Plan

- Review of Last lecture
- Section 13. Homomorphisms (continued)

Review of Last lecture

Definition 13.1. A map $\phi : G \to G'$ from a group G to a group G' is a **homomorphism** if it satisfies the property

$$\phi(ab) = \phi(a)\,\phi(b)$$

for all $a, b \in G$.

If group G has operation $*$ and group G' has operation $\star$, the condition for $\phi : G \to G'$ being a homomorphism is

$$\phi(a * b) = \phi(a) \star \phi(b)$$

for all $a, b \in G$.

If G has multiplication and G' has addition $+$, a map $\phi : G \to G'$ is a homomorphism if

$$\phi(ab) = \phi(a) + \phi(b)$$

for all $a, b \in G$.

Example. $(\mathbb{R}_{>0}, \cdot)$ is a group, $(\mathbb{R}, +)$ is a group,

$$\phi : \mathbb{R}_{>0} \to \mathbb{R}, \quad \phi(a) = \log a$$

is a homomorphism, because

$$\log(ab) = \log a + \log b$$

Example. If $f : G \to G'$ and $g : G' \to G''$ are homormorphisms of groups, then the composition $g \circ f : G \to G''$ is a homomorphism.

Example. Find a homomorphism $\phi : \mathbb{R}^* \to \mathbb{R}$ such that $\phi(2) = 2$
$f : \mathbb{R}^* \to \mathbb{R}_{>0}, \quad f(x) = |x|$ is a homomorphism

$g : \mathbb{R}_{>0} \to \mathbb{R}, g(y) = c \log y$ is a homomorphism for any $c > 0$.

$(g \circ f)(x) = c \log |x|$ is a homomorphism from $\mathbb{R}^*$ to $\mathbb{R}$,
$(g \circ f)(2) = 2, c \log 2 = 2$, so $c = \frac{2}{\log 2}$.

$$\phi : \mathbb{R}^* \to \mathbb{R}, \quad \phi(x) = \frac{2}{\log 2} \log |x|.$$

The answer is not unique.

Theorem

(Theorem 13.12). Let $\phi : G \to G'$ be a homomorphism of groups. Then
(1) If $e \in G$ is the identity element, $\phi(e) = e'$ is the identity element in G'.
(2) If $a \in G$, $\phi(a^{-1}) = \phi(a)^{-1}$.
(3) If $H \subseteq G$ is a subgroup, then $\phi(H)$ is a subgroup of G'.
(4) If $K' \subseteq G'$ is a subgroup, then $\phi^{-1}(K')$ is a subgroup of G.

A group homomorphism preserves the identity element and inverse.
A group homomorphism sends a subgroup to a subgroup.
The inverse image of a subgroup under a homomorphism is a subgroup.

Definition 13.13. Let $\phi : G \to G'$ be a homomorphism of groups. The subgroup

$$\phi^{-1}(e') = \{a \in G \mid \phi(a) = e'\}$$

is called the **kernel of** ϕ, denoted by $Ker(\phi)$.

Example. Let $\phi : \mathbb{C}^* \to \mathbb{C}^*$ be the homomorphism given by $\phi(a) = a^n$, where n is a fixed positive integer. Then

$$Ker(\phi) = \{a \in \mathbb{C}^* \mid \phi(a) = 1\} = \{a \in \mathbb{C}^* \mid a^n = 1\} = U_n$$

Theorem

(Theorem 13.15) Let $\phi : G \to G'$ be a group homomorphism, and let $H = Ker(\phi)$. Let $b \in G'$.

$$\phi^{-1}(b) = \{a \in G \mid \phi(a) = b\}$$

has two cases. Case 1. $\phi^{-1}(b) = \emptyset$. Case 2. $\phi^{-1}(b)$ is NOT empty, let $a \in \phi^{-1}(b)$, then

$$\phi^{-1}(b) = aH$$

Example. $\phi : \mathbb{R}^* \to \mathbb{R}^*$ given by $\phi(a) = |a|$ is a homomorphism.
$H = Ker(\phi) = \{1, -1\}$. For $b \in \mathbb{R}^*$,
if $b < 0$, then $\phi^{-1}(b) = \emptyset$,
if $b > 0$, then

$$\phi^{-1}(b) = \{b, -b\} = bH$$

Theorem 13.15 has the following analog in linear algebra. Let A be the $m \times n$ matrix over real numbers, $b \in \mathbb{R}^m$, consider the equation

$$Ax = b \tag{1}$$

where x is a variable column vector in $\mathbb{R}^n$.

Let V be the solution set of the corresponding homomogeneous equation $Ax = 0$,
Then equation (1) either has no solution, or every solution can be written as

$$h + v$$

where h is a given solution of (1) and v is an arbitrary solution of $Ax = 0$, in other words, if the solution of (1) exists, the solution set if

$$h + V$$

Example. $\phi : \mathbb{C}^* \to \mathbb{C}^*$ given by $\phi(z) = |z|$,

$$Ker(\phi) = \{z \in \mathbb{C}^* \mid |z| = 1\} = U$$

U, in the complex plane, is the circle centered at the origin with radius 1.

