Semester 2 Assessment, 2016

School of Mathematics and Statistics

MAST20018 Discrete Mathematics and Operations Research

Writing time: 3 hours

Reading time: 15 minutes

This is an OPEN BOOK exam

This paper consists of 8 pages (including this page)

Authorised materials:

- One double sided A4 page of handwritten notes.
- Hand-held calculator.

Instructions to Students

- You may remove this question paper at the conclusion of the examination
- This examination consists of **10** questions. You should attempt all questions.
- The total number of marks is **100**. Marks for individual questions are shown.

Instructions to Invigilators

- Students may remove this question paper at the conclusion of the examination
- Each candidate should be issued with an examination booklet, and with further booklets as needed.

Question 1 (6 marks)

Consider the following LP:

$$\max z = \frac{1}{2}x + \frac{3}{2}y$$

$$\begin{aligned} x + 3y &\leq 6 \\ x + y &\geq 4 \\ x, y &\geq 0 \end{aligned}$$

(a) Sketch the set S of feasible solutions given by the constraints.

(b) List the extreme points of S.

(c) Find the optimal solution(s) for the LP.

Question 2 (8 marks)

(a) Convert the following LP to canonical form:

$$\max z = x_1 + 2x_2 + x_4$$

$$\begin{aligned} x_1 + 3x_2 - x_3 + x_4 &\leq 5 \\ x_1 + 7x_2 + x_3 \qquad &\geq 4 \\ 4x_1 + 2x_2 \qquad + x_4 &= 3 \\ x_i &\geq 0, \quad i = 1, 2, 3, 4 \end{aligned}$$

(b) Set up, but do not solve, the initial Simplex tableau for the LP from (a) for the Two-phase Simplex method.

Question 3 (12 marks)

A local health food store packages three types of snack foods – Chewy, Crunchy, and Nutty – by mixing sunflower seeds, raisins and peanuts. The specifications for each mixture are given in the accompanying table.

Mixture	*Sunflower seeds*	*Raisins*	*Peanuts*	*Selling price (per kg)*
Chewy		At least 60%	At most 20%	\$2.00
Crunchy	At least 60%			\$1.60
Nutty	At most 20%		At least 60%	\$1.20

The suppliers of the ingredients can deliver each week at most 100 kg of sunflower seeds at \$1.00/kg, 80 kg of raisins at \$1.50/kg, and 60 kg of peanuts at \$0.80/kg. The store would like to determine a mixing scheme that maximises its profit. The store sets up the following decision variables:

Let $x_{ij} :=$ the amount of the ith ingredient in the jth mixture (in kg), where

$$\begin{aligned} \text{Ingredient 1} &= \text{Sunflower seeds} \\ \text{Ingredient 2} &= \text{Raisins} \\ \text{Ingredient 3} &= \text{Peanuts} \\ \text{Mixture 1} &= \text{Chewy} \\ \text{Mixture 2} &= \text{Crunchy} \\ \text{Mixture 3} &= \text{Nutty} \end{aligned}$$

Therefore, in terms of the x_{ij}, the weekly amount paid by the store to the suppliers is

$$\sum_{j=1}^{3} x_{1j} + 1.5 \sum_{j=1}^{3} x_{2j} + 0.8 \sum_{j=1}^{3} x_{3j}$$

(a) Write down an expression, in terms of the x_{ij}, for the profit (revenue minus costs) of the store.

(b) Write down the three inequalities which specify the limitations on the amounts the suppliers can supply of each ingredient

(c) Write down the five inequalities which specify the requirements on the mixing ratios, as given in the above table.

Do NOT solve the linear program.

Question 4 (10 marks)

Consider the following LP:

$$\max z = 5x_1 + 2x_2 + 3x_3$$

$$\begin{aligned} x_1 + 5x_2 + 2x_3 &\leq b_1 \\ x_1 - 5x_2 - 6x_3 &\leq b_2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

where b_1 and b_2 are constants. For specific values of b_1 and b_2, the optimal tableau is

BV	x_1	x_2	x_3	s_1	s_2	RHS
x_1	1	b	2	1	0	30
s_2	0	c	-8	-1	1	10
z	0	a	7	d	e	150

where a, b, c, d and e are constants. Using the algebra of the Simplex method, and showing all working, determine

(a) The values of b_1 and b_2.

(b) The optimal dual solution.

(c) The values of a, b, and c.

Question 5 (10 marks)

(a) Suppose that some feasible linear programming problem has an unbounded solution. Explain why the dual of the program must be infeasible.

(b) Show, *without* using the Simplex method, that $x = (5/26, 5/2, 27/26)$ is an optimal solution to the following linear programming problem:

$$\min z = 9x_1 + 14x_2 + 7x_3$$

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &\leq 6 \\ 5x_1 + 4x_2 + x_3 &\leq 12 \\ 2x_2 \quad\quad &\leq 5 \\ x_i &\in \mathbb{R}, \quad i = 1, 2, 3 \end{aligned}$$

Question 6 (5 marks)

Consider the following problem:

$$\max z = x_1 + 2x_2 + x_3 + x_4$$

$$\begin{array}{rcl} 2x_1 + x_2 + 3x_3 + x_4 & \leq & 8 \\ 2x_1 + 3x_2 \qquad\quad + 4x_4 & \leq & 12 \\ 3x_1 + x_2 + 2x_3 \qquad\quad & \leq & 18 \\ x_i & \geq & 0, \quad 1 \leq i \leq 4 \end{array}$$

