$$\frac{\partial\phi}{\partial a} = \frac{\partial\phi}{\partial\xi} \qquad \frac{\partial\phi(\xi,\eta)}{\partial x} = \frac{\partial\phi(\xi,\eta)}{d\xi}\frac{d\xi}{\partial x}$$

$=$

对 $(x,y)\to(a,b)$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$dx = \frac{\partial x}{\partial a}da + \frac{\partial x}{\partial b}db$$

$$dy = \frac{\partial y}{\partial a}da + \frac{\partial y}{\partial b}db$$

Homework 1 $(a,b)\to(\xi,\eta)$

Numerical Method

EN 530.766

$$\phi_a = \frac{\partial\phi}{\partial a} = \frac{\partial\phi}{\partial\xi}\frac{\partial\xi}{\partial a} + \frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial a}$$

$$\phi_{ab} = \frac{\partial}{\partial b}\left(\frac{\partial\phi}{\partial a}\right)$$

$$dz = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial a}da + \frac{\partial x}{\partial b}db\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial a}da + \frac{\partial y}{\partial b}db\right)$$
$$= \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial a}\right)da + \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial b}\right)db$$
$$= \frac{\partial z}{\partial a}da + \frac{\partial z}{\partial b}db$$

1) Consider the following PDE

$$a\Phi_{xx} + b\Phi_{xy} + c\Phi_{yy} + d\Phi_x + e\Phi_y + f\,\Phi = g(x,y).$$

Transform the above equation from (x,y) to (ξ,η) and show that the transformed equation can be written as

$$A\Phi_{\xi\xi} + B\Phi_{\xi\eta} + C\Phi_{\eta\eta} = H(\Phi_\xi, \Phi_\eta, \Phi, \xi, \eta)$$

Obtain expressions for $A, B, C, \& H$. You should use chain-rule differentiation to transform the equations.

2) Classify and determine the characteristics of:

a) $u_{xx} - x^2 y u_{yy} = 0 \qquad y > 0$

b) $e^{2x}u_{xx} + 2e^{x+y}u_{xy} + e^{2y}u_{yy} = 0$

c) $2u_{xx} - 4u_{xy} - 6u_{yy} + u_x = 0$

$16 + \overset{48}{\cancel{4\times 2\times 6}} = 64$

Plot the family (or families) of characteristics.

3) Develop finite-difference approximations for dT/dx at (i) for a <u>non-uniform grid</u>. Assume that $\Delta x_i = x_{i+1} - x_i$, $\Delta x_{i-1} = x_i - x_{i-1}$, and $\Delta x_{i-1} \neq \Delta x_i$.
   a. Develop expressions that employ the following stencils: (T_{i-1}, T_i, T_{i+1}).
      Hint: use Taylor series expansion
   b. Show the first two terms in the truncation error and determine the order of the truncation error.

4) Develop a <u>second-order accurate</u> finite difference formulation for $\left.d^3u/dx^3\right)_i$ on a uniform grid using <u>central differencing</u>. Show the leading-order term in the truncation error.

5) Consider the function

$$f(x) = \frac{\sin x}{x^3}$$

Consider the <u>first-order forward difference</u>, <u>second-order central difference</u> and <u>fourth-order central-difference</u> approximations to the first derivative. Plot the absolute value of the difference between computed and exact derivative (i.e. the truncation error) for x=4.0 for different grid sizes (Δx) and show that the error changes with grid size as expected (order of accuracy). Employ <u>at least five different grid sizes</u>.

Note: A log-log plot is the most appropriate way of showing the order of accuracy.

HW1 Haobo Zhao

Question:

1) $a\phi_{xx}+b\phi_{xy}+c\phi_{yy}+d\phi_x+e\phi_y+f\phi=g(x,y) \longrightarrow A\phi_{\xi\xi}+B\phi_{\xi\eta}+C\phi_{\eta\eta}=H(\phi_\xi,\phi_\eta,\phi,\xi,\eta)$

Answer: $a\phi_{xx}+b\phi_{xy}+c\phi_{yy}=g(x,y)-d\phi_x-e\phi_y-f\phi=H(\phi_x,\phi_y,\phi,x,y)$

