Social choice

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We will be looking at two main social choice problems:

Fair division
What is the mathematically fairest way to distribute a set of objects amongst a set of people?

Voting theory
How do we design a "perfect" voting system? Is this even possible?

These two problems will introduce you to the exciting and far-reaching field of game theory. Game theory is "the study of strategic decision making. Specifically, it is the study of mathematical models of conflict and cooperation between intelligent rational decision-makers".

Fair division schemes

An easy problem

If we have 36 identical wrapped lollies, how can we divide them fairly among 9 children?

A harder problem

If a long lost relative passes away who has an estate consisting of

- 2 homes
- 3 cars
- 5 antique pieces of furniture
- \$150 000 in cash

how do we divide this fairly among 4 family members when there is no will?

Fair Division Schemes

A fair division problem consists of N players $P_1, P_2, \ldots, P_N$ and a set S of goods to be divided fairly.

Definition

Divided fairly means that, in the opinion of each player, the value of the goods received is at least $1/N$ the total value of S.

Other examples are when proportioning common property to individuals who part ways, and sharing food at a banquet dinner.

We begin by considering three desirable properties of fair division schemes. Following this we introduce possible methods of finding a fair division.

§1 – Desirable properties of fair division schemes

Envy-free: An envy-free division is one in which no party would prefer another party's share over their own.

Pareto optimal: A Pareto optimal division is one in which no other division would make a particular party better off without making someone else worse off.

Equitable: An equitable division is one in which the portion each party receives, judged by their own valuation, is the same.

§2 – Fair division - the discrete case

The discrete case refers to the items themselves being indivisible, or at the least not conveniently divisible, for example a diamond ring.

§2.1 – Method of markers

This is suited to the fair distribution of a large number of goods of low value to a small number of people.

The goods are conveniently considered as arranged on a line. Let the number of players be N.

Step 1. Each player secretly divides the line of items into N segments, each of which they consider as a fair share. A practical way to do this is to position labelled markers.

Example: There are three players A, B, C and goods

$$g_1\ g_2\ g_3\ g_4\ g_5\ g_6\ g_7\ g_8\ g_9\ g_{10}\ g_{11}\ g_{12}$$

The fair share division of the goods for each player may be

$$g_1\ g_2 \Big|_A\ g_3\ g_4\ g_5\ g_6\ g_7\ g_8\ g_9 \Big|_A\ g_{10}\ g_{11}\ g_{12}$$

$$g_1\ g_2\ g_3\ g_4 \Big|_B\ g_5\ g_6\ g_7\ g_8\ g_9\ g_{10} \Big|_B\ g_{11}\ g_{12}$$

$$g_1\ g_2\ g_3 \Big|_C\ g_4\ g_5\ g_6\ g_7\ g_8 \Big|_C\ g_9\ g_{10}\ g_{11}\ g_{12}$$

Step 2. First segment: Locate the leftmost marker in the line, and give the corresponding player everything to the left of it.

Example: Player A will receive goods $g_1 g_2$.

Remove all the markers belonging to that player, as well as the goods received:

$$g_3 \Big|_C \, g_4 \Big|_B \, g_5 \, g_6 \, g_7 \, g_8 \Big|_C \, g_9 \, g_{10} \Big|_B \, g_{11} \, g_{12}$$

Step 3. Subsequent segments: Look for the markers for the next segment. Of the right-hand-side markers, choose the leftmost. All goods in this segment (as marked by the corresponding player) go to the corresponding player. Repeat until only one segment remains.

Example: This marker belongs to C, who then receives the goods

$$g_4\ g_5\ g_6\ g_7\ g_8.$$

Removing all the markers belonging to that player (as well as the goods received) leaves us with

$$g_3\ g_9\ g_{10}\,\Big|_B\, g_{11} g_{12}$$

Step 4. Final segment: Only one player remains. Locate the rightmost marker belonging to this player. Allocate all items to the right of that marker to the corresponding player.

Example: There is only one marker left, so the items

$$g_{11}\, g_{12}$$

are allocated to player B.

Step 5. The leftovers can be distributed according to some predetermined way (e.g. a raffle).

Exercise:

1. Is the method of markers envy free?
2. Is the method of markers Pareto optimal?
3. Is the method of markers equitable?

Solution:

1. No. Only one counter-example required. Suppose we had $A : 1|2|345, B : 1|23|45, C : 123|4|5$, A would receive 1, B would get 23 and C would have 5. A would envy B.
2. No. The method only requires an incomplete specification of the preferences of the players. Therefore suppose in the above example B doesn't place any value on 2, but A did, if B gave 2 to A, A would be better off, where B would not be affected.
3. Yes. Each player receives $1/N$ according to their valuation system.

§2.2 – Method of sealed bids

Step 1. Each player makes a sealed bid, giving their estimate of the dollar value of each item. Each player's fair share is computed by dividing the total of their bid by the number of players.

Step 2. Each item goes to the respective highest bidder (as in a traditional auction, with ties broken in some predetermined way) and the player adds/subtracts from the pot the difference between their fair share and total value of the allocated item.

Step 3. Any cash left in the pot (excess) at the end of the process is divided evenly among the players.

To play:

- each player must have enough money to cover the difference between their fair share and the total value they place on the goods
- each player must accept money as a substitute for the goods
- each player must bid independently of all other players

The rules of the method of sealed bids ensures that each player receives at least their fair share.

Example

Suppose there are 3 players (people) A, B, C and two items I_1, I_2. Let the bids of the 3 players (in dollars) be $(2100, 2700)$, $(1500, 3000)$, $(1800, 3500)$. Specify the distribution of the goods, and the cash payments.

Solution

Form a table of the bids of each player, as well as their total bid, fair share and pot contribution.

	A	B	C
I_1	$\boxed{2100}$	1500	1800
I_2	2700	3000	$\boxed{3300}$
Total	4800	4500	5100
Fair share	1600	1500	1700
Pot	500	- 1500	1600

The pot contains an excess of 600. This is distributed evenly, so that each player gets 200.

