Plan

- Section 16. Group Action on a Set

The notion of group action is an important tool in the application of group theory to the study of symmetries.
Before formally introducing the notion, we look at the following example.

We have group S_n. By the very definition of S_n, for every $\sigma \in S_n$, every $i \in X = \{1, 2, \ldots, n\}$, σ sends i to an element $\sigma(i)$ in X.
So we have a map

$$S_n \times X \to X$$

$$(\sigma, i) \mapsto \sigma(i)$$

So we have a map

$$S_n \times X \to X.$$

This map satisfies the following properties:

(1) $e(x) = x$ for all $x \in X$.
(2)

$$(\sigma_1\sigma_2)(x) = \sigma_1(\sigma_2(x))$$

for all $x \in X$ and $\sigma_1, \sigma_2 \in S_n$.

This is an example of group S_n action on the set X.

Definition 16.1. Let X be a set and G a group. An **action of G on X** is a map

$$*: G \times X \to X,$$

we will write the image of (g, x) as $g * x$ or simply gx such that

(1) $ex = x$ for all $x \in X$.
(2) $(g_1 g_2)x = g_1(g_2 x)$ for all $g_1, g_2 \in G$ and $x \in X$.

We also say X **is a G-set** or G **acts on** X.

Example. $G = GL(2, \mathbb{R})$, this group is the group of invertible linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ itself. We have the natural map:

$$* : GL(2, \mathbb{R}) \times \mathbb{R}^2 \to \mathbb{R}^2$$

$(g, v) \mapsto gv$, where gv is the matrix multiplication (2×2 matrix multiple 2×1 matrix, the result is a 2×1 matrix)

We have
(1) $I_2 v = v$ for all $v \in \mathbb{R}^2$
(2) For all $g_1, g_2 \in GL(2, \mathbb{R})$, all $v \in \mathbb{R}^2$,

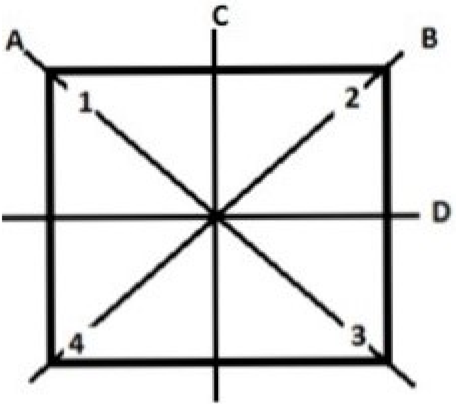
$$(g_1 g_2)v = g_1(g_2 v)$$

The above map is an action of $GL(2, \mathbb{R})$ on $\mathbb{R}^2$.

More generally, for every positive integer n, the group $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$ by matrix multiplication.

The following example relates the group action with symmetry groups.

Action of Symmetry Group of a Square



Action of Symmetry Group of a Square (continued)

Let G be the symmetry group, G has the following 8 elements:

R_0: doing nothing,
R_{90}: rotation anti-clockwisely by 90 degree
R_{180}: rotation anti-clockwisely by 180 degree
R_{270}: rotation anti-clockwisely by 270 degree
L_A: reflection about line A
L_B: reflection about line B
L_C: reflection about line C
L_D: reflection about line D

Action of Symmetry Group of a Square (continued)

Let $X = \{1, 2, 3, 4\}$ be the set of vertices of the square.

The G acts on X. That is, we have a map $G \times X \to X$ satisfying the group action axioms (1) (2).

Here are some cases of the action:

$$L_A 1 = 1, \quad L_A 2 = 4, \quad L_A 3 = 3, \quad L_A 4 = 2$$

$$R_{90} 1 = 4, \quad R_{90} 2 = 1, \quad R_{90} 3 = 2, \quad R_{90} 4 = 3$$

Action of Symmetry Group of a Square (continued)

The symmetry group acts also on other sets: Let E be a set of edges of the square, so

$$E = \{12, 23, 34, 41\}$$

$$L_A\, 12 = 41, \quad L_A\, 23 = 34, \quad L_A\, 34 = 23, \quad L_A\, 41 = 12$$

$$R_{90}\, 12 = 41, \quad R_{90}\, 23 = 12, \quad R_{90}\, 34 = 23, \quad R_{90}\, 41 = 34$$

Let X be the set of all points in the square, then G acts on X, since each symmetry moves a point in the square to another point in the square.

Example. Let H be a subgroup of G, then G acts on G/H, the set of left cosets on H: For $g \in G$, $aH \in G/H$,

$$g(aH) = (ga)H$$

Let G act on set X, for each $x \in X$, we introduce

$$G_x = \{g \in G \mid gx = x\}$$

That is, G_x is the subset of G that consists of all elements in G fixing x.

Theorem 16.12 Suppose X is a G-set, $x \in X$, then G_x is a subgroup of G

The subgroup G_x is called the **isotropy subgroup** of x.

Example. Let $G = S_3$ act on $X = \{1, 2, 3\}$. The isotropy subgroup of 2 has two elements:

$$G_2 = \{e, (13)\}$$

Example. Let S_4 act on $X = \{1, 2, 3, 4\}$. Find the isotropy subgroup G_4 of 4. Answer G_4 has the following six elements

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

Definition 16.15. Let X be a G-set, $x \in X$, the **orbit of** x is the following subset Gx of X:

$$Gx = \{gx \mid g \in G\}$$

Example. $G = S_5$ acts on $X = \{1, 2, 3, 4, 5\}$, The orbit of 1 is

$$G1 = \{1, 2, 3, 4, 5\}.$$

The orbit of 2 is

$$G2 = \{1, 2, 3, 4, 5\}.$$

The end