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Optimal Control Laboratory:
Optimal Controller Design with the LQG method

Q1

Suppose we have the plant

$$\dot{x} = Ax + Bu + \upsilon_1$$
$$y = Cx + \upsilon_2$$

The noise terms are white noise with covariance Qbar and Rbar respectively.
The aim is to minimize the cost function

$$J = E\left[\int_0^\infty \left(x^T Q x + u^T R u\right) dt\right]$$

Where Q=M^TM , Qbar=WW^T are designed parameters.
Thus, in order to use LQG method to stabilize a system, it required that:

1. The system can be described by a linear state space model
2. The pair [A,B] should be stabilizable
3. The pair [A,C] should be detectable
4. The pair [A,M] should be detectable
5. The pair [A,W] should be stabilizable

The quadrotor system with good designed parameters of Q and Qbar can satisfy the above requirements. The details are showed below.

For this quadrotor system, the linear state space model can be derived by linearization around the hover position. It should be noticed that the linear model can only satisfy the occasion where deviation around the hover position is small enough.

$$\dot{x} = Ax + Bu$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \\ \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \theta \\ \psi \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{lk_f}{J_x} & 0 & -\frac{lk_f}{J_x} \\ \frac{lk_f}{J_y} & 0 & -\frac{lk_f}{J_y} & 0 \\ \frac{k_{tn}}{J_z} & -\frac{k_{tn}}{J_z} & \frac{k_{tn}}{J_z} & -\frac{k_{tn}}{J_z} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$$

The output y should be the vector of yaw , pitch and roll angle, which can be measured by the encoders.

$y = Cx$

$$\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \theta \\ \psi \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

It is obvious that the system are stabilizable and detectable. Then we should design the optimal observer and state feedback controller using LQG, to make the overall system stable.

Designing the Qbar as BB^T, in other hands, W=B. Thus the pair [A W] is also stabilizable. Since the angle can be measured and the angular velocity can be computed, the pair [A M] is also detectable. Therefore, all the assumptions of LQG method are satisfied.

The above LQG method can improve the stability through pole placement. However , we want the quadrotor can also track some references. Thus the integral term can be added to the controller. The extra states of error are introduced.

$$z = \int_0^t (y - r) dt$$

$$\begin{bmatrix} z_\phi \\ z_\theta \\ z_\psi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \theta \\ \psi \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} - \begin{bmatrix} \phi_d \\ \theta_d \\ \psi_d \end{bmatrix}$$

The augmented state space can be represented as

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u - \begin{bmatrix} 0 \\ I \end{bmatrix} r$$

In conclusion, the LQG with integral action is appropriate for the quadrotor system with small reference around the hover position.

Q2

In practice the PID and MPC are two common-used controller for the quadrotor system.

The PID controller is easy to implement and tune. It does not need much information about the model. It can be implemented on both linear and nonlinear system. Moreover, there are only three parameters to tune. The tuning criterion is directly based on the dynamics performance. For this quadrotor system, we can use the angle error as feedback, and design a PID controller. In other words, the input voltage is directly determined by the error in angle.

MPC was developed based on LQ control[3]. It is a kind of online optimization which considers to minimize the cost function of both the error in output and input energy within a prediction horizon.

$$J=\sum_{j=N_1}^{N_2}(y_{k+j}-r_{k+j})^T Q(y_{k+j}-r_{k+j})+\sum_{j=1}^{N_u}u_{k+j-1}^T R u_{k+j-1}$$

It can be interpreted that MPC aims to track reference with minimize input in a prediction horizon. MPC has advantage on dealing with the change in reference due to the use of adjustable prediction horizon. It can also deal with the constraint of the system. Since the MPC and LQG are both methods of state feedback, the MPC is also feasible to this quadrotor system.

Q3

It can be found that the selection of Rbar is essential to the stability of the hardware system. Rbar represents the variance in the output noise. If we select a small Rbar, which means we trust the measurement a lot. Then, the simulation result should be good while the hardware performance is nightmare due to noise. We can find that the smaller the Rbar, the larger the poles of (A-LC) is, in the other words, the faster the observer converges to the output in the simulation environment. We want the poles of observer is 10 times faster than the poles of state feedback. Therefore, the desired diagonal terms in Rbar can be selected as 0.01.