In summary:

Player A gets Item I_1, and pays $500 - 200 = 300$ cash.
Player B gets $1500 + 200 = 1700$ in cash.
Player C gets Item I_2 and pays $1600 - 200 = 1400$ in cash.

Note that the amount of cash payed and received exactly balances.

Example

Let there be 1 item and 3 players denoted A, B, C. Let the amounts bid for the item be b_A, b_B, b_C and suppose $b_A > b_B$, $b_A > b_C$. Compute the fair division allocations according to the method of sealed bids.

Solution

The fair shares are $b_A/3$, $b_B/3$, $b_C/3$ respectively.
The item goes to Player A, who has to pay $2b_A/3$ to the pot.
Player B receives $b_B/3$ from the pot, and Player C receives $b_C/3$ from the pot.
The excess in the pot is then

$$\frac{2b_A}{3} - \frac{(b_B + b_C)}{3} = \frac{2b_A - b_B - b_C}{3} (\geq 0).$$

Hence each player receives a further

$$\frac{2b_A - b_B - b_C}{9}$$

in cash.

In summary

Player A gets the item and $-\frac{2b_A}{3} + \frac{2b_A - b_B - b_C}{9}$ in cash.
Player B gets $\frac{b_B}{3} + \frac{2b_A - b_B - b_C}{9}$ in cash.
Player C gets $\frac{b_C}{3} + \frac{2b_A - b_B - b_C}{9}$ in cash.

Simplifying, this reads

Player A gets the item and pays $\frac{4b_A + b_B + b_C}{9}$ in cash.
Player B gets $\frac{2b_A + 2b_B - b_C}{9}$ in cash.
Player C gets $\frac{2b_A - b_B + 2b_C}{9}$ in cash.

Observe that the amount of cash paid and received exactly balances.

Exercise

In the above setting, suppose Player A had prior knowledge of the bids of Players B and C. What bid will still get Player A the item, and furthermore minimise the amount of cash to be paid (assume all bids must be integer units)?

Solution

Let $\tilde{b}_A$ be the new bid of Player A. Then $\tilde{b}_A = \max(b_B, b_C) + 1$. This follows since Player A must bid higher than b_B and b_C and thus a bid higher than $\max(b_A, b_B)$ is required. Furthermore, the amount of cash to be paid by Player A contains a term $4\tilde{b}_A/9$, which must be made as small as possible, so the smallest allowed value greater than $\max(b_B, b_C)$ must be chosen, which is $\max(b_B, b_C) + 1$.

Exercise

In the above setting, suppose Player C had prior knowledge of the bids of Players A and B, and places less value on the item than Player A. What bid of Player C will still get the item to Player A and furthermore maximise the amount of cash Player C will receive, assuming all bids must be integer units?

Solution

Let $\tilde{b}_C$ be the new bid of Player C. Then $\tilde{b}_C = b_A - 1$. To understand this, first note that Player C must bid less than Player A to ensure that Player A wins. The cash to Player C contains a term $2\tilde{b}_C/9$ which must be made as big as possible, so the greatest value less than b_A must be chosen, which is $b_A - 1$.

Definition

In the method of sealed bids, $(b_1, b_2, \ldots, b_N)$ is a Nash equilibrium strategy profile if each Player i cannot increase their expected return by changing b_i when the other player's bids remain the same.

With one item and 3 players, under the assumption that only Player A wants the item instead of cash, the above exercises show that

$$(b_A, b_B, b_C) = (v_A, v_A - 1, v_A - 1)$$

is a Nash equilibrium strategy profile.

Desirable properties and the method of sealed bids

Exercise:

1. Is the method of sealed bids envy free?
2. Is the method of sealed bids Pareto optimal?
3. Is the method of sealed bids equitable?

Solution:

1. No. For example, with one item and 3 players, if the item goes to Player A, and Player B receives $b_B/3$ from the pot, and Player C receives $b_C/3$ from the pot, then Player B will envy Player C if $b_C > b_B$.
2. Yes. Changing the distribution to the advantage of one player would mean that at least one other player would give up some cash and so be worse off.
3. No. This is because the excess cash in the pot is divided evenly rather than via the ratio implied by the fair share.

§2.3 – Adjusted winner method

This method is also relevant to dispute resolution.

Here there are two Players A and B, and $m > 1$ items to be distributed.

Each item is given a score $a_1, a_2, \dots, a_m$ by Player A, and a score $b_1, b_2, \dots, b_m$ by Player B.

It is assumed that

(i) $a_i, b_i \geq 0$
(ii) $\sum a_i = \sum b_i$ (= some predetermined value)

Step 1. The "winning" phase. Each player receives the items they value more than the other player. The total score of both, denoted v_A, v_B is computed.

Step 2. The "adjusting" phase. Items are transferred, one at a time, from the richer player to the poorer player, starting with the items of the richer player with ratio a_i/b_i closer to the value 1. The process continues until both have the same score, or the roles of the richer and poorer players are reversed.

In the latter case, suppose that after the handover of, say, Item r we have $v_A < v_B$. Item r is then split, with Player A getting the fraction:

$$x_r = \frac{b_r + v_B^* - v_A^*}{a_r + b_r}$$

where v_A^* and v_B^* are the scores obtained by the two players with Item r excluded.

