Plan

- Review of Last Lecture
- Section 16. Group Action on a Set (continued)

Review of Last Lecture

Definition 16.1. Let X be a set and G a group. An **action of G on X** is a map

$$* : G \times X \to X,$$

we will write the image of (g, x) as $g * x$ or simply gx such that

(1) $e * x = x$ for all $x \in X$.
(2) $(g_1 g_2) * x = g_1 * (g_2 * x)$ for all $g_1, g_2 \in G$ and $x \in X$.

We also say G acts on X or X is a G-set.

If the action is written as gx instead of $g * x$,the two axioms are

$$ex = x, \quad (g_1 g_2)x = g_1(g_2 x)$$

Example. $G = GL(2, \mathbb{R})$ acts on $\mathbb{R}^2$ as follows.

For $g \in GL(2, \mathbb{R})$, $x \in \mathbb{R}^2$,
gx is a just the matrix multiplication of g by x.

Note that g is a 2×2 matrix, x is a 2×1 matrix, the multiplication gx is a 2×1 matrix.

Example. More generally, for every positive integer n, the group $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$ by matrix multiplication.

Example. The group S_n acts on $X = \{1, 2, \ldots, n\}$, for $g \in S_n$, $i \in X$, $gi = g(i)$, the outcome of the permutation g applying to i.

Let G act on set X, for each $x \in G$, we introduce

$$G_x = \{g \in G \mid gx = x\}$$

That is, G_x is the subset of G that consists of all elements in G fixing x.

Theorem 16.12. Suppose X is a G-set, $x \in X$, then G_x is a subgroup of G

The subgroup G_x is called the **isotropy subgroup** of x.

Example. Let $G = S_3$ act on $X = \{1, 2, 3\}$. The isotropy subgroup of 2 has two elements:

$$G_2 = \{e, (13)\}$$

Definition 16.15. Let X be a G-set, $x \in X$, the **orbit of** x is the following subset Gx of X:

$$Gx = \{gx \mid g \in G\}$$

Example. $G = S_5$ acts on $X = \{1, 2, 3, 4, 5\}$. The orbit of 1 is

$$G1 = \{1, 2, 3, 4, 5\}.$$

The orbit of 2 is

$$G2 = \{1, 2, 3, 4, 5\}.$$

Section 16. Group Action on a Set (continued)

Let X be a G-set, for any $a, b \in X$, the orbits Ga and Gb are either equal or disjoint.

And the set X is a disjoint union of orbits.

Example. S_n acts on $X = \{1, 2, \ldots, n\}$, there is only one orbit. For every $i \in X$,

$$G\,i = X$$

Example. S_5 acts on $X = \{1, 2, 3, 4, 5\}$. Let

$$Y = X \times X = \{(i, j) \,|\, 1 \leq i, j \leq 5\}$$

Y has 25 elements. S_5 acts on Y as

$$g(i, j) = (gi, gj)$$

An example is, for $g = (325)$ (cycle of length 3)

$$g(1, 5) = (g1, g5) = (1, 3)$$

Questoins: how many orbits are there?

Answer: there are two orbits:

$$S_5(1,2) = \{(i,j) \mid 1 \leq i,j \leq 5, i \neq j\}$$

and

$$S_5(1,1) = \{(1,1),(2,2),(3,3),(4,4),(5,5)\}$$

The first orbit has 20 elements, the second has 5 elements.

Theorem 16.16. *Let G be a finite group, X be a G-set. Then for every $x \in X$,*

$$|Gx| \cdot |G_x| = |G|.$$

Sketch of Proof. We define a map $\psi : G/G_x \to Gx$ by

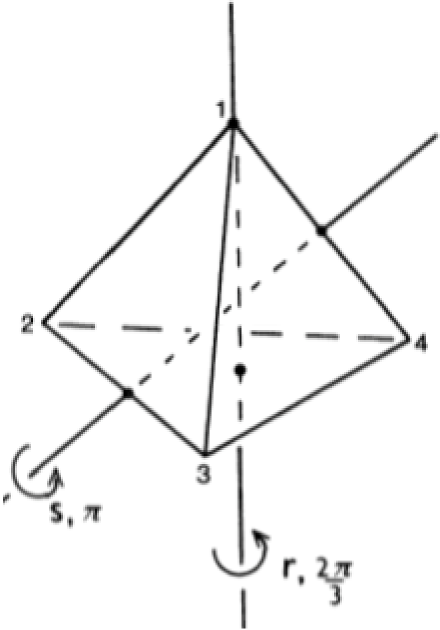
$$\psi(gG_x) = gx.$$

Then we prove
(1). ψ is well-defined.
(2). ψ is 1-1 and onto.

(2) implies that $|Gx| = |G/G_x| = \frac{|G|}{|G_x|}$.

Example. Application of Group Action.

Let G be the symmetry group of a regular tetrahedron, find the order $|G|$?


1
2
4
3
s, π
r, 2π/3

$|G| = 12.$
G acts on the set $X = \{1, 2, 3, 4\}$ of vertices.
It is easy to see that the orbit

$$G1 = \{1, 2, 3, 4\} = X$$

An explanation of the above fact is that a regular tetrahedron is very symmetric so all the vertices have equal footing.

The isotropy subgroup G_1 has 3 elements,

By Theorem 16.16, we have

$$|G| = |G1|\,|G_1| = 4 \cdot 3 = 12$$

The following are the 12 elements in G:

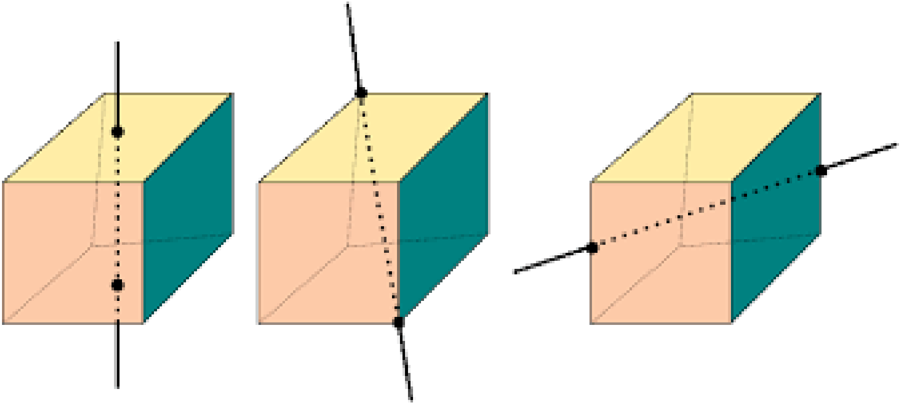
e: the identity symmetry,
$R(1,120), R(1,240)$: the rotations along axis 1 by angles 120 and 240 respectively,
$R(2,120), R(2,240)$: the rotations along axis 1 by angles 120 and 240 respectively,
$R(3,120), R(3,240)$: the rotations along axis 1 by angles 120 and 240 respectively,
$R(4,120), R(4,240)$: the rotations along axis 1 by angles 120 and 240 respectively,
There are 3 extra symmetries that we denote by

$$(14)(23), (12)(34), (13)(24)$$

the notations for these three elements agree with their permutations on the set of vertices $\{1,2,3,4\}$

Example. Application of Group Action.

Let G be the symmetry group of a cube, find the order $|G|$?



Let G act on the set X of the faces of the cube, $|X| = 6$. Let $1 \in X$ be the top face.
Then the orbit $G1 = X$, the isotropy subgroup G_1 has order 4, i.e., $|G_1| = 4$.

By Theorem 16.16,

$$|G| = |G1|\,|G_1| = 6 \cdot 4 = 24$$

The end