$a\phi_{xx}+b\phi_{xy}+c\phi_{yy}=H(\phi_x,\phi_y,\phi,x,y)\longrightarrow A\phi_{\xi\xi}+B\phi_{\xi\eta}+C\phi_{\eta\eta}=H(\phi_\xi,\phi_\eta,\phi,\xi,\eta)$

then $a\phi_{xx}+b\phi_{xy}+c\phi_{yy}=A\phi_{\xi\xi}+B\phi_{\xi\eta}+C\phi_{\eta\eta}+H'$, $H(\phi_\xi,\phi_\eta,\phi,\xi,\eta)=H(\phi_x,\phi_y,\phi,x,y)-H'$

for A, B, C:

for $\phi(a,b)$, $\phi_{ab}=\frac{\partial}{\partial b}\phi_a=\frac{\partial}{\partial b}\left(\frac{\partial\phi}{\partial\xi}\xi_a+\frac{\partial\phi}{\partial\eta}\eta_a\right)$

$=\xi_a\frac{\partial\phi_\xi}{\partial b}+\underline{\xi_{ab}\phi_\xi}+\eta_a\frac{\partial\phi_\eta}{\partial b}+\underline{\eta_{ab}\phi_\eta}$

$=\xi_a\left(\frac{\partial\phi_\xi}{\partial\xi}\xi_b+\frac{\partial\phi_\xi}{\partial\eta}\eta_b\right)+\eta_a\left(\frac{\partial\phi_\eta}{\partial\xi}\xi_b+\frac{\partial\phi_\eta}{\partial\eta}\eta_b\right)+\underline{\xi_{ab}\phi_\xi+\eta_{ab}\phi_\eta}$

$=\xi_a\xi_b\phi_{\xi\xi}+(\xi_a\eta_b+\xi_b\eta_a)\phi_{\xi\eta}+\eta_a\eta_b\phi_{\eta\eta}+\xi_{ab}\phi_\xi+\eta_{ab}\phi_\eta$

then $\phi_{xx}=\xi_x^2\phi_{\xi\xi}+2\xi_x\eta_x\phi_{\xi\eta}+\eta_x^2\phi_{\eta\eta}+\xi_{xx}\phi_\xi+\eta_{xx}\phi_\eta$

$\phi_{xy}=\xi_x\xi_y\phi_{\xi\xi}+(\xi_x\eta_y+\xi_y\eta_x)\phi_{\xi\eta}+\eta_x\eta_y\phi_{\eta\eta}+\xi_{xy}\phi_\xi+\eta_{xy}\phi_\eta$

$\phi_{yy}=\xi_y^2\phi_{\xi\xi}+2\xi_y\eta_y\phi_{\xi\eta}+\eta_y^2\phi_{\eta\eta}+\xi_{yy}\phi_\xi+\eta_{yy}\phi_\eta$

$a\phi_{xx}+b\phi_{xy}+c\phi_{yy}=A\phi_{\xi\xi}+B\phi_{\xi\eta}+C\phi_{\eta\eta}+H'(\phi_\xi,\phi_\eta)$

get $\begin{cases} A=a\xi_x^2+b\xi_x\xi_y+c\xi_y^2 \\ B=2a\xi_x\eta_x+b(\xi_x\eta_y+\xi_y\eta_x)+2c\xi_y\eta_y \\ C=a\eta_x^2+b\eta_x\eta_y+c\eta_y^2 \end{cases}$

for H: $\phi_x=\frac{\partial\phi}{\partial\xi}\frac{\partial\xi}{\partial x}+\frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial x}=\xi_x\phi_\xi+\eta_x\phi_\eta$

$\phi_y=\xi_y\phi_\xi+\eta_y\phi_\eta$

$AU_{xx}+BU_{xy}+CU_{yy}=H$

$A\left(\frac{\partial y}{\partial x}\right)^2-B\left(\frac{\partial y}{\partial x}\right)+C=0$

B^2-4AC

$\frac{dy}{dx}=\frac{B\pm\sqrt{B^2-4AC}}{2A}$

$H(\phi_\xi,\phi_\eta,\phi,\xi,\eta)=g(x,y)-d\phi_x-e\phi_y-f\phi-H'$

$=g(\xi,\eta)-(d\xi_x+e\xi_y+a\xi_{xx}+b\xi_{xy}+c\xi_{yy})\phi_\xi-(d\eta_x+e\eta_y+a\eta_{xx}+b\eta_{xy}+c\eta_{yy})\phi_\eta-f\phi$

