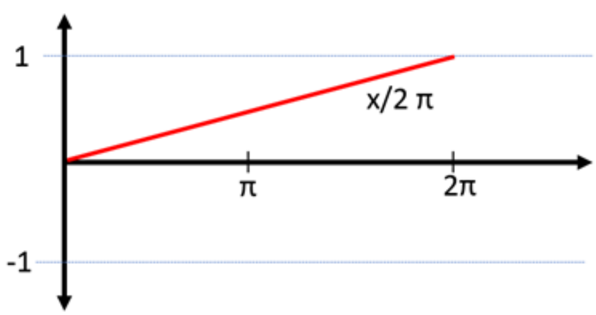
Numerical Methods

Homework #2
EN 530.766

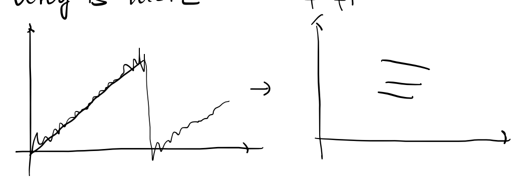
1) Derive the highest accuracy central compact difference scheme for the second derivative that employs 3 points.

2) Fourier series
   a) Derive the Fourier series for the function shown below and plot the Fourier spectrum (modulus of the complex amplitude for each wave number term) versus wavenumber on a log-log plot.
   b) Truncate the Fourier series beyond
      a. k=8
      b. k=16
      c. k=32

   and calculate the function from this truncated series. Compare the original function and the function from the truncated series. Comment on this comparison for the various cases. You might find it useful to plot the difference between the two curves.



why is more



3) Derive the truncation errors for the Forward and Backward Euler schemes when applied to the 1-D unsteady heat conduction equation.

Un- NOT Conditionally Stable.

4) Use the Forward Euler scheme to numerically solve (i.e. write a computer code) the unsteady heat conduction for the temperature T in a long thin bar (thermal diffusion coefficient = 1) (0 ≤ x ≤ 1) subject to the following boundary and initial conditions

$T_t = \alpha T_{xx}$

$T_t = T_{xx}$

$$T(x,0) = 0 \quad ; \quad T(0,t>0+) = 1; \quad T(1,t)=0$$

$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2}$

$T_i^{n+1} = \frac{1}{\Delta x^2}(T_{i+1}^n + T_{i-1}^n) + \left(\frac{1}{\Delta t} - \frac{2}{\Delta x^2}\right)T_i^n$

   a. Use a grid with 20 segments (i.e. Δx=1/20)
   b. Compare numerical solution against exact solution for Δt = 1/1000, 1/800, 1/400 and 1/200. Make the comparison at a few chosen times during the development of the solution.
   c. Comment on the results.

$i = [2, \dots n-1]$

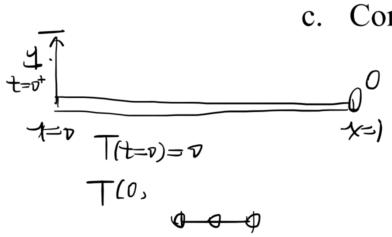
$i=1$ $\quad i=n$

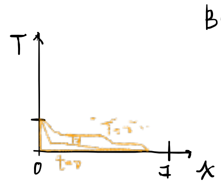
$T^{n=0} = [0, \dots 0]$

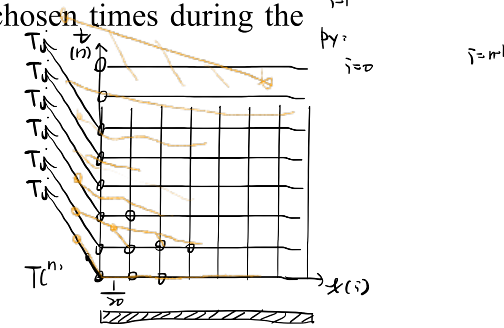
BC

$T^{n=1} = [1, 0, \dots, 0]$

$T^n = [1, ?, ?, 0]$







Haobo Zhao Numerical Methods HW 2.

1) Derive the highest accuracy central compact difference scheme for the second derivative that employs 3 points.

for points not be effected by boundary.

use f_i, f_{i-1}, f_{i+1} and derivatives $f_i'', f_{i-1}'', f_{i+1}''$ to create compact scheme to pursue highest matrix:

$$a f_i + b f_{i+1} + c f_{i-1} + d f_i'' + e f_{i-1}'' + k f_{i+1}'' = 0$$

Deploy Taylor table:

		f_i	f_i'	$f_i^{(2)}$	$f_i^{(3)}$	$f_i^{(4)}$	$f_i^{(5)}$	$f_i^{(6)}$	$f_i^{(7)}$
	af_i	a	0	0	0	0	0	0	
f	bf_{i+1}	b	$b\Delta x$	$b\frac{\Delta x^2}{2}$	$b\frac{\Delta x^3}{6}$	$b\frac{\Delta x^4}{24}$	$b\frac{\Delta x^5}{120}$	$b\frac{\Delta x^6}{720}$	
	cf_{i-1}	c	$-c\Delta x$	$c\frac{\Delta x^2}{2}$	$-c\frac{\Delta x^3}{6}$	$c\frac{\Delta x^4}{24}$	$-c\frac{\Delta x^5}{120}$	$c\frac{\Delta x^6}{720}$	
f''	df_i''	0	0	d	0	0	0	0	
	ef_{i+1}''	0	0	e	$e\Delta x$	$e\frac{\Delta x^2}{2}$	$e\frac{\Delta x^3}{6}$	$e\frac{\Delta x^4}{24}$	
	kf_{i-1}''	0	0	k	$-k\Delta x$	$k\frac{\Delta x^2}{2}$	$-k\frac{\Delta x^3}{6}$	$k\frac{\Delta x^4}{24}$	
	Σ	0	0	0	0	0	0	0	

