Lid Driven Cavity Problem

Haobo Zhao

530.767 CFD-HW4

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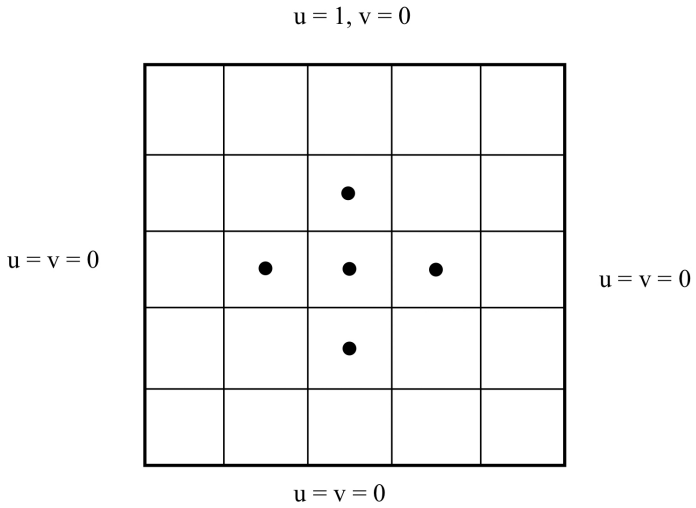
In this project, we developed a N-S solver for Lid Driven Cavity problem, which use 2^{nd} order central difference scheme in space and a second order (AB2 for convection and CN2 for viscous) fractional step in time. By separate update of the face velocity ("semi-staggered" approach), we could approach zero divergence with the compact stencil for pressure. We also used ghost point method to obtain boundary condition. By compare our result with the reference result, we could say our result is robust.

Contents

1 Question Review



Consider the 1×1 cavity shown above. The cavity has a lid that moves to the right with unit velocity. This "driven cavity" problem is considered a benchmark in CFD and is often used to validate the fidelity and accuracy of Navier-Stokes solvers. Accurate solutions to this problem have been obtained by a number of researchers. One often quoted work in this area is that of Ghia et al. (*Journal of Computational Physics*, Vol. 48, page 387, 1982)[2]. Others include Botella, O. and Peyret, R., 1998. Benchmark spectral results on the lid-driven cavity flow. *Computers & Fluids*, 27(4), pp.421-433.[1]

With this HW, you will start developing the code that you will also use for the Final Project

1. Simulate the driven cavity flow with $Re = 100$ on uniform grids with 16, 32, 64, 128 points in each direction. Simulate the flow until a steady state is reached. Use a 2^{nd} order central difference scheme in space and a second order (AB2 for convection and CN2 for viscous) fractional step in time. The basic (non-Van-Kan version of fractional step is acceptable). Use the separate update of the face velocity ("semi-staggered" approach) so that zero divergence can be satisfied with the compact stencil for pressure.

2. Compare your computed results (velocity profiles etc) against those from previous published studies for all grids and comment on the comparison.

3. Compute the flow at $Re = 1000$ and show that on a sufficiently fine mesh, your results match published results.

2 N-S Solver

2.1 Discretization of N-S Equation

The general steps solving the N-S equation is showing below:

(I): Using the data from the current time step and the previous time step to get u^*, v^*. Then use u^*, v^* to get U^*, V^*.
(II): Using U^*, V^*, solving the pressure poisson equation (PPE), get p^{n+1}.
(III): Using p^{n+1}, and u^*, v^*, get u^{n+1}, v^{n+1}.
Using p^{n+1}, and U^*, V^*, get U^{n+1}, V^{n+1}.

2.1.1 Step (I):

Using the the data we got from current time step and the previous time step get u^*, v^*. Then use u^*, v^* get U^*, V^*.

Update u^*, v^*:

The first step is going to update u^*, v^*:

$$\frac{\vec{u}^* - \vec{u}^n}{\Delta t} = -[\frac{3}{2}\nabla \cdot (\vec{U}^n\vec{u}^n) - \frac{1}{2}\nabla \cdot (\vec{U}^{n-1}\vec{u}^{n-1})] + \frac{1}{Re}\nabla^2\frac{\vec{u}^* + \vec{u}^n}{2} \tag{1}$$

which is:

$$\underbrace{[1 - \frac{\Delta t}{2Re}\nabla^2]\vec{u}^*}_{(1)} = -\frac{3}{2}\underbrace{[\Delta t\nabla \cdot (\vec{U}^n\vec{u}^n)]}_{(2)} + \frac{1}{2}\underbrace{[\Delta t\nabla \cdot (\vec{U}^{n-1}\vec{u}^{n-1})]}_{(3)} + \underbrace{[1 + \frac{\Delta t}{2Re}\nabla^2]\vec{u}}_{(4)} \tag{2}$$

Could say, (2), (3) are convection term, and (4) is the diffusion term of the eqaution. Could use this equation to update u^*, v^*.

Where,

$$(1) = \left\{1 - \frac{\Delta t}{2Re}\nabla^2\right\}u^* = P^* - \left[\frac{E^* + W^* - 2P^*}{\Delta x^2} + \frac{N^* + S^* - 2P^*}{\Delta y^2}\right]\frac{\Delta t}{2Re}$$

$$(2) = \frac{\Delta t}{\Delta}\left((U_e^n\frac{E^n + P^n}{2} - U_w^n\frac{W^n + P^n}{2}) + (V_n^n\frac{N^n + P^n}{2} - V_s^n\frac{S^n + P^n}{2})\right)$$

$$(3) = \frac{\Delta t}{\Delta}\left((U_e^{n-1}\frac{E^{n-1} + P^{n-1}}{2} - U_w^{n-1}\frac{W^{n-1} + P^{n-1}}{2}) + (V_n^{n-1}\frac{N^{n-1} + P^{n-1}}{2} - V_s^{n-1}\frac{S^{n-1} + P^{n-1}}{2})\right)$$

$$(4) = P^n + \frac{\Delta t}{2Re\Delta^2}(E^n + W^n + N^n + S^n - 4P^n)$$

Could found in this expression, the formula to calculate u^*, v^* is same, only input u or v in to the equation:

P means (i, j), W means $(i-1, j)$, E means $(i+1, j)$, S means $(i, j-1)$, N means $(i, j+1)$.

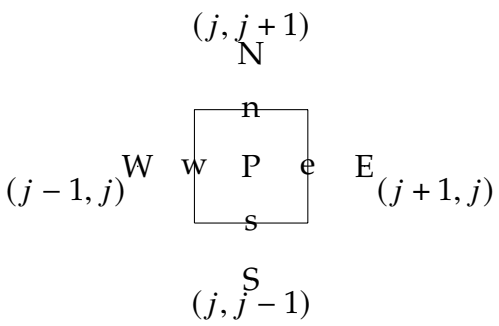


Figure 1: Illustration of a grid layout with directional points.

Put the result in, could get:

$$aW + bP + cE + eS + fN = d$$

The left hand side is the unknown characters, and the right hand side b is known from previous and current time step. Could see, this equation in our system means $Ax = d$, where A is consist of a,b,c,e,f, is a pentadiagonal matrix:

$$\begin{bmatrix} 1 & 0 & & & 0 & & & & \\ a & b & c & & & f & & & \\ & a & b & c & & & f & & \\ & & a & b & c & & & 0 & \\ 0 & & & \ddots & \ddots & \ddots & & & \\ & e & & & a & b & c & & \\ & & e & & & a & b & c & \\ & & & 0 & & & 0 & 1 & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \\ d_{n-2} \\ d_{n-1} \\ d_{j \cdot i} \end{bmatrix}$$

Solve this system, we could get u^*, and v^*

Update U^*, V^*:

$$U_e^* = \frac{u_p^* + u_E^*}{2}, V_n^* = \frac{v_P^* + v_N^*}{2}$$

2.1.2 Step (II):

Update p^{n+1}:

$$\nabla^2 p = \frac{\nabla \cdot \vec{u}^*}{\Delta t} = \frac{\frac{U_e^* - U_P^*}{\Delta x} + \frac{V_n^* - V_s^*}{\Delta y}}{\Delta t}$$

$$\frac{P_E + P_W - 2P_P}{\Delta x^2} + \frac{P_N + P_S - 2P_P}{\Delta y^2} = \frac{\frac{U_e^* - U_w^*}{\Delta x} + \frac{V_n^* - V_s^*}{\Delta y}}{\Delta t}$$

$$P_E + P_W + P_N + P_S - 4P_P = \frac{\Delta}{\Delta t}\left(U_e^* - U_w^* + V_n^* - V_s^*\right) = RHS$$

Similarly, this equation also give us a pentadiagonal system:

$$\begin{bmatrix} 1 & 0 & & & 0 & & & & \\ a & b & c & & & f & & & \\ & a & b & c & & & f & & \\ & & a & b & c & & & 0 & \\ 0 & & & \ddots & \ddots & \ddots & & & \\ & e & & & a & b & c & & \\ & & e & & & a & b & c & \\ & & & 0 & & & 0 & 1 & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \\ d_{n-2} \\ d_{n-1} \\ d_n \end{bmatrix}$$

2.1.3 Step (III):

Update u^{n+1}, v^{n+1}:
The equation to update the u^{n+1}, v^{n+1} is showing below:

$$\vec{u}^{n+1} = \vec{u}^* - \Delta t(\nabla p)_{cc}$$

where we could calculate the cell center pressure divergence $(\nabla p)_{cc}$, and:

$$\begin{cases} u^{n+1} = u^* - \Delta t \cdot \frac{P_E - P_W}{2\Delta x} \\ v^{n+1} = v^* - \Delta t \cdot \frac{P_N - P_S}{2\Delta y} \end{cases}$$

Update U^{n+1}, V^{n+1}:
The equation to update the U^{n+1}, V^{n+1} is showing below:

$$\vec{U}^{n+1} = \vec{U}^* - \Delta t(\nabla p)_{fc}$$

where we could calculate the face center pressure divergence $(\nabla p)_{fc}$, and:

$$\begin{cases} U_e^{n+1} = U_e^* - \Delta t \cdot \frac{P_E - P_P}{\Delta x} \\ V_n^{n+1} = V_n^* - \Delta t \cdot \frac{P_N - P_P}{\Delta y} \end{cases}$$

2.1.4 LineSOR implementation

In the discretization steps (I) and (II) of the Navier-Stokes equations, we obtain a pentadiagonal matrix system. However, our Tridiagonal Matrix Algorithm (TDMA) can only solve systems represented by tridiagonal matrices. To address this, we transfer the additional diagonal lines, e and f, to the right-hand side and continue iterating. This process is repeated until the results satisfy the original requirements of the pentadiagonal system.

The origional pentadiagonal system:

$$\begin{bmatrix} 1 & 0 & & & 0 & & & \\ a & b & c & & & f & & \\ & a & b & c & & & f & \\ & & a & b & c & & & 0 \\ 0 & & & \ddots & \ddots & \ddots & & \\ & e & & & a & b & c & \\ & & e & & & a & b & c \\ & & & 0 & & & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{n-2} \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \\ d_{n-2} \\ d_{n-1} \\ d_n \end{bmatrix}$$

For each line j, the system could be represented as:

$$\left[\begin{array}{c|c|c} E & T & F \end{array}\right] \begin{bmatrix} [u_{j-1}^{n+1}] \\ [u_j^{n+1}] \\ [u_{j+1}^{n+1}] \end{bmatrix} = \left[[d_j] \right]$$

Could say, E mans operating the previous line (j-1) to current line (j), and F means operating on the next line (j+1) to current line. Where $\left[\begin{array}{c|c|c} E & T & F \end{array}\right]$ is:

$$\left[\begin{array}{cccccc|cccccc|cccccc} 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & e_2 & 0 & \cdots & 0 & 0 & a_2 & b_2 & c_2 & \ddots & \vdots & 0 & 0 & f_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & e_3 & \cdots & 0 & 0 & 0 & a_3 & \ddots & \ddots & 0 & 0 & 0 & 0 & f_3 & \cdots & 0 & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & c_2 & 0 & 0 & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e_{n-1} & 0 & 0 & \ddots & \ddots & 0 & b_{n-1} & c_{n-1} & 0 & 0 & 0 & \cdots & f_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right]$$

LineSOR:

For LineSOR, we move the two diagonal line (e and f) to the right hand side:

$$\left[\begin{array}{c|c|c} 0 & T & 0 \end{array}\right] \begin{bmatrix} [u_{j-1}^{k+1}] \\ [u_j^{k+1}] \\ [u_{j+1}^{k+1}] \end{bmatrix} = \left[[d_j] \right] - \left[\begin{array}{c|c|c} E & 0 & F \end{array}\right] \begin{bmatrix} [u_{j-1}^{k}] \\ [u_j^{k}] \\ [u_{j+1}^{k}] \end{bmatrix}$$

Which is:

$$[T] \left[[u_j^{k+1}] \right] = \left[[d_j] \right] - [E] \left[[u_{j-1}^{k}] \right] - [F] \left[[u_{j+1}^{k}] \right]$$

Where we could use TDMA to solve this equation. As this result is different from directly solve the equation we previously got, we need it keep iterating, until the error small enough:

$$Error = \left[\begin{array}{c|c|c} E & T & F \end{array}\right] \begin{bmatrix} [u_{j-1}^{k+1}] \\ [u_j^{k+1}] \\ [u_{j+1}^{k+1}] \end{bmatrix} - \left[[d_j] \right] < 1e-6$$

2.2 Setting

2.2.1 Solver field

As in the discretization of N-S equation, we have u, v, p, and U, V fields. To handle boundary condition, we use ghost point. For convenient, we set all the fields in same size: $(N+2) \cdot (N+2)$, the u, v, p fields are:

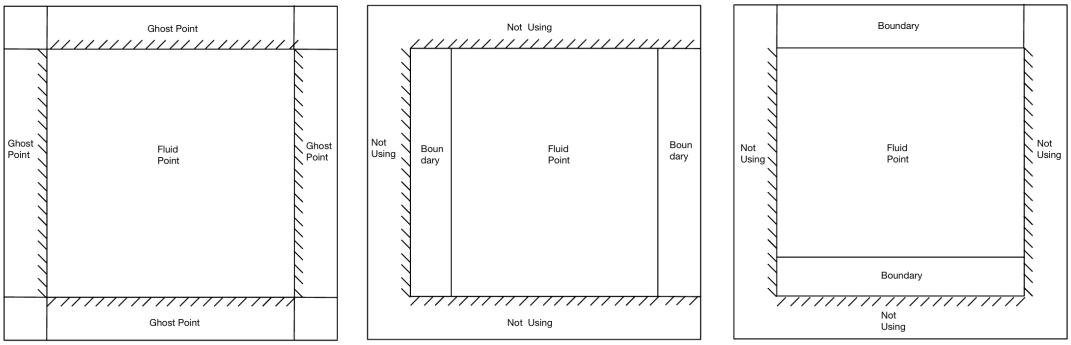


Figure 2: u/v/p, U, and V field

The left one is for u, v, and p field, where they are defined at cell center. It contains ghost point, which is in the solid part near to the boundary. U field is in the middle, which is 1 column, and 2 rows less than u/v/p field. V field is the right one, where it is 1 row, and 2 column less than u/v/p field.

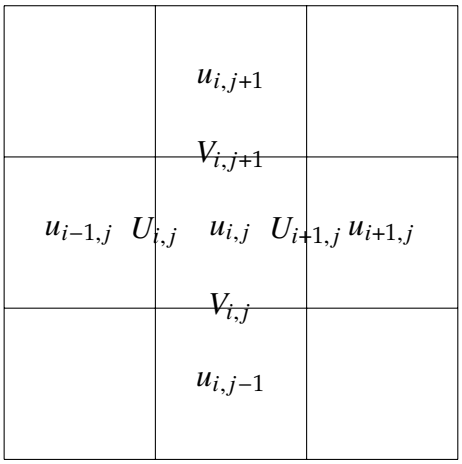


Figure 3: Illustration of a grid layout with u/v/p, U, and V.

2.2.2 Boundary condition and implementation

First, let us clarify boundary condition: Both 4 sides are in dirichlet boundary condition. Which means at the boundary point:

$$\frac{u_p + u_{gs}}{2} = u_B$$

2.3 Solver Architecture

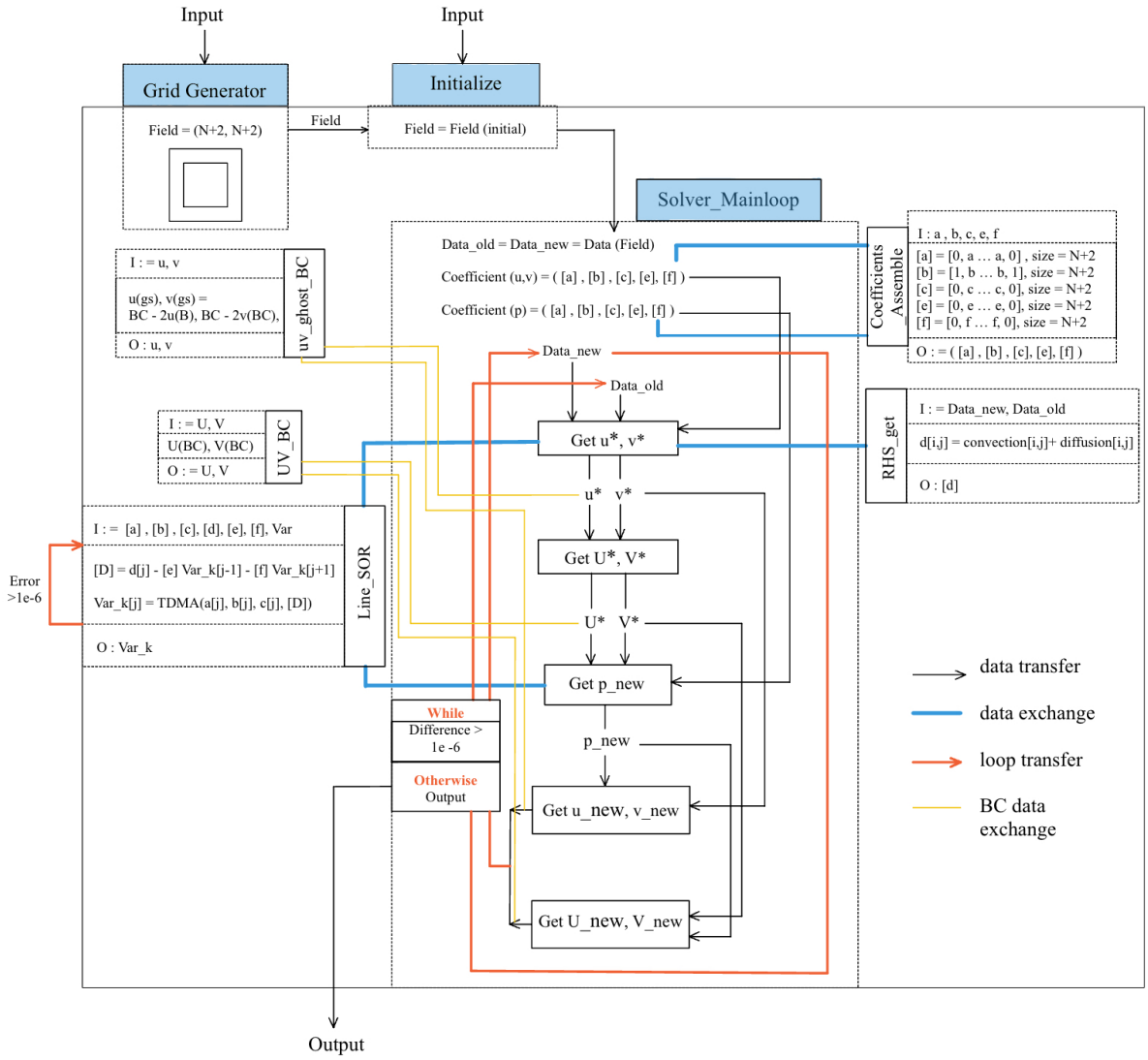


Figure 4: Solver Architecture

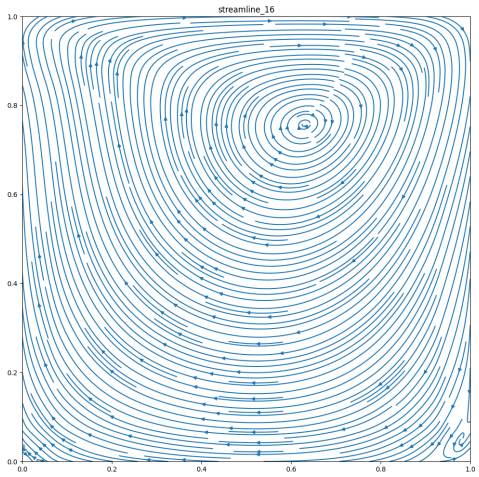
This diagram shows our Solver structure. By Using grid generator to generate fields for u, v, p, and U, V, transfer the data to initialize the initial value, then use the main loop to doing time iteration till the fields get to steady state, we could output the result.

- **I/O**: I means function input, and O means function output.
- **Data Transfer Lines**: The diagram uses different colored lines to indicate various types of data movement, while the lines with arrow means one-way data transfer, which means value assignment, and the lines without arrow means data exchange, where there is data input to function, but there is also data get back from function.

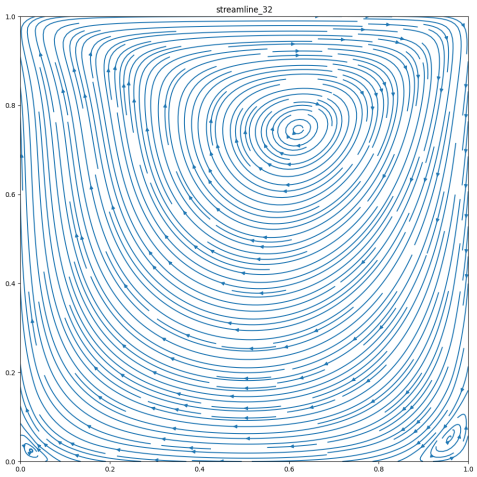
3 Result for Re=100

3.1 Stream lines at Re=100

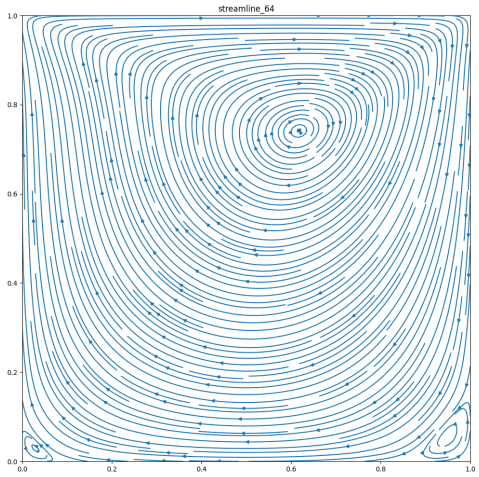
For Re=100, we got the result after the flow reaches steady state, where we use the total difference of u, v, and p smaller than 1e-6, we could get the streamlines results in different grid size showing below:



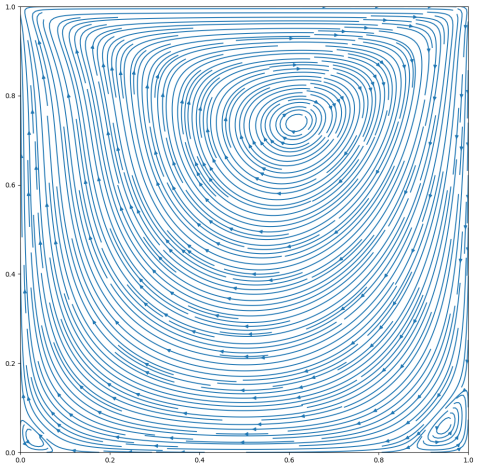
(a) N=16



(b) N=32



(c) N=64



(d) N=128

Figure 5: Stream line visualizations at different grid resolutions

For general view, we could see there is a big vortex appear at the biased center of the cavity, and the small vortex represent at the left bottom and a relatively larger vortex appear at the right bottom of the cavity.

3.1.1 Streamline comparison of different grid size result

Compare different grid size result, we could find that for N=16 grid, the vortex at left bottom is not showing, and the vortex on the right bottom is small. Where as the grid become finer, the vortex shows up and the vortex at the right bottom become larger.

3.2 Result comparison to the reference

To make our result is robust, we compare the result with the reference (Ghia et al, 1982)[2]. The results is showing below:

3.2.1 Streamline comparison

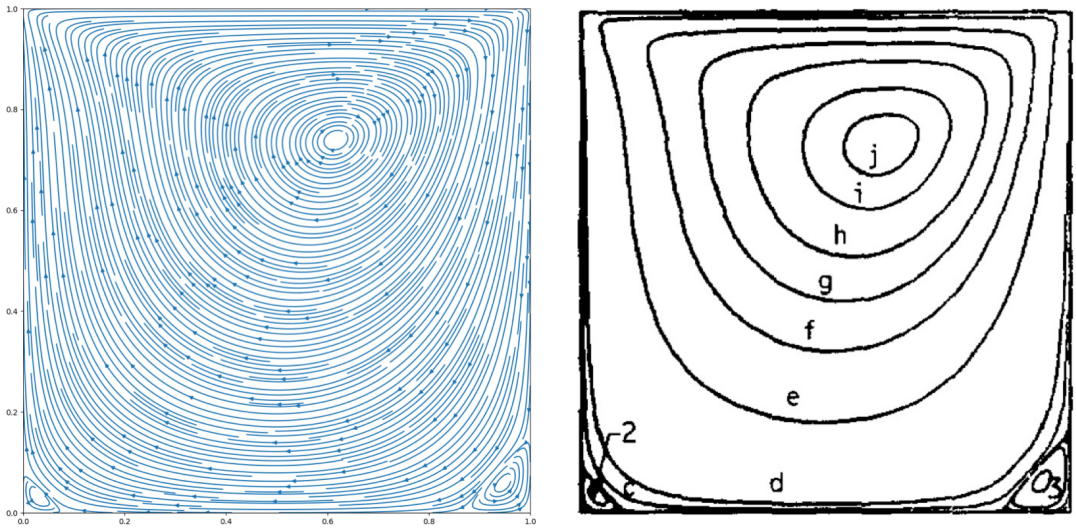


Figure 6: left: Result Streamline, right: Ghia(1982)[2]'s result

Compare out N=128 result to the Ghia (1982)[2] 's result, we could find the results is similar, where the there shows three vortex, and the stream lines are similar.

3.2.2 Velocity at geometry center

Measure our result's accuracy need us to compare the result with the reference (Ghia et al. 1982)[2]. By compare the result at geometry center, we could check the accuracy of our result.

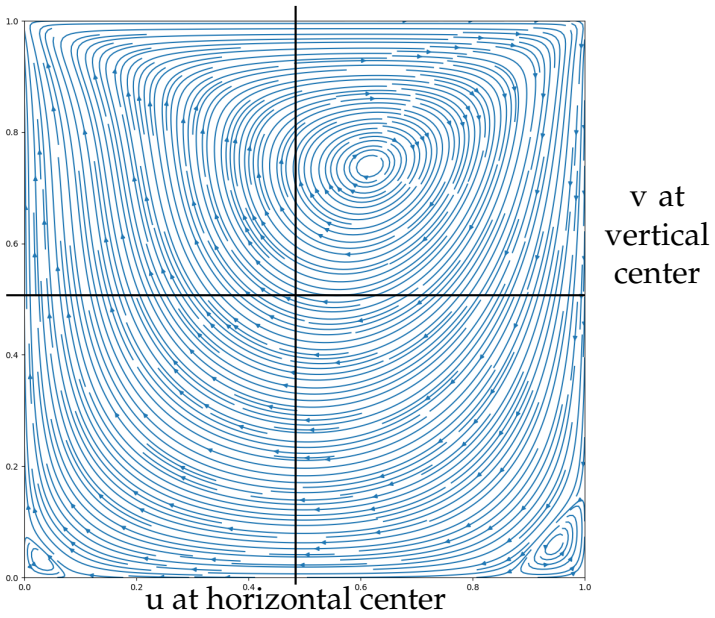


Figure 7: velocity at geometry center

We use the the horizontal velocity u at horizontal center (at same vertical line, middle at horizontal), and vertical velocity v at vertical center (at same horizontal line, middle at horizontal). The general result is showing below:

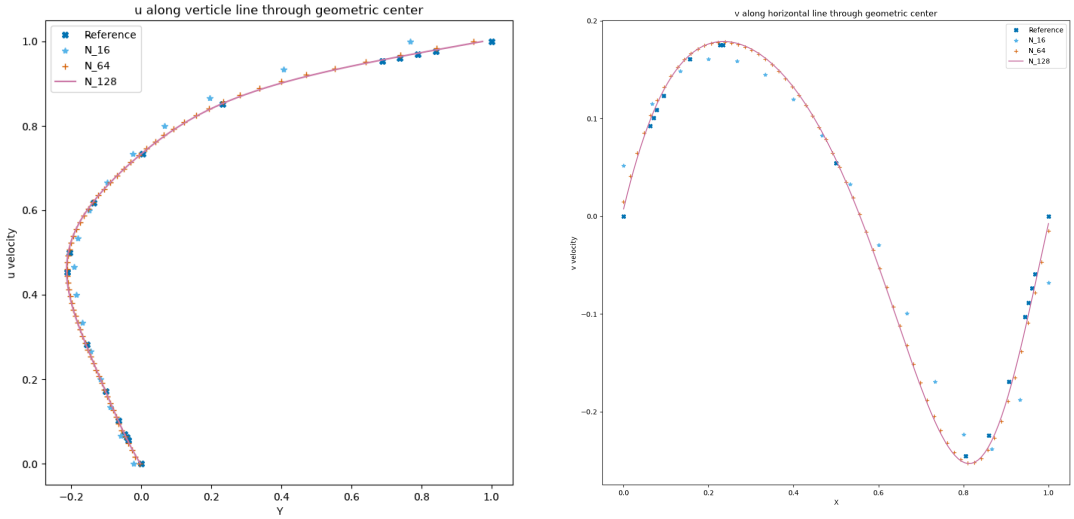


Figure 8: u along vertical line at middle of x (left), v along horizontal line at middle of y (right). Reference is Ghia (1982)[2]'s result. There is also enlarged figures showing down below.

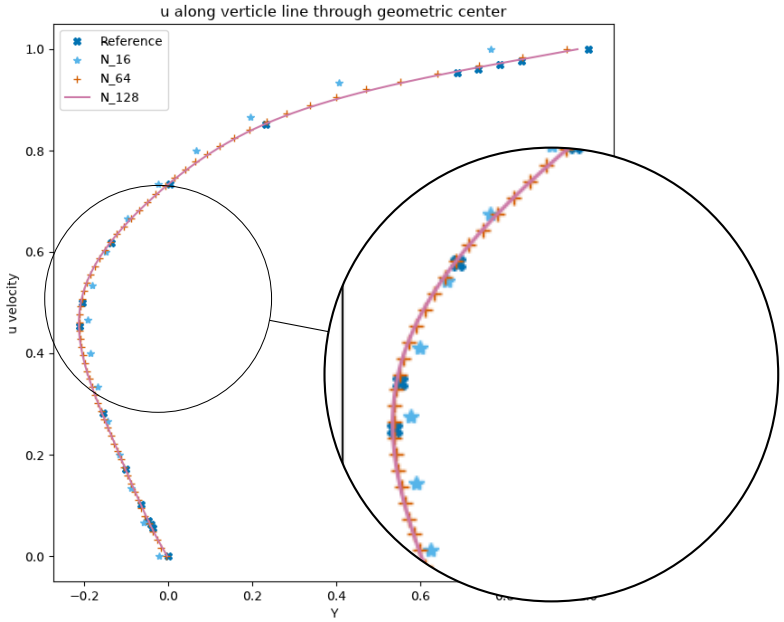


Figure 9: u along vertical line at middle of x (large)

This figures shows velocity u at the horizontal middle of the domain. Could say, in general the result is matching. Also compare different grid size, could find the coarsest grid, N=16, biased largest from reference result. Conversely, the finest grid, N=128, match pretty good.

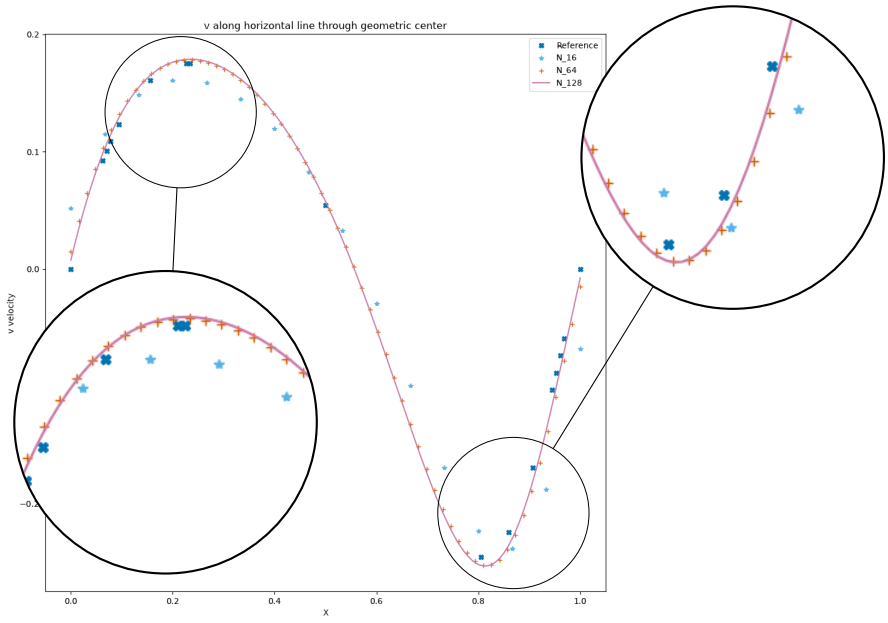


Figure 10: v along horizontal line at middle of y (large)

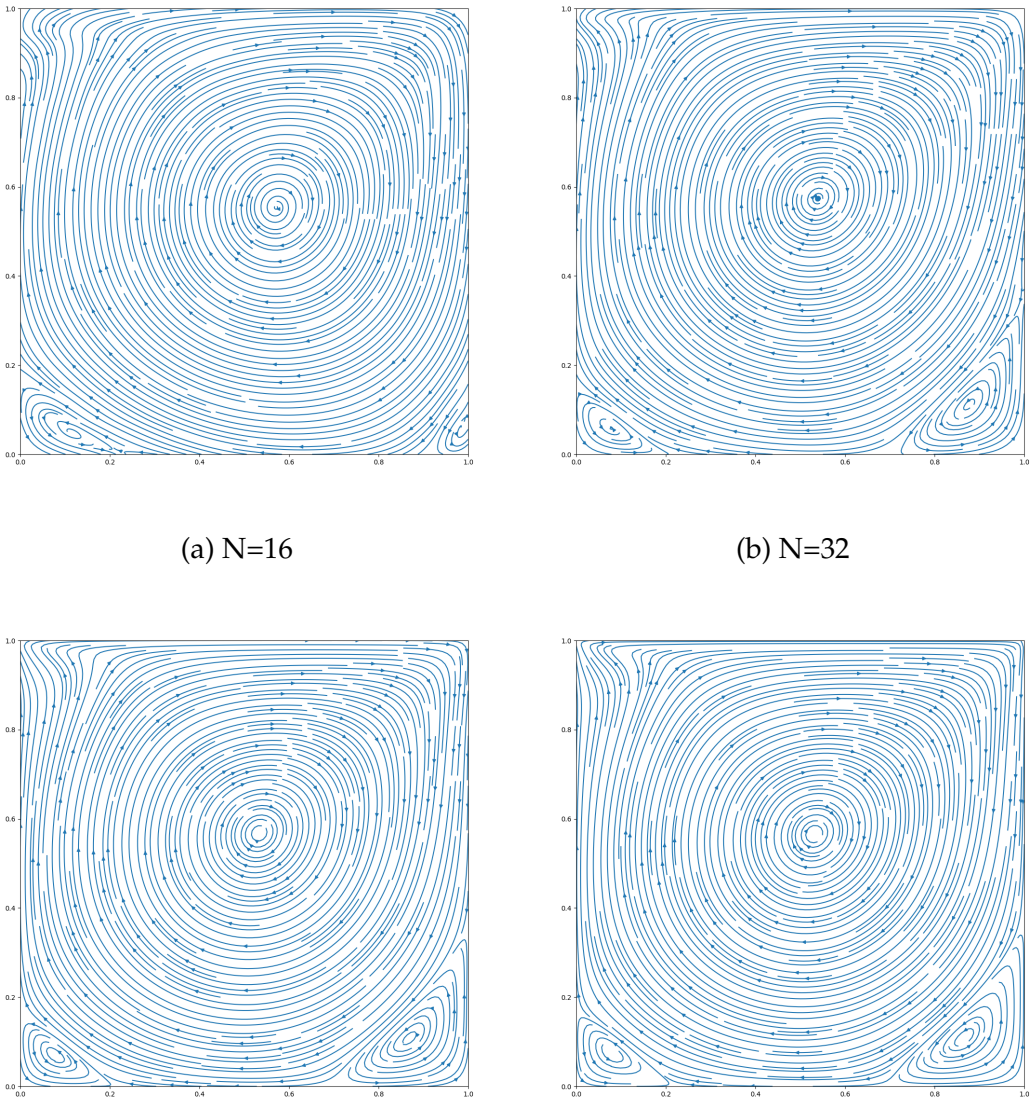
Could find in these solution, that the N = 16 grid is the most rough grid, where the difference with the paper result (N=128) is the largest. The finest grid (N=128) us the most accurate grid, where the result match the result from paper.

As the grid become finer, the result become more accurate.

4 Result for Re=1000

4.1 Stream lines at Re=1000

For Re=1000, the streamlines result is showing below:



(a) N=16

(b) N=32

(c) N=64

(d) N=128

Figure 11: Stream line visualizations for Re=1000 at different grid resolutions

For Re=1000, could see the main vortex is still at the center, while the two small

voetex at the left bottom and right bottom is also present, but compare to the result at Re=100, could also find there is a small curvature at the top left, while it is showing more obvious at the finer grid.

4.2 Result comparison to the reference

4.2.1 Velocity at geometry center

Measure our result's accuracy need us to compare the result with the reference (Ghia et al. 1982)[2]. By compare the result at geometry center, we could check the accuracy of our result.

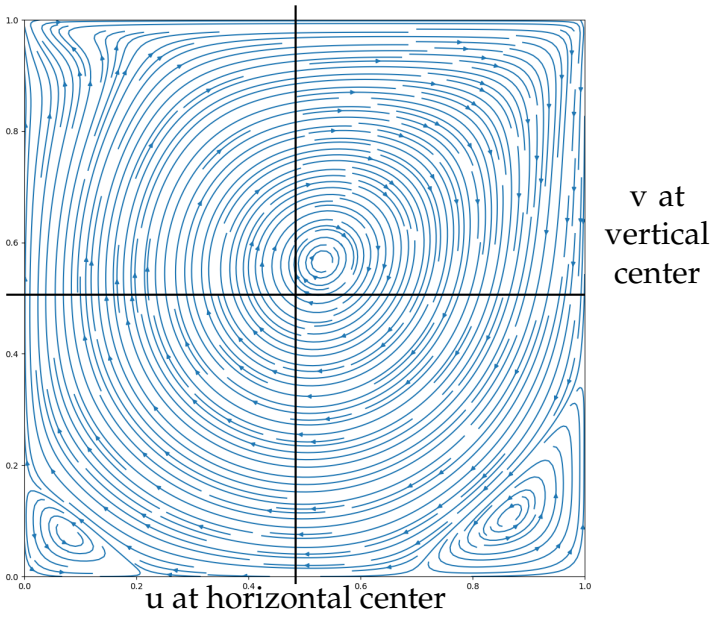


Figure 12: velocity at geometry center

At Re=1000, we use the the horizontal velocity u at horizontal center (at same vertical line, middle at horizontal), and vertical velocity v at vertical center (at same horizontal line, middle at horizontal). The general result is showing below:

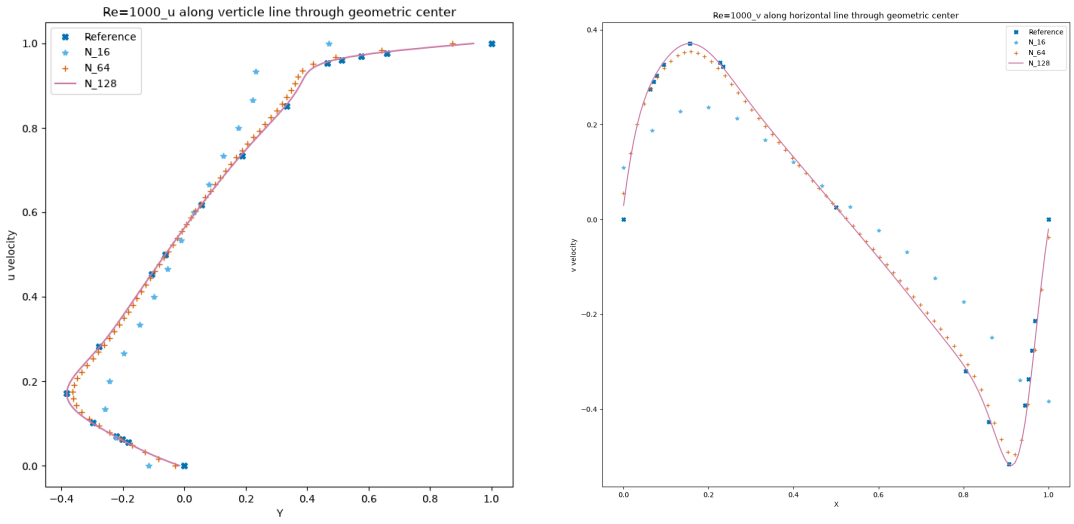


Figure 13: u along vertical line at middle of x (left), v along horizontal line at middle of y (right). There is also enlarged figures showing down below.

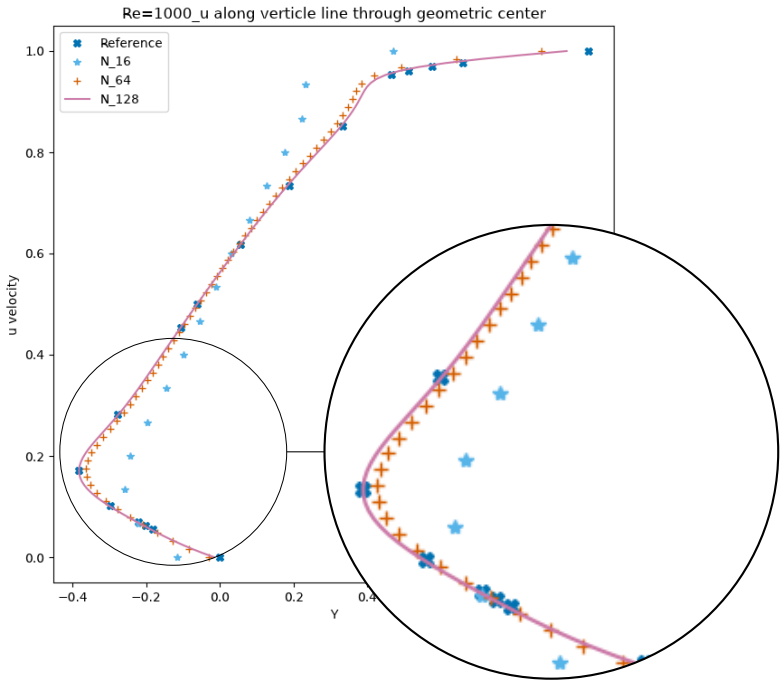


Figure 14: u along vertical line at middle of x (large)

This figures shows velocity u at the horizontal middle of the domain. Could say, in general the result is matching. Also compare different grid size, could find the coarse grid, N=16, biased largest from reference result. Conversely, the finest grid, N=128, match pretty good.

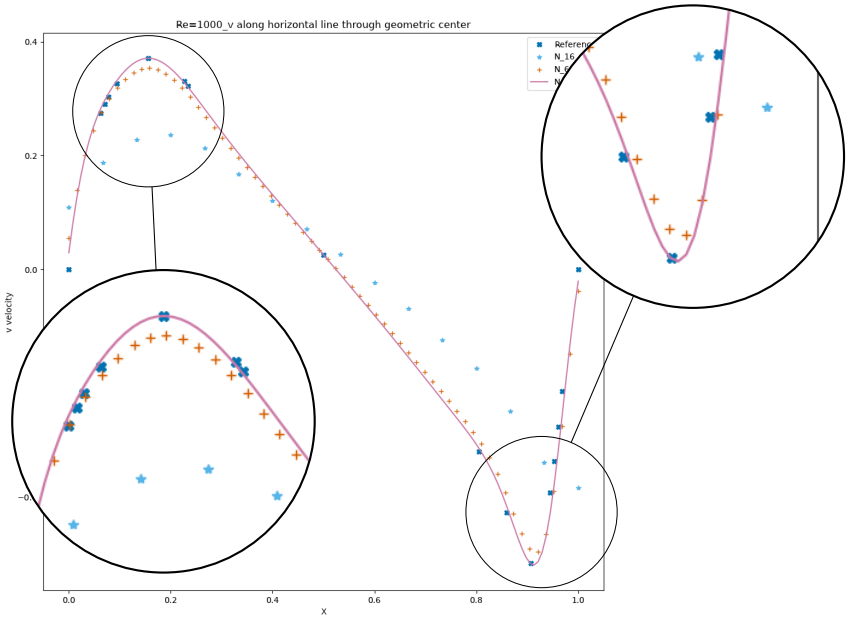


Figure 15: v along horizontal line at middle of y (large)

We also have very similar observation on the vertical velocity v on the vertical geometry center. Could find as the grid become finer, the result match better.

4.2.2 Streamline comparison

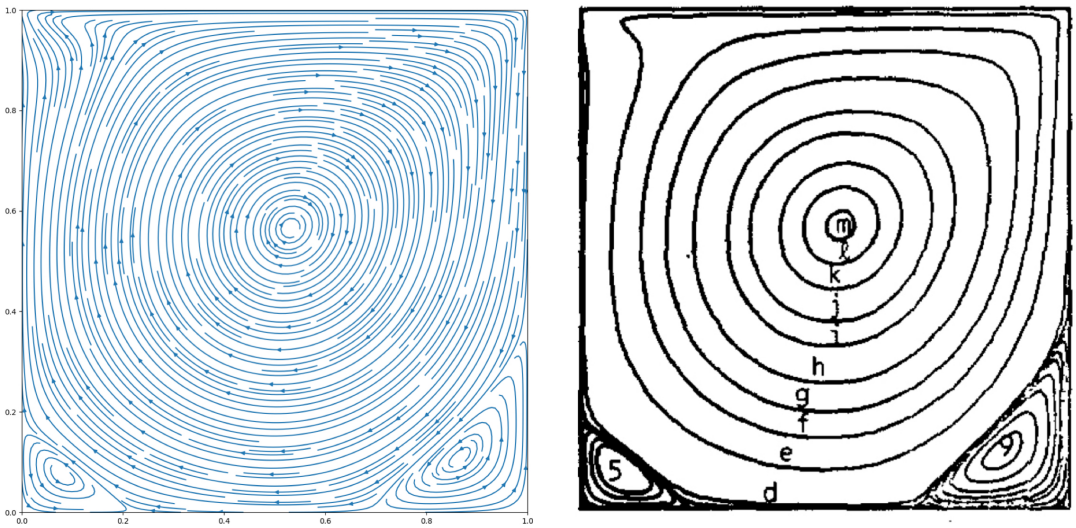


Figure 16: left: Result Streamline, right: Ghia(1982)[2]'s result

Could say the results are matching, where the vortex location and the shape are similar.

References

[1] O. Botella and R. Peyret. Benchmark spectral results on the lid-driven cavity flow. *Computers & Fluids*, 27(4):421–433, 1998.

[2] U Ghia, K.N Ghia, and C.T Shin. High-re solutions for incompressible flow using the navier-stokes equations and a multigrid method. *Journal of Computational Physics*, 48(3):387–411, 1982.

Appendix

Our basic Solver code is showing below:

```
# This solver is a new attempt using same size of u, U... ,
# keep updating ghost point and dont need to change formula

import numpy as np
import math
import matplotlib.pyplot as plt


def TDMA(a, b, c, d):  # TDMA Solver, input abcd in same length
    # a[0], c[-1] not been used
    do = d.copy()
    ao = a.copy()
    bo = b.copy()
    co = c.copy()
    N = len(d)
    xo = np.zeros(N)
    for rowi in range(1,N):
        k = ao[rowi]/bo[rowi-1]
        bo[rowi] -= co[rowi-1]*k
        do[rowi] -= do[rowi-1]*k
    xo[N-1] = do[N-1]/bo[N-1]
    for rowi in range(N-2,-1,-1):
        xo[rowi] = (do[rowi]-co[rowi]*xo[rowi+1])/bo[rowi]
    return xo


def general_grid_generator(len_x, Nx, Ny):
    # Creating same size of u, v, U, V
    # where col 0, row 0, row -1, of U not used
    # where row 0, col 0, col -1, of V not used
    initial_Value = 0.01

    dx = len_x/Nx
    grid = np.full((Nx+2, Ny+2), initial_Value)
    u,v,U,V = data_copy(grid, grid, grid, grid)
    p = np.copy(grid)
    return u, v, U, V, p, dx

def data_copy(u, v, U, V):
    return np.copy(u), np.copy(v), np.copy(U), np.copy(V)

def Ghost_BC(u,v): # updating ghost point based on BC condition
    # (u_indomain + u_ghost)/2 = Boundary_value
    # u_ghost = 2 * Boundary_value - u_indomain
    u[-1,:] = 2*1 - u[-2,:] # upper BC
    v[-1,:] =  - v[-2,:]

    u[0, :] =  - u[1,:] # bottom BC
    v[0, :] =  - v[1,:]

    u[1:-1, 0] =  - u[1:-1,1] # left BC
    v[1:-1, 0] =  - v[1:-1,1]

    u[1:-1, -1] =  - u[1:-1,-2] # right BC
```

```
    v[1:-1, -1] =  - v[1:-1,-2]
    return u, v


def Ghost_BC_p(p): # updating ghost point for p based on BC condition
    p[-1,:] =   p[-2,:] # upper BC
    p[-1,:] =   p[-2,:]

    p[0, :] =   p[1,:] # bottom BC
    p[0, :] =   p[1,:]

    p[1:-1, 0] =   p[1:-1,1] # left BC
    p[1:-1, 0] =   p[1:-1,1]

    p[1:-1, -1] =   p[1:-1,-2] # right BC
    p[1:-1, -1] =   p[1:-1,-2]
    return p



def UV_BC(U, V):
    U[1:-1,1] = 0
    U[1:-1,-1] = 0
    V[1,1:-1] = 0
    V[-1,1:-1] = 0
    return U,V



def coeff_assemble(rows_cols, a_e, b_e, c_e, e_e, f_e):
    rows, cols = rows_cols[0], rows_cols[1]
    # input elements of abcdef, get line matrix of abcdef
    a = np.full((rows, cols), a_e ) # abc in same length of Ncols of u
    b = np.full((rows, cols), b_e )
    c = np.full((rows, cols), c_e )

    e = np.full((rows, cols), e_e ) # ef in same length of Nrows of u
    f = np.full((rows, cols), f_e )
    a[:,0]= a[:,-1] = c[:,0] = c[:,-1] = 0
    b[:,0] = b[:,-1] = 1
    e[:,-1] =e[:,0] = f[:,-1]= f[:,0]=0
    return a,b,c, e,f



def RHS_uv(f, f_old, U, V, U_old, V_old, c0, r):
    # Update Right Hand Side
    f_P = f[1:-1,1:-1]
    f_W = f[1:-1,:-2]
    f_E = f[1:-1,2:]
    f_S = f[:-2,1:-1]
    f_N = f[2:,1:-1]

    f_P_old = f_old[1:-1,1:-1]
    f_W_old = f_old[1:-1,:-2]
    f_E_old = f_old[1:-1,2:]
    f_S_old = f_old[:-2,1:-1]
    f_N_old = f_old[2:,1:-1]
```

```
113
114     convection_new = c0*( U[1:-1,2:] * (f_E + f_P)/2 -
115                           U[1:-1,1:-1] * (f_W + f_P)/2 +
116                           V[2:,1:-1]  * (f_N + f_P)/2 -
117                           V[1:-1,1:-1] * (f_S + f_P)/2 )
118
119     convection_old = c0*( U_old[1:-1,2:] * (f_E_old + f_P_old)/2 -
120                           U_old[1:-1,1:-1] * (f_W_old + f_P_old)/2 +
121                           V_old[2:,1:-1]  * (f_N_old + f_P_old)/2 -
122                           V_old[1:-1,1:-1] * (f_S_old + f_P_old)/2 )
123
124     diffusion = f_P + r*(f_E + f_W + f_N + f_S -4*f_P)
125     return -3/2*convection_new + 1/2*convection_old + diffusion
126
127 def Not_Use_UV_BC(U,V):
128     U[0,:] = U[-1,:] = U[:,0] = V[:,0] = V[:,-1] = V[0,:] = -10000
129     return U,V
130
131 def Residual(u, a, b, c, d, e, f):
132     Res = np.zeros_like(u)
133     Res[1:-1,1:-1] = (a[1:-1,1:-1] * u[1:-1,:-2] +
134                       b[1:-1,1:-1] * u[1:-1,1:-1] +
135                       c[1:-1,1:-1] * u[1:-1,2:] +
136                       e[1:-1,1:-1] * u[:-2,1:-1] +
137                       f[1:-1,1:-1] * u[2:,1:-1] -
138                       d[:,:])
139     # print(Res)
140     return Res
141
142
143 def Point_GS(a,b,c,d,e,f,u):
144     u_k = np.copy(u)
145     u_k_new = np.copy(u)
146     n_rows = np.size(u,0)
147     w = 1.9
148     D = np.zeros_like(a[1,:])
149     Res = 100
150     k = 0
151     while Res>1e-6:
152         u_k = np.copy(u_k_new)
153         u_k_new[1:-1,1:-1] = (-a[1:-1,1:-1] * u_k_new[1:-1,:-2] +
154                               -c[1:-1,1:-1] * u_k[1:-1,2:] +
155                               -e[1:-1,1:-1] * u_k_new[:-2,1:-1] +
156                               -f[1:-1,1:-1] * u_k[2:,1:-1] +
157                               d[:,:])/b[1:-1,1:-1]
158         u_k_new[1:-1] = u[1:-1]*(1-w) + u_k_new[1:-1]*w # SOR term
159
160         Res = np.linalg.norm(Residual(u_k_new, a, b, c, d, e, f))
161         # print("Res=")
162         # print(Res)
163         if Res > 1e+200:
164             print("NOT CONVERGE NOT CONVERGE NOT CONVERGE NOT CONVERGE")
165             print(Res)
166             quit()
167         u = np.copy(u_k)
168         k+=1
169     # print("finish iteration")
170
```

```
    return u_k_new, k



def LineSOR(a,b,c,d,e,f, u):
    # input abcdef, and u, get D
    # then use TDMA get new u
    # abc is in same size of N_rows of u, ef in same size of N_cols of u
    # ! Input d is LIKE inner field of u, need to cut each line to use !
    u_k = np.copy(u)
    u_k_new = np.copy(u)
    n_rows = np.size(u,0)
    w = 1.9
    D = np.zeros_like(a[1,:])
    Res = 100
    while Res>1e-6:
        u_k = np.copy(u_k_new)
        for j in range(1,n_rows-1): # the number is based on d, which is (N-1)x(N-1)
            # D = np.zeros_like(a[1,:])
            D[1:-1] = d[j-1,:] - e[j,1:-1]*u_k_new[j-1,1:-1] - f[j,1:-1]*u_k[j+1,1:-1]
            D[0], D[-1] = u[j,0], u[j,-1] # ghost point = ghost point
            u_k_new[j,:] = TDMA(a[j,:],b[j,:],c[j,:],D) # TDMA solve uk each line
            # u_k_new[j,:] = u[j,:]*(1-w) + TDMA(a,b,c,D)*w # SOR term

        Res = np.linalg.norm(Residual(u_k_new, a, b, c, d, e, f))
        # print("Res=")
        # print(Res)
        if Res > 1e+20:
            print("NOT CONVERGE NOT CONVERGE NOT CONVERGE NOT CONVERGE")
            quit()
        u = np.copy(u_k)
    # print("finish iteration")
    return u_k_new



def Solver(u, v, U, V, p, t, dt, dx, Re):
    r = dt/(2*Re*dx**2)
    c0 = dt/(dx)

    rows_cols = np.shape(u)
    a_uv,b_uv,c_uv,e_uv,f_uv  = coeff_assemble(rows_cols, -r, 1+4*r, -r, -r, -r)
    a_p,b_p,c_p,e_p,f_p  = coeff_assemble(rows_cols, 1, -4, 1, 1, 1)

    U, V = UV_BC(U, V)
    u, v = Ghost_BC(u,v)

    U, V = Not_Use_UV_BC(U, V)

    u_star, v_star, U_star, V_star = data_copy(u, v, U, V)
    u_new, v_new, U_new, V_new = data_copy(u, v, U, V)
    u_old, v_old, U_old, V_old = data_copy(u, v, U, V)

```

```
    # n_max = int(t/dt)
    Difference = 100

    n=0
    while Difference>1e-6:
        U, V = UV_BC(U, V) # Setup BC

        # Step 1: get u_star, v_star, U_star, V_star
        u, v = Ghost_BC(u,v)
        u_star, v_star = Ghost_BC(u_star,v_star)

        # get u_star:
        d = RHS_uv(u, u_old, U, V, U_old, V_old, c0, r)
        u_star = LineSOR(a_uv,b_uv,c_uv, d , e_uv,f_uv, u)


        u_star, v_star = Ghost_BC(u_star,v_star)

        # get v_star:
        d = RHS_uv(v, v_old, U, V, U_old, V_old, c0, r)
        v_star = LineSOR(a_uv,b_uv,c_uv, d , e_uv,f_uv, v)

        u_star, v_star = Ghost_BC(u_star,v_star)

        # get U_star, V_star
        U_star[1:-1,2:] =  (u_star[1:-1,1:-1] + u_star[1:-1,2:])/2
        V_star[2:,1:-1] =  (v_star[2:,1:-1] + v_star[1:-1,1:-1])/2

        U_star, V_star = UV_BC(U_star, V_star) # Setup BC


        # Step 2: get New Pressure P
        d = dx/dt * (
            U_star[1:-1, 2:] - U_star[1:-1, 1:-1] +
            V_star[2:, 1:-1] - V_star[1:-1, 1:-1]
        )
        p_new,k = Point_GS(a_p,b_p,c_p,d, e_p,f_p, p)
        p_new = Ghost_BC_p(p_new)


        # Step 3: get u_new, v_new, U_new , V_new
        # update u_new, v_new
        u_new[1:-1,1:-1] = u_star[1:-1, 1:-1] - dt/dx*(p_new[1:-1,2:] -
    p_new[1:-1,:-2])/2
        v_new[1:-1,1:-1] = v_star[1:-1, 1:-1] - dt/dx*(p_new[2:,1:-1] -
    p_new[:-2,1:-1])/2
        u_new, v_new = Ghost_BC(u_new,v_new)
        # update U_new , V_new
        U_new[1:-1,2:] = U_star[1:-1,2:] - dt/dx*(p_new[1:-1,2:]-p_new
    [1:-1,1:-1])
        V_new[2:,1:-1] = V_star[2:,1:-1] - dt/dx*(p_new[2:,1:-1]-p_new
    [1:-1,1:-1])
        U_new, V_new = UV_BC(U_new, V_new) # Setup BC

        # print('new done')

        u_old, v_old, U_old, V_old = data_copy(u, v, U, V)
```

```
        u, v, U, V = data_copy(u_new, v_new, U_new, V_new)
        N = rows_cols[1]-2
        Difference = np.sum(np.abs(u - u_old)+np.abs(v - v_old)+np.abs(
    p_new - p))/N**2

        p = np.copy(p_new)

        # print(Big_Res)
        n +=1
        print("time=%.3f, Difference=%.9f, p iteration times=%d"%((n+1)*dt,
    Difference,k))
    return u, v, p



def draw_streamline(u, v, len_x, len_y ,label):
    rows, cols = np.shape(u)
    X = np.linspace(0, len_x, cols)
    Y = np.linspace(0, len_y, rows)
    plt.figure(figsize=(12, 12))
    plt.streamplot(X, Y, u, v, density =3)
    plt.axis([0,1,0,1])
    plt.title(f'streamline_{label}')
    plt.savefig(f'streamline_{label}.png')
    # plt.show()


def draw_velocity_contour(u, v, len_x, len_y,label):
    levels=100
    rows, cols = np.shape(u)
    X, Y = np.linspace(0, len_x, cols), np.linspace(0, len_y, rows)
    velocity_magnitude = np.sqrt(u**2 + v**2)
    plt.figure(figsize=(12, 12))
    plt.contourf(X, Y, velocity_magnitude, levels=levels, cmap='jet')
    plt.colorbar(label='Velocity magnitude')
    plt.xlabel('X')
    plt.ylabel('Y')
    plt.title(f'velocity_contour_{label}')
    plt.savefig(f'velocity_contour_{label}.png')
    # plt.show()


def draw_u_contour(u, v, len_x, len_y ,label):
    levels=100
    rows, cols = np.shape(u)
    X, Y = np.linspace(0, len_x, cols), np.linspace(0, len_y, rows)
    plt.figure(figsize=(12, 12))
    plt.contourf(X, Y, u, levels=levels, cmap='jet')
    plt.colorbar(label='u magnitude')
    plt.xlabel('X')
    plt.ylabel('Y')
    plt.title(f'u_contour_{label}')
    plt.savefig(f'u_contour_{label}.png')
    # plt.show()



def save(u,v,p,label):
```

```
    with open(f'output_N={label}.txt', 'w') as f:
        f.write(f'For grid size N= {label}:\n')
        f.write('Array u:\n')
        f.write(','.join(map(str, u)) + '\n\n')
        f.write('Array v:\n')
        f.write(','.join(map(str, v)) + '\n\n')
        f.write('Array p:\n')
        f.write(','.join(map(str, p)) + '\n')



def main():
    N = 32
    t = 20



    dt = 0.02

    len_x = 1
    len_y = 1
    Nx = Ny = N
    Re = 100

    u, v, U, V, p, dx = general_grid_generator(len_x, Nx, Ny)

    u,v, p =Solver(u, v, U, V, p, t, dt, dx, Re)

    print(u)
    print(v)
    print(p)

    save(u,v,p,N)



    draw_streamline(u[1:-1,1:-1],v[1:-1,1:-1], len_x, len_y , N)
    draw_velocity_contour(u[1:-1,1:-1],v[1:-1,1:-1], len_x, len_y , N)
    draw_u_contour(u[1:-1,1:-1],v[1:-1,1:-1], len_x, len_y , N)




if __name__ == '__main__':
    main()
```

Listing 1: Lid Driven Cavity Solver

```
import numpy as np
import re
import matplotlib.pyplot as plt

def load_2d_arrays_with_numpy(label):
    filename = f'output_N={label}.txt'
    try:
        with open(filename, 'r') as file:
            content = file.read()
    except FileNotFoundError:
        print("No files")
```

```
        return None, None, None

    def extract_2d_array(array_content):
        array_data = []
        matches = re.findall(r'\[(.*?)]', array_content.replace('\n', ''), re.DOTALL)
        for match in matches:
            formatted_match = ' '.join(match.split())
            numbers = np.fromstring(formatted_match, sep=' ')
            array_data.append(numbers)
        return np.array(array_data)

    u_data = re.search(r'Array u:\s*((?:\[\s*.*?\s*\][,\s]*)+)', content, re.DOTALL)
    v_data = re.search(r'Array v:\s*((?:\[\s*.*?\s*\][,\s]*)+)', content, re.DOTALL)
    p_data = re.search(r'Array p:\s*((?:\[\s*.*?\s*\][,\s]*)+)', content, re.DOTALL)

    if not (u_data and v_data and p_data):
        print("ERROR U,V,P")
        return np.array([]), np.array([]), np.array([])

    u = extract_2d_array(u_data.group(1))
    v = extract_2d_array(v_data.group(1))
    p = extract_2d_array(p_data.group(1))

    return u, v, p

def smalluvp(u,v,p):
    return u[1:-1,1:-1],v[1:-1,1:-1],p[1:-1,1:-1]

def Center(u,v,p, label):
    u,v,p = smalluvp(u,v,p)

    center = int(label/2)
    u_x_center = (u[:,center] + u[:,center-1])/2
    v_x_center = (v[:,center] + v[:,center-1])/2
    p_x_center = (p[:,center] + p[:,center-1])/2

    u_y_center = (u[center,:] + u[center-1,:])/2
    v_y_center = (v[center,:] + v[center-1,:])/2
    p_y_center = (p[center,:] + p[center-1,:])/2

    return u_x_center, v_x_center, p_x_center\
    ,u_y_center, v_y_center, p_y_center


# def drawing(u_x_c, u_y_c):


def get_ue_x_c():
    ye_0 = [1.00000, 0.9766, 0.9688, 0.9609, 0.9531, 0.8516, 0.7344, 0.6172, 0.5000, 0.4531, 0.2813, 0.1719, 0.1016, 0.0703, 0.0625, 0.0547, 0.0000]
    ue_x_c_0 = [1.00000, 0.84123, 0.78871, 0.73722, 0.68717, 0.23151, 0.00332, -0.13641, -0.20581, -0.21090, -0.15662, -0.10150, -0.06434, -0.04775, -0.04192, -0.03717, 0.00000]
```

```
    return ye_0[::-1], ue_x_c_0[::-1]

def get_ue_y_c():
    xe_0 = [1.0000, 0.9688, 0.9609, 0.9531, 0.9453, 0.9063, 0.8594, 0.8047,
     0.5000, 0.2344, 0.2266, 0.1563, 0.0938, 0.0781, 0.0703, 0.0625, 0.0000]
    xe_y_c_0 = [0.00000, -0.05906, -0.07391, -0.08864, -0.10313, -0.16914,
    -0.22445, -0.24533, 0.05454, 0.17527, 0.17507, 0.16077, 0.12317,
    0.10890, 0.10091, 0.09233, 0.00000]
    return xe_0[::-1], xe_y_c_0[::-1]


# def drawing(u_x_c_16, u_x_c_64, u_x_c_128):
#     ye, ue_x_c = get_ue_x_c()
#     y = np.linspace(0,N*1/N, N)
#     plt.figure(figsize=(8, 8))
#     plt.plot(ye, ue_x_c, label='Paper Result')
#     plt.plot(y, u_x_c, label='Result')
#     plt.xlabel('X')
#     plt.ylabel('u velocity')
#     plt.legend()
#     # plt.title('u Contour')
#     # plt.savefig(f'u_contour_{label}.png')
#     plt.show()


def drawing_x_c(u_x_c_16, u_x_c_64, u_x_c_128):
    ye, ue_x_c_vertical = get_ue_x_c()
    y_16 = np.linspace(0,1, 16)
    y_64 = np.linspace(0,1, 64)
    y_128 = np.linspace(0,1, 128)
    plt.figure(figsize=(8, 8))
    plt.plot( ue_x_c_vertical, ye, '-x', label='Paper Result', color= 'red'
    )
    plt.plot( u_x_c_16,y_16, '-x', label='Result_16',color= 'blue')
    plt.plot( u_x_c_64,y_64, '-x', label='Result_64',color= 'green')
    plt.plot( u_x_c_128,y_128, '-', label='Result_128',color= 'navy')
    plt.xlabel('Y')
    plt.ylabel('u velocity')
    plt.legend()
    plt.title('u along verticle line through geometric center')
    plt.savefig('u along verticle line through geometric center')
    # plt.show()




def drawing_y_c(v_y_c_16, v_y_c_64, v_y_c_128):
    xe, ue_y_c_vertical = get_ue_y_c()
    plt.figure(figsize=(12,12))
    x_16 = np.linspace(0,16*1/16, 16)
    x_64 = np.linspace(0,64*1/64, 64)
    x_128 = np.linspace(0,128*1/128, 128)

    plt.plot(xe, ue_y_c_vertical, '-x', label='Paper Result',color= 'red')
    plt.plot(x_16, v_y_c_16, '-x', label='Result_16',color= 'blue')
    plt.plot(x_64, v_y_c_64, '-x', label='Result_64',color= 'green')
    plt.plot(x_128, v_y_c_128, '-', label='Result_128',color= 'navy')
```

```
    plt.xlabel('X')
    plt.ylabel('v velocity')
    plt.legend()
    plt.title('v along horizontal line through geometric center')
    plt.savefig('v along horizontal line through geometric center')
    # plt.show()


def center_get(N):
    u, v, p = load_2d_arrays_with_numpy(N)
    return Center(u,v,p, N)









def main():
    # N = 128
    # u, v, p = load_2d_arrays_with_numpy(N)
    # print(u[-2,:])
    # u_x_c, v_x_c, p_x_c, u_y_c, v_y_c, p_y_c = Center(u,v,p, N)

    u_x_c_128, v_x_c_128, p_x_c_128, u_y_c_128, v_y_c_128, p_y_c_128 =
    center_get(128)
    u_x_c_64, v_x_c_64, p_x_c_64, u_y_c_64, v_y_c_64, p_y_c_64 = center_get
    (64)
    u_x_c_16, v_x_c_16, p_x_c_16, u_y_c_16, v_y_c_16, p_y_c_16 = center_get
    (16)

    # drawing_x_c(u_x_c_16, u_x_c_64, u_x_c_128)
    # drawing_y_c(v_y_c_16, v_y_c_64, v_y_c_128)



if __name__ == '__main__':
    main()
```

Listing 2: Post Operator: