nu}}y)}_{\text{wall effect on}}\right]$

viscous damping through y

$y=0$ wall point cancelled velocity.

Maximum velocity $\sim\sqrt{\frac{\nu}{\omega}}$

$\frac{dP_0}{dx}\sin\omega t$

ⓕ Pulsatile duct flow R ω

$\vec{u}=(u,0,0)$

$\frac{\partial u}{\partial x}=0$

$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})$

$u=U_0f(r,t,\nu,\omega,R)$

$\frac{u}{U_0}=f(\frac{r}{R},\omega t,\frac{R}{\sqrt{\frac{\nu}{\omega}}})$

$\delta\sim y$

Womersly # $W_0=\frac{R}{\sqrt{\frac{\nu}{\omega}}}$ $\left\{\begin{array}{l}\text{large } W_0\\ \text{Small } W_0 \to \text{Quasi Steady}\end{array}\right.$

11-7

Navier-Stokes.

- Unidirectional flow. $\vec{u}=(u,\cancel{v},\cancel{w})$

  Cont. $u=f(y,z,t)$

  x-mom. $\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\nabla_\perp^2 u$ $(=\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})$

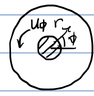
  y-mom. $0=\frac{\partial P}{\partial y}$

  No pressure gradient on y $p=p(x,t)$

- Circular flows
- Non-dimensional parameters
- Boundary layers

Circular flow.

Circular flow. $\vec{u}=(u_\phi, \cancel{u_r}, \cancel{u_x})$



Continuity $\frac{1}{r}\frac{\partial}{\partial r}(r \cancel{u_r})+\frac{1}{r}\frac{\partial}{\partial \phi}u_\phi=0$

$u_\phi=u_\phi(r,t,x)$

r-Momentum $-\frac{u_\phi^2}{r}=-\frac{1}{\rho}\frac{\partial P}{\partial r} \Rightarrow \frac{\partial P}{\partial r}=\rho\frac{u_\phi^2}{r}$

ϕ-Momentum. $\frac{\partial u_\phi}{\partial t}=-\frac{1}{\rho r}\frac{\partial P}{\partial \phi}+\frac{\nu}{r^2}\frac{\partial}{\partial r}\left[r^3\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right)\right]$

$\neq f(\phi)$ $\Rightarrow P$ at most linear in ϕ $\neq g(\phi)$

For periodicity $p \neq f(\phi)$

$P=C_1\phi+C_2$

$P(\phi=0)=P(\phi=2\pi) \rightarrow C_1=0$

$\nabla\phi$.

$\nabla^2\phi=0$ $\nabla\cdot\vec{u}=0$

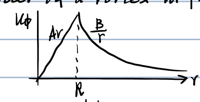
$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0$

$u_r=\frac{\partial\phi}{\partial r}$

$u_\theta=\frac{1}{r}\frac{\partial\phi}{\partial\theta}$

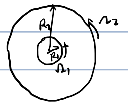
Assume Steady $u_\phi=\underbrace{Ar}_{\text{solid body rotation}}+\underbrace{\frac{B}{r}}_{\text{Ideal vortex}}$ $\frac{\Gamma}{2\pi r}$

Model of a vortex in free space



$\delta \propto \sqrt{\nu t}$

For 2 concentric/rotating cylinders (Circular Couette)



$A=\frac{\Omega_2R_2^2-\Omega_1R_1^2}{R_2^2-R_1^2}$

$B=\frac{(\Omega_2-\Omega_1)R_1^2R_2^2}{R_2^2-R_1^2}$

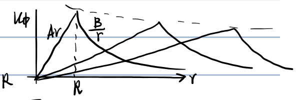
$u_\phi=\begin{cases}Ar & r\le R\\ \frac{B}{r} & r\ge R\end{cases}$ at $r=R$, $AR=\frac{B}{R}$

$A=\frac{B}{R^2}=\frac{\Gamma}{2\pi R^2}$

$u_\phi=\begin{cases}\frac{\Gamma}{2\pi R^2}r & r\le R\\ \frac{\Gamma}{2\pi r} & r\ge R\end{cases}$

$R \propto \sqrt{\nu t}$

$L=\frac{L^2}{T}T$



$AR=\frac{\Gamma}{2\pi R^2}R=\frac{\Gamma}{2\pi R}=\frac{\Gamma}{2\pi\sqrt{\nu t}}$

Vorticity $\vec{\omega}=\nabla\times\vec{u}$

$\omega_z=(\nabla\times\vec{u})_z=\frac{1}{r}\left(\frac{\partial}{\partial r}(ru_\phi)-\cancel{\frac{\partial u_r}{\partial\phi}}\right)$

$\omega_z=2A$ (2-times of solid body rotation)

Assume $A=0$ & compute circulation

$\Gamma=\int_0^{2\pi}u_\phi\, r\,d\phi=B\cdot 2\pi$

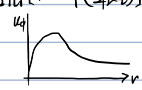
$B=\frac{\Gamma}{2\pi}$

Viscous diffusion of a line vortex

$\frac{\partial}{\partial t}(ru_\phi)=\nu\left[\frac{\partial^2}{\partial r^2}(ru_\phi)-\frac{1}{r}\frac{\partial}{\partial r}(ru_\phi)\right]$

$ru_\phi=\frac{\Gamma}{2\pi}f(t,\nu,r)=\frac{\Gamma}{2\pi}f(\eta)$, $\eta=\frac{r}{2\sqrt{\nu t}}$

$u_\phi=\frac{\Gamma}{2\pi r}\left[1-\exp\left(\frac{-r^2}{4\nu t}\right)\right]$



(a) Lim $r\gg\sqrt{\nu t}$

$u_\phi=\frac{\Gamma}{2\pi r}$

(b) Lim $r\ll\sqrt{\nu t}\rightarrow 0$

$u_\phi\sim\frac{\Gamma}{2\pi r}\left[1-\left(1-\frac{r^2}{4\nu t}\right)\right]=\frac{\Gamma}{2\pi}\frac{r}{4\nu t}=\frac{\Gamma r}{2\pi R^2}$

$\nabla_a u_b \nabla_c u_d$

$\epsilon_{iac}\nabla_a\nabla_c u_d$

$\epsilon_{iac}\nabla_c\nabla_a u_d$

Dimensionless group

$\begin{cases}\nabla\cdot\vec{u}=0\\ \text{Momentum Eqn.}\end{cases} \Rightarrow \begin{cases}\nabla^*\cdot\vec{u}^*=0\\ \frac{\partial\vec{u}^*}{\partial t^*}+\vec{u}^*\cdot\nabla^*\vec{u}^*=-\frac{1}{\rho^*}\nabla^*p^*+\nu^*\nabla^{*2}\vec{u}^*+\vec{g}^*\end{cases}$ $\nu^*=\frac{\mu^*}{\rho^*}$

Dimensional	$\vec{u}^*$	$\vec{x}^*$	t^*	∇^*p^*	$\vec{g}^*$
Dimensional unit	$\mathcal{U}$	$\mathcal{L}$	θ	$\frac{1}{\mathcal{L}}\Delta P$	$\mathcal{G}$
undimensional number	$\vec{u}$	$\vec{x}$	t	∇p	$\vec{g}$

Continuity: $\left(\frac{1}{\mathcal{L}}\mathcal{U}\right)\nabla\cdot\vec{u}=0 \rightarrow \nabla\cdot\vec{u}=0$

Momentum $\left(\frac{\mathcal{U}}{\theta}\right)\frac{\partial\vec{u}}{\partial t}+\left(\frac{\mathcal{U}^2}{\mathcal{L}}\right)\vec{u}\cdot\nabla\vec{u}=-\left(\frac{\Delta P}{\rho\mathcal{L}}\right)\nabla p+\left(\frac{\mu\mathcal{U}}{\rho\mathcal{L}^2}\right)\nabla^2\vec{u}+(\mathcal{G})\vec{g}$