6 unknowns $\rightarrow$ use 6 matries.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & \Delta x & -\Delta x & 0 & 0 & 0 \\ 0 & \frac{\Delta x^2}{2} & \frac{\Delta x^2}{2} & 1 & 1 & 1 \\ 0 & \frac{\Delta x^3}{6} & -\frac{\Delta x^3}{6} & 0 & \Delta x & -\Delta x \\ 0 & \frac{\Delta x^4}{24} & \frac{\Delta x^4}{24} & 0 & \frac{\Delta x^2}{2} & \frac{\Delta x^2}{2} \\ 0 & \frac{\Delta x^5}{120} & -\frac{\Delta x^5}{120} & 0 & \frac{\Delta x^3}{6} & -\frac{\Delta x^3}{6} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A \cdot x = 0$$

$$\sim A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & \Delta x & -\Delta x & 0 & 0 & 0 \\ 0 & \frac{\Delta x^2}{2} & \frac{\Delta x^2}{2} & 1 & 1 & 1 \\ 0 & \frac{\Delta x^3}{6} & -\frac{\Delta x^3}{6} & 0 & \Delta x & -\Delta x \\ 0 & \frac{\Delta x^4}{24} & \frac{\Delta x^4}{24} & 0 & \frac{\Delta x^2}{2} & \frac{\Delta x^2}{2} \\ 0 & \frac{\Delta x^5}{120} & -\frac{\Delta x^5}{120} & 0 & \frac{\Delta x^3}{6} & -\frac{\Delta x^3}{6} \end{bmatrix} \overset{\times\frac{1}{\Delta x^k}}{\sim} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & \Delta x & -\Delta x & 0 & 0 & 0 \\ 0 & \frac{\Delta x^2}{2} & \frac{\Delta x^2}{2} & 1 & 1 & 1 \\ 0 & \frac{\Delta x^2}{6} & -\frac{\Delta x^2}{6} & 0 & 1 & -1 \\ 0 & \frac{\Delta x^2}{24} & \frac{\Delta x^2}{24} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{\Delta x^2}{120} & -\frac{\Delta x^2}{120} & 0 & \frac{1}{6} & -\frac{1}{6} \end{bmatrix} \overset{\times k}{\sim} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & \frac{\Delta x^2}{2} & \frac{\Delta x^2}{2} & 1 & 1 & 1 \\ 0 & \frac{\Delta x^2}{6} & -\frac{\Delta x^2}{6} & 0 & 1 & -1 \\ 0 & \frac{\Delta x^2}{12} & \frac{\Delta x^2}{12} & 0 & 1 & 1 \\ 0 & \frac{\Delta x^2}{20} & -\frac{\Delta x^2}{20} & 0 & 1 & -1 \end{bmatrix} \begin{matrix} ① \\ ② \\ ③ \\ ④ \\ ⑤ \\ ⑥ \end{matrix}$$

④ $-= \frac{\Delta x^2}{6}$ ②
⑥ $-= \frac{\Delta x^2}{20}$ ②
⑥ $-=$ ④
⑤ $-= \frac{1}{6}$ ③
⑤ $\times = -6$
⑤ $+= 5\times$ ④

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & \frac{\Delta x^2}{2} & \frac{\Delta x^2}{2} & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} ① \\ ② \\ ③ \\ ④ \\ ⑤ \\ ⑥ \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & \frac{\Delta x^2}{2} & \frac{\Delta x^2}{2} & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -10 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} ① \\ ② \\ ③ \\ ④ \\ ⑤ \\ ⑥ \end{matrix}$$

④ $\leftrightarrow$ ⑤
③ $-= \frac{\Delta x^2}{2}$ ②
③ $\times = \frac{1}{\Delta x^2}$
③ $-= \frac{1}{\Delta x^2}$ ⑤
③ $-= \frac{1}{\Delta x^2}$ ④
② $+=$ ③
① $-=$ ② $+$ ③

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{-24}{\Delta x^2} \\ 0 & 1 & 0 & 0 & 0 & \frac{12}{\Delta x^2} \\ 0 & 0 & 1 & 0 & 0 & \frac{12}{\Delta x^2} \\ 0 & 0 & 0 & 1 & 0 & -10 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} ① \\ ② \\ ③ \\ ④ \\ ⑤ \\ ⑥ \end{matrix} \quad \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ k \end{bmatrix} = \begin{bmatrix} \frac{24}{\Delta x^2} \\ \frac{-12}{\Delta x^2} \\ \frac{-12}{\Delta x^2} \\ 10 \\ 1 \\ 1 \end{bmatrix}$$

let $k=1$

$$\frac{24}{\Delta x^2} f_i - \frac{12}{\Delta x^2} f_{i+1} - \frac{12}{\Delta x^2} f_{i-1} + 10 f_i'' + f_{i-1}'' + f_{i+1}'' = 0$$

(1) ANSWER:

$$\frac{24}{\Delta x^2} f_i - \frac{12}{\Delta x^2} f_{i+1} - \frac{12}{\Delta x^2} f_{i-1} + 10 f_i'' + f_{i-1}'' + f_{i+1}'' = 0 \quad \left(\text{or } \frac{-12}{5\Delta x^2} f_i + \frac{6}{5\Delta x^2} f_{i+1} + \frac{6}{5\Delta x^2} f_{i-1} = f_i'' + \frac{1}{10} f_{i-1}'' + \frac{1}{10} f_{i+1}'' \right)$$

2) Fourier series
   a) Derive the Fourier series for the function shown below and plot the Fourier spectrum (modulus of the complex amplitude for each wave number term) versus wavenumber on a log-log plot.
   b) Truncate the Fourier series beyond
      a. k=8
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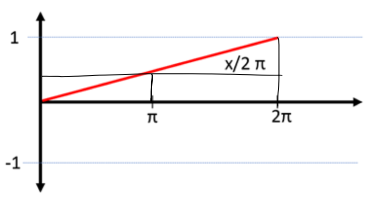
   and calculate the function from this truncated series. Compare the original function and the function from the truncated series. Comment on this comparison for the various cases. You might find it useful to plot the difference between the two curves.



a) fourier: $f(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} A_n e^{inx\left(\frac{2\pi}{L}\right)} + A_0$

$$A_n = \frac{1}{L}\int_0^L f(x)\, e^{-inx\left(\frac{2\pi}{L}\right)} dx \qquad A_0 = \frac{1}{2}$$