The selection of R is the weighting of the input energy. The larger the R is, the smaller energy will be consumed. We want the voltage is O(10) while the angle(rad) and angular velocity are both O(0.01), so the KK should be O(1000) in this case. It should be noticed that the scale between Q and R in the cost function decides the value of input voltage. Suppose the elements in Q is O(10000), the appropriate R can be 0.01.

The selection of Q is tricky because it has some relationships with the performance. Much effort is paid on tuning the elements in Q in order to get a desired dynamics performance in hardware rather than the ideal simulation environment.

The final selected parameters are

$$Q=\begin{bmatrix}400000 &&&&&&&& \\ & 400000 &&&&&&& \\ && 400000 &&&&&& \\ &&& 20 &&&&& \\ &&&& 20 &&&& \\ &&&&& 20 &&& \\ &&&&&& 80000 && \\ &&&&&&& 80000 & \\ &&&&&&&& 80000\end{bmatrix}$$

$$R = \begin{bmatrix} 0.01 & & \\ & 0.01 & \\ & & 0.01 \end{bmatrix}$$

$$\overline{Q} = \begin{bmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0.0042 & & \\ & & & & 0.3587 & \\ & & & & & 0.3587 \end{bmatrix}$$

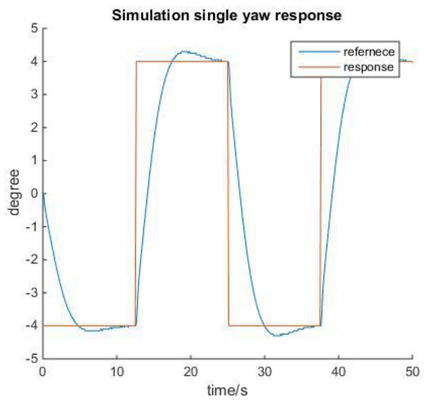
$$\overline{R} = \begin{bmatrix} 0.001 & & \\ & 0.001 & \\ & & 0.001 \end{bmatrix}$$

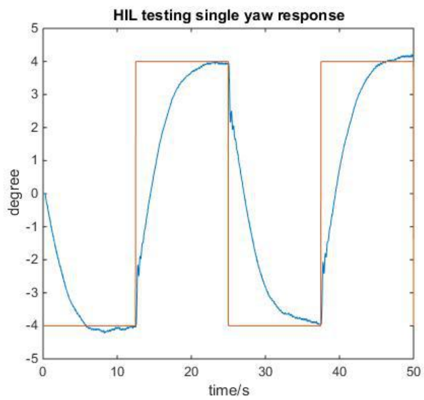
$$KK = \begin{bmatrix} -3302.2 & 4540 & 0 & -226.2 & 108.3 & 0 & -2000 & 2826.4 & 0 \\ -3302.2 & -4540 & 0 & -226.2 & -108.3 & 0 & -2000 & 2826.4 & 0 \\ 3302.2 & 0 & 4540.1 & 226.2 & 0 & 108.3 & 2000 & 0 & 2828.4 \\ 3302.2 & 0 & -4540.1 & 226.2 & 0 & -108.3 & 2000 & 0 & -2828.4 \end{bmatrix}$$

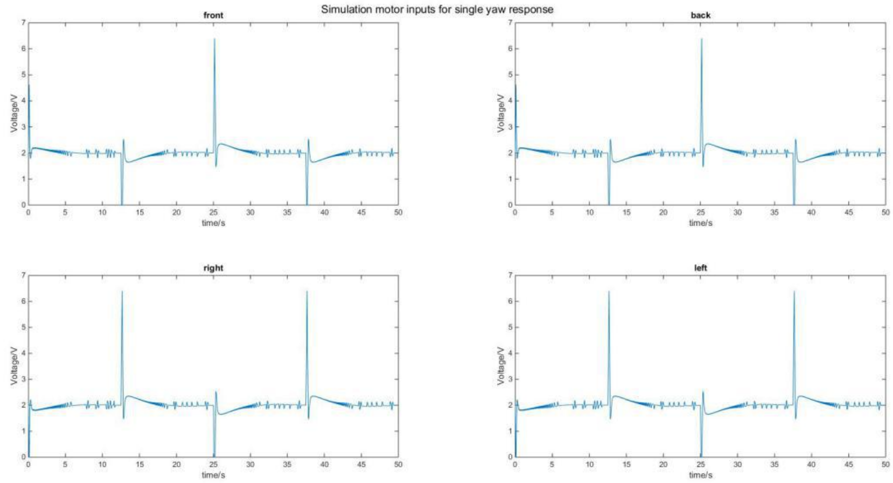
$$L = \begin{bmatrix} 2.0305 & & \\ & 6.1645 & \\ & & 6.1545 \\ 2.0615 & & \\ & 18.9387 & \\ & & 18.9387 \end{bmatrix}$$

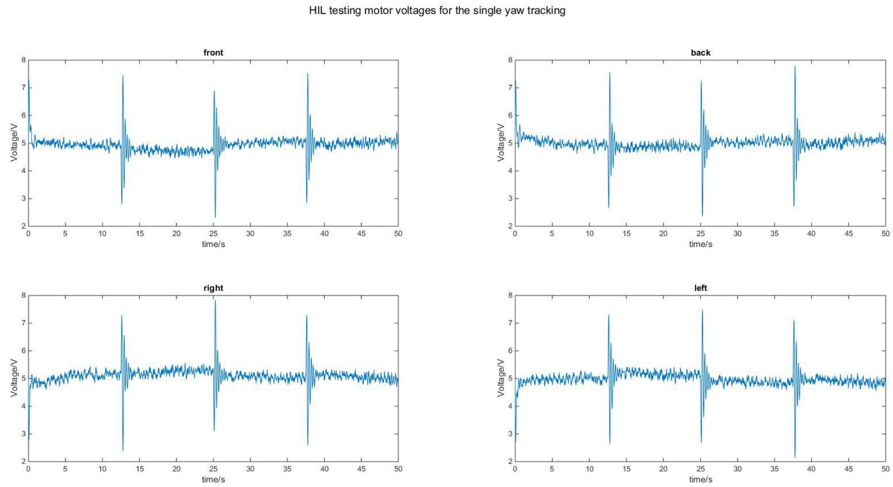
Q4

1.Single yaw tracking

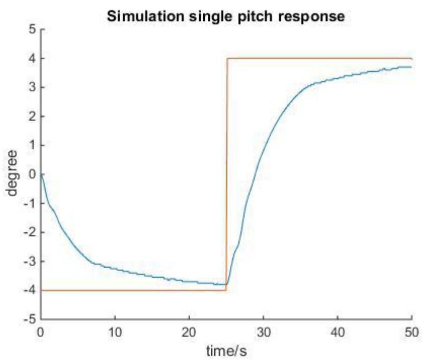


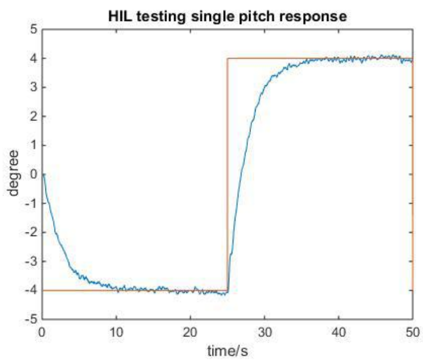


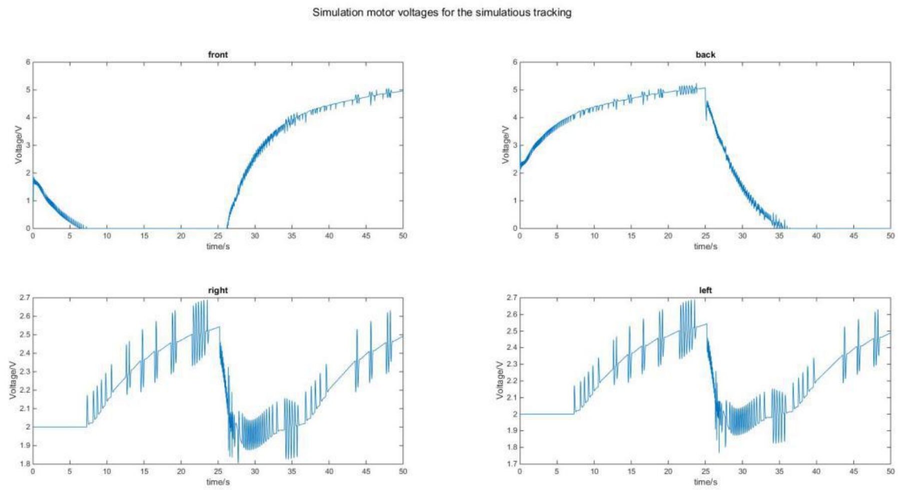


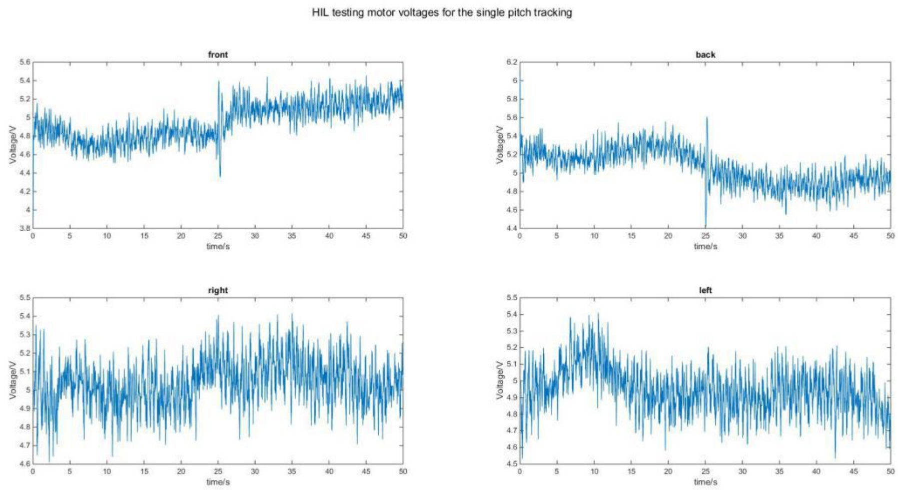


2.Single pitch tracking

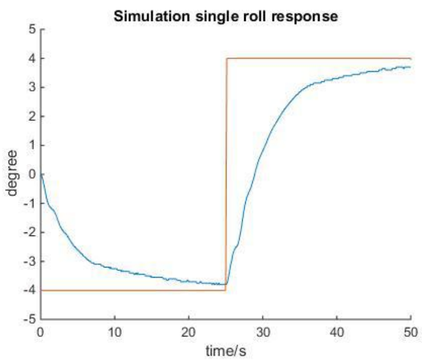


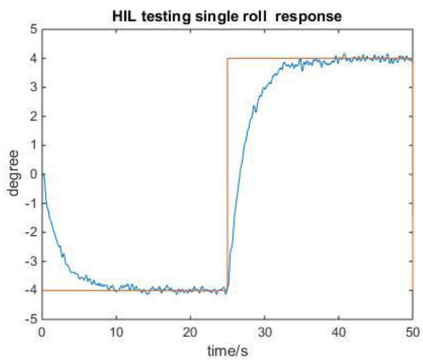


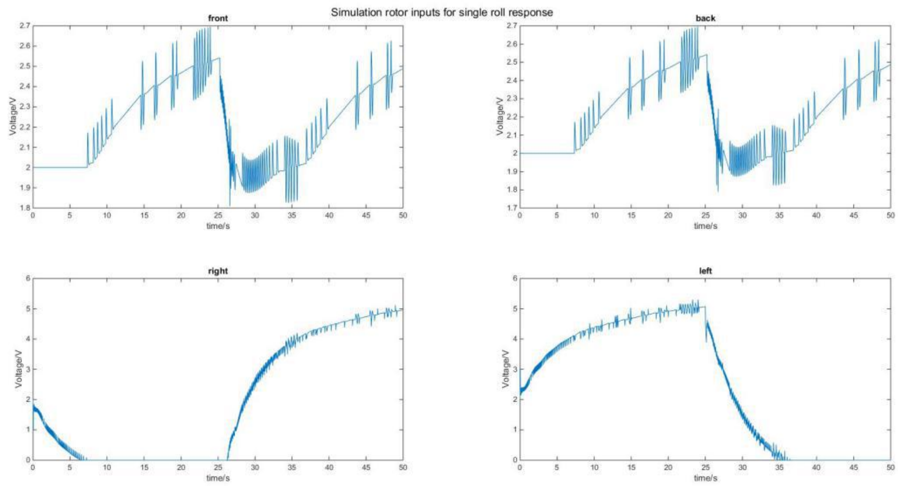


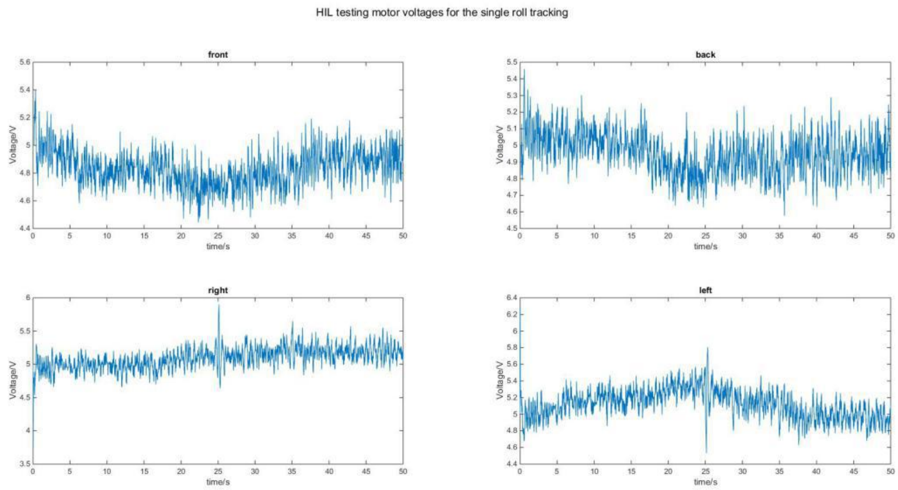


3.Single roll tracking

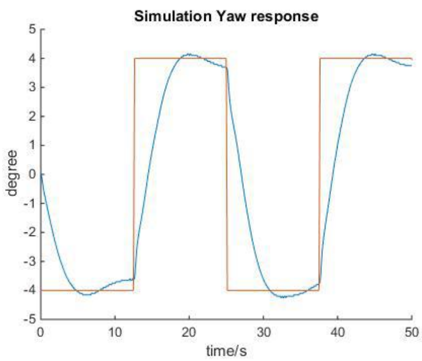


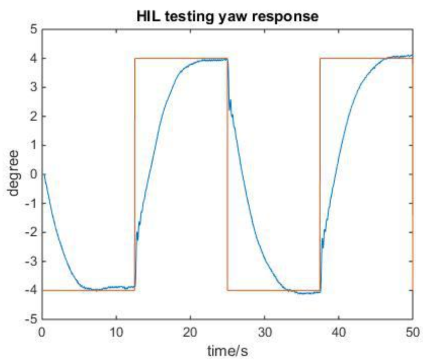


