530.767 CFD
Spring 2024
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This is CFD Homework

Contents

1 Question Review

1. We will explore the finite-difference solution to the simple 1-D wave equation. Consider the following equation:

$$u_t + u_x = 0; \quad \text{on} \quad 0 \leq x \leq 2\pi \tag{1}$$

with

$$u(0,t) = u(2\pi, t)$$
$$\frac{\partial^n u}{\partial x^n}(0,t) = \frac{\partial^n u}{\partial x^n}(2\pi, t), \quad n > 0 \quad \text{(i.e. periodic BC)}$$

$$u(x, 0) = \sin(mx)$$

(i) Discretize the above equation with an explicit scheme in time on a mesh with $\Delta x = \frac{2\pi}{20}$ and $\Delta t = 0.001$. Use the following spatial discretization schemes:

- First-order upwind
- Second-order upwind (using $i, i-1, i-2$)
- Third order upwind (using $i, i-1, i-2, i-3$)
- Third-order upwind biased (using $i+1, i, i-1$ and $i-2$)

and obtain the numerical solution for $m = 2, 4, 6,$ and 8. You should integrate the discretized equations long enough in time so that the effects of the truncation error are apparent.

(ii)Derive the exact solution of the PDE.

(iii)Derive and plot the modified wavenumber curves (check the lecture notes) for the four schemes; use the modified wavenumber analysis to explain the observed behavior of the numerical solution and its comparison to the exact solution.

2. Examination of aliasing error in solutions of time-dependent differential equations. Consider a domain of size 2π and a grid with $\Delta x = \frac{2\pi}{20}$ and the following two equations:

$$u_t + u^2 = 0$$

and

$$u_t + \sin(3x)u_x = 0$$

Let the initial condition be $u(x) = \sin(x) + 0.5\sin(4x)$ and use the Forward Euler scheme with a time step of 0.10.

For both these equations plot and compare the unaliased (assuming that the aliasing error is zero) and fully aliased energy spectra for the resolved scales at time steps from 1 to 10. Explain the key characteristics of the results that you observe. Spectra can be obtained by running a FFT on your simulation results.

You can obtain an unaliased spectra by running the same simulation on a finer grid; you might also need a small time-step size for the finer grid.

2 1. Finite Difference for 1D Wave Equations.

2.1 Finite Difference Schemes

The PDE is showing below:

$$u_t + u_x = 0; \quad \text{on} \quad 0 \le x \le 2\pi \tag{2}$$

2.1.1 1st Upwind Scheme

As the equation above, the solve

2.1.2 2nd Upwind Scheme

2.1.3 3rd Upwind Scheme

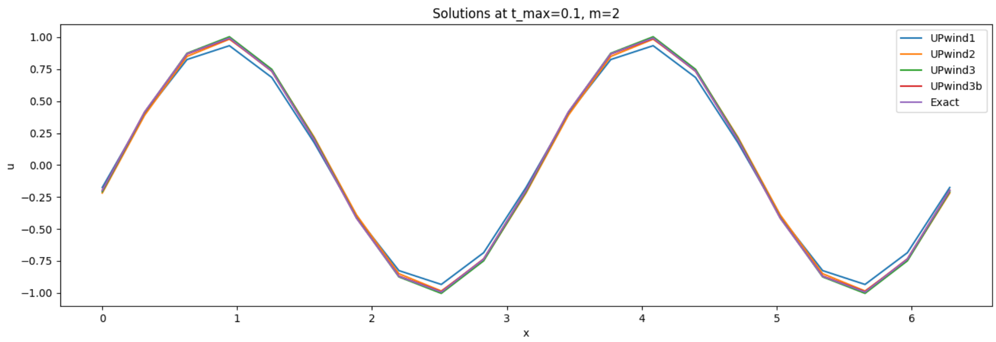
2.1.4 3rd biased Upwind Scheme

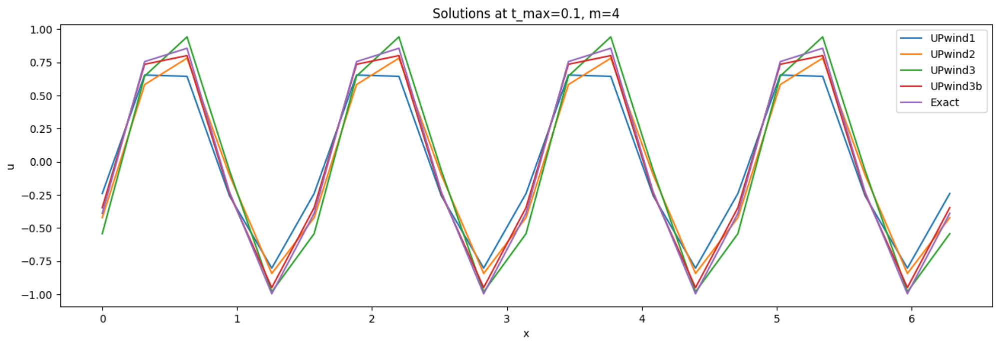
2.2 Solver Algorithm

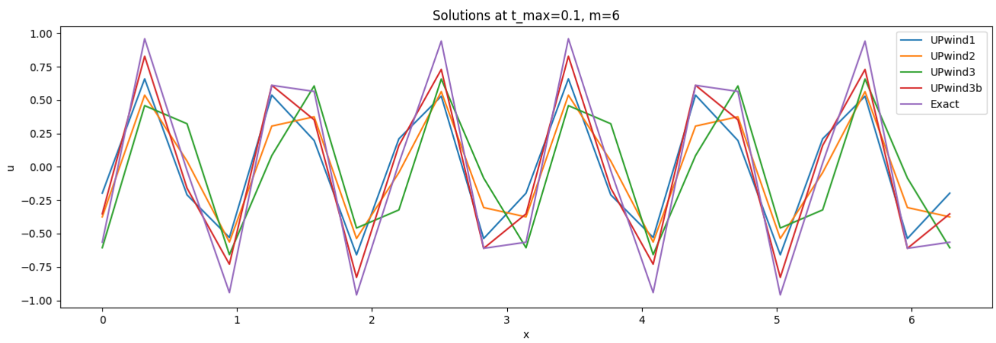
2.3 (i) Result–Solutions for different schemes

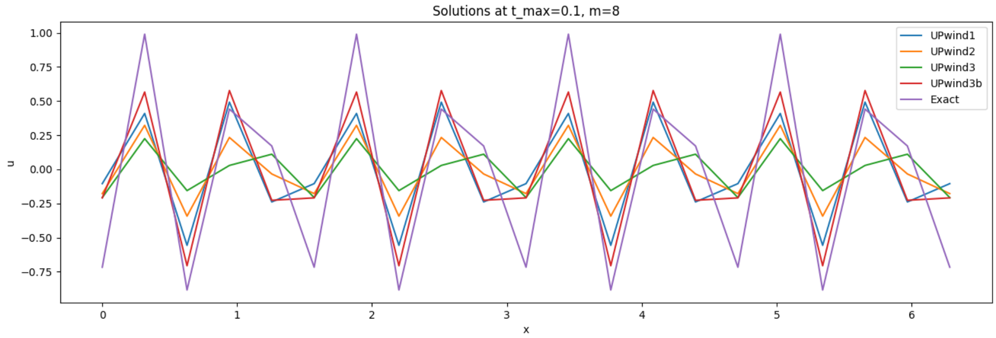
For the solution, the result is showing below:

For t_{max} = 0.1, the result for different m is showing below:

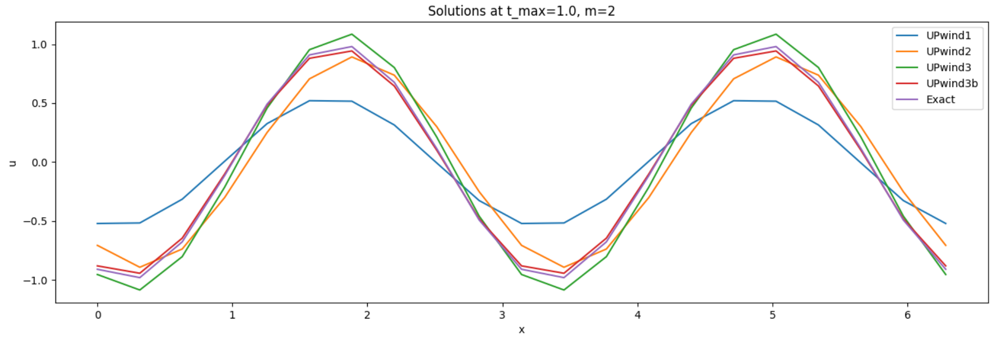


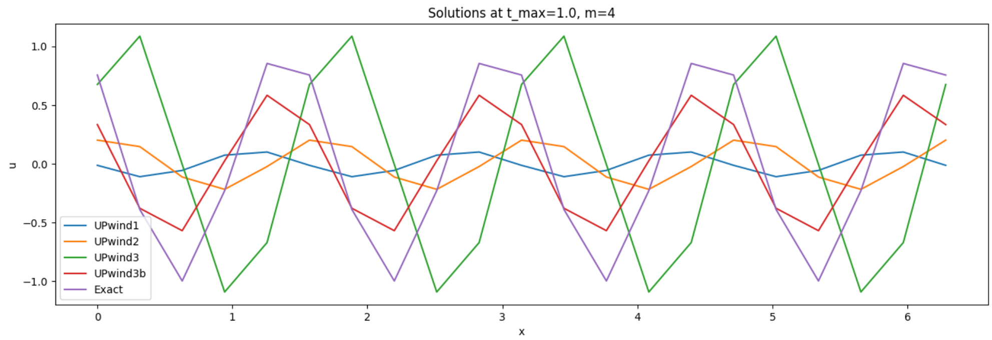


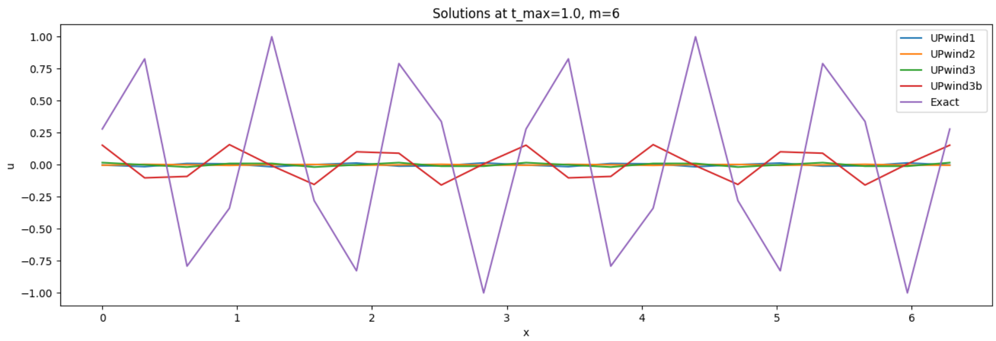


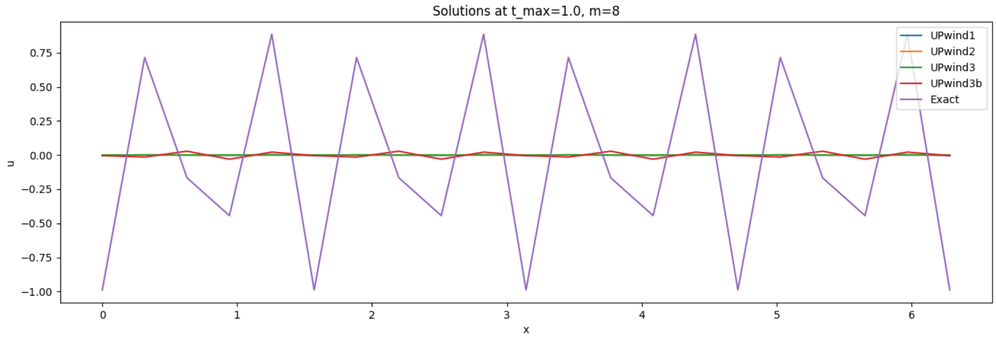


For $t_{max} = 1$, the results for different m is showing below:









2.4 (ii) Exact Solution

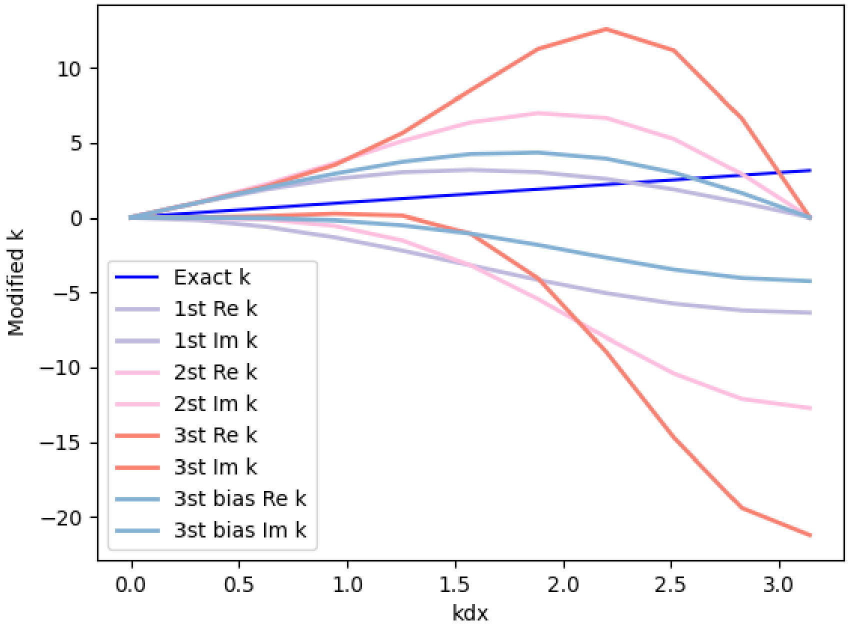
2.5 (iii) Modified Wavenumber Curves and Analysis

2.5.1 Modified Wavenumbers

$$
\begin{aligned}
\hat{u}'_j &= \frac{u_j - \hat{u}_{j-1}}{\Delta x} \\
&= \frac{\hat{u}e^{ikx} - \hat{u}e^{ik(x-\Delta x)}}{\Delta x} \\
&= \frac{\hat{u}e^{ikx}(1 - e^{-ik\Delta x})}{\Delta x} \\
&= \frac{\hat{u}}{\Delta x}(1 - e^{-ik\Delta x})\hat{u} \\
&= \frac{\hat{u}}{\Delta x}\left(1 - \frac{\sin(k\Delta x/2)}{k\Delta x/2}\right)e^{ikx}
\end{aligned}
\tag{3}
$$

$$
k' = -i\left(\frac{1 - e^{-ik\Delta x}}{\Delta x}\right) = -\frac{3}{\Delta x} + i\frac{\sin(k\Delta x) + \sin(k\Delta x)}{\Delta x} = \pm\frac{\sin(k\Delta x)}{\Delta x} + i\left(\frac{\sin(k\Delta x) - 1}{\Delta x}\right) \tag{4}
$$

2.5.2 Curves and Analysis

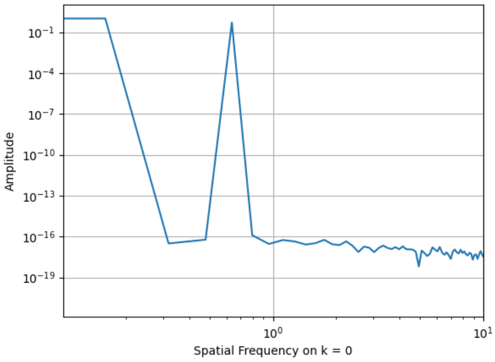


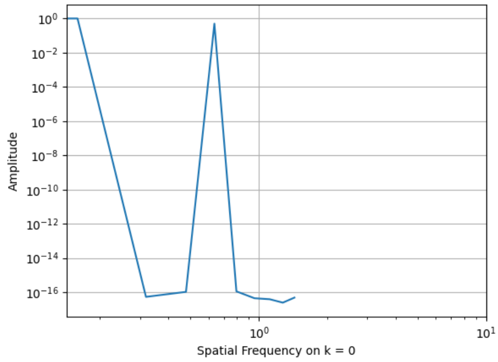
For the modified wavenumber, the result is showing above. The Exact wavenumber is ”Exact k” in the picture, which don’t have imaginary part. For the modified wavenumbers, each scheme’s wavenumber have two parts in the same color, the real part named ”Re k” in the chart, while the imaginary part named ”Im k” for each scheme.