2) Question:

a) $u_{xx}-x^2yu_{yy}=0,\ y>0$

linear

$b^2-4ac=4x^2y\geq 0$ as $y>0$

$x\neq0$ hyperbolic. $x=0$ parabolic

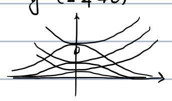
characteristic eqn. $\left(\frac{dy}{dx}\right)^2-x^2y=0$

$x\neq0$: $h=\frac{\pm\sqrt{4x^2y}}{2}=\pm x\sqrt{y}=\frac{dy}{dx}$

$x^2=\pm4\sqrt{y}+C$

$y=\left(\pm\frac{x^2}{4}+C\right)^2$

$x=0$: $y=C'$



b) $e^{2x}u_{xx}+2e^{x+y}u_{xy}+e^{2y}u_{yy}=0$

linear

$b^2-4ac=4e^{2(x+y)}-4e^{2(x+y)}=0$

parabolic.

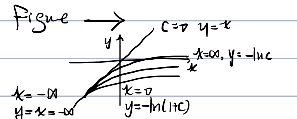
$e^{2x}\left(\frac{dy}{dx}\right)^2-2e^{x+y}\left(\frac{dy}{dx}\right)+e^{2y}=0$

$\frac{dy}{dx}=\frac{2e^{x+y}}{2e^{2x}}=e^{y-x}$

$e^{-y}=e^{-x}+C$

$y=-\ln(e^{-x}+C)$

Figue $\longrightarrow$



c) $2u_{xx}-4u_{xy}-6u_{yy}+u_x=0$

$2u_{xx}-4u_{xy}-6u_{yy}=H(u_x)$

linear

$b^2-4ac=16+48=64>0$

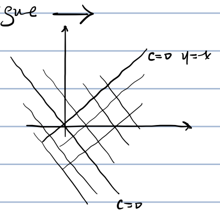
hyperbolic.

$2\left(\frac{dy}{dx}\right)^2+4\left(\frac{dy}{dx}\right)-6=0$

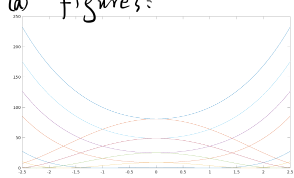
$\frac{dy}{dx}=\frac{b\pm\sqrt{b^2-4ac}}{2a}=\frac{-4\pm8}{4}=-1\pm2$

$y=(-1\pm2)x+C$

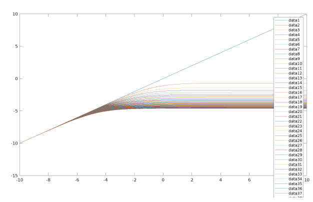
Figue $\longrightarrow$



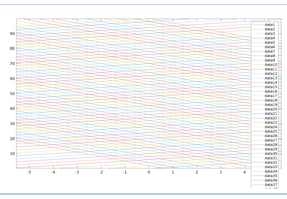
(a) figures:



(b)



(c)



3) $\frac{dT}{dx}=aT_{i+1}+bT_{i-1}+cT_i$

$aT_{i+1}=aT_i+a(x_{i+1}-x_i)T_i'+a\frac{(x_{i+1}-x_i)^2}{2}T_i''+a\frac{(x_{i+1}-x_i)^3}{6}T_i'''+a\frac{(x_{i+1}-x_i)^4}{24}T_i^{(4)}+a\sum_{n=5}^{\infty}\frac{\Delta x_i^n}{n!}T_i^{(n)}$

$bT_{i-1}=bT_i-b(x_i-x_{i-1})T_i'+b\frac{(x_i-x_{i-1})^2}{2}T_i''-b\frac{(x_i-x_{i-1})^3}{6}T_i'''+b\frac{(x_i-x_{i-1})^4}{24}T_i^{(4)}+b\sum_{n=5}^{\infty}\frac{(-\Delta x_{i-1})^n}{n!}T_i^{(n)}$

