THE UNIVERSITY OF MELBOURNE

Student Number

Semester 2 Assessment, 2017

School of Mathematics and Statistics

MAST20018 Discrete Mathematics and Operations Research

Writing time: 3 hours

Reading time: 15 minutes

This is an OPEN BOOK exam

This paper consists of 5 pages (including this page)

Authorised Materials

- Mobile phones, smart watches and internet or communication devices are forbidden.
- One double sided A4 page of handwritten notes.
- Authorised hand-held calculator.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- This examination consists of 11 questions.
- You should attempt all questions. Marks for individual questions are shown.
- The total number of marks available is 130.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.

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Question 1 (12 marks)

(a) My diet requires that all food I eat come from one of the four "basic food groups" (chocolate cake, ice cream, soda, and cheesecake). At present, the following four foods are available: brownies, chocolate ice cream, cola, and pineapple cheesecake. Each brownie costs 50c, each scoop of chocolate ice cream costs 20c, each bottle of cola costs 30c, and each piece of pineapple cheesecake costs 80c. Each day, I must ingest at least 500 calories, 6 ounces of chocolate, 10 ounces of sugar, and 8 ounces of fat. The nutritional content per unit food is shown in the table below. Formulate a linear programming model that can be used to satisfy my daily nutritional requirements at minimum cost.

Food type	*Calories*	*Choc. (ounces)*	*Sugar (ounces)*	*Fat (ounces)*
Brownie	400	3	2	2
Choc. ice cream (1 scoop)	200	2	2	4
Cola (1 bottle)	150	0	4	1
Cheesecake (1 piece)	500	0	4	5

(b) By referring to the four Assumptions of Linear programming, discuss what the limitations of the above model are with respect to this diet problem. Write at most two sentences per assumption.

Question 2 (10 marks)

(a) Sketch the feasible region for the following set of constraints:

$$\begin{aligned} 2x_1 + x_2 &\leq 100 \\ x_1 + x_2 &\leq 80 \\ x_1 &\leq 40 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(b) Using the graphical method, find the optimal solution when minimising the function $f(x_1, x_2) = 3kx_1 + 2kx_2$, where $k \in \mathbb{R}$, over the above feasible region.

Question 3 (10 marks)

Consider the following LP

$$\min z = 2x_1 + 3x_2$$

$$\begin{aligned} \frac{1}{2}x_1 + \frac{1}{4}x_2 &\leq 4 \\ x_1 + 3x_2 &\geq 36 \\ x_1 + x_2 &= 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(a) Convert the LP to canonical form.

(b) Run Phase 1 of the Simplex algorithm on the LP.

(c) What do you conclude about the LP.

Question 4 (15 marks)

Consider the LP

$$\max z = x_1 + 4x_2$$

$$\begin{aligned} x_1 + 2x_2 + s_1 \quad\quad &= 6 \\ 2x_1 + x_2 \quad\quad + s_2 &= 8 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$

For the LP, the optimal basis is $\{x_2, s_2\}$. Using the algebra of the Simplex method, compute the final Simple tableau of this LP.

Question 5 (10 marks)

Show, using Duality theory, that $x_1 = \frac{5}{26}, x_2 = \frac{5}{2}, x_3 = \frac{27}{26}$ is an optimal solution to the following linear programming problem:

$$\max z = 9x_1 + 14x_2 + 7x_3$$

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &\leq 6 \\ 5x_1 + 4x_2 + \ \ x_3 &\leq 12 \\ 2x_2 \quad\quad &\leq 5 \\ x_1, x_2, x_3 &\in \mathbb{R} \end{aligned}$$

Question 6 (10 marks)

Consider the following LP in canonical form and its final Simplex tableau:

$$\max z = 3x_1 + 4x_2$$

$$\begin{aligned} x_1 + x_2 + x_3 \quad\quad &= 4 \\ 2x_1 + x_2 \quad\quad + x_4 &= 5 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

