Supplementary Material of "Tightly-coupled Visual/Inertial/Map Integration with Observability Analysis for Reliable Localization of Intelligent Vehicles"

This is the derivation of the observability analysis about the proposed framework.

A. IMU-driven System Kinematic Model

The raw measurements of IMU gyroscope $\widetilde{\boldsymbol{\omega}}$ and accelerometer $\widetilde{\mathbf{a}}$ are modeled as

$$\begin{aligned} \widetilde{\boldsymbol{\omega}}(t) &= \boldsymbol{\omega}(t) + \mathbf{b}_g(t) + \mathbf{n}_g(t) \\ \widetilde{\mathbf{a}}(t) &= \mathbf{R}_w^b(\mathbf{a}(t) - \mathbf{g}^w) + \mathbf{b}_a(t) + \mathbf{n}_a(t) \end{aligned}, \tag{1}$$

where, $\boldsymbol{\omega}$ and $\mathbf{a}$ are the acceleration and angular velocity of the body frame respect to the inertial frame. $\mathbf{g}^w = [0, 0, g]^T$ is the Gravity acceleration in the world frame, $\mathbf{b}_g$ and $\mathbf{b}_a$ are the gyroscope and acceleration bias, which are called random walk and are modeled as Gaussian distributions with zero mean and covariance $\boldsymbol{\sigma}_{b_g}^2$ and $\boldsymbol{\sigma}_{b_a}^2$, respectively. The $\mathbf{n}_g$ and $\mathbf{n}_a$ are measurement Gaussian white noise, shown as $\mathbf{n}_g \sim \mathcal{N}(0, \boldsymbol{\sigma}_g^2)$, $\mathbf{n}_a \sim \mathcal{N}(0, \boldsymbol{\sigma}_a^2)$ [1]. The kinematic model can be described as [2]

$$\begin{aligned} \dot{\mathbf{p}}_b^w(t) &= \mathbf{v}_b^w(t) \\ \dot{\mathbf{v}}_b^w(t) &= \mathbf{R}_b^w(t)\mathbf{a}(t) \\ \dot{\mathbf{q}}_b^w(t) &= \frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega}(t))\mathbf{q}_b^w(t), \\ \dot{\mathbf{b}}_g(t) &= \mathbf{0}_{3\times1} \\ \dot{\mathbf{b}}_a(t) &= \mathbf{0}_{3\times1} \end{aligned} \tag{2}$$

where,

$$\boldsymbol{\Omega}(\boldsymbol{\omega}) = \begin{bmatrix} -\lfloor\boldsymbol{\omega}\rfloor_\times & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^\top & 0 \end{bmatrix}, \lfloor\boldsymbol{\omega}\rfloor_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{3}$$

B. Observability Analysis

First, the proposed system can be modeled into a nonlinear affine expression as follows:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \sum_{i=1}^{n} \mathbf{f}_i(\mathbf{x})\mathbf{u}_i \\ \mathbf{z}_p = \mathbf{h}_1(\mathbf{x}) \\ \mathbf{z}_l = \mathbf{h}_2(\mathbf{x}) \end{cases} \tag{4}$$

where, $\mathbf{x}$ is the states, $\mathbf{u} = [\widetilde{\boldsymbol{\omega}}, \widetilde{\mathbf{a}}]$ is the IMU inputs. $\dot{\mathbf{x}}$ is from IMU system kinematic model in (2), the derivative of the extrinsic parameters $\dot{\mathbf{q}}_c^b = \mathbf{0}_{3\times1}, \dot{\mathbf{p}}_c^b = \mathbf{0}_{3\times1}$, and the derivative of the point feature 3D position $\dot{\mathbf{P}}_k^w = \mathbf{0}_{3\times1}$. $\mathbf{z}_p$ is the point feature-based measurement model, which can be seen in our paper. $\mathbf{z}_l$ is the line feature-based measurement model. Therefore, the complete model in (4) can be expressed by

$$
\begin{aligned}
&\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \mathbf{f}_1(\mathbf{x})\widetilde{\boldsymbol{\omega}} + \mathbf{f}_2(\mathbf{x})\widetilde{\mathbf{a}} \\
\Longrightarrow &\begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{v}} \\ \dot{\mathbf{s}}_1 \\ \dot{\mathbf{b}}_g \\ \dot{\mathbf{b}}_a \\ \dot{\mathbf{p}}_c^b \\ \dot{\mathbf{s}}_2 \\ \dot{\mathbf{P}}_k^w \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ \mathbf{g}^w - \mathbf{R}_b^w \mathbf{b}_a \\ -\mathbf{D}_1 \mathbf{b}_g \\ \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times1} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_3 \\ \mathbf{0}_3 \\ \mathbf{D}_1 \\ \mathbf{0}_3 \\ \mathbf{0}_3 \\ \mathbf{0}_3 \\ \mathbf{0}_3 \\ \mathbf{0}_3 \end{bmatrix} \widetilde{\boldsymbol{\omega}} + \begin{bmatrix} \mathbf{0}_3 \\ \mathbf{R}_b^w \\ \mathbf{0}_3 \\ \mathbf{0}_3 \\ \mathbf{0}_3 \\ \mathbf{0}_3 \\ \mathbf{0}_3 \\ \mathbf{0}_3 \end{bmatrix} \widetilde{\mathbf{a}} \\
&\mathbf{z}_p = \pi\left(\mathbf{R}_b^c\left(\mathbf{R}_w^b\left(\mathbf{P}_k^w - \mathbf{p}_b^w\right) - \mathbf{p}_c^b\right)\right) \\
&\mathbf{z}_l = \pi\left(\mathbf{R}_b^c\left(\mathbf{R}_w^b\left(\mathbf{P}_l^w - \mathbf{p}_b^w\right) - \mathbf{p}_c^b\right)\right)
\end{aligned} \tag{5}
$$

where, $\mathbf{P}_l^w$ means the endpoint of the line segment, $\mathbf{s}_1$ corresponds to the $\mathbf{q}_b^w$, $\mathbf{s}_2$ means the $\mathbf{q}_c^b$. $\mathbf{s}$ is the Cayley-Gibbs-Rodrigues (CGR) parameterization representation of the orientation to facilitate the derivation calculation, and $\mathbf{s} = \mathbf{n}\tan\frac{\theta}{2}$ with $\mathbf{n}$ is the rotation axis and θ is the axial angle [3][4]. The partial derivative of $\mathbf{q}_b^w(t)$ respect to time is

$$
\begin{aligned}
\dot{\mathbf{s}}_b^w(t) &= \mathbf{D}(\widetilde{\omega} - \mathbf{b}_g) \\
\mathbf{D} \triangleq \frac{\partial \mathbf{s}}{\partial \theta} &= \frac{1}{2}(\mathbf{I} + \lfloor \mathbf{s} \rfloor_\times + \mathbf{s}\mathbf{s}^\top)
\end{aligned}. \tag{6}
$$