The total score of Player A is then

$$v_A = v_A^* + x_r a_r = \frac{v_A^* b_r + a_r b_r + v_B^* a_r}{a_r + b_r}.$$

Player B gets fraction $(1 - x_r)$ of Item r, so their total score is

$$v_B = v_B^* + (1 - x_r) b_r = \frac{v_A^* b_r + a_r b_r + v_B^* a_r}{a_r + b_r}.$$

Hence

$$v_A = v_B.$$

Note that x_r can be computed by solving the equation

$$v_A^* + x_r a_r = v_B^* + (1 - x_r) b_r$$

Example

A box of 8 chocolates is to be divided up amongst two people according to the adjusted winner method, with a total number of points of 100. The scores are

chocolate	A	B
1	5	15
2	12	10
3	10	5
4	3	2
5	25	20
6	25	13
7	10	15
8	10	20

We now extend the table according to the steps in the adjusted winner method.

chocolate	A	B	Winning phase	ratio	Adjusting phase
1	5	15	B		
2	12	10	A	1.2	B
3	10	5	A	2	
4	3	2	A	1.5	
5	25	20	A	1.25	
6	25	13	A	1.83	
7	10	15	B		
8	10	20	B		

After the winning phase, $v_A = 75$ and $v_B = 50$, so items must be transferred to Player B, starting with the one with the smallest ratio. This is chocolate number 2, which must therefore be transferred to Player B.

After the first adjusting phase, the total scores are $v_A = 63$, $v_B = 60$. Since $v_A > v_B$, another item must be transferred to B. Chocolate 5 has the smallest ratio. However transferring it all to Player B makes $v_A < v_B$ and so only a fraction must be transferred. Not including Chocolate 5 the total scores are $v_A^* = 38$ and $v_B^* = 60$. We require that Player A transfer a fraction $1 - x$ of chocolate 5, keeping a fraction x, so that

$$v_A^* + xa_5 = v_B^* + (1 - x)b_5.$$

Hence we must solve

$$38 + 25x = 60 + 20(1 - x)$$

which implies $x = 42/45 = 0.93$. So the final allocation is
Player A: Chocolates 3, 4, 6 and 93% of 5.
Player B: Chocolates 1, 2, 7 and 8 and 7% of 5.

Desirable properties and the adjusted winner method.

Exercise:

1. Is the adjusted winner method envy free?
2. Is the adjusted winner method Pareto optimal?
3. Is the adjusted winner method equitable?

Solution:

1. Yes. Each player receives the same number of points according to their own value system. However, if we tally the scores using another player value system, every player must win less than the owner of the value system. Therefore, no player ever envies another.
2. Yes. The proof is detailed, not examinable.
3. Yes. Both players receive the same number of points.

§3 – Fair division - the continuous case

In the above examples, the objects to be divided were regarded as being (preferably) indivisible. Now, we turn our attention to the case the object is continuous. The object is referred to as a cake.

§3.1 – Cut and choose method (applicable to $N = 2$)

This method is also called the divider-chooser method

Step 1. Player A cuts the cake into two equal pieces, according to his/her measure.

Step 2. Player B chooses whichever has the greater value according to his/her measure.

Step 3. Player A takes the remaining piece.

By this strategy, Player A is guaranteed to get exactly $\frac{1}{2}$ (in his/her measure) and Player B is guaranteed to get greater than or equal to $\frac{1}{2}$ (in his/her measure).

Exercise

What does this tell us about the cut and choose method?

Solution

This tells us that the cut and choose method is envy free.

Exercise

1. Is the cut and choose method equitable?
2. Is the cut and choose method Pareto Optimal?

Solution

1. No. Generally the second player will receive more than 50%.
2. No. Suppose the cake was half vanilla and half chocolate. If the first player likes both equally they might cut the cake so each slice is half vanilla half chocolate. If the second player hates chocolate, they would be willing to give some (or all!) chocolate to the first player, making the first player better off and not making themselves worse off.

§3.2 – Lone divider method ($N = 3$)

This can be considered an extension of the divide and choose method to 3 (or more) players. Here there is one divider and two choosers.

The divider slices the cake into 3 equal pieces. Each chooser declares which piece is considered acceptable (i.e., contains at least $1/3$ of the total value).

An acceptable piece for each player can always be found. Why?

Case 1: At least one chooser declares more than one piece acceptable. The choosers can then be allocated an acceptable piece, and the divider gets what is left.

Case 2: Each chooser declares a different piece acceptable. They get the respective pieces.

Case 3: Both choosers select one and the same piece. The divider is given one of the two undeclared pieces (randomly selected). The remaining two pieces are put back together the cut and choose method is played by the two choosers.

Desirable properties and the lone divider method ($N = 3$).

Exercise:

1. Is the lone divider method envy free?
2. Is the lone divider method Pareto optimal?
3. Is the lone divider method equitable?

Solution:

1. No. For example, suppose after the cut, both declare just one piece acceptable, and it is the same. The divider gets one of the two unacceptable pieces, which have the same value from the divider's perspective.

   But now the two remaining pieces are put back together, and re-cut. It is possible that the original cutter may envy one of the new cuts.

2. No. Suppose we have a cake, $1/3$ chocolate, $1/3$ vanilla and $1/3$ strawberry. Suppose that Adam only likes chocolate, Bette only likes vanilla and Con only likes strawberry. Suppose Adam is the cutter, he must divide the cake into three pieces each containing $1/3$ of chocolate (since he may not know the preferences of the other players).

   As in the cut and choose method, at the end of the procedure Adam could acquire more chocolate cake from Bette or Con, making himself better off and them no worse off.

3. No. Generally the very last chooser will receive more than $1/3$ of the total.

Example

Player A is the cutter. From the viewpoint of Player B, the three pieces have values in ratio $1 : 2 : 3$ respectively, while from the viewpoint of Player C the pieces have values $3 : 2 : 1$.

Player B has assigned values $1/6$, $1/3$ and $1/2$ to each piece, so Pieces 2 and 3 are acceptable.

Player C has assigned values $1/2$, $1/3$ and $1/6$ to each piece, so Pieces 1 and 2 are acceptable.

Player B chooses Piece 3 and Player C chooses Piece 1, leaving Piece 2 for Player A.

§3.3 – Last diminisher method (N players)

Randomly assign an order of play, $P_1, P_2, \ldots, P_N$.