For $f(x) = \frac{x}{2\pi}$ in $x \in [0, 2\pi]$, $L = 2\pi$

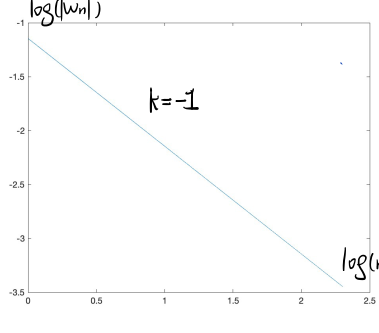
$n \neq 0$:
$$A_n = \frac{1}{2\pi}\int_0^{2\pi} \frac{x}{2\pi} e^{-inx} dx = \frac{1}{4\pi^2}\int_0^{2\pi} -\frac{1}{in} x\, d e^{-inx}$$
$$= \frac{i}{4\pi^2 n}\left[x e^{-inx}\Big|_0^{2\pi} - \int_0^{2\pi} e^{-inx} dx \right]$$
$$= \frac{1}{4\pi^2}\left[\frac{i}{n}(x\cos nx - ix\sin nx)\Big|_{x=0}^{x=2\pi} - e^{-inx}\Big|_0^{2\pi} \right]$$
$$= \frac{1}{4\pi^2}\left[\frac{i}{n}(2\pi) - [\cos nx - i\sin nx]_{x=0}^{x=2\pi} \right]$$
$$= \frac{i}{2\pi n} \qquad A_n$$

$n=0$: $A_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{x}{2\pi} dx = \frac{1}{2}$

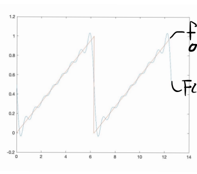
$$F(x) = \frac{1}{2} + \sum_{\substack{-\infty \\ n\neq 0}}^{\infty} \frac{i}{2\pi n} e^{inx}$$
$$= \frac{1}{2} + \sum_{\substack{-\infty \\ n\neq 0}}^{\infty} \frac{1}{2\pi n}(i\cos nx - \sin nx)$$

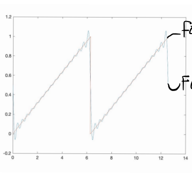
$$f = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin nx}{\pi n} \qquad \frac{\sin nx}{\pi n} \quad \frac{\sin nx}{\pi n}.$$

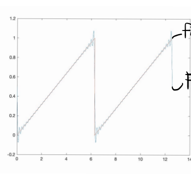
$$W(n) = \begin{cases} \frac{1}{2} & n=0 \\ \frac{1}{\pi n} & n=1,2,3\ldots \end{cases} \qquad n=0,1,\ldots$$



(b) $F(x)\Big|_{n=8} = \frac{1}{2} - \sum_{n=1}^{8} \frac{\sin(nx)}{\pi n}$ $\quad F(x)\Big|_{n=16} = \frac{1}{2} - \sum_{n=1}^{16} \frac{\sin(nx)}{\pi n}$ $\quad F(x)\Big|_{n=16} = \frac{1}{2} - \sum_{n=1}^{32} \frac{\sin(nx)}{\pi n}$





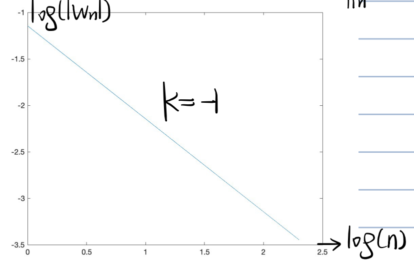


can see for bigger n, they are closer, means simulation more accurate

(2) ANSWER:

(a) $f = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin nx}{\pi n}$, $W(n) = \begin{cases} \frac{1}{2} & n=0 \\ -\frac{1}{\pi n} & n=1,2,3\ldots \end{cases} \rightarrow |W(n)| = \frac{1}{\pi n}$

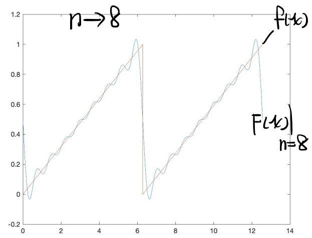
$\log(|W(n)|) = \log\left(\frac{1}{\pi n}\right) = -\log(\pi) - \log(n)$

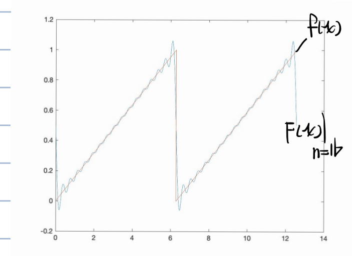


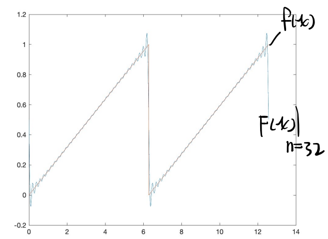
(b) Trun-8: $F(x)\big|_{n=8} = \frac{1}{2} - \sum_{n=1}^{8} \frac{\sin(nx)}{\pi n}$

Trun-16: $F(x)\big|_{n=16} = \frac{1}{2} - \sum_{n=1}^{16} \frac{\sin(nx)}{\pi n}$

Trun-32: $F(x)\big|_{n=16} = \frac{1}{2} - \sum_{n=1}^{32} \frac{\sin(nx)}{\pi n}$



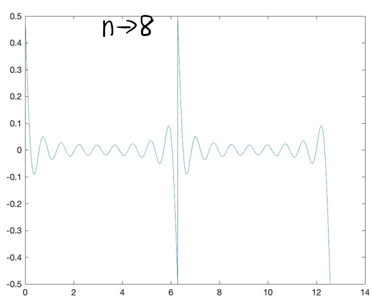


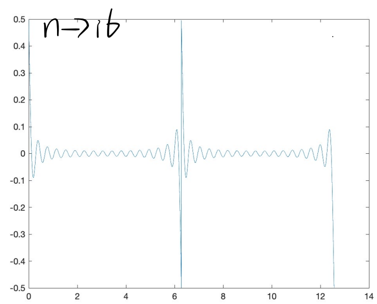


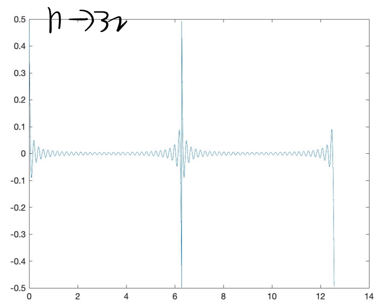
Can see with bigger n, they are closer, means simulation more accurate

$\text{Error } F\big|_{n=8} > \text{Error } F\big|_{n=16} > \text{Error } F\big|_{n=32}$

ERRORS = F(x) − f(x)







with bigger n the error wave "moves towards" the period beging and end point, err

Please see Appendix if pictures not clear enough (end of document)

3) Derive the truncation errors for the Forward and Backward Euler schemes when applied to the 1-D unsteady heat conduction equation.