$\underbrace{\left(\frac{\mathcal{L}}{\mathcal{U}\theta}\right)}_{St}\frac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\nabla\vec{u}=-\underbrace{\left(\frac{\Delta P}{\rho\mathcal{U}^2}\right)}_{Eu}\nabla p+\underbrace{\left(\frac{\mu}{\rho\mathcal{U}\mathcal{L}}\right)}_{Re}\nabla^2\vec{u}+\underbrace{\left(\frac{\mathcal{G}\mathcal{L}}{\mathcal{U}^2}\right)}_{Fr}\vec{g}$

Dimensionless groups

$\nabla^*\cdot\vec{u}^*=0$

$\frac{\partial\vec{u}^*}{\partial t^*}+\vec{u}^*\cdot\nabla^*\vec{u}^*=-\frac{1}{\rho^*}\nabla^*p^*+\nu^*\nabla^{*2}\vec{u}^*+\vec{g}^*$

$\underbrace{\vec{u}^*}_{\text{dimensional}}=\mathcal{U}\underbrace{\vec{u}}_{\text{nondimensional}}$, $\vec{x}^*=\mathcal{L}\vec{x}$, $t^*=\theta t$, $\nabla^*p^*=\frac{\Delta P}{\mathcal{L}}\nabla p$, $\vec{g}^*=\mathcal{G}\vec{g}$

Cont. $\nabla^*\cdot\vec{u}^*=0$

$\frac{1}{\mathcal{L}}\nabla\cdot(\mathcal{U}\vec{u})=0$

$\frac{\mathcal{U}}{\mathcal{L}}\nabla\cdot\vec{u}=0$

$\nabla\cdot\vec{u}=0$

Mom $\underbrace{\left(\frac{\mathcal{L}}{\mathcal{U}\theta}\right)}_{\frac{1}{St}}\frac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\nabla\vec{u}=-\underbrace{\left(\frac{\Delta P}{\rho\mathcal{U}^2}\right)}_{Eu}\nabla p+\underbrace{\left(\frac{\mu}{\rho\mathcal{U}\mathcal{L}}\right)}_{\frac{1}{Re}}\nabla^2\vec{u}+\underbrace{\left(\frac{\mathcal{G}\mathcal{L}}{\mathcal{U}^2}\right)}_{\frac{1}{Fr}}\vec{g}$

$Re=\frac{\rho\mathcal{U}\mathcal{L}}{\mu}=\frac{\rho\mathcal{U}^2}{\mu(\mathcal{U}/\mathcal{L})}$ ~ Momentum flux / ~ Viscous flux

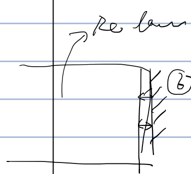
$Re=\frac{\rho\mathcal{U}\mathcal{L}}{\mu}=\frac{\rho\mathcal{L}^2/\mu}{\mathcal{L}/\mathcal{U}}=\frac{\mathcal{L}^2/\nu}{\mathcal{L}/\mathcal{U}}=\frac{T_{\text{diffusion}}}{T_{\text{advection}}}$ ~ diffusion time / ~ advection time.

Re-eqn $\longrightarrow$ $\frac{Re}{St}\frac{\partial\vec{u}}{\partial t}+Re\,\vec{u}\cdot\nabla\vec{u}=-Eu\,Re\,\nabla p+\nabla^2\vec{u}+\frac{Re}{Fr}$

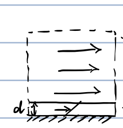
Lim $Re\rightarrow 0$ ignore $\vec{u}\cdot\nabla\vec{u}$ term. Eqn. Nonlinear.

Lim Re large. Conditionally drop $\nabla^2\vec{u}$

Re law



(b) $Re=\frac{\delta}{\mu}$



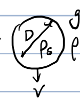
Can drop ∇^2u. Re large. (D) D

d Re small (d) cannot drop ∇^2u. (also means viscosity) (boundary layer)

Buckingham Pi Thm.

- Identifying "N" revelant varibles
- Number of dimensions "m"
- There are (N-m) non-dimensional groups
- Select "m" varibles from π groups from all other (N-m) Varibles.

Ex.



