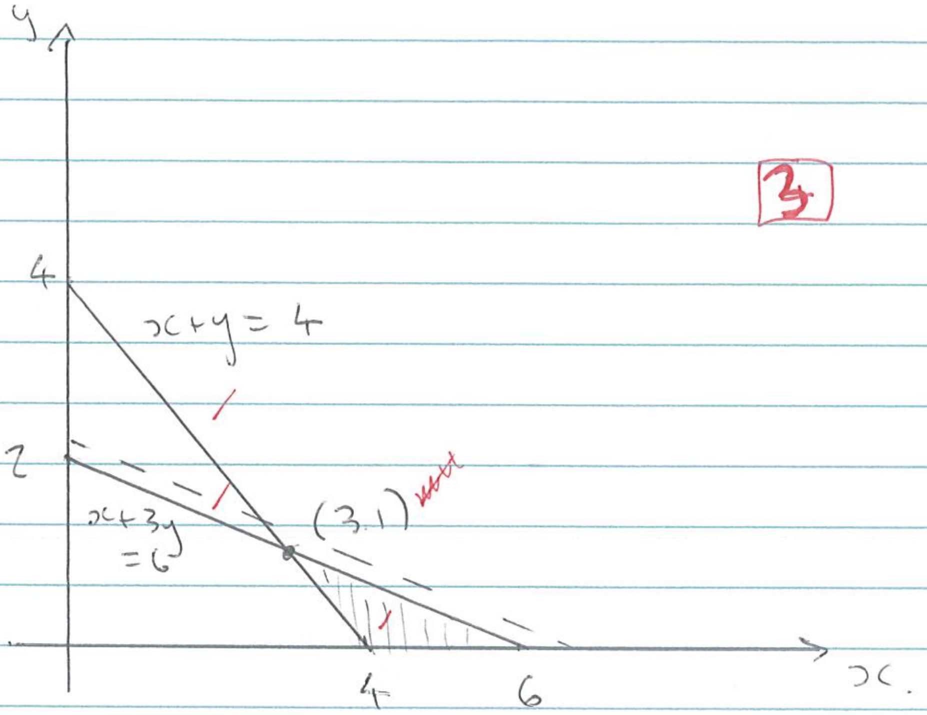
MAST 20018 - Exam 2016 - Solutions

Q 1

a)



$\boxed{3}$

-1 mark if any labels are missing

b) $(6,0), (4,0), (3,1).$ $\boxed{1}$

c.) Optimal solutions $x^* \in \{\lambda(3,1)+(1-\lambda)(6,0) : \lambda \in [0,1]\}.$

$z^* = 3$ $\boxed{2}$

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Q2

a) $\max Z = x_1 + 2x_2 + x_4$

$$\begin{aligned} x_1 + 3x_2 - x_3 + x_4 + x_5 &= 5 \\ x_1 + 7x_2 + x_3 - x_6 + y_1 &= 4 \\ 4x_1 + 2x_2 + x_4 + y_2 &= 3 \end{aligned}$$

$x_i \geq 0 \quad i = 1 \ldots 6$

$y_1, y_2 \geq 0.$

$\boxed{5}$

b) $\min w = y_1 + y_2$

BV	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	RHS
x_5	1	3	−1	1	1	0	0	0	5
y_1	1	7	1	0	0	−1	1	0	4
y_2	4	2	0	1	0	0	0	1	3
w	0	0	0	0	0	0	−1	−1	0

$\boxed{3}$

$\Rightarrow$

BV	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	RHS
x_5	1	3	−1	1	1	0	0	0	5
y_1	1	7	1	0	0	−1	1	0	4
y_2	4	2	0	1	0	0	0	1	3
w	5	9	1	1	0	−1	0	0	7

Not necessary to restore canonical form

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Q3.