$cT_i=cT_i$

$a+b+c=0$ $\quad a\frac{(x_{i+1}-x_i)^2}{2}+b\frac{(x_i-x_{i-1})^2}{2}=0$

$a(x_{i+1}-x_i)-b(x_i-x_{i-1})=1$

$T=a\frac{(x_{i+1}-x_i)^3}{6}T_i'''+a\frac{(x_{i+1}-x_i)^4}{24}T_i^{(4)}+a\sum_{n=5}^{\infty}\frac{\Delta x_i^n}{n!}T_i^{(n)}$
$\quad-b\frac{(x_i-x_{i-1})^3}{6}T_i'''+b\frac{(x_i-x_{i-1})^4}{24}T_i^{(4)}+b\sum_{n=5}^{\infty}\frac{(-\Delta x_{i-1})^n}{n!}T_i^{(n)}$

$$\begin{pmatrix}1&1&1\\ \Delta x_i&-\Delta x_{i-1}&0\\ \frac{\Delta x_i^2}{2}&\frac{\Delta x_{i-1}^2}{2}&0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$Ax=0$

$$A=\begin{pmatrix}1&1&1\\ x_{i+1}-x_i&-(x_i-x_{i-1})&0\\ \frac{x_{i+1}^2-2x_{i+1}x_i+x_i^2}{2}&\frac{x_i^2-2x_ix_{i-1}+x_{i-1}^2}{2}&0\end{pmatrix}\Longrightarrow\begin{cases} a=\frac{x_i-x_{i-1}}{(x_{i+1}-x_i)(x_{i+1}-x_{i-1})}=\frac{\Delta x_{i-1}}{\Delta x_i(\Delta x_i+\Delta x_{i-1})} \\ b=-\frac{x_{i+1}-x_i}{(x_{i+1}-x_{i-1})(x_i-x_{i-1})}=-\frac{\Delta x_i}{\Delta x_{i-1}(\Delta x_i+\Delta x_{i-1})} \\ c=\frac{x_{i+1}-2x_i+x_{i-1}}{(x_{i+1}-x_i)(x_i-x_{i-1})}=\frac{\Delta x_i-\Delta x_{i-1}}{\Delta x_i\Delta x_{i-1}}\end{cases}$$

Then. $\frac{dT}{dx}=\frac{\Delta x_{i-1}T_{i+1}}{\Delta x_i(\Delta x_i+\Delta x_{i-1})}-\frac{\Delta x_iT_{i-1}}{\Delta x_{i-1}(\Delta x_i+\Delta x_{i-1})}+\frac{(\Delta x_i-\Delta x_{i-1})T_i}{\Delta x_i\Delta x_{i-1}}$

$T=\frac{T_i'''}{6}(a\Delta x_i^3-b\Delta x_{i-1}^3)+\frac{T_i^{(4)}}{24}(a\Delta x_i^4+b\Delta x_{i-1}^4)+a\sum_{n=5}^{\infty}\frac{\Delta x_i^n}{n!}T_i^{(n)}+b\sum_{n=5}^{\infty}\frac{(-\Delta x_{i-1})^n}{n!}T_i^{(n)}$

$=\frac{T_i'''}{6}\left(\frac{\Delta x_{i-1}\cdot\Delta x_i^{\cancel{3}2}}{\cancel{\Delta x_i}(\Delta x_i+\Delta x_{i-1})}+\frac{\Delta x_i\cdot\Delta x_{i-1}^{\cancel{3}2}}{\cancel{\Delta x_{i-1}}(\Delta x_i+\Delta x_{i-1})}\right)+\frac{T_i^{(4)}}{24}\left(\frac{\Delta x_{i-1}\cdot\Delta x_i^{\cancel{4}3}}{\cancel{\Delta x_i}(\Delta x_i+\Delta x_{i-1})}-\frac{\Delta x_i\cdot\Delta x_{i-1}^{\cancel{4}3}}{\cancel{\Delta x_{i-1}}(\Delta x_i+\Delta x_{i-1})}\right)+\sum_5^{\infty}T_i^{(n)}$

$=\frac{T_i'''}{6}\cdot\frac{\Delta x_i\Delta x_{i-1}\cancel{(\Delta x_i+\Delta x_{i-1})}}{\cancel{(\Delta x_i+\Delta x_{i-1})}}+\frac{T_i^{(4)}}{24}\frac{\Delta x_i\Delta x_{i-1}(\Delta x_i-\Delta x_{i-1})\cancel{(\Delta x_i+\Delta x_{i-1})}}{\cancel{(\Delta x_i+\Delta x_{i-1})}}+\Sigma$

