# Plan

- Section 10. Cosets and the Theorem of Lagrange (Review and Continue)
- Section 11. Direct Products and Finitely Generated Abelian Groups.

Let $H$ be a subgroup of $G$.

**Definition 10.2.** For $a \in G$, the subset

$$aH = \{ah \mid h \in H\}$$

is called the **left coset of $H$ containing $a$**.

**Example.** $G = S_3 = \{e, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$.

The set of even permutations in $S_3$ is a subgroup $A_3$ called the alternating group of 3 letters.

$$A_3 = \{e, (1,2,3), (1,3,2)\}$$

There are 2 left cosets of $A_3$.

$$eA_3 = (1,2,3)A_3 = (1,3,2)A_3 = \{e, (1,2,3), (1,3,2)\}$$
$$(1,2)A_3 = (1,3)A_3 = (2,3)A_3 = \{(1,2), (1,3), (2,3)\}$$

Let $H$ be a finite subgroup of group $G$. Then every very left coset has the same number of elements as $H$. Let

$$H = \{h_1, h_2, \ldots, h_k\}$$

(so $H$ has $k$ elements), then

$$aH = \{ah_1, ah_2, \ldots, ah_k\}$$

For any two left cosets $aH$ and $bH$, either $aH = bH$ or $aH \cap bH = \emptyset$.

## Theorem

**Theorem 10.10** *(Lagrange Theorem) Let $H$ be a subgroup of a finite group $G$. Then the order of $H$ is a divisor of the order of $G$.*

*Proof.* $G$ is a disjoint union of left cosets of $H$:

$$G = a_1H \sqcup a_2H \sqcup \cdots \sqcup a_mH$$

so

$$|G| = |a_1H| + |a_2H| + \cdots + |a_mH|$$

Since every left coset has the same number of elements as $H$, so $|a_1H| = |a_2H| = \cdots = |a_mH| = |H|$. So we have $|G| = m|H|$. This proves $|H|$ is a divisor of $|G|$.

We used the concept of left cosets in the proof of Lagrange Theorem. An alternative approach is to use the concept of right cosets.

Everything we proved for left cosets generalizes to right cosets.

**Corollary 10.11.** *Every group of prime order is cyclic.*

*Proof.* Let $G$ be a group with $|G| = p$ a prime number. Choose $a \in G$ such that $a \neq e$. Consider the cyclic subgroup $\langle a \rangle$ generated by $a$. By Lagrange Theorem, $|\langle a \rangle|$ is a divisor of $|G| = p$. Since $p$ is a prime, so

$$|\langle a \rangle| = 1, \quad \text{or} \quad |\langle a \rangle| = p$$

Since $\langle a \rangle$ contains $a$ and $e$, so

$$|\langle a \rangle| \geq 2.$$

This implies

$$|\langle a \rangle| = p$$

So $\langle a \rangle = G$. This proves $G$ is a cyclic group.

## Theorem

*(**Theorem 10.12**) The order of an element of a finite group is a divisor of the order of the group.*

Recall $a \in G$, where $G$ is a finite group. The order of $a$ is the smallest positive integer $n$ with $a^n = e$, is also equal to the order of the cyclic subgroup $\langle a \rangle$ generated by $a$. Theorem 10.12 follows from the Lagrange Theorem.

**Example.** $S_3 = \{e, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$.
$|S_3| = 6$.

$e$ has order 1;
$(1,2), (1,3), (2,3)$ have order 2;
$(1,2,3), (1,3,2)$ have order 3;
they are all divisors of 6.

**Difinition 10.13.** Let $H$ be a subgroup of a group $G$. The number of left cosets of $H$ in $G$ is called the **index** $(G : H)$ of $H$ in $G$.

We have

$$(G : H) = \frac{|G|}{|H|}.$$

**Example.** $(S_n : A_n) = 2$.

**Example.** There are two possible group tables for order 4 group.

One group is a cyclic group with order 4;

the other has elements $G = \{e, a, b, c\}$ with the property

$$a^2 = b^2 = c^2 = e.$$

# Section 11. Direct Products and Finitely Generated Abelian Groups.

**Definition 11.1.** Let $S_1, \ldots, S_n$ be sets. The **Cartesian product** of the sets, denoted by $S_1 \times S_2 \times \cdots \times S_n$, is the set of all ordered $n$-tuples

$$(a_1, a_2, \ldots, a_n)$$

with $a_1 \in S_1, a_2 \in S_2, \ldots, a_n \in S_n$.

## Theorem

*(**Theorem 11.2**) Let $G_1, G_2, \ldots, G_n$ be groups, we define a binary operation on $G_1 \times G_2 \times \cdots \times G_n$ by*

$$(a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n) = (a_1 b_1, a_2 b_2, \ldots, a_n b_n)$$

*Then $G_1 \times G_2 \times \cdots \times G_n$ is a group under this operation. This group is called the* **direct product of the groups** $G_i$.

**Example.** $\mathbb{Z}_2 = \{0, 1\}$, modulo 2 integer group
$\mathbb{Z}_3 = \{0, 1, 2\}$, modulo 3 integer group

The direct product group $\mathbb{Z}_2 \times \mathbb{Z}_3$ has 6 elements.

**Example.** If $G$ is a cyclic group of order $m$ and $G'$ is a cyclic group of order $n$. Assume $m$ and $n$ are relatively prime, then the direct product group $G \times G'$ is a cyclic group.

*Proof.* Let $G = \langle a \rangle$, $G' = \langle b \rangle$, so $a$ has order $m$ and $b$ has order $n$. Consider $(a, b) \in G \times G'$, it has order $mn$, which is $|G \times G'|$, so $G \times G'$ is cyclic.

**Definition.** Let $G$ be a group, $S$ be a subset of $G$, we say $S$ **generates** $G$ if every element $g \in G$ can be written as

$$g = a_1^{k_1} a_2^{k_2} \cdots a_m^{k_m}$$

for some $m$ and $a_1, \ldots, a_m \in S$ (not necessarily distinct) and $k_1, \ldots, k_m \in \mathbb{Z}$.

**Example.** $\mathbb{Z} \times \mathbb{Z}$. $S = \{(1,0),(0,1)\}$, then $S$ generates $\mathbb{Z} \times \mathbb{Z}$.

**Example.** A group $G$ is called a **finitely generated group** if there exists a finite subset $S$ that generates $G$.

The end