① $u_i^{n+1} = u_i^n + \Delta t\, \partial_t u_i^n + \frac{\Delta t^2}{2} \partial_t^2 u_i^n + \frac{\Delta t^3}{6} \partial_t^3 u_i^n + \ldots$

② $-u_i^{n} = -u_i^{n+1} + \Delta t\, \partial_t u_i^{n+1} - \frac{\Delta t^2}{2} \partial_t^2 u_i^{n+1} + \frac{\Delta t^3}{6} \partial_t^3 u_i^{n+1} + \ldots$

PDE: $u_t = \alpha u_{xx}$

Forward Euler: $\frac{u_i^{n+1} - u_i^n}{\Delta t} - \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x} = u_t - \alpha u_{xx} + \tau_{FE}$

Backward Euler: $\frac{u_i^{n+1} - u_i^n}{\Delta t} - \alpha \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x} = u_t - \alpha u_{xx} + \tau_{BE}$

		u_i^k	$\partial_x u^k$	$\partial_x^2 u^k$	$\partial_x^3 u^k$	$\partial_x^4 u^k$	$\partial_x^5 u^k$	$\partial_x^6 u^k$
③	$\frac{1}{\Delta x} u_{i+1}^k$	$\frac{1}{\Delta x}$	1	$\frac{\Delta x}{2}$	$\frac{\Delta x^2}{6}$	$\frac{\Delta x^3}{24}$	$\frac{\Delta x^4}{120}$	$\frac{\Delta x^5}{720}$
④	$\frac{1}{\Delta x} u_i^k$	$\frac{1}{\Delta x}$	0	0	0	0		
③	$\frac{1}{\Delta x} u_{i-1}^k$	$\frac{1}{\Delta x}$	−1	$\frac{\Delta x}{2}$	$-\frac{\Delta x^2}{6}$	$\frac{\Delta x^3}{24}$	$-\frac{\Delta x^4}{120}$	$\frac{\Delta x^5}{720}$

$k=n$, $k=n+1$
for FE   for BE

$\tau_{FE} = \frac{u_i^{n+1} - u_i^n}{\Delta t} - \partial_t u_i^n - \alpha\left(\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x} - \partial_x^2 u_i^n\right)$

① ↦ $= \left(\frac{\Delta t}{2} \partial_t^2 u_i^n + \frac{\Delta t^2}{6} \partial_t^3 u_i^n + \ldots\right) - \alpha\left(\frac{\Delta x^2}{12} \partial_x^4 u_i^n + \frac{\Delta x^4}{360} \partial_x^6 u_i^n + \ldots\right)$

$= \left(\frac{\Delta t}{2} \partial_t^2 u + \frac{\Delta t^2}{6} \partial_t^3 u + \ldots\right) - \alpha\left(\frac{\Delta x^2}{12} \partial_x^4 u + \frac{\Delta x^4}{360} \partial_x^6 u + \ldots\right)$

$\tau_{BE} = \frac{u_i^{n+1} - u_i^n}{\Delta t} - \partial_t u_i^{n+1} - \alpha\left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x} - \partial_x^2 u_i^{n+1}\right)$

① ↦ $= \left(-\frac{\Delta t}{2} \partial_t^2 u_i^{n+1} + \frac{\Delta t^2}{6} \partial_t^3 u_i^{n+1} + \ldots\right) - \alpha\left(\frac{\Delta x^2}{12} \partial_x^4 u_i^{n+1} + \frac{\Delta x^4}{360} \partial_x^6 u_i^{n+1} + \ldots\right)$

$= \left(-\frac{\Delta t}{2} \partial_t^2 u + \frac{\Delta t^2}{6} \partial_t^3 u + \ldots\right) - \alpha\left(\frac{\Delta x^2}{12} \partial_x^4 u + \frac{\Delta x^4}{360} \partial_x^6 u + \ldots\right)$

(3) ANSWER:

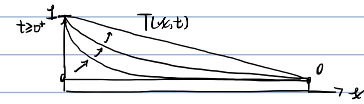
Forward Euler: $\tau_{FE} = \left(\frac{\Delta t}{2} \partial_t^2 u + \frac{\Delta t^2}{6} \partial_t^3 u + \ldots\right) - \alpha\left(\frac{\Delta x^2}{12} \partial_x^4 u + \frac{\Delta x^4}{360} \partial_x^6 u + \ldots\right) = O(\Delta t, \Delta x^2)$

Backward Euler: $\tau_{BE} = \left(-\frac{\Delta t}{2} \partial_t^2 u + \frac{\Delta t^2}{6} \partial_t^3 u + \ldots\right) - \alpha\left(\frac{\Delta x^2}{12} \partial_x^4 u + \frac{\Delta x^4}{360} \partial_x^6 u + \ldots\right) = O(\Delta t, \Delta x^2)$

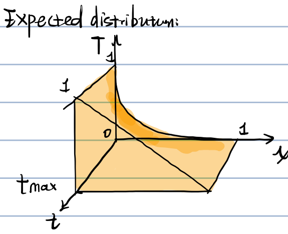
4) Use the Forward Euler scheme to numerically solve (i.e. write a computer code) the unsteady heat conduction for the temperature T in a long thin bar (thermal diffusion coefficient = 1) $(0 \le x \le 1)$ subject to the following boundary and initial conditions

$$T(x,0) = 0 \quad ; \quad T(0,t>0+) = 1; \quad T(1,t)=0$$

a. Use a grid with 20 segments (i.e. $\Delta x=1/20$)
b. Compare numerical solution against exact solution for Δt = 1/1000, 1/800, 1/400 and 1/200. Make the comparison at a few chosen times during the development of the solution.
c. Comment on the results.