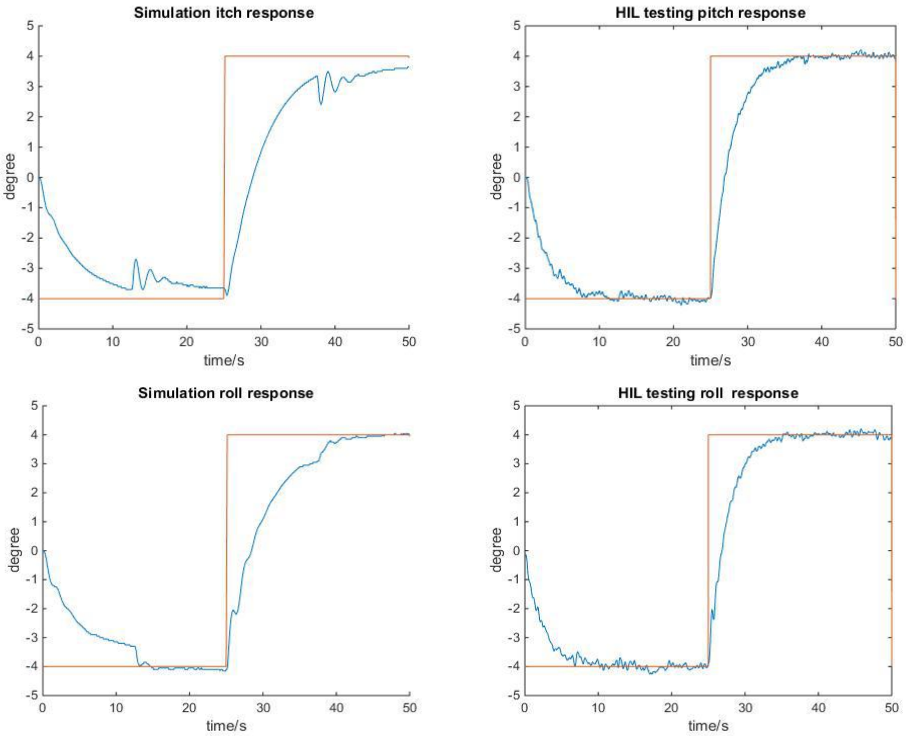


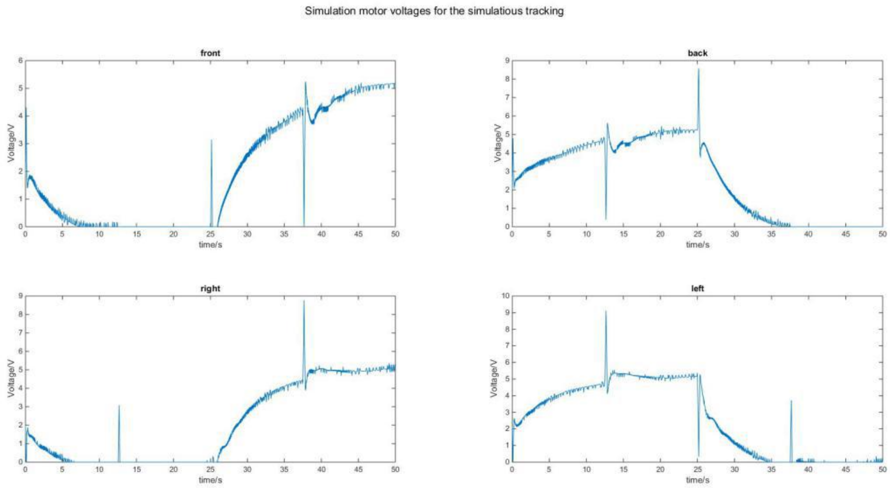


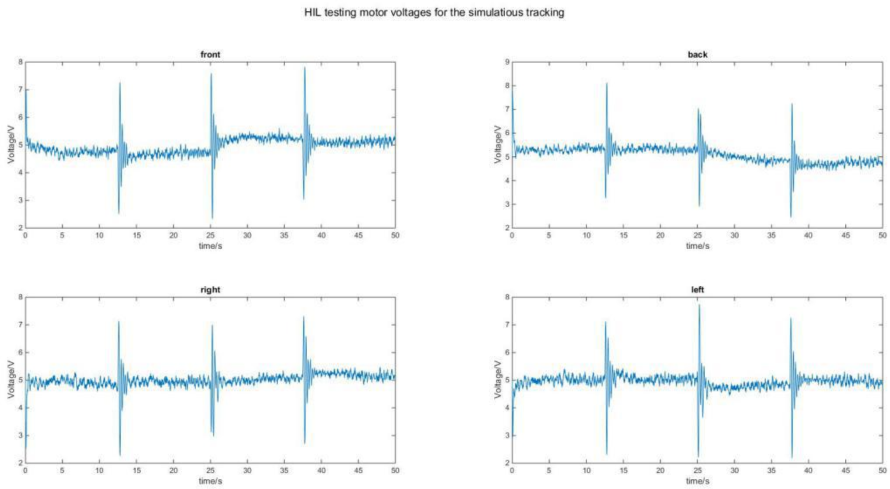
4.Simultaneous tracking






Simulation itch response
HIL testing pitch response
Simulation roll response
HIL testing roll response
degree
time/s