Step 1. Player P_1 cuts piece P of value $1/N$.

Step 2. For each of the other players in turn, if P has value $> 1/N$, they "trim" it to the value exactly $1/N$ (for practical purposes, a trimmer would simply indicate the trim by marking rather than cutting).

Step 3. The last player to trim P gets it (if nobody trims P, then Player P_1 gets it) and he/she is out of the game.

Step 4. Repeat by letting a remaining player with the smallest index be the next cutter (i.e., if P_1 is still playing then he/she will cut again).

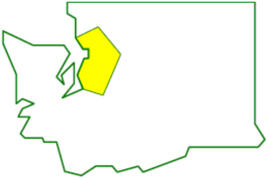
Step 5. The last two remaining players can divide what's left by using the cut and choose method.

This is fair because the first player will get a piece of value exactly $1/N$ and the last $N-1$ players will get $\geq 1/N$.

Last diminisher example

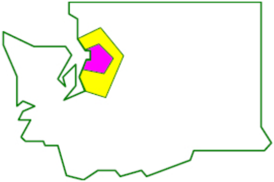
Example: Suppose that four salespeople are dividing up Washington State into sales regions; each will get one region to work in. They pull names from a hat to decide play order.

Round 1. The first salesman, Bob, draws a region around Seattle, the most populous area of the state. The piece Bob cuts and automatically lays claim to is shown in yellow.



Last diminisher example...

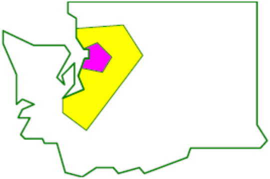
The second salesman, Henry, felt that this region was worth more than 25%, each player's fair share. Because of this, Henry opts to trim this piece. The new piece is shown in pink. The trimmings (in yellow) return to the to-be-divided portion of the state. Henry automatically lays claim to this smaller piece since he trimmed it.



The third saleswoman, Marjo, feels this piece is worth less than 25% and passes, as does the fourth saleswoman, Beth. Since both pass, the last person to trim it, Henry, receives the piece.

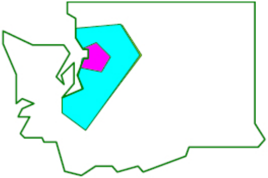
Last diminisher example...

Round 2. The second round begins with Bob laying claim to a piece, shown again in yellow. Henry already has a piece, so is out of the process now. Marjo passes on this piece, feeling it is worth less than a fair share.



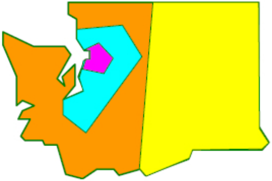
Last diminisher example...

Beth, on the other hand, feels the piece as currently drawn is worth 35%. Beth is in an advantageous position, being the last to make a decision. Even though Beth values this piece at 35%, she can cut a very small amount and still lay claim to it. So Beth barely cuts the piece, resulting in a piece (blue) that is essentially worth 35% to her.



Last diminisher example...

Round 3. At this point, Bob and Marjo are the only players without a piece. Since there are two of them, they can finish the division using the divider-chooser method. They flip a coin, and determine that Marjo will be the divider. Marjo draws a line dividing the remainder of the state into two pieces. Bob chooses the Eastern piece, leaving Marjo with the Western half.



§3.4 – Envy free cake division ($N = 3$)

Step 1. A cuts into $1/3$'s

Step 2. B "trims" the largest piece (B's perspective) so that it's now equal to the second largest piece.

Step 3. Let C pick a piece, then B, then A. The game requires B to take the trimmed piece if C does not. Out of Players B and C, let T be the player who took the trimmed piece, and let T^* be the player who didn't.

Step 4. To deal with the trimmings, let T^* cut them into $1/3$'s

Step 5. Let the players pick from the remaining pieces in the order T, A, T^*.

Voting theory

Why do we need a mathematical theory about something so simple as voting?

How difficult could it be to find a simple, fair and consistent procedure for determining the outcome of an election?

When there are only two candidates (or alternatives) it is easy. How?

As we'll see, it is very different when there are more than two candidates and we wish to rank each of them in order of preference.

The Family Holiday

Suppose there is a family deciding where to take a holiday:

Dad loves the outback but hates the beach (too sandy!). He would enjoy a trip to the QLD theme parks but not as much as the outback.
Mum loves the beach but hates the outback. She wouldn't mind the theme parks but would prefer the beach.
Like her Mum, Alice has her heart set on the beach and doesn't care at all for the outback. The theme parks are ok in her eyes too.
Harry loves the theme parks more than anything! He doesn't really like the outback but can't stand the beach.

If the family can only choose one of the beach, the theme parks or the outback, where should they go?

For the previous example

The set up: We have alternatives, and voters who rank the alternatives. Dad's rankings of alternatives, $\{O, T, B\}$ (outback, theme parks, beach), can be indicated by the notation

$$O > T > B$$

with the outback the 1st choice, theme parks the 2nd choice, and the beach the 3rd choice. Pairwise, this means $O > T$, $T > B$, $O > B$ (transitivity).

Representing the whole family we have

$$
\begin{aligned}
D: &\quad O > T > B \\
M: &\quad B > T > O \\
A: &\quad B > T > O \\
H: &\quad T > O > B
\end{aligned}
$$

One way to choose a destination is to ignore all but the first choice. Two common examples are

- Majority rule The alternative which (who) receives more than half the votes is elected.
- Plurality method The alternative which (who) receives the largest number of votes is elected.

Do we have a winning location for the family holiday using the majority rule? How about the plurality method?

By using the plurality method Dad and Harry end up going on a holiday they hate! Is there a better option?

In this topic we will consider 5 different voting methods (although there are many more). We will learn about how they work and examine the strengths and weaknesses of each.

More specifically we will investigate how each voting system stacks up against the 4 fairness criteria.

We will answer the question: Is there a perfect voting system?