For the derivation of the observability analysis, we refer to the Theorem in [2]. Here is the theorem: *Theorem 1:* Assume that there exists a nonlinear transformation $\boldsymbol{\beta}(\mathbf{x}) = [\beta_1(\mathbf{x})^\top, \beta_2(\mathbf{x})^\top, \cdots, \beta_t(\mathbf{x})^\top]$. These bases are functions of the state variable $\mathbf{x}$, and the number of basis elements t, is defined so as to fulfill:

$$
\begin{aligned}
&(C1) \qquad \boldsymbol{\beta_1}(\mathbf{x}) = \mathbf{z}(\mathbf{x}) \\
&(C2) \qquad \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{x}} \cdot \mathbf{f}_i = 0; i = 0, \cdots, l \\
&(C3) \qquad system: \begin{cases} \dot{\boldsymbol{\beta}} = \mathbf{g}_0(\boldsymbol{\beta}) + \sum_{i=1}^{l} \mathbf{g}_i(\boldsymbol{\beta})\mathbf{u}_i \\ \mathbf{z} = \mathbf{h} = \beta_1(\mathbf{x}) \end{cases},
\end{aligned} \tag{7}
$$

where $\mathbf{g}_i(\beta) = \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{x}} \cdot \mathbf{f}_i = 0; i = 0, \cdots, l$ is obervable. Then:

(i) The observability matrix of (5) can be factorized as

$$
\mathcal{O} = \Xi \cdot \mathbf{B}, \tag{8}
$$

where Ξ is the observability matrix of system (7) and $\mathbf{B} \triangleq \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{x}}$.

(ii) $null(\mathcal{O}) = null(\mathbf{B})$.

Turning back to our problem, we need to apply Theorem 1 to derive the observability of system (5). The first step is to find all basis functions based on the assuming condition (C1) and (C2). According to the (C1), we get the first basis function

$$
\beta_1 = \mathbf{z}_p = \frac{1}{p_z}\begin{bmatrix} p_x \\ p_y \end{bmatrix}, \tag{9}
$$

where, $\mathbf{z}_p$ is the pixel coordinate of a feature point under the normalized camera model. Next, condition (C2) is used to find other basis functions.

We start by computing the span of $\partial\beta_1$ with respect to $\mathbf{x}$, i.e.

$$\begin{aligned}\frac{\partial\beta_1}{\partial\mathbf{x}} &= \begin{bmatrix}\frac{\partial\beta_1}{\partial\mathbf{p}} & \frac{\partial\beta_1}{\partial\mathbf{v}} & \frac{\partial\beta_1}{\partial\theta_1}\frac{\partial\theta_1}{\partial\mathbf{s}_1} & \frac{\partial\beta_1}{\partial\mathbf{b}_g} & \frac{\partial\beta_1}{\partial\mathbf{b}_a} & \frac{\partial\beta_1}{\partial\mathbf{p}_c^b} & \frac{\partial\beta_1}{\partial\theta_2}\frac{\partial\theta_2}{\partial\mathbf{s}_2} & \frac{\partial\beta_1}{\partial\mathbf{p}_k^w}\end{bmatrix} \\ &= \frac{\partial\mathbf{z}_p}{\partial\mathbf{P}_k^c}\cdot\frac{\partial\mathbf{P}_k^c}{\partial\mathbf{x}} \\ &= \begin{bmatrix}\frac{1}{p_z} & 0 & -\frac{p_x}{p_z^2} \\ 0 & \frac{1}{p_z} & -\frac{p_y}{p_z^2}\end{bmatrix}\begin{bmatrix}\mathbf{C}_2\mathbf{C}_1 & \mathbf{0}_3 & \mathbf{C}_2\lfloor\mathbf{P}_k^b\rfloor_\times\frac{\partial\theta_1}{\partial\mathbf{s}_1} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{C}_2 & \lfloor\mathbf{P}_k^c\rfloor_\times\frac{\partial\theta_2}{\partial\mathbf{s}_2} & -\mathbf{C}_2\mathbf{C}_1\end{bmatrix}\end{aligned} \quad . \tag{10}$$

Where, $\mathbf{C}_1 = \mathbf{R}_w^b$, $\mathbf{C}_2 = \mathbf{R}_b^c$, $\frac{\partial\mathbf{z}_p}{\partial\mathbf{P}_k^c}$ comes from the normalized camera intrinsic projection model. Regarding as $\frac{\partial\beta}{\partial\theta}$, it can be solved by a small-perturbation model about the derivation of $\mathbf{Rp}$ respect to the rotation $\mathbf{R}$ [5], it is

$$\frac{\partial\mathbf{Rp}}{\partial\varphi} = -\lfloor\mathbf{Rp}\rfloor_\times, \tag{11}$$

among that, $\mathbf{p}$ is a space point, $\mathbf{R}$ is a rotation matrix, $\varphi = \Delta\mathbf{R}$ is the small perturbation and equal to θ, $\lfloor\cdot\rfloor_\times$ means the notation of the skew symmetric matrix, same with (**??**).

Then, the $\frac{\partial\beta_1}{\partial\mathbf{x}}\mathbf{f}_0$ is

$$\begin{aligned}\frac{\partial\beta_1}{\partial\mathbf{x}}\mathbf{f}_0 &= \begin{bmatrix}\frac{1}{p_z} & 0 & -\frac{p_x}{p_z^2} \\ 0 & \frac{1}{p_z} & -\frac{p_y}{p_z^2}\end{bmatrix}(\mathbf{C}_2\mathbf{C}_1\mathbf{v} - \mathbf{C}_2\lfloor\mathbf{P}_k^b\rfloor_\times\mathbf{b}_g) \\ &= \begin{bmatrix}\mathbf{I}_2 & -\beta_1\end{bmatrix}(\frac{1}{p_z}\mathbf{C}_2\mathbf{C}_1\mathbf{v} - \lfloor\begin{bmatrix}\beta_1 \\ 1\end{bmatrix} + \frac{1}{p_z}\mathbf{C}_2\mathbf{p}_c^b\rfloor_\times\mathbf{C}_2\mathbf{b}_g),\end{aligned} \tag{12}$$