Expected distribution:



eqn: $\partial_t T = \alpha \partial_{xx} T$

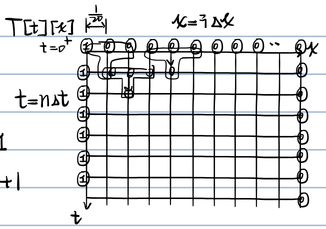
$\alpha = 1$

$\partial_t T = \partial_x^2 T$

Forward Euler: $\dfrac{T_i^{n+1} - T_i^n}{\Delta t} = \dfrac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2}$

Iteration fn: $T_i^{n+1} = \frac{\Delta t}{\Delta x^2}(T_{i+1}^n + T_{i-1}^n) + (1 - \frac{2\Delta t}{\Delta x^2}) T_i^n$

Build up solving sys: (Algorithm):



Input: $\to \Delta x, \Delta t \quad (\Delta x = \frac{1}{20})$
$\to t_{max}$

Setup T: xpoints number: $N_x = x\text{long} = \frac{1}{\Delta x} + 1$
tpoints number: $N_t = t\text{long} = \frac{t_{max}}{\Delta t} + 1$
$T = T[n][i]$
long: (N_t) (N_x)

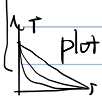
$t \in (0, t_{max})$

$n_t = \frac{t_{max}}{\Delta t}$

setup IC:
T[0][i] = 0
T[n][Nx-1] = 0
T[n][0] = 1

Iteration: For $i \in [0, N_x)$: $(\Delta i = 1)$
For $n \in [0, N_t)$: $(\Delta n = 1)$
$T[n+1][i] = \frac{\Delta t}{\Delta x^2}(T[n][i+1] + T[n][i-1]) + (1 - \frac{2\Delta t}{\Delta x^2}) T[n][i]$

Output T[n1][i]:



plot
$\leftarrow (x, T[N_t]) \quad t = N_t \cdot \Delta t$
$\leftarrow (x, T[\text{int}(\frac{N_t}{2})]) \quad t = \text{int}(\frac{N_t}{2}) \Delta t$
$\leftarrow (x, T[\text{int}(\frac{N_t}{4})]) \quad t = \text{int}(\frac{N_t}{4}) \Delta t$
$\leftarrow (x, T[\text{int}(\frac{N_t}{8})]) \quad t = \text{int}(\frac{N_t}{8}) \Delta t$

Error:

Setup exact value
Texact = Texact[n][i]

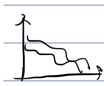
Texact[n][Nx-1] = 0
Texact[n][0] = 1

Come from Exact Soln

For $i \in [0, N_x)$:
For $n \in [0, N_t)$:
Texact[n][i] = $1 - i\Delta x$
For $k \in [1, \infty)$: large number, use 1000
$\sum_k$ Texact[n][i] += $\frac{-2}{k\pi} e^{-k^2\pi^2(n\Delta t)} \sin(k\pi(i\Delta x))$

$E = T[n][i] - \text{Texact}[n][i]$

Output T[n1][i]:



$\leftarrow (x, E[N_t]) \quad t = N_t \cdot \Delta t$
$\leftarrow (x, E[\text{int}(\frac{N_t}{2})]) \quad t = \text{int}(\frac{N_t}{2}) \Delta t$
$\leftarrow (x, E[\text{int}(\frac{N_t}{4})]) \quad t = \text{int}(\frac{N_t}{4}) \Delta t$
$\leftarrow (x, E[\text{int}(\frac{N_t}{8})]) \quad t = \text{int}(\frac{N_t}{8}) \Delta t$

Exact soln:

$T(t,x) = T_{steady} - T_{un\text{-}steady}$

$T_{steady} = T(t \to \infty, x) = T_s(x)$

$T_{unsteady} = T_u(t,x)$

As $\partial_t T = \partial_x^2 T$ linear eqn.

get $\begin{cases} \partial_t T_s = \partial_x^2 T_s \\ \partial_t T_u = \partial_x^2 T_u \end{cases}$

① For $T_s(x)$ $\quad T_s(x) = T(t \to \infty, x)$

BC for T_s $\begin{cases} T_s(0) = T(t \to \infty, 0) = 1 \\ T_s(1) = T(t \to \infty, 1) = 0 \end{cases}$

BC for T: $\begin{cases} T(t, x=0) = 1 \\ T(t, x=1) = 0 \end{cases} \quad (t>0)$

As. $\partial_t T_s = \partial_x^2 T_s$

$0 = \frac{d^2}{dx^2} T_s(x)$

$T_s(x) = C_1 x + C_2$

BC for T_u: $\begin{cases} T_u(t, x=0) = T(t, x=0) - T_s(0) = 0 \\ T_u(t, x=1) = T(t, x=1) - T_s(1) = 0 \end{cases}$

BC: $\begin{cases} T_s(0) = 1 = C_2 \\ T_s(1) = 0 = C_1 + C_2 \end{cases} \to \begin{cases} C_1 = -1 \\ C_2 = 1 \end{cases} \to T_s(x) = 1 - x$

BC for t=0:

$T(t=0, x) = \delta(x) = T_s(x) + T_u(t=0, x)$

$T_u(t=0, x) = \delta(x) + x - 1$

② For $T_u(t,x)$: Seperate variables:

$T_u(t,x) = A(t) B(x)$

As. $\partial_t T_u = \partial_x^2 T_u$

$\frac{1}{A(t)} \frac{dA(t)}{dt} = \frac{1}{B(x)} \frac{d^2 B(x)}{dx^2}$

$\underset{H(t)}{\shortparallel} \quad = \quad \underset{L(x)}{\shortparallel} = C$

let $C = -\lambda^2$, then get

$\frac{dA(t)}{A(t)} = -\lambda^2 dt \quad , \quad \frac{d^2 B(x)}{B(x)\, dx^2} = -\lambda^2$

$A(t) = Ce^{-\lambda^2 t} \quad$ try $B(x) = e^{kx}$

$k^2 = -\lambda^2$

$k = \pm i\lambda$

$B(x) = C_1 e^{i\lambda x} + C_2 e^{-i\lambda x}$

Euler formula:

$B(x) = \alpha \cos \lambda x + \beta \sin \lambda x$

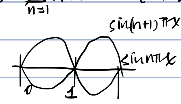
$T_u(t,x) = e^{-\lambda^2 t}(\alpha \cos \lambda x + \beta \sin \lambda x)$