a) $$P = 2\sum_{i=1}^{3} x_{i1} + 1.6\sum_{i=1}^{3} x_{i2} + 1.2\sum_{i=1}^{3} x_{i3} - \sum_{j=1}^{3} x_{1j} - 1.5\sum_{j=1}^{3} x_{2j} - 0.8\sum_{j=1}^{3} x_{3j}$$ 3

b) $\sum_{j=1}^{3} x_{1j} \le 100$, $\sum_{j=1}^{3} x_{2j} \le 80$, $\sum_{j=1}^{3} x_{3j} \le 60$ 3

c)

$0.6x_{11} - 0.4x_{21} + 0.6x_{31} \le 0$

$-0.2x_{11} - 0.2x_{21} + 0.8x_{31} \le 0$

$-0.4x_{12} + 0.6x_{22} + 0.6x_{32} \le 0$ 6

$0.8x_{13} - 0.2x_{23} - 0.2x_{33} \le 0$

$0.6x_{13} + 0.6x_{23} - 0.4x_{33} \le 0$.

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$x_{21} \ge 0.6(x_{21} + x_{11} + x_{31})$

$x_{31} \le 0.2(x_{11} + x_{21} + x_{31})$

$x_{12} \ge 0.6(x_{12} + x_{22} + x_{32})$

$x_{13} \le 0.2(x_{13} + x_{23} + x_{33})$

$x_{33} \ge 0.6(x_{13} + x_{23} + x_{33})$

Q4.

a)
$$A_B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad A_B^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

~~KBF~~ $x_B^* = A_B^{-1} b$

$$\begin{bmatrix} 30 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$ // $\boxed{3}$

$\therefore \quad b_1 = 30 \qquad b_2 \cancel{+} -30 = 10$

$b_2 = 40$

b). $y^* = (y_1, y_2, y_3, y_4, y_5)$

$= (d, e, 0, a, 7)$

$e = 0$ since s_2 is a bv.

Now $c_B A_B^{-1} = (d, e)$ // $\boxed{3}$

and $c_B = (5, 0)$

$$(5,0) \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = (5,0)$$

$\therefore d = 5$

c).

$$c_B A_B^{-1} A - c = (0, a, 7).$$

$$(5,0)\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 5 & 2 & 1 & 0\\ 1 & -5 & -6 & 0 & 1\end{bmatrix} - (5,2,3,0,0)$$

$$= (5,0)\begin{bmatrix}1 & 5 & 2 & 1 & 0\\ 1 & -5 & -6 & 0 & 1\end{bmatrix} - (5,2,3,0,0)$$

$\boxed{4}$

$$= (5,25,10,5,0) - (5,2,3,0,0)$$

$$= (0,23,7,5,0)$$

$\therefore$ $a = 23$

Also.

$$A_B^{-1} A = \begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 5 & 2\\ 1 & -5 & -6\end{bmatrix}$$

$$= \begin{bmatrix}1 & 5 & 2\\ 0 & -10 & -8\end{bmatrix}$$

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$\therefore$ $b = 5$, $c = -10$

Q5

a) Every point in the feasible region has an objective value that is an upper-bound on the primal solution.

∴ If there is no upper-bound to the primal, there can be no feasible point in the dual.

$\boxed{4}$

b).

check feasibility in primal

$$2 \cdot \frac{5}{26} + \frac{5}{2} + 3 \cdot \frac{27}{26}$$

$$= \frac{10 + 65 + 81}{26} = \frac{156}{26} = 6 \quad \therefore \; s_1 = 0$$

$$5 \cdot \frac{5}{26} + 4 \cdot \frac{5}{2} + \frac{27}{26} = 12 \quad \therefore \quad s_2 = 0.$$

$$2 \cdot \left(\frac{5}{2}\right) = 5 \quad \therefore \quad s_3 = 0.$$

Dual $\quad \min \; 6y_1 + 12y_2 + 5y_3$

$$2y_1 + 5y_2 \geq 9$$
$$y_1 + 4y_2 + 2y_3 \geq 14$$
$$3y_1 + y_2 \geq 7$$

$$y_1, y_2, y_3 \geq 0.$$

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$x_1 > 0$, $x_2 > 0$, $x_3 > 0$

$\therefore$ $t_1 = 0$ $t_2 = 0$ $t_3 = 0$.