since

$$\mathbf{C}_2\lfloor\mathbf{P}_k^b\rfloor_\times = \mathbf{C}_2\lfloor\mathbf{R}_c^b\mathbf{P}_k^c + \mathbf{P}_c^b\rfloor_\times = \lfloor\mathbf{C}_2\mathbf{C}_2^\top\mathbf{P}_k^c + \mathbf{C}_2\mathbf{P}_c^b\rfloor_\times\mathbf{C}_2 = \lfloor\mathbf{P}_k^c + \mathbf{C}_2\mathbf{P}_c^b\rfloor_\times\mathbf{C}_2 \tag{13}$$

based on the skew symmetric matrix operation rule:

$$\lfloor\mathbf{Ua}\rfloor_\times = \mathbf{U}\lfloor\mathbf{a}\rfloor_\times\mathbf{U}^\top \Longrightarrow \lfloor\mathbf{Ua}\rfloor_\times\mathbf{U} = \mathbf{U}\lfloor\mathbf{a}\rfloor_\times, \tag{14}$$

with $\mathbf{U}$ is a rotation matrix, $\mathbf{a}$ is a vector.

It is obvious that $\frac{\partial\beta_1}{\partial\mathbf{x}}\mathbf{f}_0$ cannot be expressed by the basis function β_1, so we add other basis functions as:

$$\begin{aligned}\beta_2 &\triangleq \frac{1}{p_z} \\ \beta_3 &\triangleq \mathbf{C}_1\mathbf{v} \\ \beta_4 &\triangleq \mathbf{C}_2 \\ \beta_5 &\triangleq \mathbf{P}_c^b \\ \beta_6 &\triangleq \mathbf{b}_g\end{aligned} \quad , \tag{15}$$

then, we have

$$\frac{\partial\beta_1}{\partial\mathbf{x}}\mathbf{f}_0 = \begin{bmatrix}\mathbf{I}_2 & -\beta_1\end{bmatrix}(\beta_2\beta_4\beta_3 - \lfloor\begin{bmatrix}\beta_1 \\ 1\end{bmatrix} + \beta_2\beta_4\beta_5\rfloor_\times\beta_4\beta_6) \tag{16}$$

The $\frac{\partial\beta_1}{\partial\mathbf{x}}\mathbf{f}_1$ is

$$\frac{\partial\beta_1}{\partial\mathbf{x}}\mathbf{f}_1 = \begin{bmatrix}\frac{1}{p_z} & 0 & -\frac{p_x}{p_z^2} \\ 0 & \frac{1}{p_z} & -\frac{p_y}{p_z^2}\end{bmatrix}\mathbf{C}_2\lfloor\mathbf{P}_k^b\rfloor_\times = \begin{bmatrix}\mathbf{I}_2 & -\beta_1\end{bmatrix}\beta_4\lfloor\begin{bmatrix}\beta_1 \\ 1\end{bmatrix} + \beta_2\beta_4\beta_5\rfloor_\times\beta_4 \tag{17}$$

We do not need to add any other functions.

For $\frac{\partial \beta_1}{\partial \mathbf{x}}\mathbf{f}_2$, we have

$$\frac{\partial \beta_1}{\partial \mathbf{x}}\mathbf{f}_2 = \mathbf{0}. \tag{18}$$

So far, we have complated the calculation of the first basis function β_1. Next, we will apply this calculation process to other basis functions.

For basis function $\beta_2 = \frac{1}{p_z}$, the span space is

$$\begin{aligned}\frac{\partial \beta_2}{\partial \mathbf{x}} &= \begin{bmatrix} \frac{\partial \beta_2}{\partial \mathbf{p}} & \frac{\partial \beta_2}{\partial \mathbf{v}} & \frac{\partial \beta_2}{\partial \theta_1}\frac{\partial \theta_1}{\partial \mathbf{s}_1} & \frac{\partial \beta_2}{\partial \mathbf{b}_g} & \frac{\partial \beta_2}{\partial \mathbf{b}_a} & \frac{\partial \beta_2}{\partial \mathbf{p}_c^b} & \frac{\partial \beta_2}{\partial \theta_2}\frac{\partial \theta_2}{\partial \mathbf{s}_2} & \frac{\partial \beta_2}{\partial \mathbf{p}_k^w} \end{bmatrix} \\ &= \frac{\partial \beta_2}{\partial \mathbf{P}_k^c} \cdot \frac{\partial \mathbf{P}_k^c}{\partial \mathbf{x}} \\ &= -\frac{1}{p_z^2}\mathbf{e}_3^\top \begin{bmatrix} \mathbf{C}_2\mathbf{C}_1 & \mathbf{0}_3 & \mathbf{C}_2\lfloor \mathbf{P}_k^b \rfloor_\times \frac{\partial \theta_1}{\partial \mathbf{s}_1} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{C}_2 & \lfloor \mathbf{P}_k^c \rfloor_\times \frac{\partial \theta_2}{\partial \mathbf{s}_2} & -\mathbf{C}_2\mathbf{C}_1 \end{bmatrix}\end{aligned}, \tag{19}$$

where $\mathbf{e}_3$ is the third col of a 3×3 identity matrix. $[\mathbf{e}_1 \ \mathbf{e}_2 \mathbf{e}_3] = \mathbf{I}_3$.

Similar to (37), (17), (18), we have

$$\begin{aligned}\frac{\partial \beta_2}{\partial \mathbf{x}}\mathbf{f}_0 &= -\beta_2 \mathbf{e}_3^\top (\beta_2\beta_4\beta_3 - \lfloor \begin{bmatrix} \beta_1 \\ 1 \end{bmatrix} + \beta_2\beta_4\beta_5 \rfloor_\times \beta_4\beta_6) \\ \frac{\partial \beta_2}{\partial \mathbf{x}}\mathbf{f}_1 &= -\beta_2 \mathbf{e}_3^\top \beta_4 \lfloor \begin{bmatrix} \beta_1 \\ 1 \end{bmatrix} + \beta_2\beta_4\beta_5 \rfloor_\times \beta_4 \\ \frac{\partial \beta_2}{\partial \mathbf{x}}\mathbf{f}_2 &= \mathbf{0}.\end{aligned} \tag{20}$$