$=\frac{T_i'''}{6}\Delta x_i\Delta x_{i-1}+\frac{T_i^{(4)}}{24}\Delta x_i\Delta x_{i-1}(\Delta x_i-\Delta x_{i-1})+\Sigma$

first two terms of truncation error

4) central difference $\Delta_c T_i = \frac{T_{i+\frac{1}{2}} - T_{i-\frac{1}{2}}}{\Delta x}$

$$\Delta_c^2 T_i = \frac{\Delta_c T_{i+\frac{1}{2}} - \Delta_c T_{i-\frac{1}{2}}}{\Delta x}$$

$$\frac{d^3T}{dx^3} \approx \Delta_c^3 T_i = \frac{\Delta_c^2 T_{i+\frac{1}{2}} - \Delta_c^2 T_{i-\frac{1}{2}}}{\Delta x} = \frac{\frac{\Delta_c T_{i+\frac{1}{2}+\frac{1}{2}} - \Delta_c T_{i-\frac{1}{2}+\frac{1}{2}}}{\Delta x} - \frac{\Delta_c T_{i-\frac{1}{2}+\frac{1}{2}} - \Delta_c T_{i-\frac{1}{2}-\frac{1}{2}}}{\Delta x}}{\Delta x} = \frac{\Delta_c T_{i+1} - \Delta_c T_i - \Delta_c T_i + \Delta_c T_{i-1}}{\Delta x^2}$$

$$= \frac{\Delta_c T_{i+1} - 2\Delta_c T_i + \Delta_c T_{i-1}}{\Delta x^2} = \frac{\frac{T_{i+1+\frac{1}{2}} - T_{i+1-\frac{1}{2}}}{\Delta x} - 2\frac{T_{i+\frac{1}{2}} - T_{i-\frac{1}{2}}}{\Delta x} + \frac{T_{i-1+\frac{1}{2}} - T_{i-1-\frac{1}{2}}}{\Delta x}}{\Delta x^2} = \frac{T_{i+\frac{3}{2}} - 3T_{i+\frac{1}{2}} + 3T_{i-\frac{1}{2}} - T_{i-\frac{3}{2}}}{\Delta x^3}$$

$$= \frac{\frac{T_{i+2}+\cancel{T_{i+1}}}{2} - \frac{\cancel{3T_{i+1}}+\cancel{3T_i}}{2} + \frac{\cancel{3T_i}+\cancel{3T_{i-1}}}{2} - \frac{T_{i-2}+\cancel{T_{i-1}}}{2}}{\Delta x^3} = \frac{T_{i+2} - 2T_{i+1} + 2T_{i-1} - T_{i-2}}{2\Delta x^3}$$

C. ① $\underline{2}$. Δx^2 save one point?

①	$o(\Delta x)$	1	1	✓

forward. $f^{(n)}$ $o(\Delta x^r)$ #point: $n+r$

central. $f^{(n)}$ $o(\Delta x^r)$ #point: $n+r-1$

① 3 2

$3+r-1$

Deploy Taylor Table:

	T_i	T_i'	T_i''	T_i'''	$T_i^{(4)}$	$T_i^{(5)}$	$T_i^{(6)}$
T_{i+2}	1	$2\Delta x$	$2\Delta x^2$	$\frac{4}{3}\Delta x^3$	$\frac{2}{3}\Delta x^4$	$\frac{4}{15}\Delta x^5$	$\frac{4}{45}\Delta x^6$
$-2T_{i+1}$	-2	$-2\Delta x$	$-\Delta x^2$	$-\frac{\Delta x^3}{3}$	$-\frac{\Delta x^4}{12}$	$-\frac{\Delta x^5}{60}$	$-\frac{\Delta x^6}{360}$
$2T_{i-1}$	2	$-2\Delta x$	Δx^2	$-\frac{\Delta x^3}{3}$	$\frac{\Delta x^4}{12}$	$-\frac{\Delta x^5}{60}$	$\frac{\Delta x^6}{360}$
$-T_{i-2}$	-1	$2\Delta x$	$-2\Delta x^2$	$\frac{4}{3}\Delta x^3$	$-\frac{2}{3}\Delta x^4$	$\frac{4}{15}\Delta x^5$	$-\frac{4}{45}\Delta x^6$
$2\Delta x^3 \Delta_c^3 T = \Sigma$	0	0	0	$2\Delta x^3$	0	$\frac{1}{2}\Delta x^5$	0
$\Delta_c^3 T$	0	0	0	1	0	$\frac{1}{4}\Delta x^2$	0

leading term: $\frac{1}{4}\Delta x^2$

5) Taylor table:

	T_i	T_i'	T_i''	T_i'''	$T_i^{(4)}$	$T_i^{(5)}$	$T_i^{(6)}$
aT_i	$1 \cdot a$	0	0	0	0	0	
bT_{i-1}	$1 \cdot b$	$-\Delta x b$	$\frac{\Delta x^2}{2}b$	$-\frac{\Delta x^3}{6}b$	$\frac{\Delta x^4}{24}b$	$-\frac{\Delta x^5}{120}b$	$\frac{\Delta x^6}{720}b$
cT_{i+1}	$1 \cdot c$	$\Delta x c$	$\frac{\Delta x^2}{2}c$	$\frac{\Delta x^3}{6}c$	$\frac{\Delta x^4}{24}c$	$\frac{\Delta x^5}{120}c$	$\frac{\Delta x^6}{720}c$
dT_{i-2}	$1 \cdot d$	$-2\Delta x d$	$2\Delta x^2 d$	$-\frac{4}{3}\Delta x^3 d$	$\frac{2}{3}\Delta x^4 d$	$-\frac{4}{15}\Delta x^5 d$	$-\frac{4}{45}\Delta x^6 d$
eT_{i+2}	$1 \cdot e$	$2\Delta x e$	$2\Delta x^2 e$	$\frac{4}{3}\Delta x^3 e$	$\frac{2}{3}\Delta x^4 e$	$\frac{4}{15}\Delta x^5 e$	$\frac{4}{45}\Delta x^6 e$

1st Forward difference: $\Delta_F T_i = aT_i + cT_{i+1} = T_i' + o(\Delta x) = T_i' + k\Delta x T_i^{(2)} + o(\Delta x^2)$

let. $aT_i + cT_{i+1} = T_i' + k\Delta x T_i^{(2)} + ..$

$$\begin{matrix} kT_i = 0 \\ k_1 T_i' = T_i' \end{matrix} \begin{cases} a + c = 0 \\ c\Delta x = 1 \end{cases} \Rightarrow \begin{cases} a = -\frac{1}{\Delta x} \\ c = \frac{1}{\Delta x} \end{cases} \qquad k\Delta x T_i^{(2)} = \frac{\Delta x^2}{2} \cdot \frac{1}{\Delta x} T_i^{(2)} = \frac{1}{2}\Delta x^2 T_i^{(2)}$$

$$\Delta_F T_i = \frac{T_{i+1} - T_i}{\Delta x} \qquad \Delta_F \Big|_{x=4} T = \frac{\frac{\sin(4+\Delta x)}{(4+\Delta x)^3} - \frac{\sin(4)}{(4)^3}}{\Delta x}$$

2nd Central order difference: $\underset{(2)}{\Delta_c} T_i = aT_i + bT_{i-1} + cT_{i+1} = T_i' + o(\Delta x^2) = T_i' + k\Delta x^2 T_i^{(3)} + o(x^4)$

let $aT_i + bT_{i-1} + cT_{i+1} = T_i' + k\Delta x^2 T_i^{(3)} + ..$

$$\begin{matrix} k_1 T_i = 0 \\ k_2 T_i' = T_i' \\ k_3 T_i^{(2)} = 0 \end{matrix} \begin{cases} a + b + c = 0 \\ (-b + c)\Delta x = 1 \\ b + c = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = \frac{-1}{2\Delta x} \\ c = \frac{1}{2\Delta x} \end{cases} \qquad \begin{matrix} k\Delta x T_i^{(2)} = 0 \checkmark \\ k\Delta x^2 T_i^{(3)} = \frac{\Delta x^2}{6} T_i^{(3)} \end{matrix}$$

$$\underset{(2)}{\Delta_c} T_i = \frac{-T_{i-1} + T_{i+1}}{2\Delta x} \qquad \underset{(2)}{\Delta_c} \Big|_{x=4} T = \frac{-\frac{\sin(4-\Delta x)}{(4-\Delta x)^3} + \frac{\sin(4+\Delta x)}{(4+\Delta x)^3}}{2\Delta x}$$