BV	x_1	x_2	x_3	x_4	RHS
x_2	1	1	1	0	4
x_4	1	0	-1	1	1
z	1	0	4	0	16

Now consider the parametric LP, where $\alpha \in [0, 1/3]$:

$$\max z = 3(\alpha + 1)x_1 + 4(1 - \alpha)x_2$$

$$\begin{aligned} x_1 + x_2 + x_3 \quad\quad &= 4 \\ 2x_1 + x_2 \quad\quad + x_4 &= 5 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(i) Showing all calculations, provide the range of α for which x_2 and x_4 are the basic variables in the optimal solution.

(ii) Showing all calculations, provide the range of α for which x_1 and x_2 are the basic variables in the optimal solution.

(iii) Describe an infinite set of optimal solutions for the parametric program when $\alpha = 1/7$.

Question 7 (8 marks)

A *cycle* in a graph G is a path $v_0, v_1, ..., v_p, v_0$ where (v_i, v_{i+1}) is an edge of G for all $0 \le i \le p-1$, and (v_p, v_0) is also an edge of G. A *chord* of a cycle is an edge of G which joins two distinct, non-adjacent nodes on the cycle. In *chordal* graphs every cycle has a chord, and therefore chordal graphs are also called *triangulated* graphs. Prove that interval graphs are chordal. Note, proof by example is not sufficient.

Question 8 (20 marks)

(a) Draw the bipartite graph corresponding to the matrix

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Clearly label the nodes of your bipartite graph if the columns of the matrix are $y_1, ..., y_4$ and the rows are $x_1, ..., x_4$.

(b) Find a matching in your bipartite graph from (a) which includes all vertices of maximum degree.

(c) Repeat the process in (b) in order to find a set of four disjoint matchings in your bipartite graph.

(d) Suppose you are organising a swing dancing event for 6 participants. You have been tasked to match everyone with a partner for the event. There are three leaders, L_1, L_2, L_3 and three followers F_1, F_2, F_3. By carefully observing the participants' dancing skills over the last couple of weeks, you have compiled a compatibility table as follows:

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 8 \\ 0 & 4 & 1 \end{bmatrix}$$

where the entry in row i and column j represents the compatibility of couple (F_i, L_j). The score ranges from 0 to 9, with 0 being the most compatible.

Using the algorithm you learnt in class, find an assignment of leaders to followers that optimises the total compatibility for the swing dancing event.

Question 9 (9 marks)

Suppose that three players, A, B, C are playing the envy-free cake division game, covered in your lectures. Player A cuts first, B is the trimmer, and C is the first to pick. Suppose that C picks the trimmed piece. Explain in detail, with reference to every player, why the game is indeed envy free.

Question 10 (15 marks)

There are four items I_1, I_2, I_3, I_4 to be distributed among three people A, B, C using the method of sealed bids. The bids are given by the following table.

bids	A	B	C
I_1	\$10,000	\$5,000	\$6,000
I_2	\$3,000	\$1,000	\$4,000
I_3	\$1500	\$1,500	\$2,000
I_4	\$1800	\$2,000	\$1,000

(a) Extend the table to include the fair share of each player, and their contribution to the pot.

(b) Compute the excess in the pot.

(c) Summarize the allocation to each player.

(d) Show, in detail and with reference to every player, whether this allocation is envy free or not.

Question 11 (11 marks)

(b) Consider the following election between three candidates by 53 voters.

$$\begin{aligned} 27 &: B > A > C \\ 24 &: C > A > B \\ 2 &: A > C > B \end{aligned}$$

(i) Show that C is the loser of this contest if the Borda count scoring method is used.

(ii) C was so embarrassed about losing the election that he/she demanded to retroactively withdraw. Now using the Borda count committee scoring method, who is the winner?

(iii) What does this say about the Borda count method?

End of Exam—Total Available Marks = 130