As B.C. for T_u: $\begin{cases} T_u(t, x=0) = 0 \to Ce^{-\lambda^2 t} \alpha = 0 \to \alpha = 0 \\ T_u(t, x=1) = 0 \to Ce^{-\lambda^2 t}(\alpha \cos \lambda + \beta \sin \lambda) = 0 \end{cases} \Rightarrow \begin{cases} \alpha = 0 \\ \sin \lambda = 0 \to \lambda = n\pi, \ n = 1, 2, 3 \cdots \end{cases}$

β can't be 0 $\qquad \lambda \neq 0$

$T_u(t,x) = Ce^{-n^2\pi^2 t} \sin(n\pi x)$

But that's one soln. whole soln: $T_u(t,x) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t} \sin(n\pi x)$



$T_u(0,x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = \delta(x) + x - 1$

$\sum_{n=1}^{\infty} \int_0^1 A_n \sin^2(n\pi x)\, dx \quad \sum_{n=1}^{\infty} \int_0^1 (\delta(x) + x - 1) \sin(n\pi x)\, dx$

$\int_0^1 A_n \frac{1 - \cos(2n\pi x)}{2} dx = 0 + \int_0^1 (x-1) \sin(n\pi x)\, dx$

$A_n \left(\frac{1}{2} - \frac{\sin(2\pi n)}{4\pi n}\right) \quad = \frac{\cos(n\pi x)}{n\pi} \Big|_{x=0}^{x=1} + \int_0^1 \frac{-x\, d\cos(n\pi x)}{n\pi}$

$= \frac{\cos n\pi}{n\pi} - \frac{1}{n\pi} - \frac{x \cos(n\pi x)}{n\pi} \Big|_0^1 + \int_0^1 \frac{\cos(n\pi x)\, dx}{n\pi}$

$= \cancel{\frac{\cos n\pi}{n\pi}} - \frac{1}{n\pi} - \cancel{\frac{\cos(n\pi)}{n\pi}} + \frac{\sin(n\pi x)}{n^2\pi^2} \Big|_0^1$

$\frac{A_n}{2}\left(1 - \underset{0}{\cancel{\frac{\sin(2\pi n)}{2\pi n}}}\right) = -\frac{1}{n\pi}\left(1 - \underset{0}{\cancel{\frac{\sin n\pi}{n\pi}}}\right)$

$\frac{A_n}{2} = -\frac{1}{n\pi}$

$T_u(t,x) = \sum_{n=1}^{\infty} \frac{-2}{n\pi} e^{-n^2\pi^2 t} \sin(n\pi x)$

$T = T_s + T_u = 1 - x + \sum_{n=1}^{\infty} \frac{-2}{n\pi} e^{-n^2\pi^2 t} \sin(n\pi x)$

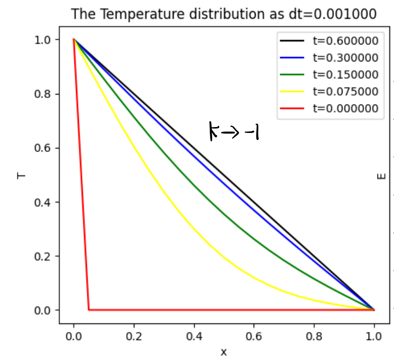
Exact soln:

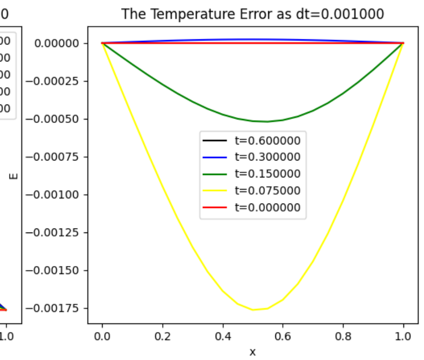
$$\left[\, T(t,x) = 1 - x + \sum_{n=1}^{\infty} \frac{-2}{n\pi} e^{-n^2\pi^2 t} \sin(n\pi x) \,\right]$$

Solution: $t_{max} = 0.5$ (s) $\Delta x = \frac{1}{20}$

Soln. Difference with Exact soln: Iteration fn: $T_i^{n+1} = \frac{\Delta t}{\Delta x^2}(T_{i+1}^n + T_{i-1}^n) + (1 - \frac{2\Delta t}{\Delta x^2})T_i^n$

$\Delta t = \frac{1}{1000}$:



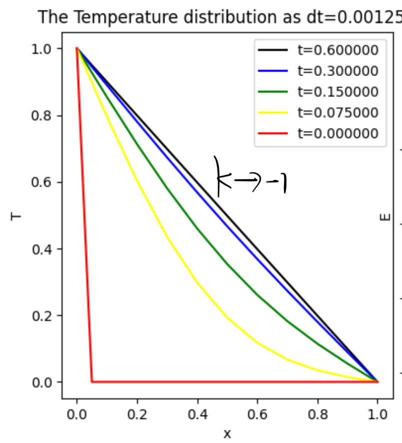


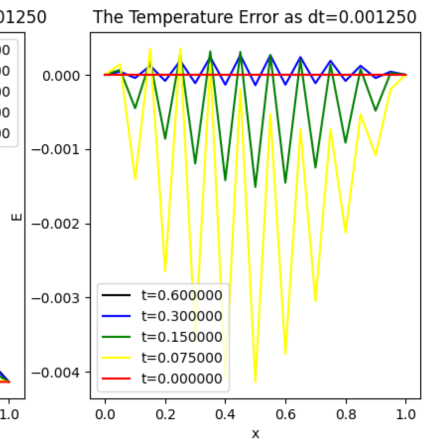
Exact soln:

$$T(t,x) = 1 - x + \sum_{n=1}^{\infty} \frac{-2}{n\pi} e^{-n^2\pi^2 t} \sin(n\pi x)$$

$\Delta t = 1/1000$'s error seems very small

$\Delta t = \frac{1}{800}$:





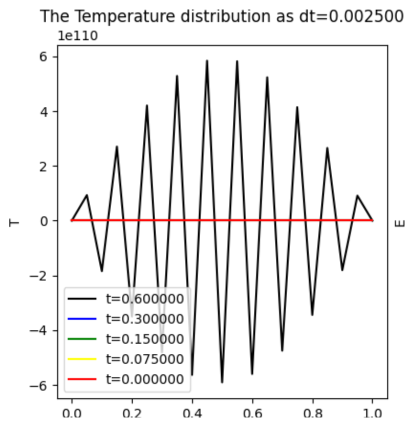
We can notice that $\Delta t = \frac{1}{1000}$ and $\Delta t = \frac{1}{800}$ figure can perform simulation very well.