$\phi^{-1}(b)$ is empty if b is not a positive real number.

If $b > 0$, then $\phi^{-1}(b) = b\,Ker(\phi) = bU$

Corollary 13.18. A group homomorphism $\phi : G \to G'$ is a one-to-one map if and only of $Ker(\phi) = \{e\}$.

Example. If $\phi : G \to G'$ is a homomorphism of finite groups and it is surjective, prove that $|G'|$ is a divisor of $|G|$.

Let $H = Ker(\phi)$, $|H| = n$. Since ϕ is surjective, for every $b \in G'$, by Theorem 13.15, $\phi^{-1}(b) = aH$ is left coset of H for any a satisfying $\phi(a) = b$. So the map ϕ is n to 1 map, therefore $|G| = n|G'|$. This proves $|G'|$ is a divisor of $|G|$.

Definition 13.19. A subgroup H of a group G is called a **normal subgroup** if for all $g \in G$, $gH = Hg$.

If G is an abelian group, every subgroup is normal.

Example. A_n is a normal subgroup of S_n. For every $\sigma \in S_n$, if σ is even, σA_n and $A_n\sigma$ are both A_n, i.e., $\sigma A_n = A_n\sigma = A_n$.
If σ is odd, then σA_n is the set of all odd permutations, $A_n\sigma$ is also the set of all odd permutations. So $\sigma A_n = A_n\sigma$.

Lemma. A subgroup H of G is normal iff for every $g \in G$, $h \in H$, $ghg^{-1} \in H$,

Proof. If H is normal, for $g \in G$, $h \in H$, we want to prove $ghg^{-1} \in H$. Since H is normal, so $gH = Hg$, so $gh \in gH = Hg$, this means $gh = h'g$ for some $h' \in H$. So $ghg^{-1} = h'gg^{-1} = h' \in H$.

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Proof (continued) Conversely, if for every $g \in G$, $h \in H$, $ghg^{-1} \in H$, we want to prove H is normal , i.e., $aH = Ha$ for all $a \in G$.

For arbitrary $ah \in aH$, $h \in H$. $ah = aha^{-1}a$, since $aha^{-1} \in H$, so $ah = aha^{-1}a \in Ha$. This proves

$$aH \subseteq Ha.$$

For arbitrary $ha \in Ha$, $h \in H$, $ha = aa^{-1}ha$, since $a^{-1}ha \in H$, $ha = a(a^{-1}ha) \in aH$. This proves

$$Ha \subseteq aH.$$

So $aH = Ha$.

Example. Let $SL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) \,|\, \det(A) = 1\}$, Prove that $SL(n,\mathbb{R})$ is a normal subgroup of $GL(n,\mathbb{R})$.

First we consider group homomorphism

$$\phi : GL(n,\mathbb{R}) \to \mathbb{R}^*, \quad \phi(A) = \det(A)$$

We see that $SL(n,\mathbb{R}) = Ker(\phi)$, so $SL(n,\mathbb{R})$ is a subgroup. For arbitrary $g \in GL(n,\mathbb{R})$, $h \in SL(n,\mathbb{R})$, we want to show $ghg^{-1} \in SL(n,\mathbb{R})$ (with the previous lemma in mind).

$$\det(ghg^{-1}) = \det(g)\det(h)\det(g^{-1}) = \det(g)\det(h)\det(g)^{-1} = \det(h) = 1$$

This proves $ghg^{-1} \in SL(n,\mathbb{R})$. By Lemma, $SL(n,\mathbb{R})$ is a normal subgroup.

Corollary 13.20. If $\phi : G \to G'$ is a homomorphism of groups, then $Ker(\phi)$ is a normal subgroup of G.

Proof. By Lemma, it is enough to prove that for arbitrary $g \in G$, $h \in Ker(\phi)$, $ghg^{-1} \in Ker(\phi)$.

$$\phi(ghg^{-1}) = \phi(g)\phi(h)\phi(g^{-1}) = \phi(g)\phi(g)^{-1} = e'$$

This proves $ghg^{-1} \in Ker(\phi)$.

A map $\phi : G \to G'$ from group G to group G' is called an **isomorphism** (of groups) if ϕ is a homomorphism and is one-to-one and onto.

Two groups G and G' are called to be **isomorphic** if there exists an isomorphism $\phi : G \to G'$.

Example. The groups $(\mathbb{R}_{>0}, \cdot)$ and $(\mathbb{R}, +)$ are isomorphic. Because

$$\phi : \mathbb{R}_{>0} \to \mathbb{R}, \quad \phi(a) = \log a$$

is an isomorphism.

The end