$\therefore$ $2y_1 + 5y_2 = 9$
$3y_1 + y_2 = 7$

$-13y_1 = -26$
$y_1 = 2$
$y_2 = 1$

$y_1 + 4y_2 + 2y_3 = 14$

$\therefore$ $2 + 4 + 2y_3 = 14$
$y_3 = \cancel{5/2}\ 4$

$w = 6y_1 + 12y_2 + 5y_3$

$= 6(2) + 12(1) + 5 \cdot \cancel{5/2}\ 4$
$= 12 + 12 + \frac{20}{2} = \cancel{36\frac{1}{2}}\ 44$

$\boxed{6}$

$z = 9(5/26) + 14(5/12) + 7(\frac{27}{26})$

$= 44$

$\therefore$ $x^* = (5/26, 5/2, 27/26)$ is optimal.

Full marks (6) if proved its not optimal

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Q6

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_3	$4/9$	0	1	$-1/9$	$1/3$	$-1/9$	0	$4/3$
x_2	$2/3$	1	0	$4/3$	0	$1/3$	0	4
x_7	$13/9$	0	0	$-10/9$	$-2/3$	$-1/9$	1	$34/3$
z	$7/9$	$-\delta$ ✓	0	$14/9$	$1/3$	$5/9$	0	$28/3$

new z-row = z-row + δ. Row 2

= x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
$7/9 + \frac{2\delta}{3}$	0	0	$14/9 + \frac{4\delta}{3}$	$1/3$	$5/9 + \frac{\delta}{3}$	0	$28/3 + 4\delta$ ✓

$\therefore$ For optimality, $\frac{7}{9} + \frac{2\delta}{3} \geq 0$ ... (1)

and $\frac{14}{9} + \frac{4\delta}{3} \geq 0$ ... (2)

and $\frac{5}{9} + \frac{\delta}{3} \geq 0$ ... (3)

$\delta \geq -\frac{5}{3}$

(1) $\delta \geq -\frac{7}{6}$

(2) $\delta \geq -\frac{14}{12}$ ✓

(3) $\delta \geq -\frac{5}{\cancel{9}\,3}$

$\therefore \quad \delta \geq \cancel{-\frac{5}{9}\,3}$

$\delta \geq -\frac{7}{6}$

$\boxed{5}$

$\therefore$ since $c_2 = 2$ we have

/5

$2 + \delta \geq 2 - \cancel{\frac{5}{9}\,3}\ 7/6$

$= \cancel{\frac{13}{9}}\ \ 5/6$

Q7.

a) Node 1 = 0.

Node 5 = 0

Node 3 = Node 2.

Don't need.

$$B(3) = \frac{\mu_{14}(3)}{\mu_{14}} + \frac{\mu_{15}(3)}{\mu_{15}} \cancel{+\mu}$$

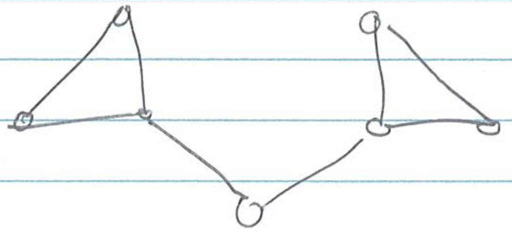
$$= \frac{1}{2} + \frac{1}{2} = 1$$

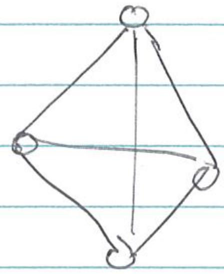
$$B(4) = \frac{\mu_{15}(4)}{\mu_{15}} \cancel{+\mu_{14}} \frac{\mu_{25}(4)}{\mu_{25}} + \frac{\mu_{35}(4)}{\mu_{35}}$$