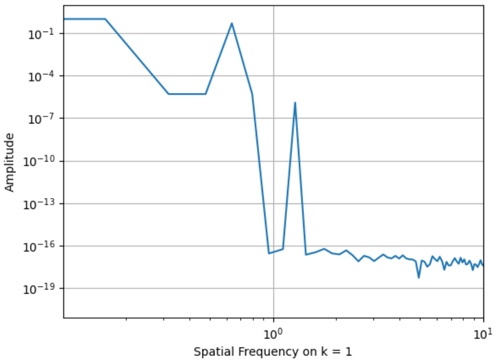
It is easily to be observed that, for the modified wavenumber’s real part, the 3rd UPwind scheme is much larger than others, which is also far away from the exact k, which could explain it cannot well-simulate the equation, espically on the lone time rigion. On the other hand, 3rd biased UPwind scheme is much closer to the exact k than the unbiased one, which could explain its simulation result is closer to the exact solution.

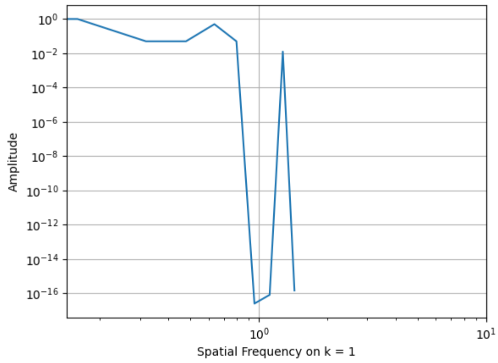
For the modified wavenumber’s imaginary part, the 3rd biased scheme’s Im k is much smaller than the others, which could explain its good simulation result.