For basis function $\beta_3 = \mathbf{C}_1\mathbf{v}$, we have

$$\begin{aligned}\frac{\partial \beta_3}{\partial \mathbf{x}} &= \begin{bmatrix} \frac{\partial \beta_3}{\partial \mathbf{p}} & \frac{\partial \beta_3}{\partial \mathbf{v}} & \frac{\partial \beta_3}{\partial \theta_1}\frac{\partial \theta_1}{\partial \mathbf{s}_1} & \frac{\partial \beta_3}{\partial \mathbf{b}_g} & \frac{\partial \beta_3}{\partial \mathbf{b}_a} & \frac{\partial \beta_3}{\partial \mathbf{p}_c^b} & \frac{\partial \beta_3}{\partial \theta_2}\frac{\partial \theta_2}{\partial \mathbf{s}_2} & \frac{\partial \beta_3}{\partial \mathbf{p}_k^w} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0}_3 & \mathbf{C}_1 & \lfloor \mathbf{C}_1\mathbf{v} \rfloor_\times \frac{\partial \theta_1}{\partial \mathbf{s}_1} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix}\end{aligned}, \tag{21}$$

then,

$$\frac{\partial \beta_3}{\partial \mathbf{x}}\mathbf{f}_0 = \mathbf{C}_1\mathbf{g}^w - \mathbf{b}_a - \lfloor \mathbf{C}_1\mathbf{v} \rfloor_\times \mathbf{b}_g, \tag{22}$$

here, we need to define two new basis functions as

$$\begin{aligned}\beta_7 &\triangleq \mathbf{C}_1\mathbf{g}^w \\ \beta_8 &\triangleq \mathbf{b}_a\end{aligned}, \tag{23}$$

so, we have

$$\begin{aligned}\frac{\partial \beta_3}{\partial \mathbf{x}}\mathbf{f}_0 &= \beta_7 - \beta_8 - \lfloor \beta_3 \rfloor_\times \beta_6 \\ \frac{\partial \beta_3}{\partial \mathbf{x}}\mathbf{f}_1 &= \lfloor \mathbf{C}_1\mathbf{v} \rfloor_\times = \lfloor \beta_3 \rfloor_\times \\ \frac{\partial \beta_3}{\partial \mathbf{x}}\mathbf{f}_2 &= \mathbf{I}_3\end{aligned} \tag{24}$$

For basis function $\beta_4 = \mathbf{C}_2$, we have

$$\begin{aligned}\frac{\partial \beta_4}{\partial \mathbf{x}} &= \begin{bmatrix} \frac{\partial \beta_4}{\partial \mathbf{p}} & \frac{\partial \beta_4}{\partial \mathbf{v}} & \frac{\partial \beta_4}{\partial \theta_1}\frac{\partial \theta_1}{\partial \mathbf{s}_1} & \frac{\partial \beta_4}{\partial \mathbf{b}_g} & \frac{\partial \beta_4}{\partial \mathbf{b}_a} & \frac{\partial \beta_4}{\partial \mathbf{p}_c^b} & \frac{\partial \beta_4}{\partial \theta_2}\frac{\partial \theta_2}{\partial \mathbf{s}_2} & \frac{\partial \beta_4}{\partial \mathbf{p}_k^w} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \end{bmatrix}\end{aligned}, \tag{25}$$

then,

$$\begin{aligned}
\frac{\partial \beta_4}{\partial \mathbf{x}}\mathbf{f}_0 &= \mathbf{0} \\
\frac{\partial \beta_4}{\partial \mathbf{x}}\mathbf{f}_1 &= \mathbf{0} \\
\frac{\partial \beta_4}{\partial \mathbf{x}}\mathbf{f}_2 &= \mathbf{0}
\end{aligned} \tag{26}$$

For basis function $\beta_5 = \mathbf{P}_c^b$, we have

$$\begin{aligned}
\frac{\partial \beta_5}{\partial \mathbf{x}} &= \begin{bmatrix} \frac{\partial \beta_5}{\partial \mathbf{p}} & \frac{\partial \beta_5}{\partial \mathbf{v}} & \frac{\partial \beta_5}{\partial \theta_1}\frac{\partial \theta_1}{\partial \mathbf{s}_1} & \frac{\partial \beta_5}{\partial \mathbf{b}_g} & \frac{\partial \beta_5}{\partial \mathbf{b}_a} & \frac{\partial \beta_5}{\partial \mathbf{p}_c^b} & \frac{\partial \beta_5}{\partial \theta_2}\frac{\partial \theta_2}{\partial \mathbf{s}_2} & \frac{\partial \beta_2}{\partial \mathbf{p}_k^w} \end{bmatrix}, \\
&= \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix}
\end{aligned} \tag{27}$$

with,

$$\begin{aligned}
\frac{\partial \beta_5}{\partial \mathbf{x}}\mathbf{f}_0 &= \mathbf{0} \\
\frac{\partial \beta_5}{\partial \mathbf{x}}\mathbf{f}_1 &= \mathbf{0}. \\
\frac{\partial \beta_5}{\partial \mathbf{x}}\mathbf{f}_2 &= \mathbf{0}
\end{aligned} \tag{28}$$

For basis function $\beta_6 = \mathbf{b}_g$, we have

$$\begin{aligned}
\frac{\partial \beta_6}{\partial \mathbf{x}} &= \begin{bmatrix} \frac{\partial \beta_6}{\partial \mathbf{p}} & \frac{\partial \beta_6}{\partial \mathbf{v}} & \frac{\partial \beta_6}{\partial \theta_1}\frac{\partial \theta_1}{\partial \mathbf{s}_1} & \frac{\partial \beta_6}{\partial \mathbf{b}_g} & \frac{\partial \beta_6}{\partial \mathbf{b}_a} & \frac{\partial \beta_6}{\partial \mathbf{p}_c^b} & \frac{\partial \beta_6}{\partial \theta_2}\frac{\partial \theta_2}{\partial \mathbf{s}_2} & \frac{\partial \beta_6}{\partial \mathbf{p}_k^w} \end{bmatrix}, \\
&= \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix}
\end{aligned} \tag{29}$$

with,

$$\begin{aligned}
\frac{\partial \beta_6}{\partial \mathbf{x}}\mathbf{f}_0 &= \mathbf{0} \\
\frac{\partial \beta_6}{\partial \mathbf{x}}\mathbf{f}_1 &= \mathbf{0}. \\
\frac{\partial \beta_6}{\partial \mathbf{x}}\mathbf{f}_2 &= \mathbf{0}
\end{aligned} \tag{30}$$

