Plan

- Review of Last lecture
- Section 13. Homomorphisms (completed)
- Section 14. Factor Groups

Review of Last lecture

Theorem

(Theorem 13.15) Let $\phi : G \to G'$ be a group homomorphism, and let $H = Ker(\phi)$. Let $b \in G'$.

$$\phi^{-1}(b) = \{a \in G \mid \phi(a) = b\}$$

has two cases for $\phi^{-1}(b)$.
Case 1. $\phi^{-1}(b) = \emptyset$.
Case 2. $\phi^{-1}(b)$ is NOT empty, for any $a \in \phi^{-1}(b)$, then

$$\phi^{-1}(b) = aH$$

Example. Let $\phi : \mathbb{R}^* \to \mathbb{R}^*$ be the homomorphism given by

$$\phi(x) = |x|,$$

Then

$$Ker(\phi) = \{x \in \mathbb{R}^* \mid \phi(x) = 1\} = \{1, -1\}$$

If $b < 0$, then $\phi^{-1}(b) = \emptyset$.
If $b > 0$,

$$\phi^{-1}(b) = \{x \in \mathbb{R}^* \mid \phi(x) = b\} = \{x \in \mathbb{R}^* \mid |x| = b\} = \{b, -b\} = b\,Ker(\phi)$$

Corollary 13.18. A group homomorphism $\phi : G \to G'$ is a one-to-one map if and only if $Ker(\phi) = \{e\}$.

Definition 13.19. A subgroup H of a group G is called a **normal subgroup** if for all $g \in G$, $gH = Hg$.

Lemma. A subgroup H of G is normal iff for every $g \in G$, $h \in H$, $ghg^{-1} \in H$,

Corollary 13.20. If $\phi : G \to G'$ is a homomorphism of groups, then $Ker(\phi)$ is a normal subgroup of G.

A map $\phi : G \to G'$ from group G to group G' is called an **isomorphism** (of groups) if ϕ is a homomorphism and is one-to-one and onto.

Two groups G and G' are called to be **isomorphic** if there exists an isomorphism $\phi : G \to G'$.

Example. The groups $(\mathbb{R}_{>0}, \cdot)$ and $(\mathbb{R}, +)$ are isomorphic. Because

$$\phi : \mathbb{R}_{>0} \to \mathbb{R}, \quad \phi(a) = \log a$$

is an isomorphism.

Example. The groups $\mathbb{Z}$ and $H = \{2^n \mid n \in \mathbb{Z}\}$ are isomorphic. Note that the first group is additive, the second group is multiplicative.

$$\phi : \mathbb{Z} \to H, \qquad \phi(n) = 2^n$$

is an isomorphism.

Every two infinite cyclic groups are isomorphic.

The groups $\mathbb{Z}_n$ and U_n are isomorphic.

Recall $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$, the binary operation is the addition modulo n,

$$U_n = \{a \in \mathbb{C}^* \mid a^n = 1\} = \{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$$

where $\alpha = e^{\frac{2\pi i}{n}}$.

The map $\phi : \mathbb{Z}_n \to U_n$ given by

$$\phi(k) = \alpha^k$$

is an isomorphism.

Any two cyclic groups of equal order are isomorphic.

Every group of prime order p is isomorphic to $\mathbb{Z}_p$.

A group of order 4 is either isomorphic to $\mathbb{Z}_4$ or isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

But $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are NOT isomorphic.

Theorem

*(**Theorem 14.4 + Corollary 14.5**) Let G be a group, H be a normal subgroup of G, G/H be the set of all left cosets of H, then*
(1) the binary operation on G/H given by

$$(aH)(bH) = (ab)H$$

is well-defined.
(2). G/H is a group under the above binary operation.

The symbol G/H denotes the set of all left cosets of H, (since H is normal, G/H is also the set of all right cosets of H, this fact will not be used in our discussion).

Elements in G/H are left cosets, aH, bH. However the ways to represent a left coset in the form aH is not unique, that is, it may happens that $a_1 \neq a_2$ but $a_1 H = a_2 H$.

Example. $G = \mathbb{Z}$, $H = 10\mathbb{Z}$, cosets $12 + 10\mathbb{Z}$ and $2 + 10\mathbb{Z}$ are equal.

(1) in the above theorem means that if $a_1 H = a_2 H$ and $b_1 H = b_2 H$, then $(a_1 b_1)H = (a_2 b_2)H$.

The group G/H in the Theorem 14.4 is called the **factor group** (or **quotient group**) of G by H.

Example. $G = \mathbb{Z}$, $H = 3\mathbb{Z}$,

$$\mathbb{Z}/3\mathbb{Z} = \{0 + \mathbb{Z}, 1 + \mathbb{Z}, 2 + \mathbb{Z}\}$$

The factor group of $\mathbb{Z}$ by $3\mathbb{Z}$ is just $\mathbb{Z}_3$.

More generally, the factor group of $\mathbb{Z}$ by $n\mathbb{Z}$ is just $\mathbb{Z}_n$.

For $n \geq 2$, the factor group of S_n by A_n has only two elements

$$S_n/A_n = \{A_n, (12)A_n\}$$

The binary operation is given by

$$\begin{aligned} A_n \cdot A_n &= A_n \\ A_n \cdot (12)A_n &= (12)A_n \\ (12)A_n \cdot A_n &= (12)A_n \\ (12)A_n \cdot (12)A_n &= A_n \end{aligned}$$

The end