ρ,μ g

$v=f(D,\rho_s,\rho,\mu,g)$

$F(v,D,\rho_s,\rho,\mu,g)=0$

$\frac{L}{T}$ L $\frac{M}{L^3}$ $\frac{M}{L^3}$ $\frac{M}{LT}$ $\frac{L}{T^2}$

$N=6$. $V\ D\ \rho_s\ \rho\ \mu\ g$

$m=3$ $L\ T\ M$

Select 3 varibles capture (M,L,T)

#N revelant varibles

#m dimensions

#(N-M) undimension groups

Dimensionless group

$F(a,b,c\ldots)=0$ #M varibles.

Basic dimension: $[M]\ [L]\ [T]$ 3.

#M-3 π groups Ex. $\pi_1=ab^\alpha c^\beta\ldots$

$\Sigma[L]=0$

$\Sigma[T]=0$

$\Sigma[M]=0$

$\Pi_1 = f(\Pi_2, \Pi_3, \dots)$

D, ρ, μ

$L \quad \frac{M}{L^3} \quad \frac{M}{LT}$

$\Pi_4 = V D^\alpha \rho^\beta \mu^\gamma [=] \frac{L}{T} L^\alpha \left(\frac{M}{L^3}\right)^\beta \left(\frac{M}{LT}\right)^\gamma$

$\begin{cases} L^0: 1+\alpha-3\beta-\gamma=0 \\ T^0: -1-\gamma=0 \\ M^0: \beta+\gamma=0 \end{cases} \rightarrow \begin{cases} \gamma=-1 \\ \beta=1 \\ \alpha=1 \end{cases}$

$\Pi_4 = \frac{VD\rho}{\mu} = Re.$

$\Pi_2 = \rho_s \rho^\alpha D^\beta M^\gamma = \frac{\rho_s}{\rho}$

$\Pi_3 = g \rho^\alpha D^\beta M^\gamma = \frac{\rho\sqrt{gD}\,D}{\mu} = \frac{\rho g^{\frac{1}{2}} D^{\frac{3}{2}}}{\mu} = Re$

Boundary layers

Navier-Stokes $\longrightarrow$ Boundary layer eqns.

Integral B-L-eqn.

Navier-Stokes (* mean non-dimensional)

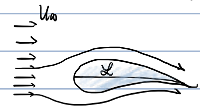
$$\frac{\partial u^*}{\partial t^*} + u^* \cdot \nabla_* u^* = -\nabla_* p^* + \frac{1}{Re}\nabla_*^2 u^*$$

Lim $Re \gg 1$, mainly drop viscous item. $\longrightarrow$! Math. problem $u^{(n)} + u = 0$ $\rightarrow n$ BC.

No longer satisfy no penetration & no slip.

"Small region" cannot drop $\frac{1}{Re}$ part

Boundary layer $\leftarrow$ / cannot ignore small part around object



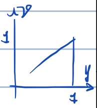
Ex. $\epsilon \nu'' + \nu' = A$ $\quad \epsilon = \frac{1}{Re}$

BC $\begin{cases} \nu(y=0)=0 \\ \nu(y=1)=1 \end{cases}$

$\epsilon \frac{\partial^2 \nu}{\partial y^2} + \frac{\partial \nu}{\partial y} = A$

① $\cancel{\epsilon \frac{\partial^2 \nu}{\partial y^2}} + \frac{\partial \nu}{\partial y} = A$

ϵ very small. $\nu = Ay + B$



BC②: $1 = A + B \rightarrow B = 1 - A$

$\nu_o = Ay + (1-A)$

② as $\eta = \frac{y}{\delta}$:

$\epsilon \frac{\partial^2 \nu}{\partial y^2} + \frac{\partial \nu}{\partial y} = A$

$\epsilon \left[\frac{\partial}{\partial \eta}\left(\frac{\partial \nu}{\partial \eta}\frac{\partial \eta}{\partial y}\right)\right]\frac{\partial \eta}{\partial y} + \frac{\partial \nu}{\partial \eta}\frac{\partial \eta}{\partial y} = A$

$\epsilon \frac{1}{\delta^2}\frac{\partial^2 \nu}{\partial \eta^2} + \frac{1}{\delta}\frac{\partial \nu}{\partial \eta} = A$

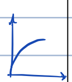
$\frac{\epsilon}{\delta}\frac{\partial^2 \nu}{\partial \eta^2} + \frac{\partial \nu}{\partial \eta} = \cancel{\delta A}$, δ small

$\epsilon \sim \delta$

$\nu = C_1 e^{-\eta} + C_2$

$\nu(y=0) = 0 \rightarrow C_1 + C_2 = 0.$

$\nu_i = d(1 - e^{-\frac{y}{\epsilon}})$



got $\lim_{y \to 0} \nu_o = \lim_{\epsilon \to \infty} \nu_i$

$1 - A = d$

$\nu_i = (1-A)(1-e^{-\frac{y}{\epsilon}})$

$\nu_o = (1-A) + Ay$

$\nu = \nu_o + \nu_i - \nu_{overlap}$ $\quad 1-A$

$= Ay - (1-A)(1 - e^{-\frac{y}{\epsilon}})$

Ex. $\epsilon \nu'' + \nu' = A$ $\quad$ in $0 \le y \le 1$ $\quad \epsilon \ll 1$

① $\nu(y=0) = 0$

$\nu(y=1) = 1$

$\epsilon \frac{d^2 \nu}{dy^2} + \frac{d\nu}{dy} = A$

$y = \frac{y}{L}$ $\quad$ outer layer

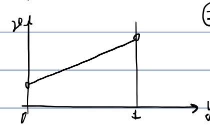
$\cancel{\epsilon \frac{d^2 \nu}{dy^2}} + \frac{d\nu}{dy} = A$

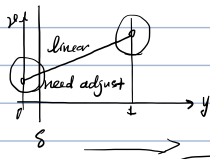
ignore

$\nu_o = Ay + c$

BC: $\nu_o(y=1) = 1 \rightarrow c = 1 - A$

$[\nu_o = Ay + (1-A)]$





② $\eta = \frac{y}{\delta}$

$\frac{d}{dy} = \frac{1}{\delta}\frac{d}{d\eta}$

$\frac{1}{\delta^2}\epsilon \frac{d^2 \nu}{d\eta^2} + \frac{1}{\delta}\frac{d\nu}{d\eta} = A$

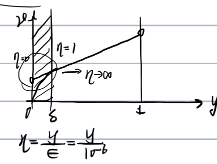
$\frac{\epsilon}{\delta} \frac{d^2 \nu}{d\eta^2} + \frac{d\nu}{d\eta} = \cancel{\delta A}$ $\quad$ δ small. $\delta \equiv \epsilon$

$\delta \sim \epsilon$ $\quad \frac{d^2 \nu}{d\eta^2} + \frac{d\nu}{d\eta} = 0$

$\eta \to \frac{y}{\epsilon}$ $\quad = C_1 e^{-\eta} + C_2 = C_1 + C_2$

$\left[\nu_i = d(1 - e^{-\eta}) = d(1 - e^{-\frac{y}{\epsilon}})\right]$

inner Solution.



$\eta = \frac{y}{\epsilon} = \frac{y}{10^{-6}}$

③ $y \gg \delta$ $\quad \lim_{\eta \to \infty} \nu_i = \lim_{y \to 0} \nu_o$

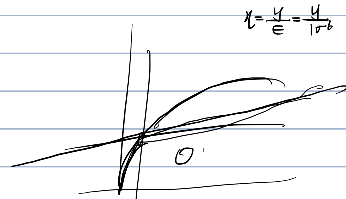
$d = 1 - A$

$y \to 0$

$\nu_i = (1-A)(1 - e^{-\frac{y}{\epsilon}})$

$\nu_o = Ay(1-A)$

ϵ ? δ ? y^0

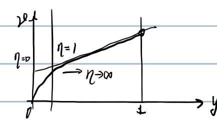


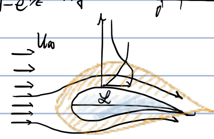
$\nu = \nu_i + \nu_o - \nu_{overlap}$

$= (1-A)(1 - e^{-\frac{y}{\epsilon}}) + Ay(1-A) - (1-A)$

$\nu = (1-A)(1 - e^{-y/\epsilon}) + Ay$

$\nu_{true} = (1-A)\frac{(1 - e^{-y/\epsilon})}{1 - e^{-1/\epsilon}} + Ay$





Back to Navier-Stokes (2D Dimensional)