Before we state the four fairness criteria we consider two simpler desirable properties.

Definition (Anonymous Voting System)

A voting system is said to be anonymous if it treats all voters equally.

Definition (Neutral Voting System)

A voting system is said to be neutral if it treats all candidates equally.

Exercises

We used the plurality method for determining the family holiday. Is this method neutral and anonymous?

Solution

The system is anonymous since if any two voters traded their voting preferences the outcome would remain the same.

The system is neutral since if each person switched their preferences in the same way (for example, beach $\rightarrow$ outback, outback $\rightarrow$ beach, theme park votes unchanged) the outcome would change in the same way (outback would now be the winner).

§4 – The four fairness criteria for voting systems

Over the years, those who study voting theory have proposed many criteria which most people would expect a “fair” preferential election method to satisfy. In this course, we will consider only four Fairness Criteria

Four fairness criteria

1. The majority criterion
2. The Condorcet criterion
3. The monotonicity criterion
4. The independence of irrelevant alternatives criterion

We begin by stating the four fairness criteria. After this we introduce voting systems and analyse them according to these criteria.

The Majority Criterion

Any candidate receiving a majority of first place votes should be the winner.

The Condorcet Criterion

Any candidate who wins head-to-head matchups with all other candidates should be the winner.

The Monotonicity Criterion

A winning candidate should not become a losing candidate if their ranking is increased by any individual voter.

The Independence of Irrelevant Alternatives Criterion

Suppose candidate X is the winning candidate, then a new alternative Y is inserted in all individual rankings. Then only X or Y should be able to win the election.

Anecdote (due to philosopher Sidney Morgenbesser)
Sam decides to order dessert. There are two choices: apple or blueberry pie. After choosing apple pie, the waiter returns a minute later to say that there is a third choice, cherry pie. Sam replies, in that case I'll have blueberry pie.

§5 – Voting System 1: The Borda count

Each candidate is given a score equal to the number of rankings above the other candidates, minus the total number of rankings below the other candidates.

The candidates are then ranked according to their Borda count scores, with the highest score being elected.

Let's return to the family holiday problem

$$\begin{aligned} D:&\quad O > T > B \\ M:&\quad B > T > O \\ A:&\quad B > T > O \\ H:&\quad T > O > B \end{aligned}$$

O is ranked above other alternatives 3 times and under another alternative 5 times. Hence the Borda count for O is $3 - 5 = -2$.

We also have $T:\ 5 - 3 = 2$ and $B:\ 4 - 4 = 0$.

T has the highest Borda count score and therefore the theme park is the winning destination.

Exercise

Suppose there are 32 voters, and the rankings are

$$\begin{aligned} 14: &\quad A > B > C \\ 11: &\quad B > C > A \\ 7: &\quad C > A > B \end{aligned}$$

Determine the winner.

Note: $14:\ A > B > C$ indicates that 14 people ordered the candidates $A > B > C$.

Solution

A is ranked over another candidate 35 times (21 over B, 14 over C), and under another candidate 29 times (11 under B and 18 under C). Hence the Borda count for A is $35 - 29 = 6$.

For B: $14 + 22 - 14 - 14 = 8$. For C: $11 + 14 - 28 - 11 = -14$.

Hence B is declared the winner, with A the runner up and C third.

For a single voter, the rule for the scores in the Borda count with n candidates is equal to a candidate getting $-(n-1)$ points for a last place vote, $-(n-1)+2$ points for a second last place vote, $-(n-1)+4$ points for a third last place vote etc.

In compiling the scores, we might as well ignore the $-(n-1)$'s which then means scoring each candidate $0, 2, 4, \ldots, 2(n-1)$ from worst to best. But these are all even numbers, so we might as well take out a factor of 2 and add 1 to get $1, 2, \ldots, n$ for the scores of the worst up to best candidate. It is this scoring system which is often used to implement the Borda count in committees. It will be distinguished from the original Borda count by calling it the Borda count committee scoring system.

Exercises

1. Using the Borda count committee scoring system, tally up the scores corresponding to the earlier example: There are 32 voters, and the rankings are

$$\begin{aligned} 14 &: A > B > C \\ 11 &: B > C > A \\ 7 &: C > A > B \end{aligned}$$

2. Suppose there are N voters and c candidates. Let S_i denote the Borda count score for candidate i (using the standard scoring method). Let A_i denote the number of times candidate i is ranked ahead of other candidates. Show

$$S_i = 2A_i - (c-1)N.$$

3. Recall what it means for a voting system to satisfy the monotonicity criterion, and show that the Borda count method satisfies it.

Exercises continued...

4. Explain why the election consisting of 100 voters and 3 candidates, with rankings $63 : E > F > G$ and $37 : F > G > E$, demonstrates that the Borda count does not satisfy the majority criterion.

§5.1 – Borda Count: Condorcet winner criterion

So far we see that Borda count satisfies the monotonicity criterion, but does not satisfy the majority criterion. How about the Condercet criterion?

Definition (Condorcet Winner)

A candidate in an election who would defeat every other candidate in a head-to-head contest is called a Condorcet winner.

Recall that a voting system that always elects a Condorcet winner, when one exists, is said to satisfy the Condorcet winner criterion.

From the voter rankings we can construct a table to display the implied results of the head-to-head contests.

We choose to use the rows as “for” and the columns as “against”. To determine the head-to-head winner, the entry in positions (ij) and (ji) need to be compared (they also need to add up to the total number of voters).

For example, the data of the previous exercise gives

	E	F	G
E	_	63	63
F	37	_	100
G	37	0	_

In the above example, according to the Borda count F was the winner, so by the method of counter example we have shown that the Borda count does not satisfy the Condorcet winner criterion.

It follows immediately from the definitions that an election satisfying the majority criterion will always elect the Condorcet winner, as was suggested by the previous example.