$$= \frac{2}{2} + \frac{1}{1} + \frac{1}{1} = 3.$$

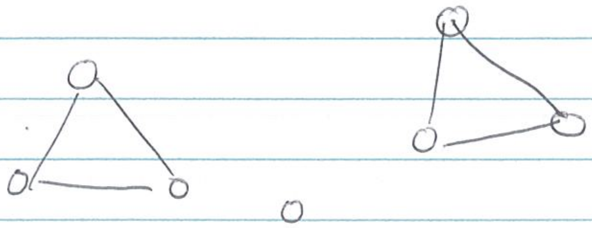
$\boxed{4}$

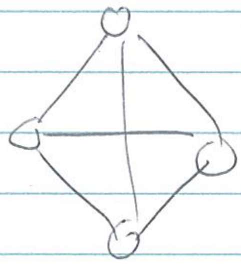
b)



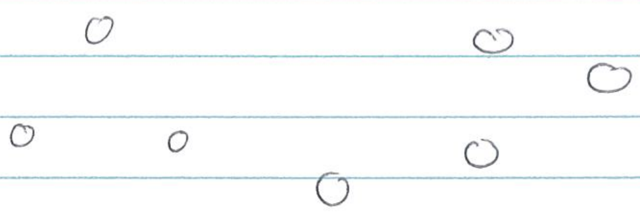


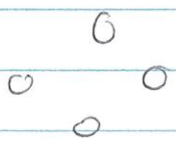
$\Rightarrow$

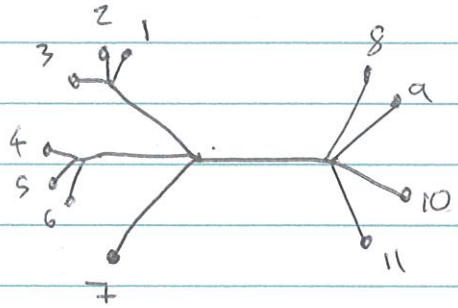




$\Rightarrow$







$\boxed{6}$

Q8

a)

For any vertex $v \in V(G)$, let $d(v)$ be the degree of v.

The sum $\sum_{v \in V(G)} d(v)$ counts every edge of G twice.

$\therefore \sum_{v \in V(G)} d(v) = 2m$, where m is the number of edges of G.

$\therefore$ Since the sum of the degrees of all even nodes is even, and an even number plus an odd number is odd, there must be an even number of odd nodes.

$\boxed{6}$

b). A graph is bipartite iff the node-set can be partitioned into two independent sets, where an independent set of nodes does not contain any pair of adjacent nodes. ///

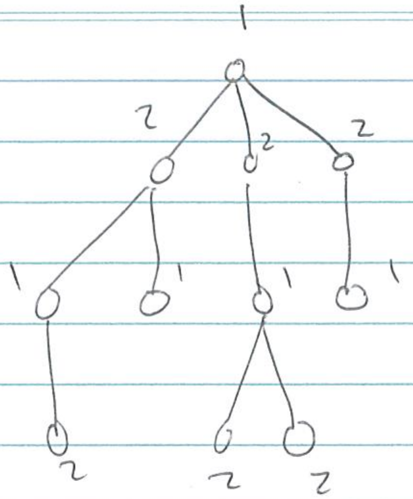
Let G be a bipartite graph with partite sets V_1 and V_2. Suppose that $C = v_1, v_2, \ldots v_k, v_1$ is a cycle of G. We may assume w.l.o.g that $v_1 \in V_1$.
Then $v_2 \in V_2$, $v_3 \in V_1$, $V_4 \in V_2$ etc. ////

$\therefore$ $k = 2s$ for some s. $\therefore$ C has even length.