For basis function $\beta_7 = \mathbf{C}_1\mathbf{g}^w$, we have

$$\begin{aligned}
\frac{\partial \beta_7}{\partial \mathbf{x}} &= \begin{bmatrix} \frac{\partial \beta_7}{\partial \mathbf{p}} & \frac{\partial \beta_7}{\partial \mathbf{v}} & \frac{\partial \beta_7}{\partial \theta_1}\frac{\partial \theta_1}{\partial \mathbf{s}_1} & \frac{\partial \beta_7}{\partial \mathbf{b}_g} & \frac{\partial \beta_7}{\partial \mathbf{b}_a} & \frac{\partial \beta_7}{\partial \mathbf{p}_c^b} & \frac{\partial \beta_7}{\partial \theta_2}\frac{\partial \theta_2}{\partial \mathbf{s}_2} & \frac{\partial \beta_7}{\partial \mathbf{p}_k^w} \end{bmatrix}, \\
&= \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 & \lfloor \mathbf{C}_1\mathbf{g}^w \rfloor_\times \frac{\partial \theta_1}{\partial \mathbf{s}_1} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix}
\end{aligned} \tag{31}$$

with,

$$\begin{aligned}
\frac{\partial \beta_6}{\partial \mathbf{x}}\mathbf{f}_0 &= -\lfloor \mathbf{C}_1\mathbf{g}^w \rfloor_\times \mathbf{b}_g = \lfloor \beta_7 \rfloor_\times \beta_6 \\
\frac{\partial \beta_6}{\partial \mathbf{x}}\mathbf{f}_1 &= \lfloor \mathbf{C}_1\mathbf{g}^w \rfloor_\times = \lfloor \beta_7 \rfloor_\times \\
\frac{\partial \beta_6}{\partial \mathbf{x}}\mathbf{f}_2 &= \mathbf{0}
\end{aligned} \tag{32}$$

For basis function $\beta_8 = \mathbf{b}_a$, we have

$$\begin{aligned}
\frac{\partial \beta_8}{\partial \mathbf{x}} &= \begin{bmatrix} \frac{\partial \beta_8}{\partial \mathbf{p}} & \frac{\partial \beta_8}{\partial \mathbf{v}} & \frac{\partial \beta_8}{\partial \theta_1}\frac{\partial \theta_1}{\partial \mathbf{s}_1} & \frac{\partial \beta_8}{\partial \mathbf{b}_g} & \frac{\partial \beta_8}{\partial \mathbf{b}_a} & \frac{\partial \beta_8}{\partial \mathbf{p}_c^b} & \frac{\partial \beta_8}{\partial \theta_2}\frac{\partial \theta_2}{\partial \mathbf{s}_2} & \frac{\partial \beta_8}{\partial \mathbf{p}_k^w} \end{bmatrix}, \\
&= \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix}
\end{aligned} \tag{33}$$

with,

$$
\begin{aligned}
\frac{\partial \beta_6}{\partial \mathbf{x}} \mathbf{f}_0 &= \mathbf{0} \\
\frac{\partial \beta_6}{\partial \mathbf{x}} \mathbf{f}_1 &= \mathbf{0}. \\
\frac{\partial \beta_6}{\partial \mathbf{x}} \mathbf{f}_2 &= \mathbf{0}
\end{aligned}
\tag{34}
$$

Besides, there is another measurement model $\mathbf{z}_l$ in the proposed method, which is similar with the basis function β_1. We define it as

$$
\beta_9 = \mathbf{z}_l = \frac{1}{p_{z_l}} \begin{bmatrix} p_{x_l} \\ p_{y_l} \end{bmatrix}, \tag{35}
$$

where, $\mathbf{z}_l$ is the pixel coordinate of a endpoint of a line feature under the normalized camera model.

Therefore, referring to the calculation of the basis function β_1, we have

$$
\begin{aligned}
\frac{\partial \beta_9}{\partial \mathbf{x}} &= \begin{bmatrix} \frac{\partial \beta_9}{\partial \mathbf{p}} & \frac{\partial \beta_9}{\partial \mathbf{v}} & \frac{\partial \beta_9}{\partial \theta_1}\frac{\partial \theta_1}{\partial \mathbf{s}_1} & \frac{\partial \beta_9}{\partial \mathbf{b}_g} & \frac{\partial \beta_9}{\partial \mathbf{b}_a} & \frac{\partial \beta_9}{\partial \mathbf{p}_c^b} & \frac{\partial \beta_9}{\partial \theta_2}\frac{\partial \theta_2}{\partial \mathbf{s}_2} & \frac{\partial \beta_9}{\partial \mathbf{p}_k^w} \end{bmatrix} \\
&= \frac{\partial \mathbf{z}_l}{\partial \mathbf{P}_l^c} \cdot \frac{\partial \mathbf{P}_l^c}{\partial \mathbf{x}} \\
&= \begin{bmatrix} \frac{1}{p_{z_l}} & 0 & -\frac{p_{x_l}}{p_{z_l}^2} \\ 0 & \frac{1}{p_{z_l}} & -\frac{p_{y_l}}{p_{z_l}^2} \end{bmatrix} \begin{bmatrix} \mathbf{C}_2\mathbf{C}_1 & \mathbf{0}_3 & \mathbf{C}_2 \lfloor \mathbf{P}_l^b \rfloor_\times \frac{\partial \theta_1}{\partial \mathbf{s}_1} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{C}_2 & \lfloor \mathbf{P}_l^c \rfloor_\times \frac{\partial \theta_2}{\partial \mathbf{s}_2} & \mathbf{0}_3 \end{bmatrix}
\end{aligned}
. \tag{36}
$$

with

$$
\begin{aligned}
\frac{\partial \beta_9}{\partial \mathbf{x}} \mathbf{f}_0 &= \begin{bmatrix} \frac{1}{p_{z_l}} & 0 & -\frac{p_{x_l}}{p_{z_l}^2} \\ 0 & \frac{1}{p_{z_l}} & -\frac{p_{y_l}}{p_{z_l}^2} \end{bmatrix} (\mathbf{C}_2\mathbf{C}_1\mathbf{v} - \mathbf{C}_2 \lfloor \mathbf{P}_l^b \rfloor_\times \mathbf{b}_g) \\
&= \begin{bmatrix} \mathbf{I}_2 & -\beta_9 \end{bmatrix} (\frac{1}{p_{z_l}} \mathbf{C}_2\mathbf{C}_1\mathbf{v} - \lfloor \begin{bmatrix} \beta_9 \\ 1 \end{bmatrix} + \frac{1}{p_{z_l}} \mathbf{C}_2 \mathbf{p}_c^b \rfloor_\times \mathbf{C}_2 \mathbf{b}_g).
\end{aligned}
\tag{37}
$$