But the Condorcet winner need not be ranked 1st by a majority of voters: Consider the example $1 : A > B > C$, $2 : B > C > A$, $1 : C > B > A$.

The majority criterion is therefore a stronger criterion than the Condorcet winner criterion.

Exercises

1. For the head to head data

	A	B	C	D
A	–	24	14	19
B	12	–	23	20
C	22	13	–	21
D	17	16	15	–

determine the head to head winners. Is there a Condorcet winner?

2. Construct an example of an election with three candidates and three voters in which there is no Condorcet winner.

3. Argue why it is not possible to have two Condorcet winners.

§5.2 – Independence of Irrelevant Alternatives

Finally we consider independence of irrelevant alternatives.

Recall that a voting system is said to satisfy the independence of irrelevant alternatives (IIA) criterion if whenever a particular candidate X wins the election, and then a new alternative Y is inserted into all the individual rankings, only X or Y can now be the winner.

Do you think the Borda count method will satisfy the IIA criterion?

Exercise

Suppose there is a contest for the best meat pie and there are three competitors, Anne, Beau, Cole. The 15 voters have the following preferences

$$\begin{aligned} 7 &: A > B > C \\ 6 &: B > A > C \\ 2 &: A > C > B \end{aligned}$$

Dom enters late and the preferences become

$$\begin{aligned} 7 &: A > B > D > C \\ 6 &: B > D > A > C \\ 2 &: D > A > C > B \end{aligned}$$

Using the Borda count method, determine the winner of each contest.

What can you conclude about the Borda count method in relation to the IIA criterion?

§5.3 – Properties of the Borda count

Let's pause to summarise how the Borda count rates against anonymity, neutrality and the four fairness criteria.

Good	Bad
neutral	no majority criterion
anonymous	no Condorcet winner criterion
monotonicity criterion	no IIA criterion

§6 – Voting System 2: Nanson method

We can modify the Borda count method so that it satisfies the Condorcet winner criterion. We call this modified method the Nanson method. We refine the Borda count as follows:

Step 1. Eliminate all candidates with zero or negative Borda count.

Step 2. Recalculate the Borda count and repeat until only one candidate remains.

Exercises

For the voter rankings

$$
\begin{aligned}
40: &\quad D > A > B > C \\
26: &\quad B > C > A > D \\
24: &\quad C > A > B > D \\
10: &\quad C > D > B > A
\end{aligned}
$$

use the Nanson method to determine the result of the election.

§6.1 – Condorcet winner criterion

The reason that the Nanson method satisfies the Condorcet winner criterion is that a candidate with a zero or negative Borda count cannot be a Condorcet winner.

This is because a Condorcet winner must win all head to head contests. In other words, a Condorcet winner must be ranked ahead of other candidates more often than ranked below, which in turn means that the Borda count must be positive.

§6.2 – Monotonicity criterion

But do our new rules satisfy the monotonicity criterion?

Yes, for 3 candidates. But for 4 candidates, consider the data from the above exercise

$$\begin{aligned} 40: &\quad D > A > B > C \\ 26: &\quad B > C > A > D \\ 24: &\quad C > A > B > D \\ 10: &\quad C > D > B > A \end{aligned}$$

Notice the effect of improving the Nanson winner A's position by replacing $10 : C > D > B > A$ by $10 : C > D > A > B$.

Exercise

Prove that the Nanson method satisfies the majority criterion. (Hint: Why will the candidate with the majority of first place votes never have a negative Borda count score?)

§6.4 – IIA criterion

Exercise

1. Who is the winner in the following election using majority rule?

$$\begin{aligned} 3 &: A > B \\ 2 &: B > A \end{aligned}$$

2. Who is the winner in the following election using the Nanson method?

$$\begin{aligned} 2 &: B > C > A \\ 2 &: A > B > C \\ 1 &: C > A > B \end{aligned}$$

3. What do 1. and 2. tell us about the Nanson method?

§6.5 – Properties of the Nanson method

Let's pause again to summarise how the Nanson method rates against anonymity, neutrality and the four fairness criteria.

Good	Bad
neutral	no monotonicity criterion
anonymous	no IIA criterion
majority criterion	
Condorcet winner criterion	

Nanson's method was formerly used by the Anglican Diocese of Melbourne and in the election of members of the University Council of the University of Adelaide. It was used by the University of Melbourne until 1983.

§7 – Voting System 3: Instant run-off

The next voting system we consider is called instant run-off. This system is used in the Australian House of Representatives elections.

To conduct an instant run-off election:

1. Each voter in the election submits a preference order.
2. If a candidate has majority 1st place votes, this candidate wins. Otherwise, go to Step 3.
3. The candidate with the least number of 1st place votes is eliminated from each voter's preference order, and the remaining candidates are moved up on each preference order, yielding a new collection of preferences for the election. Go to Step 2.

Example

Suppose we have 17 voters with the following preferences:

$$\begin{aligned} 6:&\quad G > M > D > S \\ 5:&\quad M > G > D > S \\ 4:&\quad D > S > M > G \\ 2:&\quad S > D > G > M \end{aligned}$$

Eliminate S. Then get

$$\begin{aligned} 6:&\quad G > M > D \\ 5:&\quad M > G > D \\ 4:&\quad D > M > G \\ 2:&\quad D > G > M \end{aligned}$$

Now eliminate M, since both G and D have six 1st place votes:

$$\begin{aligned} 6: &\quad G > D \\ 5: &\quad G > D \\ 4: &\quad D > G \\ 2: &\quad D > G \end{aligned}$$

Hence G is the winner.

Exercise

Verify that instant run-off satisfies the conditions of neutrality and being anonymous.

§7.1 – Monotonicity criterion

In the previous example, suppose

$$2: \quad S > D > G > M$$

is changed to

$$2: \quad S > G > D > M$$

which improves the ranking of G in the individual votes.

According to instant run-off, after eliminating S we have

$$\begin{aligned} 6:&\quad G > M > D \\ 5:&\quad M > G > D \\ 4:&\quad D > M > G \\ 2:&\quad G > D > M \end{aligned}$$

Now eliminate D (previously we eliminated M)

$$\begin{aligned} 6: &\quad G > M \\ 5: &\quad M > G \\ 4: &\quad M > G \\ 2: &\quad G > M \end{aligned}$$

Hence M is now the winner.

This example shows that instant run-off does not satisfy the monotonicity criterion.

§7.2 – Majority criterion

Exercise

Suppose at some stage of an instant run-off election one candidate receives a majority of 1st place votes.

Show that this candidate will win the election.

What does this tell us about the majority criterion?

§7.3 – Condorcet criterion

What about the Condorcet criterion in relation to an instant run-off election? Consider

$$\begin{aligned} 1: &\quad A > B > C \\ 2: &\quad B > A > C \\ 2: &\quad C > A > B \end{aligned}$$

Here we have

$$\begin{array}{cccc} & A & B & C \\ A & \_ & 3 & 3 \\ B & 2 & \_ & 3 \\ C & 2 & 2 & \_ \end{array}$$

and so A is the Condorcet winner.

But according to the instant run-off procedure, A is eliminated in the first round, leaving

$$\begin{aligned} 1: &\quad B > C \\ 2: &\quad B > C \\ 2: &\quad C > B \end{aligned}$$

giving B as the winner.

Therefore instant run-off does not satisfy the Condorcet criterion.

§7.4 – IIA criterion

Suppose we have

$$\begin{aligned} 3:&\quad A > B \\ 2:&\quad B > A \end{aligned}$$

Suppose now candidate C enters the election.

$$\begin{aligned} 2:&\quad B > C > A \\ 2:&\quad C > A > B \\ 1:&\quad A > B > C \end{aligned}$$

Use instant run-off, to determine the winner in both cases.
What do you conclude?

§7.5 – Properties of an instant run-off election

We now summarise the desirable properties of instant run-off.

Good	Bad
neutral	no monotonicity
anonymous	no Condorcet winner criterion
majority criterion	no IIA criterion

§8 – Voting System 4: Smith set

Neither instant run-off, nor Borda count nor Nanson method simultaneously satisfy monotonicity and the Condorcet winner criterion.

One way to construct a voting system which satisfies both criteria is to use the idea of a Smith set.

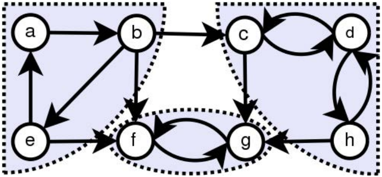
Definition (Smith set)

The Smith set is the smallest set of candidates such that every candidate inside the set beats every candidate outside the set.

Remark: In the case that there is a Condorcet winner, the Smith set consists of just that candidate. Furthermore, a Smith set cannot have exactly two members, except in a special circumstance (which is?).

In graph theoretic terms, the Smith set is a maximal strongly connected component of smallest size, where a directed edge represents one candidate beating another.

Since the graph of a head-to-head contest is a clique (when ignoring the arrows on the edges), the strongly connected components will also be cliques.



In the above example we have shown the strongly connected components. However, this graph is not a head-to-head contest graph. Why?

Example

Find the Smith Set in the following head-to-head table:

	A	B	C	D
A	–	24	14	19
B	12	–	23	20
C	22	13	–	21
D	17	16	15	–

By marking in the head-to-head winners we see that the D is beaten by all other candidates, and the Smith set is $\{A, B, C\}$.

Definition: A weak Condorcet winner is a candidate that wins or draws in head-to-head comparisons with every other candidate.

The following procedure extends the Smith set into a voting system that is monotone and satisfies the Condorcet criterion:

1. Find the Smith set. Eliminate the candidates not in the set. If there is a (strong) Condorcet winner, choose this as the overall winner and stop.
2. Find the head-to-head pair with the smallest difference. Label this pair as a draw.
3. Check if there is a weak Condorcet winner. If there is, choose this candidate as the overall winner. Otherwise return to the previous step.

Why might this method not be in popular use?

Exercise

Determine the winner of the following election according to the Smith method.

	A	B	C	D
A	–	24	14	19
B	12	–	23	20
C	22	13	–	21
D	17	16	15	–

Fairness criteria for Smith set

Monotonicity: Ranking a candidate higher will only increase their head to head wins, making the voting system monotone.

Condorcet criterion: If a candidate won all head to heads they would immediately become the winner using the Smith method.

Majority criterion: If one candidate receives the majority of the votes they immediately win all head to heads and become the winner.

IIA: A new candidate can not effect the head to head contests between the current winning candidate and the losing candidates.

Fairness criteria for Smith set

We now summarise the desirable properties of the Smith set method.

Good	Bad
neutral	not simple enough (for politicians)
anonymous	there may no winner
monotonicity criterion	
majority criterion	
Condorcet criterion	
IIA criterion	

§9 – Voting System 5: Approval voting

So far we have considered preferential voting only. Approval voting is a non-preferential voting method.

The rule:

1. each voter either approves or disapproves each candidate in the election.
2. the preference order is then determined by the number of approval votes that each candidate receives, starting with the candidate who receives the most approval votes, and ending with the candidate who receives the fewest.

This is used by the American Mathematical Society and the Institute for Operations Research and Management Sciences in their elections for offices.


Volksabstimmung und Großdeutscher Reichstag

Stimmzettel

Bist Du mit der am 13. März 1938 vollzogenen

Wiedervereinigung Österreichs mit dem Deutschen Reich

einverstanden und stimmst Du für die Liste unseres Führers

Adolf Hitler?

Ja

Nein

It is convenient to score an approved candidate one and a disapproved candidate zero.

Example

	c_1	c_2	c_3
v_1	1	0	1
v_2	0	1	1
v_3	1	0	1
v_4	0	1	0

With 3 votes of approval, which is the most, candidate c_3 is the winner.