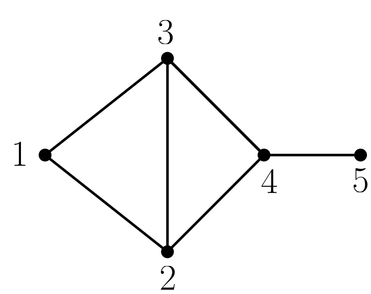
After adding slack variables x_5, x_6 and x_7 and solving by the Simplex method, we obtain the final tableau below

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_3	$4/9$	0	1	$-1/9$	$1/3$	$-1/9$	0	$4/3$
x_2	$2/3$	1	0	$4/3$	0	$1/3$	0	4
x_7	$13/9$	0	0	$-10/9$	$-2/3$	$-1/9$	1	$34/3$
z	$7/9$	0	0	$14/9$	$1/3$	$5/9$	0	$28/3$

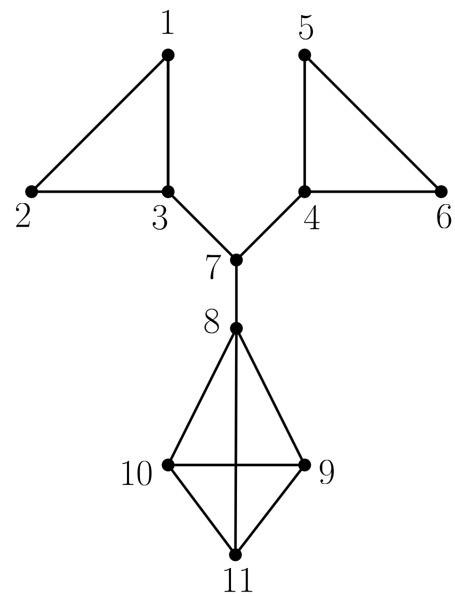
For cost coefficient c_2, calculate the range of values for which the above solution remains optimal.

Question 7 (10 marks)

(a) Calculate the node-betweenness centrality for node 4 in the following network:



(b) Construct the dendogram for the following graph using the Girvan-Newman algorithm. Note: vertices have been labelled from 1 to 11. All steps of the algorithm must be shown, but explicit calculations of edge-betweenness centrality are not necessary.

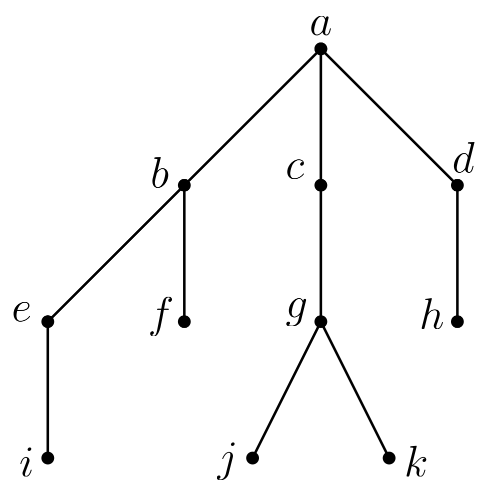


Question 8 (25 marks)

(a) Let G be any graph. A vertex v of G is called an *odd* vertex if the degree of v is odd. Otherwise, v is called an *even* vertex. Prove that any graph G has an even number of odd vertices.

(b) Provide a formal definition of bipartite graphs, and then prove that bipartite graphs cannot contain cycles of odd length.

(c) Consider the following rooted tree



i. Find an optimal vertex colouring of the tree.

ii. Starting with the matching $M = \{(e, b)\}$ in the tree, extend M into a matching of size two by first finding an augmenting path. Repeat this process until you have found a maximum matching in the tree. Explicitly show each step. Note, an edge joining two exposed nodes is classified (trivially) as an augmenting path.

(e) Sun City Hotel has received the following online bookings for the week of 26 December to 1 January.

Customer 1			Wed				
Customer 2	Mon			Thur			
Customer 3						Sat	Sun
Customer 4	Mon	Tue	Wed				
Customer 5					Fri	Sat	
Customer 6		Tue	Wed	Thur	Fri	Sat	Sun
Customer 7		Tue		Thur		Sat	

By using a vertex colouring model, find the minimum number of rooms the hotel needs to use for this week of bookings. Show all steps, including a drawing of the appropriate graph.

Question 9 (8 marks)

In June 1974, the United States and Panama agreed on a treaty after two rounds of negotiations. The following point allocations are taken from The Art and Science of Negotiation (1982) by Howard Raiffa:

Issue	United States	Panama
1 US defense rights	22	9
2 Use rights	22	15
3 Land and water	15	15
4 Expansion rights	14	3
5 Duration	11	15
6 Expansion routes	6	5
7 Compensation	4	11
8 Jurisdiction	2	7
9 Us military rights	2	7
10 Defense role of Panama	2	13

Use the Adjusted Winner method to determine which issue each party should be responsible for.

Question 10 (6 marks)

Suppose that in a survey 100 people are asked to rank their ice cream preferences with the results given below.

Flavour	33 Voters	3 Voters	10 Voters	20 Voters	7 Voters	27 Voters
Chocolate	1	1	2	3	2	3
Vanilla	2	3	1	1	3	2
Mocha	3	2	3	2	1	1

(a) Calculate the winning flavour according to the Borda count method.

(b) Calculate the winning flavour according to the Plurality criterion.

End of Exam—Total Available Marks = 100