Plan

- Section 13. Homomorphisms (continued)

Definition 13.1. A map $\phi : G \to G'$ from a group G to a group G' is a **homomorphism** if it satisfies the property

$$\phi(ab) = \phi(a)\,\phi(b)$$

for all $a, b \in G$.

The concept of homomorphism is used to compare different groups.

If group G has operation $*$ and group G' has operation $\star$, the condition for $\phi : G \to G'$ being a homomorphism is

$$\phi(a * b) = \phi(a) \star \phi(b)$$

for all $a, b \in G$.

If both G and G' have binary operation addition $+$, a map $\phi : G \to G'$ is a homomorphism if

$$\phi(a+b) = \phi(a) + \phi(b)$$

for all $a, b \in G$.

Example. $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = 2020x$$

is a homomorphism. The homomorphism condition for this ϕ is

$$2020(a+b) = 2020a + 2020b$$

If both G and G' have binary operation multiplication $\cdot$, a map $\phi : G \to G'$ is a homomorphism if

$$\phi(ab) = \phi(a)\phi(b)$$

for all $a, b \in G$.

Example. $\phi : \mathbb{C}^* \to \mathbb{C}^*$ given by

$$\phi(z) = z^3$$

is a homomorphism. The homomorphism condition for this ϕ is

$$(ab)^3 = a^3 b^3.$$

Example. $\phi : \mathbb{R}^* \to \mathbb{R}^*$ given by

$$\phi(x) = |x|$$

is a homomorphism. The homomophism condition is

$$|ab| = |a|\,|b|.$$

Example. $\det : GL(n, \mathbb{R}) \to \mathbb{R}^*$, the determinant map, is a homomorphism because of the following property of the determinant:

$$\det(AB) = \det(A)\det(B)$$

If group G has $+$ and group G' has multiplication $\cdot$, a map $\phi : G \to G'$ is a homomorphism if

$$\phi(a+b) = \phi(a)\phi(b)$$

for all $a, b \in G$.

Example. The map

$$f : \mathbb{R} \to \mathbb{R}^*, \quad f(x) = e^x$$

is a homomorphism because

$$e^{x+y} = e^x\, e^y$$

For any basis $c > 0$, the map $\phi : \mathbb{R} \to \mathbb{R}^*$ given by

$$\phi(x) = c^x$$

is a homomorphism because

$$c^{a+b} = c^a \, c^b.$$

If group G has multiplication $\cdot$ and group G' has $+$, a map $\phi : G \to G'$ is a homomorphism if

$$\phi(ab) = \phi(a) + \phi(b)$$

for all $a, b \in G$.

Example. $(\mathbb{R}_{>0}, \cdot)$ is a group, $(\mathbb{R}, +)$ is a group,

$$\phi : \mathbb{R}_{>0} \to \mathbb{R}, \quad \phi(a) = \log a$$

is a homomorphism, because

$$\log(ab) = \log a + \log b$$

For any base $c > 1$, the map $\phi : \mathbb{R}_{>0} \to \mathbb{R}$ given by

$$\phi(a) = \log_c a$$

is a homomorphism, because

$$\log_c(ab) = \log_c a + \log_c b$$

Example. If $f : G \to G'$ and $g : G' \to G''$ are homormorphisms of groups, then the composition $g \circ f : G \to G''$ is a homomorphism.

Example. Find a homomorphism $\phi : \mathbb{R}^* \to \mathbb{R}$ that is NOT a constant map.

$$\phi(a) = \log |a|$$

ϕ is the composition of the homomorphism

$$f : \mathbb{R}^* \to \mathbb{R}_{>0}, \quad f(a) = |a|$$

and the homomorphism

$$g : \mathbb{R}_{>0} \to \mathbb{R}, \quad g(b) = \log\ b$$

For arbitrary groups G and G', then constant map

$$\phi : G \to G' \quad \phi(a) = e'$$

is a homomorphism, where e' is the identity element of G'.

This homomorphism is called the **trivial homomorphism**.

The identity map $I : G \to G$, $I(a) = a$ is a homomorphism.

Example. Find a homomorphism $\phi : GL(n, \mathbb{R}) \to \mathbb{R}$ that is NOT a constant map.

$$\phi(A) = \log |\det A|$$

To study properties of homomorphisms, we recall the concept of image and inverse image of a map.

Definition 13.11. Let $\phi : X \to Y$ be a map. For $A \subseteq X$, then **image** $\phi(A)$ **of** A **in** Y **under** ϕ is

$$\phi(A) = \{\phi(a) \mid a \in A\}.$$

For $B \subseteq Y$, the **inverse image** $\phi^{-1}(B)$ **of** B **in** X is

$$\phi^{-1}(B) = \{a \in X \mid \phi(a) \in B\}.$$

Theorem

(Theorem 13.12). Let $\phi : G \to G'$ *be a homomorphism of groups. Then*
(1) If $e \in G$ *is the identity element,* $\phi(e) = e'$ *is the identity element in* G'.
(2) If $a \in G$, $\phi(a^{-1}) = \phi(a)^{-1}$.
(3) If $H \subseteq G$ *is a subgroup, then* $\phi(H)$ *is a subgroup of* G'.
(4) If $K' \subseteq G'$ *is a subgroup, then* $\phi^{-1}(K')$ *is a subgroup of* G.

Definition 13.13. Let $\phi : G \to G'$ be a homomorphism of groups. The subgroup

$$\phi^{-1}(e') = \{a \in G \mid \phi(a) = e'\}$$

is called the **kernel of** ϕ, denoted by $Ker(\phi)$.

Example. Let $\phi : \mathbb{C}^* \to \mathbb{C}^*$ be the homomorphism given by $\phi(a) = a^3$, then

$$Ker(\phi) = \{a \in \mathbb{C}^* \mid \phi(a) = 1\} = \{a \in \mathbb{C}^* \mid a^3 = 1\} = U_3$$

Next time, we will discuss the following theorem:

Theorem

(Theorem 13.15) Let $\phi : G \to G'$ be a group homomorphism, and let $H = Ker(\phi)$. Let $b \in G'$.

$$\phi^{-1}(b) = \{a \in G \mid \phi(a) = b\}$$

has two cases. Case 1. $\phi^{-1}(b) = \emptyset$. Case 2. $\phi^{-1}(b)$ is NOT empty, let $a \in \phi^{-1}(b)$, then

$$\phi^{-1}(b) = aH$$

The end