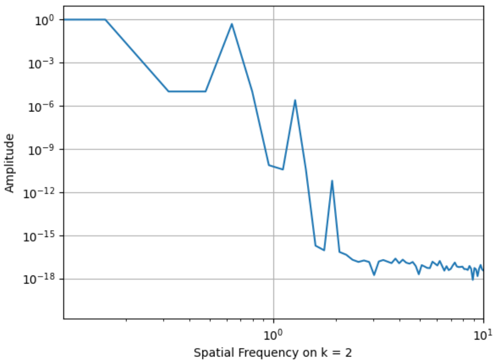
3 2. Aliased Analysis with FFT

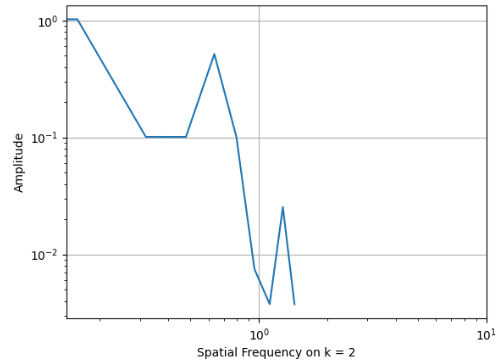


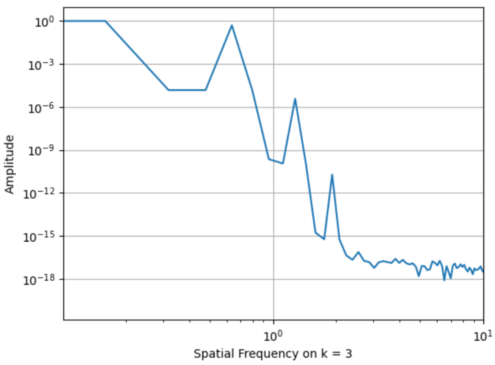


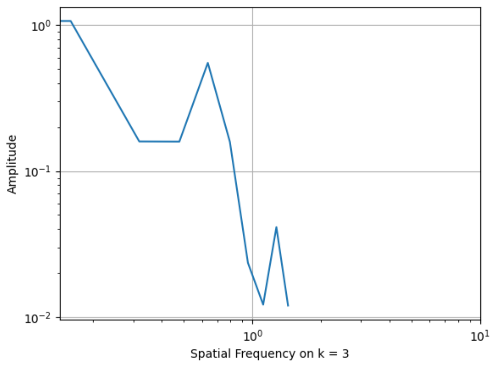









Amplitude
Spatial Frequency on k = 3


Amplitude
Spatial Frequency on k = 3

3.1

Appendix

Listing 1: Problem1, Py code for Solvers

```
import math
import numpy as np
import matplotlib.pyplot as plt
import matplotlib as mpl
import copy

class UPwind1st_Solver:
    def __init__(self, dx, dt, x_max, t_max,m):
        self.dx = dx
        self.dt = dt
        self.x_max = x_max
        self.t_max = t_max


        self.m = m  #added condition input

    def grid_generate(self):
        self.i_max = int(self.x_max/self.dx)
        self.n_max = int(self.t_max/self.dt)

        self.u = np.zeros(( self.i_max  ))


    def IC(self,m):
        for i in range(0, self.i_max):
            self.u[i] = math.sin(m*i*self.dx)

    def Iteration_Formula(self, u_W, u):
        u_new = u + (self.dt/(self.dx))*(u_W-u)
        return u_new

    def Iterative(self):
        u_next = copy.deepcopy(self.u)

        for n in range(1, self.n_max+1):
            for i in range(0,self.i_max):
                u_next[i] = self.Iteration_Formula(self.u[i-1], self.u[i])
            self.u[:] = u_next[:]
        self.u_Full = self.getTHElastBACK(self.u)
        return self.u_Full


    def getTHElastBACK(self,u_lost):
        u_Fullget = np.copy(u_lost)
        u_Fullget = np.append(u_Fullget, [u_lost[0]])
        return u_Fullget

    def plot_result(self):
        x = np.arange(0,self.i_max+1)
        plt.plot(x*(2*math.pi/20),self.u_Full)
        plt.show()


    def exactsoln(self):
        u_e = np.zeros(self.i_max+1)
        for i in range(self.i_max+1):
            x = i * self.dx
            u_e[i] = math.sin(self.m * (x - self.t_max))
        return u_e

    def plot_result_compare(self):
        x = np.arange(0,self.i_max+1)
        u_e =  self.exactsoln()
        plt.plot(x*(2*math.pi/20),self.u_Full)
        plt.plot(x*(2*math.pi/20),u_e)
```

```
        print(u_e - self.u_Full)

        plt.show()



class UPwind2nd_Solver(UPwind1st_Solver):
    def Iteration_Formula(self, u, u_W, u_WW):
        u_new = u - (3*u -4*u_W +u_WW)*self.dt/(2*self.dx)

        return u_new

    def Iterative(self):
        u_next = np.copy(self.u)

        for n in range(1, self.n_max+1):
            for i in range(0,self.i_max):
                u_next[i] = self.Iteration_Formula(self.u[i],self.u[i-1],self.u[i-2])
            self.u[:] = u_next[:]
        self.u_Full = self.getTHElastBACK(self.u)
        return self.u_Full


class UPwind3rd_Solver(UPwind1st_Solver):
    def Iteration_Formula(self, u, u_W, u_WW,u_WWW):
        u_new = u - ((11/6)*u -3*u_W +(3/2)*u_WW -(1/3)*u_WWW)*self.dt/(self.dx)

        return u_new

    def Iterative(self):
        u_next = np.copy(self.u)

        for n in range(1, self.n_max+1):
            for i in range(0,self.i_max):
                u_next[i] = self.Iteration_Formula(self.u[i],self.u[i-1],self.u[i-2],self.u[i-3])
            self.u[:] = np.copy(u_next)

        self.u_Full = self.getTHElastBACK(np.copy(self.u))
        return self.u_Full



class UPwind3rdBias_Solver(UPwind1st_Solver):
    def Iteration_Formula(self, u, u_W, u_WW,u_E):
        u_new = u - ((3)*u -6*u_W +1*u_WW +2*u_E)*self.dt/(6*self.dx)

        return u_new

    def Iterative(self):
        u_next = copy.deepcopy(self.u)

        for n in range(1, self.n_max+1):
            for i in range(0,self.i_max):
                u_next[i] = self.Iteration_Formula(self.u[i],self.u[i-1],self.u[i-2],self.u[(i+1)%(self.i_
            self.u[:] = u_next[:]
        self.u_Full = self.getTHElastBACK(self.u)
        return self.u_Full



class Total_Compare(UPwind1st_Solver):
    def __init__(self, dx, dt, x_max, t_max, m, UPwind1, UPwind2, UPwind3, UPwind3b):
        super().__init__(dx, dt, x_max, t_max, m)
        self.UPwind1 = UPwind1
        self.UPwind2 = UPwind2
        self.UPwind3 = UPwind3
        self.UPwind3b = UPwind3b

    def plot_result_compare(self):
        x = np.arange(0,self.i_max+1)

        u_e =  self.exactsoln()
```

```
        plt.plot(x*(2*math.pi/20), self.UPwind1, label = "UPwind1")
        plt.plot(x*(2*math.pi/20), self.UPwind2, label = "UPwind2")
        plt.plot(x*(2*math.pi/20), self.UPwind3, label = "UPwind3")
        plt.plot(x*(2*math.pi/20), self.UPwind3b, label = "UPwind3b")
        plt.plot(x*(2*math.pi/20),u_e, label = "Exact")

        plt.title('Solutions at t_max=%f'%self.t_max)
        plt.xlabel('x')
        plt.ylabel('u')
        plt.legend()
        plt.show()


    def Error_Compute(self, u, u_e):
        Error = np.zeros(self.i_max+1)
        for i in range(0,self.i_max+1):
                Error[i]+= abs(u[i]-u_e[i])
        return Error

    def plot_Error_compare(self):
        x = np.arange(0,self.i_max+1)
        u_e =  self.exactsoln()

        EUPwind1 = self.Error_Compute(self.UPwind1, u_e)
        EUPwind2 = self.Error_Compute(self.UPwind2, u_e)
        EUPwind3 = self.Error_Compute(self.UPwind3, u_e)
        EUPwind3b = self.Error_Compute(self.UPwind3b, u_e)

        plt.plot(x*(2*math.pi/20),EUPwind1, label = "UPwind1")
        plt.plot(x*(2*math.pi/20),EUPwind2, label = "UPwind2")
        plt.plot(x*(2*math.pi/20),EUPwind3, label = "UPwind3")
        plt.plot(x*(2*math.pi/20),EUPwind3b, label = "UPwind3b")

        plt.title('Solutions at t_max=%f'%self.t_max)
        plt.xlabel('x')
        plt.ylabel('u')
        plt.legend()
        plt.show()


def main():

    x_max = 2*math.pi
    t_max = .1

    dx = 2*math.pi/20
    dt = 0.001

    m = 2


    ####### 2st Upwind Iteration
    upwind1st = UPwind1st_Solver(dx, dt, x_max, t_max,m)
    upwind1st.grid_generate()
    upwind1st.IC(m)
    U1 = upwind1st.Iterative()

    ####### 2nd Upwind Iteration
    upwind2nd = UPwind2nd_Solver(dx, dt, x_max, t_max,m)
    upwind2nd.grid_generate()
    upwind2nd.IC(m)
    U2 = upwind2nd.Iterative()


    ####### 3rd Upwind Iteration
    upwind3 = UPwind3rd_Solver(dx, dt, x_max, t_max, m)
    upwind3.grid_generate()
    upwind3.IC(m)
    U3 = upwind3.Iterative()


    ####### 3rd bias Iteration
    upwind3b = UPwind3rdBias_Solver(dx, dt, x_max, t_max,m)
```

```python
    upwind3b.grid_generate()
    upwind3b.IC(m)
    U3b = upwind3b.Iterative()


    #### Total Compare
    Total = Total_Compare(dx, dt, x_max, t_max, m, U1, U2, U3, U3b)
    Total.grid_generate()
    Total.IC(m)
    Total.plot_result_compare()
    #Total.plot_Error_compare()


if __name__ == '__main__':
    main()
```