Inertia $\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \longrightarrow \quad \frac{U}{L}\frac{\partial u_*}{\partial x_*} + \frac{U\delta}{L\delta}\frac{\partial \hat{v}}{\partial \hat{y}} = 0$

$O(\frac{U_\infty}{L}), O(\frac{\nu}{\delta})$

$\nu \sim O(U_\infty \frac{\delta}{L})$ $\quad u = U u_*$ $\quad \frac{\partial u_*}{\partial x_*} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0$

$x = L x_*$

$v = U \frac{\delta}{L} \hat{v}$

$y = \delta \hat{y}$

$\delta = L\,O(\frac{1}{\sqrt{Re}})$

$Re = \frac{\rho v L}{\mu}$

$\delta \sim O(\sqrt{\frac{\nu L}{U}})$

Momentum.

x-direction $\quad Re = \frac{UL}{\nu}$

$$\underbrace{u_* \frac{\partial u_*}{\partial x_*}}_{O(1)} + \underbrace{\hat{v}\frac{\partial u_*}{\partial \hat{y}}}_{O(1)} = -\underbrace{\frac{\partial P_*}{\partial x_*}}_{O(1)} + \frac{1}{Re}\frac{L^2}{\delta^2}\underbrace{\left(\cancel{\frac{\delta^2}{L^2}\frac{\partial^2 u_*}{\partial x_*^2}} + \frac{\partial^2 u_*}{\partial \hat{y}^2}\right)}_{O(1)}$$

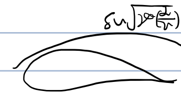
$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}$

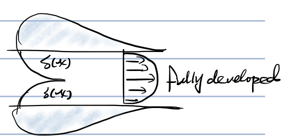
$\frac{1}{Re}\frac{L^2}{\delta^2} \sim O(1)$

$\frac{\delta}{L} \sim O(\frac{1}{\sqrt{Re}}) \sim O(\sqrt{\frac{\nu}{UL}})$

$\delta = O(\sqrt{\frac{\nu L}{U}}) = O(\sqrt{\frac{\nu x}{U}})$

$\delta \sim \sqrt{\frac{\nu x}{U}}$





y-momentum $\quad \underbrace{u_* \frac{\partial \hat{v}}{\partial x_*}}_{O(1)} + \underbrace{\hat{v}\frac{\partial \hat{v}}{\partial \hat{y}}}_{O(1)} = -\underbrace{\frac{L^2}{\delta^2}\frac{\partial P_*}{\partial \hat{y}}}_{\ll O(\frac{1}{\epsilon^2}) \gg} + \frac{1}{Re}\frac{L^2}{\delta^2}\left(\underbrace{\frac{\delta^2}{L^2}\frac{\partial^2 \hat{v}}{\partial x_*^2}}_{O(\epsilon^2)} + \underbrace{\frac{\partial^2 \hat{v}}{\partial \hat{y}^2}}_{O(1)}\right)$

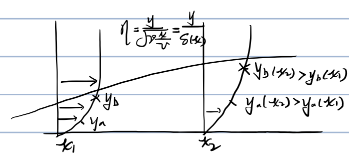
only possible $\frac{\partial P}{\partial y} = 0$

Summary

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ $\qquad \frac{\partial P}{\partial y} = 0 \Rightarrow P \neq f(y)$

$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$

The Blasius equation.