$\boxed{7}$

c)

i.



ii. (e,b)

→ ~~a~~ i, (e,b), a

→ ~~[illegible]~~ (i,e), (b,a)

→ (i,e), (b,a), (c,g)

→ (i,e), (b,a), (c,g), (d,h)

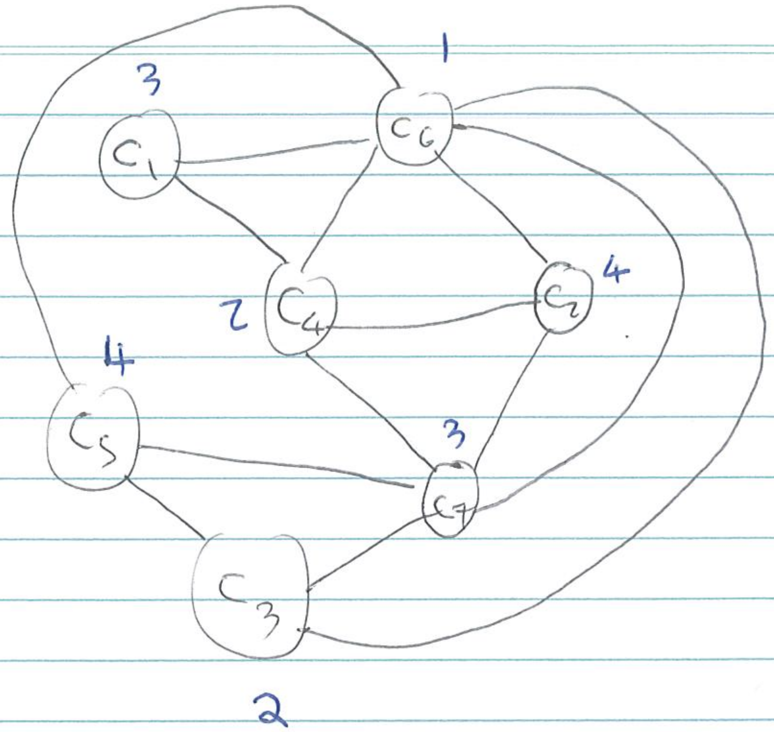
→ Augmenting path f, (b,a), (c,g), k → (f,b), (a,c), (g,k)

No more augmenting paths.

$\therefore M_{max} = \{(i,e), (f,b), (a,c), (g,k), (d,h)\}$.

$\boxed{6}$

c)



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ii 4 rooms.

Q9

The USA initially wins on issues 1, 2, 4 and 6
giving it $22 + 22 + 14 + 6 = 64$ points. We give the tie of Issue 3
to the USA, bringing it up to 79 points.

Panama wins on issues 5, 7, 8, 9, and 10,
giving it $15 + 11 + 7 + 7 + 13 = 53$ points

Issue 3 has the smallest ratio $(15/15 = 1)$.

Transferring Issue 3 to Panama switches the winner to the loser.

$\therefore$ we divide Issue 3.

$V^*_{USA} = 64 \qquad V^*_{pan} = 53$

$\therefore \quad V^*_{USA} + 15 \cdot x = V^*_{pan} + 15(1-x)$

$\therefore \quad x = \frac{2}{15}.$

$\therefore$ USA gets issues 1, 2, 4, 6 and $\frac{2}{15}$ of Issue 3.

Pan gets issues 5, 7, 8, 9, 10 and $\frac{13}{15}$ of Issue 3

$\boxed{8}$

Q10

a) Using the committee method.

Chocolate : $33\times3 + 3\times3 + 10\times2 + 20\times1 + 7\times2 + 27\times1$
$= 189$ (-22)

Vanilla : $33\times2 + 3\times1 + 10\times3 + 20\times3 + 7\times1 + 27\times2$
$= 220.$ 40

Mocha : $33\times1 + 3\times2 + 10\times1 + 20\times2 + 7\times3 + 27\times3$
$= 191$ (-18)

∴ Vanilla wins.

b) Chocolate is in position 1 : $33 + 3 = 36$ times
Vanilla is in position 1 : $10 + 20 = 30$ times
Mocha is in position 1 : $7 + 27 = 34$ times

∴ Chocolate wins by plurality

$\boxed{6}$