Here, we need to define a new basis function

$$
\beta_{10} \triangleq \frac{1}{p_{z_l}} \tag{38}
$$

then,

$$
\begin{aligned}
\frac{\partial \beta_9}{\partial \mathbf{x}} \mathbf{f}_0 &= \begin{bmatrix} \mathbf{I}_2 & -\beta_9 \end{bmatrix} (\beta_{10}\beta_4\beta_3 - \lfloor \begin{bmatrix} \beta_9 \\ 1 \end{bmatrix} + \beta_{10}\beta_4\beta_5 \rfloor_\times \beta_4\beta_6) \\
\frac{\partial \beta_9}{\partial \mathbf{x}} \mathbf{f}_1 &= \begin{bmatrix} \frac{1}{p_{z_l}} & 0 & -\frac{p_{x_l}}{p_{z_l}^2} \\ 0 & \frac{1}{p_{z_l}} & -\frac{p_{y_l}}{p_{z_l}^2} \end{bmatrix} \mathbf{C}_2 \lfloor \mathbf{P}_k^b \rfloor_\times = \begin{bmatrix} \mathbf{I}_2 & -\beta_9 \end{bmatrix} \beta_4 \lfloor \begin{bmatrix} \beta_9 \\ 1 \end{bmatrix} + \beta_{10}\beta_4\beta_5 \rfloor_\times \beta_4 \\
\frac{\partial \beta_9}{\partial \mathbf{x}} \mathbf{f}_1 &= \mathbf{0}
\end{aligned}
\tag{39}
$$

For basis function $\beta_{10} = \frac{1}{p_{z_l}}$, the span space is

$$
\begin{aligned}
\frac{\partial \beta_{10}}{\partial \mathbf{x}} &= \begin{bmatrix} \frac{\partial \beta_{10}}{\partial \mathbf{p}} & \frac{\partial \beta_{10}}{\partial \mathbf{v}} & \frac{\partial \beta_{10}}{\partial \theta_1}\frac{\partial \theta_1}{\partial \mathbf{s}_1} & \frac{\partial \beta_{10}}{\partial \mathbf{b}_g} & \frac{\partial \beta_{10}}{\partial \mathbf{b}_a} & \frac{\partial \beta_{10}}{\partial \mathbf{p}_c^b} & \frac{\partial \beta_{10}}{\partial \theta_2}\frac{\partial \theta_2}{\partial \mathbf{s}_2} & \frac{\partial \beta_{10}}{\partial \mathbf{p}_k^w} \end{bmatrix} \\
&= \frac{\partial \beta_{10}}{\partial \mathbf{P}_l^c} \cdot \frac{\partial \mathbf{P}_l^c}{\partial \mathbf{x}} \\
&= -\frac{1}{p_{z_l}^2} \mathbf{e}_3^\top \begin{bmatrix} \mathbf{C}_2\mathbf{C}_1 & \mathbf{0}_3 & \mathbf{C}_2 \lfloor \mathbf{P}_l^b \rfloor_\times \frac{\partial \theta_1}{\partial \mathbf{s}_1} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{C}_2 & \lfloor \mathbf{P}_l^c \rfloor_\times \frac{\partial \theta_2}{\partial \mathbf{s}_2} & \mathbf{0}_3 \end{bmatrix}
\end{aligned}
, \tag{40}
$$

with,

$$\begin{aligned}
\frac{\partial \beta_{10}}{\partial \mathbf{x}}\mathbf{f}_0 &= -\beta_{10}\mathbf{e}_3^\top(\beta_{10}\beta_4\beta_3 - \lfloor \begin{bmatrix}\beta_9\\1\end{bmatrix} + \beta_{10}\beta_4\beta_5\rfloor_\times \beta_4\beta_6) \\
\frac{\partial \beta_{10}}{\partial \mathbf{x}}\mathbf{f}_1 &= -\beta_{10}\mathbf{e}_3^\top\beta_4\lfloor \begin{bmatrix}\beta_9\\1\end{bmatrix} + \beta_{10}\beta_4\beta_5\rfloor_\times \beta_4 \\
\frac{\partial \beta_{10}}{\partial \mathbf{x}}\mathbf{f}_2 &= \mathbf{0}.
\end{aligned} \tag{41}$$

Until now, we obtain the new nonlinear system constructed with all basis functions $\boldsymbol{\beta}$ based on the *Theorem 1 C(3)*

$$\begin{aligned}
\dot{\boldsymbol{\beta}} = \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial t} = \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{x}}\dot{\mathbf{x}} &= \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{x}}(\mathbf{f}_0 + \mathbf{f}_1\omega + \mathbf{f}_2\mathbf{a}) = \mathbf{g}_0 + \mathbf{g}_1\omega + \mathbf{g}_2\mathbf{a} \\
\mathbf{z}_1 &= \beta_1 \\
\mathbf{z}_2 &= \beta_9
\end{aligned} \tag{42}$$

According to the *Theorem 1 (ii)*, $null(\mathcal{O}) = null(\mathbf{B})$ with $\mathbf{B} \triangleq \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{x}}$. Combining (10), (19), (21), (25), (27), (29), (31), (33), (36), (40), we get $\mathbf{B}$

$$\begin{aligned}
&\mathbf{B} \triangleq \mathbf{B}_1\mathbf{B}_2 \\
&= \begin{bmatrix}
\zeta & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \eta & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3
\end{bmatrix}
\begin{bmatrix}
\mathbf{C}_2\mathbf{C}_1 & \mathbf{0}_3 & \mathbf{C}_2\lfloor \mathbf{P}_k^b\rfloor_\times \frac{\partial \theta_1}{\partial \mathbf{s}_1} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{C}_2 & \lfloor \mathbf{P}_k^c\rfloor_\times \frac{\partial \theta_2}{\partial \mathbf{s}_2} & -\mathbf{C}_2\mathbf{C}_1 \\
\mathbf{C}_2\mathbf{C}_1 & \mathbf{0}_3 & \mathbf{C}_2\lfloor \mathbf{P}_l^b\rfloor_\times \frac{\partial \theta_1}{\partial \mathbf{s}_1} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{C}_2 & \lfloor \mathbf{P}_l^c\rfloor_\times \frac{\partial \theta_2}{\partial \mathbf{s}_2} & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{C}_1 & \lfloor \mathbf{C}_1\mathbf{v}\rfloor_\times \frac{\partial \theta_1}{\partial \mathbf{s}_1} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \lfloor \mathbf{C}_1\mathbf{g}^w\rfloor_\times \frac{\partial \theta_1}{\partial \mathbf{s}_1} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3
\end{bmatrix}
\end{aligned} \tag{43}$$