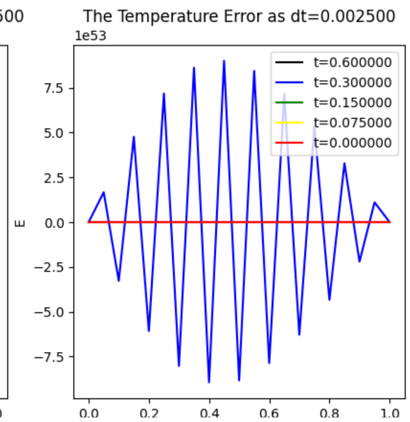
As time goes on, heat distribution is more close to steady solution

$\Delta t = 1/800$, error larger but still kind of small, But already can see is at the edge of steady

Note: for $t = 0.05$, Error diagram distribution is little different. See Appendix (end of the file)

$\Delta t = \frac{1}{400}$:





As we can see, $\Delta t = \frac{1}{400}$ and $\Delta t = \frac{1}{200}$ both have big Error.

Using Von-Neumann stability analysis:

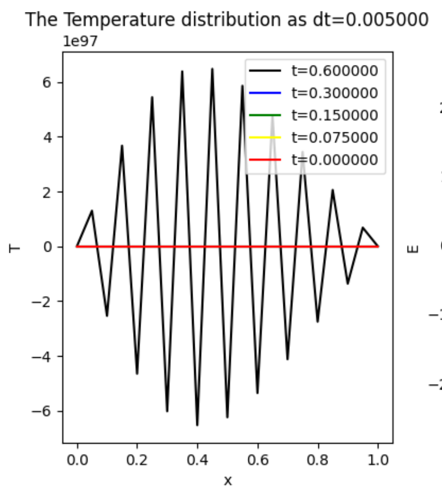
$\Delta t \le \frac{(\Delta x)^2}{2}$

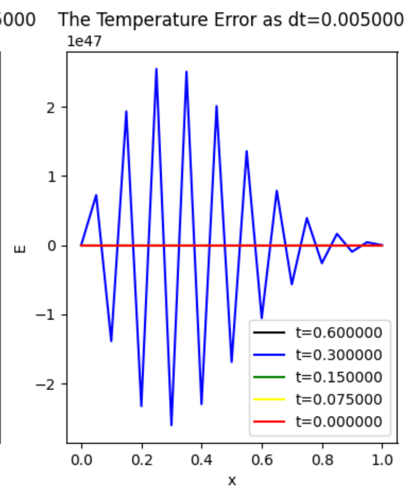
as $\Delta x = \frac{1}{20}$

$\Delta t \le \frac{1}{800}$ stable

and $\Delta t = \frac{1}{400}$ and $\frac{1}{200}$ are not stable

$\Delta t = \frac{1}{200}$:

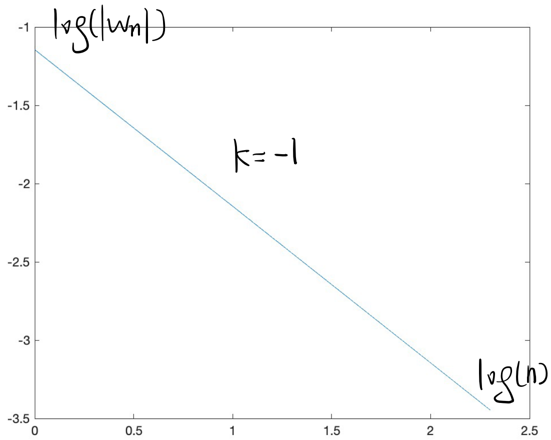


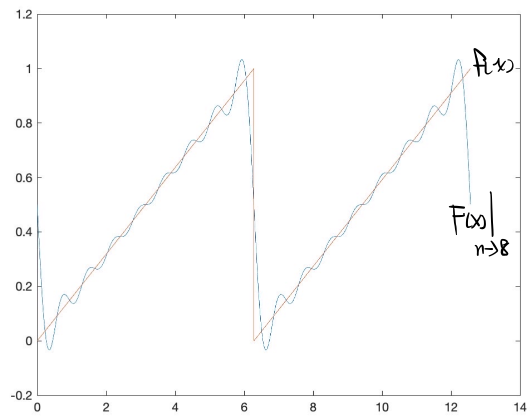


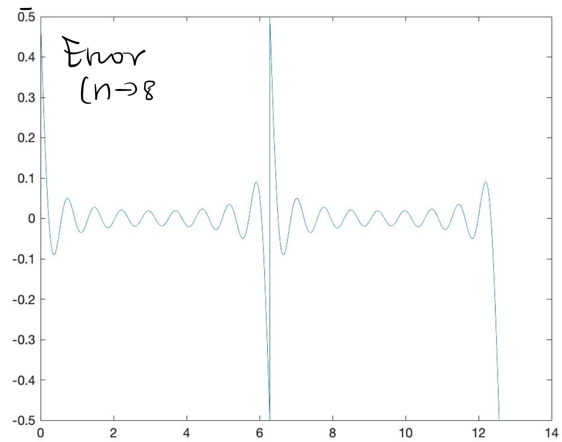
Comment:

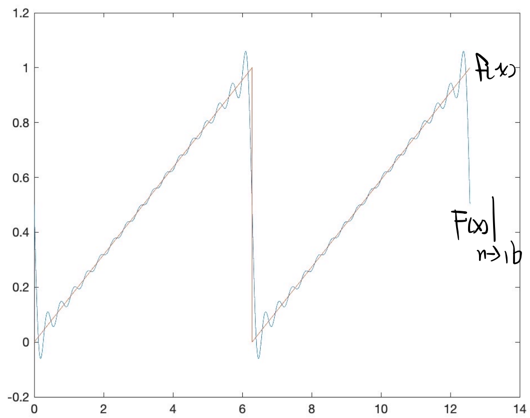
b

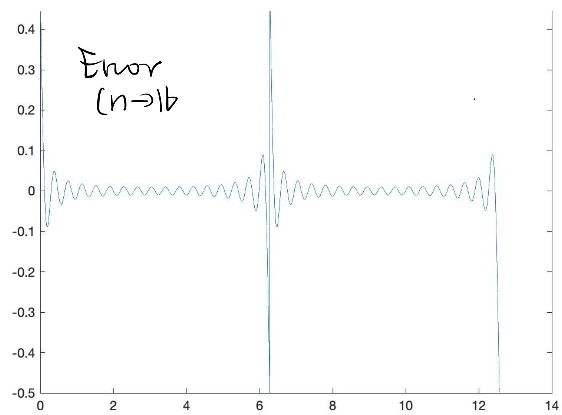
Appendix

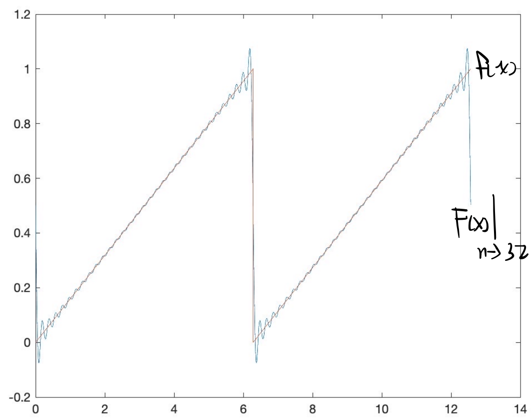

log(|wn|)
k=-1
log(n)

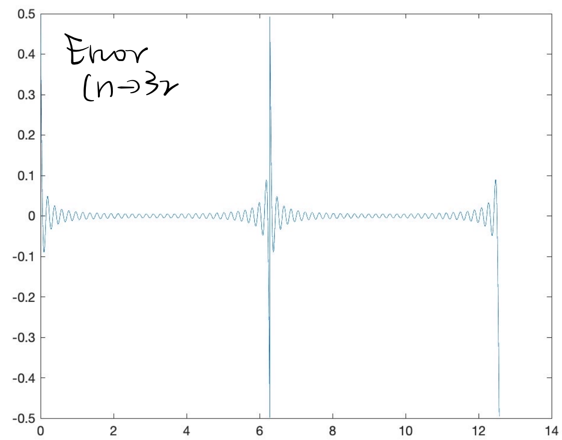

f(x)
F(x)| n→8


Error
(n→8

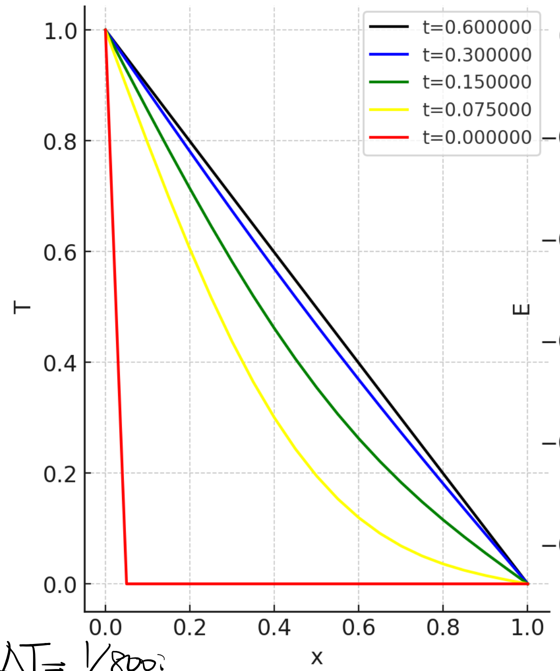

f(x)
F(x)| n→16


Error
(n→16

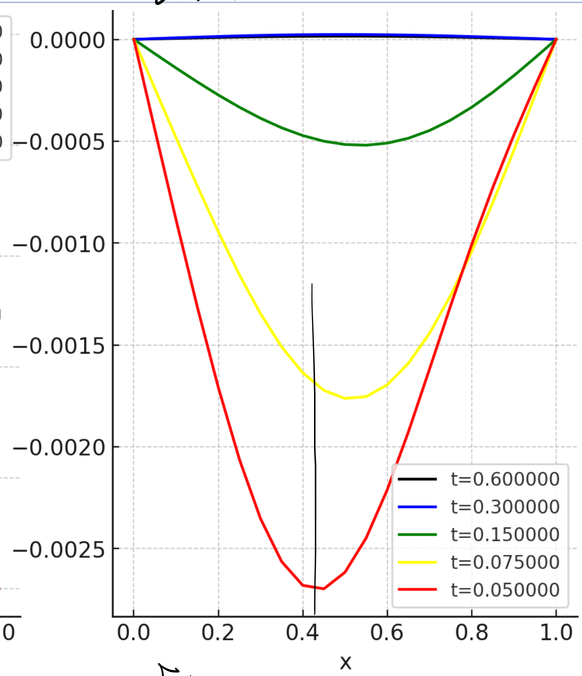

f(x)
F(x)| n→32


Error
(n→32

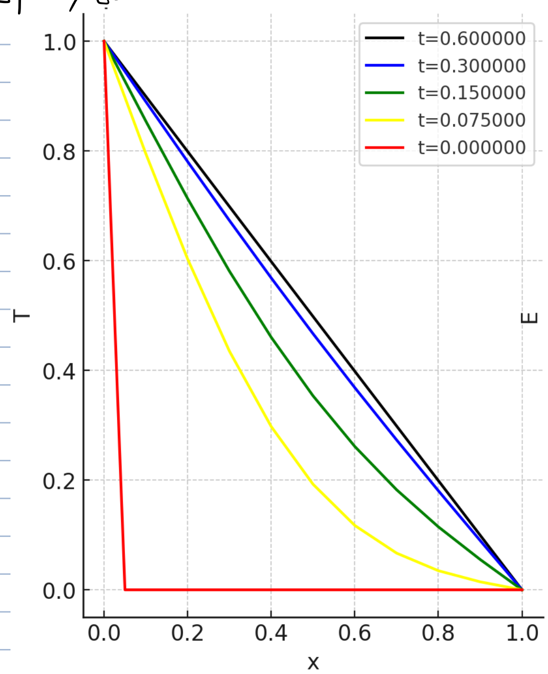
$\Delta T = \frac{1}{1000}$:



Error:



$\Delta T = 1/800$:



Error: