**Task 8.2.1**

The fixed point for this system can be computed as

$$\dot{x}_1 = x_2 = 0$$
$$\dot{x}_2 = -\beta \cdot x_2 - \gamma \cdot \sin x_1 = 0$$

In this case, the range of pitch angle can be restricted within (-π/2,π/2). Thus, the($x_1$,$x_2$)=(0,0) is an equilibrium point for this system.

The system is nonlinear due to the term of sine. It can be linearized through Taylor expansion. Thus, the Jacobian matrix around this equilibrium point can be derived as

$$J = \begin{bmatrix} 0 & 1 \\ -\gamma \cos x_1 & -\beta \end{bmatrix}_{(x_1=0,x_2=0)} = \begin{bmatrix} 0 & 1 \\ -\gamma & -\beta \end{bmatrix}$$

The system model can be linearized as

$$\dot{x} = J \cdot x$$

**Task 8.2.2**

The eigenvalue of the system can be computed as

$$\lambda^2 + \beta\lambda + \gamma = 0$$
$$\lambda_{1,2} = -\frac{\beta}{2} \pm \sqrt{\frac{\beta^2 - 4\gamma}{4}} = -\frac{\beta}{2} \pm \sqrt{\gamma - \frac{\beta^2}{4}} j$$

Thus,

$$\omega = \sqrt{\gamma - \frac{\beta^2}{4}} \approx \sqrt{\gamma}$$

Thus the frequency of the system is firstly measured and the γ can be determined. Then tune the value of β to get a desired damping ratio for this system. From the simulated results below, it can be seen the trajectories are very similar near at the beginning stage.

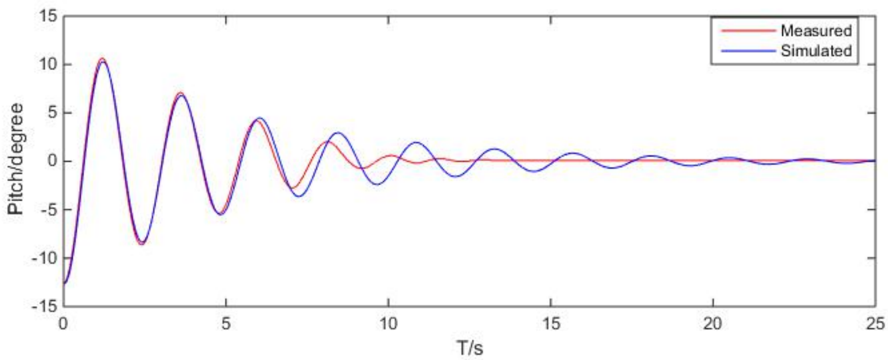


Figure 1

**Task 8.2.3**

Comparing the phase portraits of the nonlinear model and linearized model, they are very similar (topologically equivalent) around the equilibrium point. Thus the simulated trajectory in time domain fits well near zero pitch angle. However, when approaching zero, the discrepancies seems to increase. The period decreases as the oscillation decays very quickly.

According to the previous experience with this device, the friction may contribute to this phenomenon. According to the friction model, when the motion is slow ,the friction should get larger, which leads to the different trajectories around the equilibrium point.

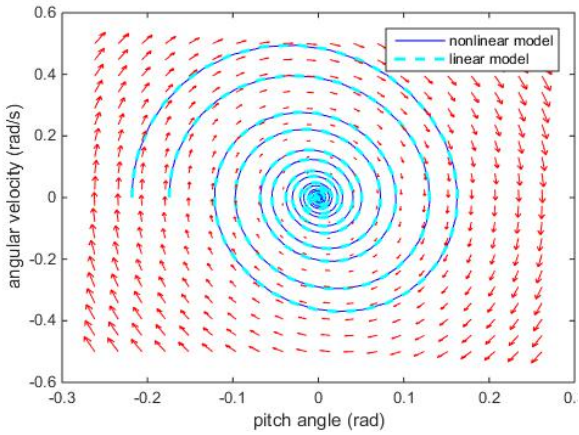


Figure 2 The Phase Portrait of Nonlinear and Linearized System

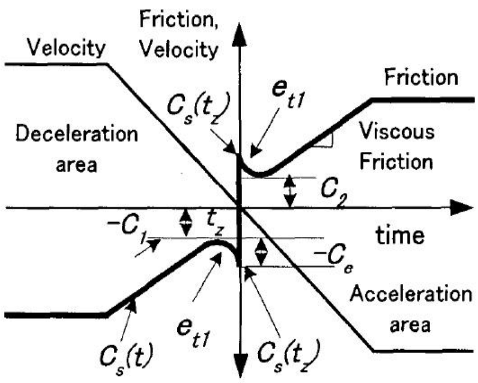


Figure 3 Conventional Friction Model

**Task 8.2.4**

The unmeasured velocity can be estimated as zero in the minimum or maximum position. Thus, the initial state ($x_1$,$x_2$) can be determined , which gives a good initial value for observer design. On the other hand, the system is slow around these points. The tracking performance could be better in these position. It is easy to tune the parameters for tracking even if the resolution of the sensor is not very high.

