Homework No.5 for Math 3121

Deadline: Dec. 4, 11:00pm

Problem 1. Multiple choice (each problem has only one correct answer, no reasons needed). (1). Which of the following rings is an integral domain?
(a). $\mathbb{Z} \times \mathbb{Z}$. (b). $\mathbb{Z}_{20}$. (c) $\mathbb{Z}$. (d). None of above

(2). If R is a commutative ring, $a \in R$, $a \neq 0$ is NOT a 0-divisor, which of the following is NOT correct?
(a) $ab = 0$ implies that $b = 0$. (b) a^{2019} is not a 0-divisor.
(c) a^{-1} exists. (d) $-a$ is not a 0-divisor.

(3). Which of the following sets is a subring of $M_2(\mathbb{R})$?
(a). $S = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R}\}$. (b). $S = \{\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} \mid x \in \mathbb{R}\}$
(c). $S = \{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{R}\}$ (d). None of the above.

(4). Which of the following elements in $\mathbb{Z}_{100}$ has a multiplicative inverse? (recall that a' is the multiplicative inverse of a if $aa' = 1$).
(a). 55 , (b). 9, (c). 40 (d). None of above.

Problem 2. Determine whether each of the following maps is a ring homomorphism (no reasons needed)
(1). $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $\phi((a, b)) = b$.
(2). $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $\phi(a, b) = ab$.
(3). $\phi : \mathbb{R} \to M_2(\mathbb{R})$ given by $\phi(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$.
(4). $\Phi : \mathbb{R} \to M_2(\mathbb{R})$ given by $\phi(a) = \begin{pmatrix} a & -a \\ 0 & 0 \end{pmatrix}$.
(5). $\Phi : \mathbb{R} \to M_2(\mathbb{R})$ given by $\phi(a) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$.

(6). $\phi : \mathbb{C} \to \mathbb{C}$ given by $\phi(a + bi) = a - bi$.
(7). $\phi : \mathbb{C} \to \mathbb{C}$ given by $\phi(z) = 2z$.
(8). Let g be given 2×2 invertible matrix, $\phi : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ given by $\phi(X) = gXg^{-1}$.

Problem 3. (no reasons needed) (1) Find a subring of $M_2(\mathbb{R})$ that is isomorphisc to $\mathbb{R}$.
(2) Find a subring R of $\mathbb{Q}$ such that R contains $\mathbb{Z}$ but $R \neq \mathbb{Z}$ and $R \neq \mathbb{Q}$.

Problem 4. Let R be a ring with unity 1. Suppose $a \in R$ has multiplicative inverse $a^{-1} \in R$, that is $aa^{-1} = a^{-1}a = 1$. Prove that the map $\phi : R \to R$ given by $\phi(x) = axa^{-1}$ is a ring homomorphism. Which of the following proofs is correct ?

Proof 1. For arbitrary $x \in R$, $\phi(x) = axa^{-1} = aa^{-1}x = x$, so $\phi(x + y) = x + y = \phi(x) + \phi(y)$, $\phi(xy) = xy = \phi(x)\phi(y)$. This proves ϕ is a ring homomorphism.

Proof 2. For arbitrary $x, y \in R$, $\phi(x+y) = a(x+y)a^{-1} = axa^{-1}+aya^{-1} = \phi(x) + \phi(y)$, $\phi(xy) = axya^{-1} = (axa^{-1})(aya^{-1}) = \phi(x)\phi(y)$. This proves ϕ is a ring homomorphism.

Proof 3. For arbitrary $x, y \in R$, $\phi(x+y) = a(x+y)a^{-1} = axa^{-1}+aya^{-1} = \phi(x) + \phi(y)$, This proves ϕ is a ring homomorphism.