Simulation motor voltages for the simulatious tracking
front
back
right
left
Voltage/V
time/s


HIL testing motor voltages for the simulatious tracking
front
back
right
left
Voltage/V
time/s

Q5

Table 5.1 Simulation results for single tracking

	Single yaw tracking	Single pitch tracking	Single roll tracking
Steady-state error	0	0.2	0.2
Rise time	3.5	10.7	10.7
Peak overshoot	0.3	0	0
Motor peak-peak voltage	6.4	5.1	5

Table 5.2 Simulation results for simultaneous tracking

	Yaw tracking	Pitch tracking	Roll tracking
Steady-state error	0.3	0.35	0
Rise time	4.5	10	12.3
Peak overshoot	0.15	0	0
Motor peak-peak voltage	8.6		

Table 5.3 HIL testing results for single tracking

	Single yaw tracking	Single pitch tracking	Single roll tracking
Steady-state error	0.08	0	0
Rise time	5.4	5.6	5.6
Peak overshoot	0	0	0
Motor peak-peak voltage	5.5	1.1	1.2

Table 5.4 HIL testing results for simultaneous tracking

	Yaw tracking	Pitch tracking	Roll tracking
Steady-state error	0.05	0	0
Rise time	5.4	6	5.3
Peak overshoot	0	0	0
Motor peak-peak voltage	5.5		

Q6

Comparing the simulation and experimental results, the true performance are better than the simulation, due to the discrepancies between the real system and mathematical model.

1. The dynamics of systems

In the hardware, the rise time of tracking the yaw is slower and the system no loner has overshoot . The friction in the joints may introduce this error. The rise time of pitch and roll are much faster than the simulation. It means the actual poles of these two state are bigger than simulation. Some parameters of system may be inaccurate.

When balance only the pitch or roll, only two of the motors are used. We can say the control of pitch or roll are independent to the other two angles. Then it is easy to control. However, when change the yaw angle, both the pitch and roll will be affected. In simulation environment, this phenomenon is obvious. The roll and pitch oscillate a lot if the yaw changes. In realistic ,due to the faster converging speed of pitch and roll, this disturbance is not obvious.

2. The tracking of reference

The steady state error is reduced in the hardware test. Since the tracking is realized by integral terms. The simulating model is continuous while the real system is discrete. Thus the integral error should be of different value, resulting in different desired integral gain to eliminate the steady state error.

In hardware, the system has small oscillation even if it reaches the steady state. This error can be derived from: 1.The electric elements: The uncertainty in motors and transmission devices will have some effect. The observation is noisy due to the uncertainties in encoders. 2.The physical elements: The mechanical structure itself should vibrate. The complexity behaviours of aerodynamics will also contribute to the uncertainty.

3. The input energy

In the hardware, all the four voltages finally converge to a value around 5 V. However, in the simulation environment, the voltages may converge to other values, like 0, 2 and 5. The overall curve also looks very different. Since the performance of system is similar, we can trust the simulation results. So it can be inferred that the true voltages are in fact recorded in different way from the simulation voltages.

The voltage is proportional to the input signal. So the phenomenon of fast growth in voltage is consistent with the fast convergence of states. Moreover, the voltage signal is noisy compared to the simulation results. The reason is , in the simulation environment, we do not consider the noise.

It should also be noticed that the initial value of states are zeros. Thus the first step is half of the later ones. If the states cannot converge to steady state, the integral error will be accumulated. This will affect the later response.

Reference:
[1]Optimal Control W.P.Heath
[2]Optimal Control Laboratory: Optimal Controller Design with the LQG method
[3]Fundamentals on Model Predictive Control Alexandru Stancu