A more general probability-perspective to prove the balanced equations

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201829078

My proof for the balanced equations:

More general probability-perspective → The formulated relation between $p_n(t+dt)$ and $p_n(t)$ → Taking the limitation of dt to zero, to figure out the balanced equations

A slight difference is in the first step so for the second step I would not take the limitation of dt subsequently for more time-saving. In the first step, we would formulate $p_0(t+dt)$ and $p_n(t+dt)$, like the formulas mentioned in our lecture:

$$p_0(t+dt) = p_0(t)(1-\lambda_0 dt) + p_1(t)\mu_1 dt \tag{1}$$

$$p_n(t+dt) = p_{n-1}(t)\lambda_{n-1}dt + p_n(t)(1-\lambda_n dt)(1-\mu_n dt) + p_{n+1}(t)\mu_{n+1}dt \tag{2}$$

The following part I would show that when starting from the probability perspective, we can obtain the same formulas as above.

1. n=0;

$p_0(t+dt)$ can be generated from two possible states, one is $p_0(t)$ with $t_{inter-arrival} \geq dt$, the other one is $p_1(t)$ with $t_{serve} \leq dt$:

$$p_0(t+dt) = p_0(t)\cdot p(t_{inter-arrival} \geq dt) + p_1(t)\cdot p(t_{serve} \leq dt)$$

$$p(t_{inter-arrival} \geq dt) = \int_{dt}^{+\infty} \lambda_0 e^{-\lambda_0 \tau} d\tau, \qquad p(t_{serve} \leq dt) = \int_0^{dt} \mu_1 e^{-\mu_1 \tau} d\tau$$

$$\begin{aligned} p_0(t+dt) &= p_0(t)\cdot\int_{dt}^{+\infty} \lambda_0 e^{-\lambda_0 \tau} d\tau + p_1(t)\cdot\int_0^{dt} \mu_1 e^{-\mu_1 \tau} d\tau \\ &= p_0(t)\cdot\left(-e^{-\lambda_0 \tau}\right)\Big|_{dt}^{+\infty} + p_1(t)\cdot\left(-e^{-\mu_1 \tau}\right)\Big|_0^{dt} \\ &= p_0(t)\cdot\left[0-\left(-e^{-\lambda_0 dt}\right)\right] + p_1(t)\cdot\left[\left(-e^{-\mu_1 dt}\right)-(-1)\right] \\ &= p_0(t)\cdot e^{-\lambda_0 dt} + p_1(t)\cdot\left(1-e^{-\mu_1 dt}\right) \end{aligned}$$

Because we are considering the infinitesimal dt→0, we can take **the Taylor expansion** of $e^{-\lambda_0 dt}$ and $e^{-\mu_1 dt}$: (with the approximation to the first order term of dt when dt is small enough, meaning $(dt)^i = 0$, when $i \ge 2$)

$$e^{-\lambda_0 dt} = 1 - \lambda_0 dt + \frac{1}{2!}(-\lambda_0 dt)^2 + \cdots \longrightarrow 1 - \lambda_0 dt$$
$$e^{-\mu_1 dt} = 1 - \mu_1 dt + \frac{1}{2!}(-\mu_1 dt)^2 + \cdots \longrightarrow 1 - \mu_1 dt$$

Substitute these two formulas back to the equation

$p_0(t+dt) = p_0(t) \cdot e^{-\lambda_0 dt} + p_1(t) \cdot \left(1 - e^{-\mu_1 dt}\right)$ then we can obtain:

$p_0(t+dt) = p_0(t)(1 - \lambda_0 dt) + p_1(t)\mu_1 dt$, which is the same as the equation (1).

2. n>=1;

$p_n(t+dt)$ can be generated from three possible states, one is $p_{n-1}(t)$ with $t_{inter-arrival} \le dt$, another one is $p_n(t)$ with $t_{inter-arrival} \ge dt$ and $t_{serve} \ge dt$, the third one is $p_{n+1}(t)$ with $t_{serve} \le dt$. Hence,

$$\begin{aligned} p_n(t+dt) = {} & p_{n-1}(t) \cdot p(t_{inter-arrival} \le dt) + p_n(t) \cdot p(t_{inter-arrival} \ge dt) \cdot p(t_{serve} \ge dt) \\ & + p_{n+1}(t)\, p(t_{serve} \le dt) \end{aligned}$$

$$\begin{aligned} p_n(t+dt) &= p_{n-1}(t) \cdot \int_0^{dt} \lambda_{n-1} e^{-\lambda_{n-1}\tau} d\tau + p_n(t) \int_{dt}^{+\infty} \lambda_n e^{-\lambda_n \tau} d\tau \int_{dt}^{+\infty} \mu_n e^{-\mu_n \tau} d\tau + p_{n+1}(t) \int_0^{dt} \mu_{n+1} e^{-\mu_{n+1}\tau} d\tau \\ &= p_{n-1}(t) \cdot \left(1 - e^{-\lambda_{n-1} dt}\right) + p_n(t) \cdot e^{-\lambda_n dt} e^{-\mu_n dt} + p_{n+1}(t) \cdot \left(1 - e^{-\mu_{n+1} dt}\right) \end{aligned}$$

Again we utilize the Taylor expansions of $e^{-\lambda_0 dt}$ and $e^{-\mu_1 dt}$, with $e^{-\lambda_n dt} = 1 - \lambda_n dt$ and $e^{-\mu_n dt} = 1 - \mu_n dt$, hence we can formula

$$p_n(t+dt) = p_{n-1}(t) \cdot \lambda_{n-1} dt + p_n(t) \cdot (1 - \lambda_n dt) \cdot (1 - \mu_n dt) + p_{n+1}(t) \cdot \mu_{n+1} dt$$

where we can obtain the equation shown as (2).

The subsequent procedure following equations $(1)(2)$ is similar to that in our lecture slide. My confusion is that based on the assumption with dt small (and we take this to zero so it is small) so that only zero or one even can occur, we had better start our story from the basic probability perspective:

$$p_0(t+dt) = p_0(t) \cdot p(t_{\text{int}er-arrival} \geq dt) + p_1(t) \cdot p(t_{serve} \leq dt)$$

and

$$p_n(t+dt) = p_{n-1}(t) \cdot p(t_{\text{int}er-arrival} \leq dt) + p_n(t) \cdot p(t_{\text{int}er-arrival} \geq dt) \cdot p(t_{serve} \geq dt) \\ + p_{n+1}(t)\, p(t_{serve} \leq dt)$$

And the equations (1)(2) which we have written directly in our lecture slide, are indeed the mathematical outcomes of the cumulated probability from the exponential distribution with the approximation at infinitesimal dt→0.