**Task 8.3.1**

The C matrix is

$$C=\begin{bmatrix}1 & 0\end{bmatrix}$$

The observability matrix is

$$O=\begin{bmatrix}C\\ CA\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$rank\left(O\right)=2$$

The computation shows that the observability matrix is full rank. Thus the system is observable.

According to the Ackermann's Formula, the results of observer gain are showed as below.

| Gain Matrix | Observer Poles |
|---|---|
| $[2.5200 \quad -6.0096]^T$ | [ -1  -2 ] |
| $[1.5200 \quad -5.5296]^T$ | [ -1+i  -1-i ] |
| $[19.5200 \quad 183.8304]^T$ | [-10+10*i  -10-10*i ] |
| $[39.5200 \quad 774.2304]^T$ | [-20+20*i  -20-20*i ] |

Chart 1 The Gain Matrix for Each Poles Placement

**Task 8.3.2**

In case 1, the tracking performance is not good. There are no overshoot in the beginning but exists steady state error on the final stage. If implemented with a disturbance, it takes time to restore the tracking.

In case 2, the two poles are conjugate. The estimated results are very similar to case 1.

In case 3, the estimated pitch angle matches well with the measured one. The tracking error is very small when implemented with a disturbance. The overshoot occurs at the beginning stage of the response due to the large gain.

In case 4, there is not so much difference in pitch angle tracking with case 3. However, the initial overshoot is violent. Furthermore, the estimated angular velocity becomes a little bit noisy.

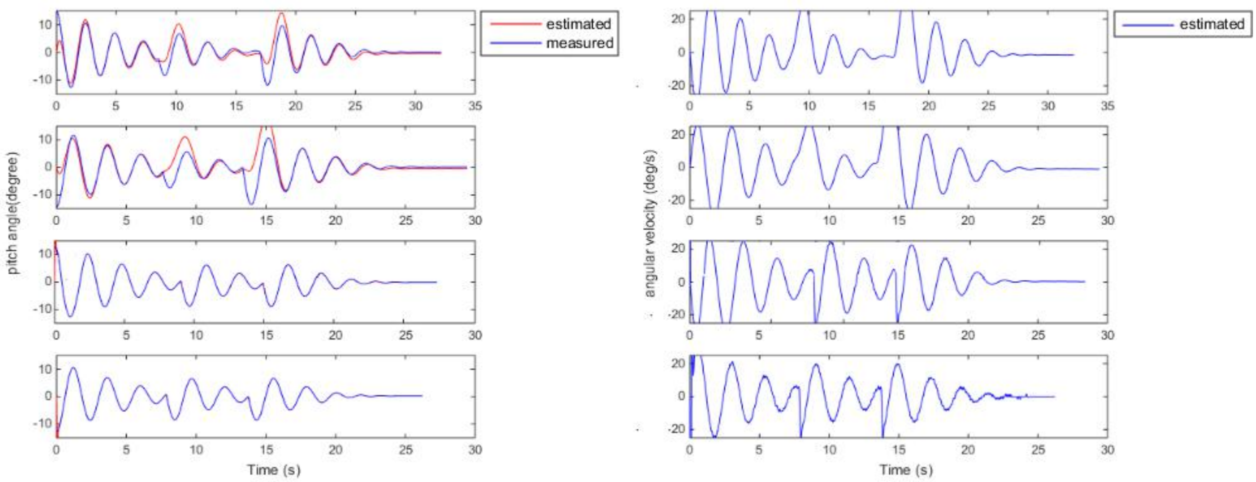


Figure 4 For the pitch angle plot: blue lines are measured ones, red lines are estimated ones
The blue lines on the right show the estimated angular velocity

**Task 8.3.3**

From the previous case, the observed velocity becomes noisy if the real part of the pole is 20 . So it can be deducted that the location of the poles should be the cause. To verify this, the faster poles are tested. The results show that if placing the poles too far in the left half plane, Figure shows the case when the poles locate in [-40+40*i -40-40*i]. At the beginning stage of the observation, the starting point is $(x_1,x_2)= (0,0)$ while the reference is $(x_1,x_2)= (-14.24,0)$. The response may oscillate a lot due to the large gain. The computed gain of the position is relative small so that the simulated position is acceptable, while the values of velocity should be very noisy. If the real part of the poles decrease to -50, the observed position and velocity get unstable. This phenomenon can be explained as the fast poles increases the bandwidth of the observer so that the model tends to be sensitive to noise.