$2\eta f$

$\phi''' + \frac{1}{2}\phi\phi'' = 0$

$\frac{u}{U} = F(\eta)$ $\quad \frac{v}{U\frac{\delta}{x}} = \frac{v}{U\sqrt{\frac{\nu x}{x^2 U}}} = \frac{v}{U\sqrt{\frac{\nu}{Ux}}} = \frac{v}{U/\sqrt{Re}} = g(\eta)$ $\quad \eta = \frac{y}{\sqrt{\frac{\nu x}{U}}}$

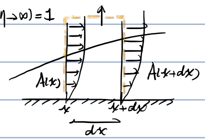
$\eta = \frac{y}{\sqrt{\frac{\nu x}{U}}}$

$$\begin{cases} \frac{1}{2}\eta f' - g' = 0 & \text{continuity} \\ -\frac{1}{2}\eta f f' + g f' = f'' & \text{x-momentum} \\ \left[\phi''' + \frac{1}{2}\phi\phi'' = 0\right] \end{cases}$$

$\phi(0) = 0$

$\phi'(0) = 0$

$\phi'(\eta \to \infty) = 1$



$\int_{A(x)}$ $\quad \int_{A(x+dx)}$

$\vec{u} \cdot \nabla u$

Integral Boundary Layer Eqn.

Continuity

$\delta x \frac{1}{\delta x}\left(-\int_{A(x)} u\,dA + \int_{A(x+dx)} u\,dA\right) + \int_{A_{top}} \vec{u}\cdot\vec{n}\,dA = 0$ $\quad$ less.

$\Delta x \frac{d}{dx}\int u\,dA + \int_{A_{top}} \vec{u}\cdot\vec{n}\,dA = 0$

$\int_A \frac{1}{2}\nabla v^2 - \int_{A(x+dx)} \frac{1}{2}u^2$

influence velocity

x-momentum. $-\int_{A(x)} u^2 dA + \int_{A(x+dx)} u^2 dA + \int U(\vec{u}\cdot\vec{n})_{top} = \int_{A(x)} P\,dA - \int_{A(x+dx)} P\,dA - \tau\,dx$

$\frac{d}{dx}\int u^2 - \frac{d}{dx}\int uU$

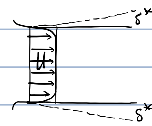
use Bernoulli for pressure

$\frac{d}{dx}\int_0^\infty u(U-u)\,dA + \frac{dU}{dx}\int_0^\infty (U-u)\,dA = \frac{\tau_w}{\rho}$

Displacement thickness

$U\delta^* = \int_0^\infty U\,dy - \int_0^\infty u\,dy$

$\delta^* = \frac{1}{U}\int_0^\infty (U-u)\,dy$



Momentum thickness

$\theta = \frac{1}{U^2}\int uU - uu\,dy$

$\theta = \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy$

$$\left[\frac{d}{dx}(U^2\theta) + U\frac{dU}{dx}\delta^* = \frac{\tau_w}{\rho}\right]$$

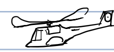
$U\frac{dU}{dx} = -\frac{1}{\rho}\frac{dP}{dx}$

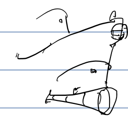
Momentum Thickness

$\delta = \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy$

Replacement Thickness

$\delta = \int_0^\infty \frac{1}{U}\left(1-\frac{u}{U}\right)dy$

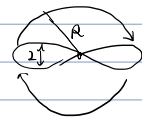




$Re\ 10^5$

δ 5×10^{-3}

L 0.2 m.



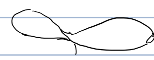
$U = \Omega R = 40$ m/s $\quad$ tip speed

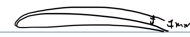
$\nu = 10^{-5}$

$Re = \frac{UL}{\nu} = \frac{40\times0.2}{10^{-5}} = 8\times10^5 \sim 10^6$

$\delta \sim \frac{L}{\sqrt{Re}} = \sqrt{\frac{\nu x}{U}} = \sqrt{\frac{10^{-5}\times.2}{4}} = \sqrt{0.5}\times10^{-3} = \frac{\sqrt{2}}{2}\times10^{-3}$

$\nu\ 2\times10^{-x}$





0.2m.

10^{-5}

$Re = \frac{UL}{\nu}$

$\delta = \sqrt{\frac{\nu x}{U}}$

	Air	Water
μ	10^{-3}	10^{-3}
ν	10^{-5}	10^{-6}
ρ	1	10^{3}