THE UNIVERSITY OF MELBOURNE

Semester 2 Assessment, 2015

School of Mathematics and Statistics

MAST20018 Discrete Mathematics and Operations Research

Writing time: 3 hours

Reading time: 15 minutes

This is an OPEN BOOK exam

This paper consists of 7 pages (including this page)

Authorised materials:

- One double sided A4 page of handwritten notes.
- Hand-held calculator.

Instructions to Students

- You may remove this question paper at the conclusion of the examination
- This examination consists of **10** questions. You should attempt all questions.
- The total number of marks is **130**. Marks for individual questions are shown.

Instructions to Invigilators

- Students may remove this question paper at the conclusion of the examination
- Each candidate should be issued with an examination booklet, and with further booklets as needed.

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Question 1 (6 marks)

Ozzy Mikks produces both interior and exterior paints from two raw materials, $M1$ and $M2$. The following table provides the data of the problem:

	Tonnes of raw material per tonne of		
	Exterior paint	*Interior paint*	*Availability (tonnes/day)*
Raw material $M1$	6	4	24
Raw material $M2$	1	2	6
Profit per tonne (\$1000)	5	4	

A market survey indicates that the daily demand for interior paint cannot exceed that of exterior paint by more than one tonne. Also, the maximum daily demand of interior paint is 2 tonnes. Ozzy Mikks wants to determine the optimum product combination of interior and exterior paints that maximises the total daily profit. Formulate, but do not solve, the problem as an LP.

Question 2 (12 marks)

(a) Find all basic feasible solutions of the following LP using the graphical method [**6 marks**]:

$$\max z = 2x_1 + 3x_2$$

`such that`

$$\begin{aligned} 2x_1 + x_2 &\leq 4 \\ x_1 + 2x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(b) Convert the LP from (a) into canonical form. [**3 marks**]

(c) How would you use your answer in (b) to find the basic feasible solutions without using the graphical method? Note: just state the method here; do not recalculate the basic feasible solutions. [**3 marks**]

Question 3 (14 marks)

(a) Convert the following LP to canonical form [**5 marks**]:

$$\min z = 2x_1 + 3x_2$$

`such that`

$$\begin{aligned} \frac{1}{2}x_1 + \frac{1}{4}x_2 &\leq 4 \\ x_1 + 3x_2 &\geq 36 \\ x_1 + x_2 &= 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(b) Implement Phase 1 of the Two-phase Simplex method for the LP from (a). [**7 marks**]

(c) What do you conclude from (b)? Why? [**2 marks**]

Question 4 (10 marks)

Consider the following LP:

$$\max z = x_1 - x_2 + 2x_3$$

`such that`

$$\begin{aligned} x_1 + x_2 + 3x_3 &\leq 15 \\ 2x_1 - x_2 + x_3 &\leq 2 \\ -x_1 + x_2 + x_3 &\leq 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Let x_4, x_5 and x_6 denote the slack variables for the respective constraints. After applying the Simplex method, a portion of the final tableau is as follows:

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_4				1	-1	-2	
x_3				0	$1/2$	$1/2$	
x_2				0	$-1/2$	$1/2$	
z				0	$3/2$	$1/2$	

Use the algebra of the Simplex method in order to complete the tableau. Note, you are NOT required to complete the RHS column.

Question 5 (17 marks)

Consider the following LP and its optimal tableau:

$$\max z = 3x_1 + x_2 - x_3$$

`such that`

$$\begin{aligned} 2x_1 + x_2 + x_3 &\leq 8 \\ 4x_1 + x_2 - x_3 &\leq 10 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

BV	x_1	x_2	x_3	s_1	s_2	RHS
x_2	0	1	3	2	-1	6
x_1	1	0	-1	$-1/2$	$1/2$	1
z	0	0	1	$1/2$	$1/2$	9

(a) Find the dual of this LP and its optimal solution without doing any new Simplex operations. [**7 marks**]

(b) Confirm the complementary slackness conditions for the LP and its dual. [**5 marks**]

(c) Find the range of values of b_2 for which the current basis remains optimal. [**5 marks**]

Question 6 (12 marks)

Consider the following problem:

$$\max z = 10x_1 + 4x_2$$

`such that`

$$\begin{aligned} 3x_1 + x_2 &\leq 30 \\ 2x_1 + x_2 &\leq 25 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Let x_3 and x_4 denote the slack variables of the respective functional constraints. After applying the Simplex method, the final tableau is

BV	x_1	x_2	x_3	x_4	RHS
x_2	0	1	-2	3	15
x_1	1	0	1	-1	5
z	0	0	2	2	110

Use the above information to solve the following parametric program for $\theta \geq 0$:

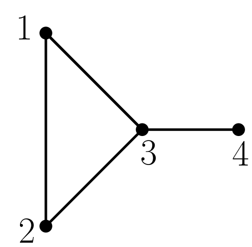
$$\max z = (10 - 2\theta)x_1 + (4 + \theta)x_2$$

`such that`

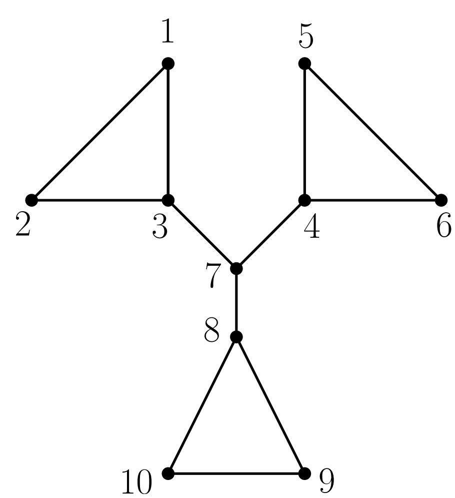
$$\begin{aligned} 3x_1 + x_2 &\leq 30 \\ 2x_1 + x_2 &\leq 25 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Question 7 (10 marks)

(a) Calculate the node-betweenness centrality for node 3 in the following network [**3 marks**]:



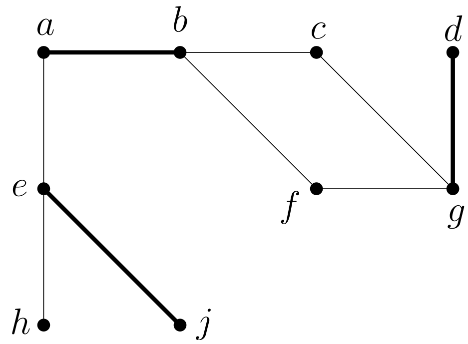
(b) Construct the dendogram for the following graph using the Girvan-Newman algorithm. Note: vertices have been labelled from 1 to 10. All steps of the algorithm must be shown, but calculations of edge-betweenness centrality are not necessary. [**7 marks**]



Question 8 (25 marks)

(a) Let G be any graph. The size of a largest matching in G is denoted by $\nu(G)$, and the size of a smallest node cover of G is denoted by $\tau(G)$. Prove that $\nu(G) \leq \tau(G)$. [**4 marks**]

(b) Consider the following graph



i. Find an optimal vertex colouring of the graph. [**2 marks**]

ii. To which of the following classes does the graph belong: chordal graphs, interval graphs, perfect graphs, cliques, bipartite graphs, trees? Give reasons for your answers. [**6 marks**]

iii. How do we know that the given matching $M = \{(a,b),(e,j),(g,d)\}$ (depicted with bold edges) is a maximum? Which theorem did you use? [**2 marks**]

iv. Find a minimum node cover of the graph. By which theorem do you know that your node cover is optimal? [**3 marks**]

v. What is the minimum number of colours needed for an edge-colouring of the graph. How do you know? [**2 marks**]

(c) Suppose you are organising a ballroom dancing event for 6 participants. You have been tasked to match everyone with a partner for the event. There are three males, $M1, M2, M3$ and three females $F1, F2, F3$. By carefully observing the participants' dancing skills over the last couple of weeks, you have compiled a compatibility table as follows:

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 8 \\ 0 & 4 & 1 \end{bmatrix}$$

where the entry in row i and column j represents the compatibility of couple (F_i, M_j). The score ranges from 0 to 9, with 0 being the most compatible.

Using the algorithm you learnt in class, find an assignment of males to females that optimises the total compatibility for the ballroom dancing event. [**6 marks**]

Question 9 (15 marks)

There are four items I_1, I_2, I_3, I_4 to be distributed among three people A, B, C using the method of sealed bids. The bids are given by the following table.

bids	A	B	C
I_1	$\$10,000$	$\$4,000$	$\$7,000$
I_2	$\$2,000$	$\$1,000$	$\$4,000$
I_3	$\$500$	$\$1,500$	$\$2,000$
I_4	$\$800$	$\$2,000$	$\$1,000$

(a) Extend the table to include the fair share of each player, and their contribution to the pot. [**2 marks**]

(b) Compute the excess in the pot. [**1 marks**]

(c) Summarize the allocation to each player. [**3 marks**]

(d) Show, in detail and with reference to every player, that this allocation is envy free. [**9 marks**]

Question 10 (9 marks)

It has been suggested that range voting should replace the preferential voting system currently used in Australian elections.

(a) Describe the method of range voting. [**3 marks**]

(b) What are the benefits of range voting? [**4 marks**]

(c) What would be the drawbacks of implementing range voting in Australia? [**2 marks**]

End of Exam—Total Available Marks = 130