4th Central order difference. $\underset{(4)}{\Delta_c} T_i = aT_i + bT_{i-1} + cT_{i+1} + dT_{i-2} + eT_{i+2} = T_i' + o(\Delta x^4) = T_i' + k\Delta x^4 T_i^{(5)} + o(\Delta x^5)$

let $aT_i + bT_{i-1} + cT_{i+1} + dT_{i-2} + eT_{i+2} = T_i' + k\Delta x^4 T_i^{(5)} + ..$

$$\begin{matrix} k_0 T_i \to k_0 = 0 \\ k_1 T_i' \to k_1 = 1 \\ k_2 T_i'' \to k_2 = 0 \\ k_3 T_i''' \to k_3 = 0 \\ k_4 T_i^{(4)} \to k_4 = 0 \end{matrix} \begin{cases} a + b + c + d + e = 0 \\ -b + c - 2d + 2e = \frac{1}{\Delta x} \\ \frac{1}{2}b + \frac{1}{2}c + 2d + 2e = 0 \\ -\frac{1}{6}b + \frac{1}{6}c - \frac{4}{3}d + \frac{4}{3}e = 0 \\ \frac{1}{24}b + \frac{1}{24}c + \frac{2}{3}d + \frac{2}{3}e = 0 \end{cases} \to \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -2 & 2 \\ 0 & \frac{1}{2} & \frac{1}{2} & 2 & 2 \\ 0 & -\frac{1}{6} & \frac{1}{6} & -\frac{4}{3} & \frac{4}{3} \\ 0 & \frac{1}{24} & \frac{1}{24} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 0 \\ \frac{1}{\Delta x} \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{b} \to \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{bmatrix} 0 \\ -\frac{2}{3}\frac{1}{\Delta x} \\ \frac{2}{3}\frac{1}{\Delta x} \\ \frac{1}{12}\frac{1}{\Delta x} \\ -\frac{1}{12}\frac{1}{\Delta x} \end{bmatrix}$$

$$\underset{(4)}{\Delta_c} T_i = \frac{-8T_{i-1} + 8T_{i+1} + T_{i-2} - T_{i+2}}{12\Delta x} \qquad \underset{(4)}{\Delta_c} \Big|_{x=4} T = \frac{-8\frac{\sin(4-\Delta x)}{(4-\Delta x)^3} + 8\frac{\sin(4+\Delta x)}{(4+\Delta x)^3} + \frac{\sin(4-2\Delta x)}{(4-2\Delta x)^3} - \frac{\sin(4+2\Delta x)}{(4+2\Delta x)^3}}{12\Delta x}$$

$$\Delta_F \Big|_{x=4} T = \frac{\frac{\sin(4+\Delta x)}{(4+\Delta x)^3} - \frac{\sin(4)}{(4)^3}}{\Delta x}$$

$$\underset{(2)}{\Delta_c} \Big|_{x=4} T = \frac{-\frac{\sin(4-\Delta x)}{(4-\Delta x)^3} + \frac{\sin(4+\Delta x)}{(4+\Delta x)^3}}{2\Delta x}$$

$$\underset{(4)}{\Delta_c} \Big|_{x=4} T = \frac{-8\frac{\sin(4-\Delta x)}{(4-\Delta x)^3} + 8\frac{\sin(4+\Delta x)}{(4+\Delta x)^3} + \frac{\sin(4-2\Delta x)}{(4-2\Delta x)^3} - \frac{\sin(4+2\Delta x)}{(4+2\Delta x)^3}}{12\Delta x}$$

$$d_{exact}\Big|_{x=4} = \left(\frac{\sin x}{x^3}\right)'\Big|_{x=4}$$

Error: $\underset{(1)}{\varepsilon_F} = d_{exact}\Big|_{x=4} - \Delta_F \Big|_{x=4} T$

$$\underset{(2)}{\varepsilon_c} = d_{exact}\Big|_{x=4} - \underset{(2)}{\Delta_c} \Big|_{x=4} T$$

$$\underset{(4)}{\varepsilon_c} = d_{exact}\Big|_{x=4} - \underset{(4)}{\Delta_c} \Big|_{x=4} T$$