where,

$$\zeta = \begin{bmatrix} \frac{1}{p_z} & 0 & -\frac{p_x}{p_z^2} \\ 0 & \frac{1}{p_z} & -\frac{p_y}{p_z^2} \\ 0 & 0 & -\frac{1}{p_z^2} \end{bmatrix}, \quad \eta = \begin{bmatrix} \frac{1}{p_{z_l}} & 0 & -\frac{p_{x_l}}{p_{z_l}^2} \\ 0 & \frac{1}{p_{z_l}} & -\frac{p_{y_l}}{p_{z_l}^2} \\ 0 & 0 & -\frac{1}{p_{z_l}^2} \end{bmatrix}. \tag{44}$$

where the first row of the matrix $\mathbf{B}_2$ comes from the combination of $\frac{\partial \beta_1}{\partial \mathbf{x}}$ and $\frac{\partial \beta_2}{\partial \mathbf{x}}$. Similarly, the second row of the matrix $\mathbf{B}_2$ is the combination of $\frac{\partial \beta_9}{\partial \mathbf{x}}$ and $\frac{\partial \beta_{10}}{\partial \mathbf{x}}$. For ease of operation, we adjust it to the second row of $\mathbf{B}_2$.

It is obvious that $\mathbf{B}_1$ is full rank. The null space of $\mathbf{B}_2$ is the same as that of $\mathbf{B}$. Assuming the left null space of $\mathbf{B}_2$ is $\mathbf{N}_L = [\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4, \mathbf{N}_5, \mathbf{N}_6, \mathbf{N}_7, \mathbf{N}_8]$ we have

$$\mathbf{0} = [\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4, \mathbf{N}_5, \mathbf{N}_6, \mathbf{N}_7, \mathbf{N}_8]
\begin{bmatrix}
\mathbf{C}_2\mathbf{C}_1 & \mathbf{0}_3 & \mathbf{C}_2\lfloor \mathbf{P}_k^b\rfloor_\times \frac{\partial \theta_1}{\partial \mathbf{s}_1} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{C}_2 & \lfloor \mathbf{P}_k^c\rfloor_\times \frac{\partial \theta_2}{\partial \mathbf{s}_2} & -\mathbf{C}_2\mathbf{C}_1 \\
\mathbf{C}_2\mathbf{C}_1 & \mathbf{0}_3 & \mathbf{C}_2\lfloor \mathbf{P}_l^b\rfloor_\times \frac{\partial \theta_1}{\partial \mathbf{s}_1} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{C}_2 & \lfloor \mathbf{P}_l^c\rfloor_\times \frac{\partial \theta_2}{\partial \mathbf{s}_2} & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{C}_1 & \lfloor \mathbf{C}_1\mathbf{v}\rfloor_\times \frac{\partial \theta_1}{\partial \mathbf{s}_1} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \lfloor \mathbf{C}_1\mathbf{g}^w\rfloor_\times \frac{\partial \theta_1}{\partial \mathbf{s}_1} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3
\end{bmatrix}. \tag{45}$$

It can be seen from (45) that $\mathbf{N}_3\mathbf{C}_1 = \mathbf{0}$, $\mathbf{N}_6\mathbf{I}_1 = \mathbf{0}$, $\mathbf{N}_8\mathbf{I}_1 = \mathbf{0}$, and $-\mathbf{N}_1\mathbf{C}_2\mathbf{C}_1 = \mathbf{0}$, so $\mathbf{N}_1 = 0$, $\mathbf{N}_3 = 0$, $\mathbf{N}_6 = 0$, and $\mathbf{N}_8 = 0$. Then $\mathbf{N}_2\mathbf{C}_2\mathbf{C}_1 = \mathbf{0} \to \mathbf{N}_2 = \mathbf{0}$, so $\mathbf{N}_5 = \mathbf{0}$ and $\mathbf{N}_4 = \mathbf{0}$ based on the sixth and seventh columns of matrix $\mathbf{B}_2$. Therefore, only the $\mathbf{N}_7$ is non-zero vector, and it can be seen from the third column of $\mathbf{B}_2$,

$$\mathbf{N}_7\lfloor \mathbf{C}_1\mathbf{g}^w \rfloor_\times \frac{\partial \theta_1}{\partial \mathbf{s}_1} = \mathbf{0}. \tag{46}$$

Since the matrix $\frac{\partial \theta_1}{\partial \mathbf{s}_1}$ is full rank, we have

$$\mathbf{N}_7 = \pm(\mathbf{C}_1\mathbf{g}^w)^\top, \tag{47}$$

because of the another skew symmetric matrix operation rule,

$$\lfloor \mathbf{a} \rfloor_\times \mathbf{a} = \mathbf{0}, \quad \mathbf{a}^\top \lfloor \mathbf{a} \rfloor_\times = \mathbf{0} \tag{48}$$

In summary, the left null space of $\mathbf{B}_2$ is

$$\mathbf{N}_L = [\mathbf{0}_{1\times3}, \mathbf{0}_{1\times3}, \mathbf{0}_{1\times3}, \mathbf{0}_{1\times3}, \mathbf{0}_{1\times3}, \mathbf{0}_{1\times3}, (\mathbf{C}_1\mathbf{g}^w)^\top, \mathbf{0}_{1\times3}]_{1\times24} \tag{49}$$

Its dimension is 1. Therefore, the dimension of the right null space of 24×24 matrix $\mathbf{B}_2$ is also 1. The right null space of $\mathbf{B}_2$ is,

$$\mathbf{N}_R = \begin{bmatrix} \lfloor \mathbf{C}_1^\top \mathbf{P}_c^b \rfloor_\times \mathbf{g}^w \\ -\lfloor \mathbf{v} \rfloor_\times \mathbf{g}^w \\ \frac{\partial \mathbf{s}_1}{\partial \theta_1}\mathbf{C}_1\mathbf{g}^w \\ \mathbf{0}_3 \\ \mathbf{0}_3 \\ \mathbf{0}_3 \\ -\frac{\partial \mathbf{s}_2}{\partial \theta_2}\mathbf{C}_2\mathbf{C}_1\mathbf{g}^w \\ \mathbf{0}_3 \end{bmatrix}_{24\times1}. \tag{50}$$

The definition of unobservable state variables is as follows: small perturbation is applied to state variables utilizing the span of right null space, state variables affected by the disturbance are unobservable [2]. It can be seen from $\mathbf{N}_R$ that the only 1 dimension right null space is related to the gravity $\mathbf{g}^w$. Here, we use $\mathbf{N}_R$ to exert a small perturbation on states, which is a small rotation disturbance around the gravity $\mathbf{R}_w^{w'}$, and we have

$$\mathbf{R}_w^{w'} \approx \mathbf{I}_3 - c\lfloor \bar{\mathbf{g}}^w \rfloor_\times, \tag{51}$$

where, c is a constant, $\bar{\mathbf{g}}^w$ is the unit vector along the direction of gravity.