To get a good performance, one useful guideline is to place the observer poles faster (in practice 5 to 10 times) than the controller poles . It enables the observer converges faster than the systems, which indicates a good tracking behaviour. The initial condition, similar to the previous case, can be given from the measurement from the maximum or minimum position, in order to avoid the violent initial transient response.

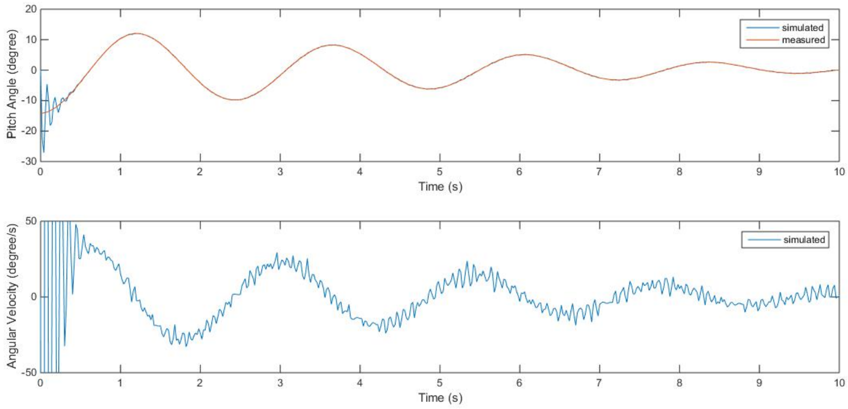


Figure 5 The observed states when the real part of poles is -40

**Task 8.4.1**

Double the model parameters to test the robustness for the observer. The results are showed in figure 6. For the first two cases, the tracking performance is much worse than the accurate model. For the latter two cases, the observed pitch angles match well with the measured one. However, the overshoot for both pitch angles and angular velocity get larger than the accurate model.

**Task 8.4.2**

Using the five-points stencil to compute the derivative of the pitch angle, i.e. the angular velocity. The red lines in Figure 6 indicate the numerical results. The numerical results is not so smooth as the observed ones due to the computation errors. Furthermore,It is obvious that the observed velocity is not consistent with the actual one due to the inaccurate system model. The large gain can forced the observed results track the measured one. However, for those unmeasured states, the estimated values could be very wrong. For the reason that the difference equations have to use the value in future, unless the predicted value can be got (e.g. Using methods like MPC.etc), these numerical methods cannot be used in real time.

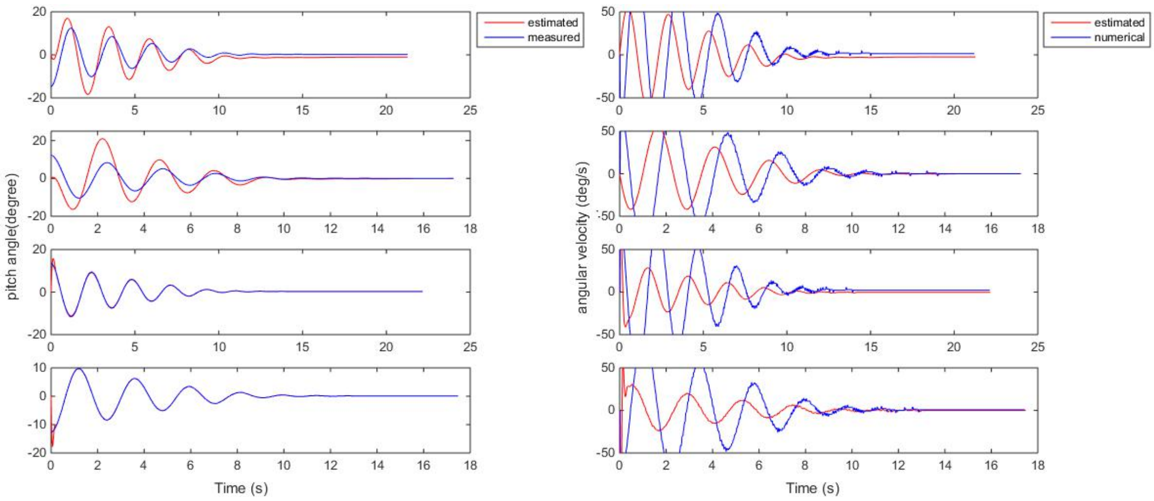


Figure 6 The measured and observed results for inaccurate model