Exercises

Show that approval voting

1. is anonymous
2. is neutral
3. satisfies the monotonicity criterion
4. satisfies the IIA criterion

Other advantages of approval voting are:

1. It gives voters more flexible options
2. It helps elect the strongest candidate
3. It increases voter turn out
4. It gives minority candidates their proper due
5. It is very practical

§9.1 – Remaining fairness criteria

Condorcet criterion

Approval voting does not satisfy the Condorcet winner criterion as there is no one-to-one correspondence between rankings and approvals. For example, a voter may prefer A to B but still approve both of them.

Majority criterion

Similarly to the previous point, the criterion "the candidate receiving the majority of first place votes", doesn't align with approval voting, since in approval voting there is generally no such thing as a first place vote.

This emphasizes that approval voting is fundamentally different to preferential voting.

§9.2 – Properties of approval voting

We now summarise the desirable properties of approval voting.

Good	Bad
neutral	no majority criterion
anonymous	no Condorcet winner criterion
monotonicity	
IIA criterion	

§9.3 – Range voting

To restore the property of the Condercet winner criterion, approval voting can be extended to range voting (score candidates continuously from 0 to 1).

Observe, though, that the majority criterion is still not satisfied. Eg. for candidates X, Y, Z with:

$$\begin{array}{ll} 51: & X = 0.99, Y = 0.5, Z = 0 \\ 49: & Y = 0.99, Z = 0.1, X = 0 \end{array}$$

There is a movement to promote range voting: rangevoting.org

What are some obvious difficulties with range voting?

§10 – A perfect voting system?

Here we end our study of voting systems from the viewpoint of desirable properties.

What we've found is that the most common voting systems are defective as far as possessing certain desirable properties.

At the beginning of the topic we asked "Is there a perfect voting system?"

The answer is no. It was found by Kenneth Arrow in 1952 that there is no consistent method of making a fair choice among three or more candidates using a preferential voting method. This result is known as Arrow's theorem.

§11 – Weighted voting systems

Finally, we consider cases when not all voters are treated equally. This happens in a dictatorship, and also on company boards where board members represent certain percentage of ownership of the company.

A weighted voting system is used to make a decision on a yes/no question, or motion. Weighted voting systems are characterised by:

- a collection of voters $v_1, \ldots, v_n$
- a collection of weights $w_1, \ldots, w_n$ associated with each voter, thought of as the number of votes held by that voter
- a quota–this is a positive number q such that a motion will pass if the sum of the weights of all the voters who vote "yes" on the motion equals or exceeds q, and will fail otherwise

Notation: If voter v_i has w_i votes $(i = 1, \dots, n)$ and the quota is q, we write $[q : w_1, w_2, \dots, w_n]$.

Exercise

Suppose there are three voters with weights $w_1 = 101$, $w_2 = 97$, $w_3 = 2$. What is the significance of a quota of 101?

Definitions

- A coalition is a collection of voters (possibly empty) in a weighted voting system.
- The weight of a coalition is the sum of all the weights of all the votes in the coalition.
- A winning coalition is a coalition that can single-handedly force a motion to pass.
- A losing coalition is a coalition that cannot single-handedly force a motion to pass.
- A minimal winning coalition is a winning coalition that would become a losing coalition if a single voter were removed from it.

Example

In the above exercise, $\{v_1\}$ is a winning coalition, and also a minimal winning coalition. $\{v_2, v_3\}$ is a losing coalition.

Exercise

What are the minimal winning coalition in the above exercise with a quota of 99? What are all the winning coalitions?

Solution

Again, $\{v_1\}$ is a minimal winning coalition. So is $\{v_2, v_3\}$. The winning coalitions are any coalition that contains $\{v_1\}$ or $\{v_2, v_3\}$ as a subset.

Definitions

The following are defined for all yes/no voting systems.

- a voter who is present in every winning coalition and absent from every losing coalition is called a dictator,
- a voter who is present in every winning coalition is said to have veto power,
- a voter who does not belong to any minimal winning coalition is called a dummy.

Exercise

The United Nations Security Council consists of 15 representatives, one from each of 15 countries. Five representatives (China, France, Great Britain, Russia and United States) are considered permanent members, and the remaining 10 change from year to year and so are non-permanent. Passage of motion requires votes in favour from all 5 permanent members and at least 4 non-permanent members.

By finding the smallest x and corresponding quota q, show that this can be viewed as a weighted voting system, with $w_{\textbf{perm}} = x$, $w_{\textbf{non perm}} = 1$.

§11.1 – Power

A concept of relevance to weighted voting systems is that of power: who has it, and how much do they have. Two ways of defining power will be discussed, the Banzhaf power index, and the Shapley-Shubik power index.

Banzhaf power index — definitions

- A candidate whose desertion of a winning coalition turns it into a losing one is called a critical voter.
- The Banzhaf power index, b_i, of voter i is defined as

$$b_i = \frac{\#\text{ of times voter } i \text{ is critical}}{\#\text{ of times any voter is critical}}.$$

- The closer b_i is to 1, the more powerful the candidate.

Example

Determine the Banzhaf power indices for the weighted voting system $[6 : 4, 3, 2]$.

Shapley-Shubik power index — definitions

- A sequential coalition is one in which the players are listed in the order that they entered the coalition (this means that there are $n!$ sequential coalitions, i.e., the number of ways to order the numbers $1, 2, \ldots, n$).
- A pivotal voter is the one in a sequential coalition who can change it from a losing to a winning one. That is, the first voter in the sequence that raises the cumulative sum to q or more, is the pivotal voter.
- The Shapley-Shubik power index p_i of voter i is defined by

$$p_i = \frac{\text{\# of times voter } i \text{ is pivotal}}{n!}$$

Example

Determine the Shapley-Shubik power indices for the weighted voting system $[6 : 4, 3, 2]$.