Taking the rotation matrix $\mathbf{C}_1 = \mathbf{R}_w^b$ as an example, the impact of $\mathbf{R}_w^{w'}$ on the rotation matrix is,

$$\begin{aligned} \mathbf{R}_{w'}^b &\approx (\mathbf{I}_3 - \lfloor \delta\theta \rfloor_\times)\mathbf{R}_w^b \\ &= (\mathbf{I}_3 - \lfloor \delta\theta \rfloor_\times)\mathbf{R}_{w'}^b\mathbf{R}_w^{w'} = (\mathbf{I}_3 - \lfloor \delta\theta \rfloor_\times)\mathbf{R}_{w'}^b(\mathbf{I}_3 - c\lfloor \bar{\mathbf{g}}^w \rfloor_\times) \\ &= \mathbf{R}_{w'}^b - \lfloor \delta\theta \rfloor_\times \mathbf{R}_{w'}^b - c\mathbf{R}_{w'}^b\lfloor \bar{\mathbf{g}}^w \rfloor_\times + c\lfloor \delta\theta \rfloor_\times \mathbf{R}_{w'}^b \lfloor \bar{\mathbf{g}}^w \rfloor_\times \\ &\approx \mathbf{R}_{w'}^b - \lfloor \delta\theta \rfloor_\times \mathbf{R}_{w'}^b - c\mathbf{R}_{w'}^b\lfloor \bar{\mathbf{g}}^w \rfloor_\times \end{aligned} \tag{52}$$

$\Longrightarrow$

$$\begin{aligned} \lfloor \delta\theta \rfloor_\times \mathbf{R}_{w'}^b &= -c\mathbf{R}_{w'}^b\lfloor \bar{\mathbf{g}}^w \rfloor_\times \\ \lfloor \delta\theta \rfloor_\times &= -c\mathbf{R}_{w'}^b\lfloor \bar{\mathbf{g}}^w \rfloor_\times {\mathbf{R}_{w'}^b}^\top = -c\lfloor \mathbf{R}_{w'}^b\bar{\mathbf{g}}^w \rfloor_\times \end{aligned} \tag{53}$$

$\Longrightarrow$

$$\begin{aligned} \delta\theta &= -c\mathbf{R}_{w'}^b\bar{\mathbf{g}}^w \\ &= -c\mathbf{R}_w^b\mathbf{R}_w^{w'}\bar{\mathbf{g}}^w = -c\mathbf{R}_w^b\bar{\mathbf{g}}^w \\ &= -\frac{c}{\parallel \mathbf{g}^w \parallel_2}\mathbf{R}_w^b\mathbf{g}^w \end{aligned} \tag{54}$$

since $\mathbf{R}_w^{w'}$ express a rotation around the direction of gravity, then $\mathbf{R}_w^{w'}\bar{\mathbf{g}}^w = \bar{\mathbf{g}}^w$. Based on (52) (53) (54), we have the perturbation of matrix $\mathbf{R}_w^b$ is

$$\delta\mathbf{s}_1 = \frac{\partial\mathbf{s}_1}{\partial\theta_1}\delta\theta = -\frac{c}{\| \mathbf{g}^w \|_2}\frac{\partial\mathbf{s}_1}{\partial\theta_1}\mathbf{R}_w^b\mathbf{g}^w = -\frac{c}{\| \mathbf{g}^w \|_2}\frac{\partial\mathbf{s}}{\partial\theta}\mathbf{C}_1\mathbf{g}^w. \tag{55}$$

For rotation matrix $\mathbf{C}_2 = \mathbf{R}_c^b$, we have the similar derivation. For states other than rotation matrix, we take state v as an example,

$$\begin{aligned}\mathbf{v}' &= \mathbf{R}_w^{w'}\mathbf{x} \approx (\mathbf{I}_3 - c\lfloor\bar{\mathbf{g}}^w\rfloor_\times)\mathbf{v} \\ &= \mathbf{x} - c\lfloor\bar{\mathbf{g}}^w\rfloor_\times\mathbf{v} = \mathbf{x} - \frac{c}{\| \mathbf{g}^w \|_2}(-\lfloor\mathbf{v}\rfloor_\times\mathbf{g}^w)\end{aligned} \tag{56}$$

therefore, the perturbation of state $\mathbf{v}$ is

$$\delta\mathbf{v} = -\frac{c}{\| \mathbf{g}^w \|_2}(-\lfloor\mathbf{v}\rfloor_\times\mathbf{g}^w) \tag{57}$$

Combining (55) and (57), it can be seen that

$$\delta\mathbf{x} = -\frac{c}{\| \mathbf{g}^w \|_2}\mathbf{N}_R \tag{58}$$

The impact of states caused by the small perturbation inducing by the rotation around the vector of gravity is shown in (58), which means that the rotation around the gravity (yaw angle) is unobservable. Among that, the first row of (58) is not strictly equal to $-\frac{c}{\|\mathbf{g}^w\|_2}(-\lfloor\mathbf{p}\rfloor_\times\mathbf{g}^w)$. However, $\mathbf{C}_1^\top\mathbf{P}_c^b = \mathbf{R}_b^w\mathbf{P}_c^b$ can be regarded as the mapping of the extrinsic translation in the global coordinate system.

In summary, the observability analysis indicates that the global x, y, z translation directions of the proposed system after fusing prior line map are observable. The only unobservable direction is corresponds to global rotation about the gravity vector (yaw angle).

References

[1] T. Qin, P. Li, and S. Shen, "Vins-mono: A robust and versatile monocular visual-inertial state estimator," *IEEE Transactions on Robotics*, vol. 34, no. 4, pp. 1004–1020, 2018.

[2] J. A. Hesch, D. G. Kottas, S. L. Bowman, and S. I. Roumeliotis, "Camera-imu-based localization: Observability analysis and consistency improvement," *The International Journal of Robotics Research*, vol. 33, no. 1, pp. 182–201, 2014.

[3] P. Furgale, T. D. Barfoot, and G. Sibley, "Continuous-time batch estimation using temporal basis functions," in *2012 IEEE International Conference on Robotics and Automation*. IEEE, 2012, pp. 2088–2095.

[4] P. C. Hughes, *Spacecraft attitude dynamics*. Courier Corporation, 2012.

[5] X. Gao, T. Zhang, Y. Liu, and Q. Yan, "14 lectures on visual slam: from theory to practice," *Publishing House of Electronics Industry, Beijing*, 2017.