Stevens Institute of Technology

SYS-611 Homework #3

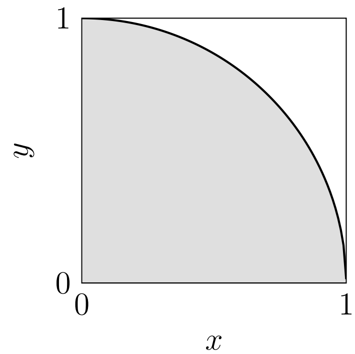
Due Sep. 28 2022

Submit the following using the online submission system: 1) Cover sheet with name, date, and collaborators, 2) Written responses in PDF format, 3) All work (e.g. .xlsx or .py files).

3.1 Simulating π Again [5 points]

In class, we performed Buffon's needle dropping experiment to simulate the value of π. This problem illustrates a second common method to simulate the value of π.

Imagine dropping grains of rice onto the area shown where the gray region defined by $x^2 + y^2 \leq 1$ represents a quarter of a unit circle (with radius 1) inside a square with area 1. If evenly distributed, the probability of a rice grain landing within the gray region is proportional to its area,



$$P\{\text{rice in gray}\} = \frac{\text{area of gray quarter circle}}{\text{area of white square}} = \frac{\pi \cdot 1^2/4}{1 \cdot 1} = \frac{\pi}{4}$$

We can simulate this problem by assigning the elementary random variables for the (X, Y) location of a randomly-dropped grain of rice:

$$X \sim \text{uniform}(0, 1)$$
$$Y \sim \text{uniform}(0, 1)$$

and compute the derived random variable Z signaling if the rice falls inside the gray area:

$$Z = \begin{cases} 4 & \text{if } \sqrt{X^2 + Y^2} \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\mu_z = E[Z] = 4 \cdot \frac{\pi}{4} = \pi$.

(a) 1 pt Identify or implement process generators for the two random variables X and Y and implement a derived state variable function Z based on the equation above.

(b) 2 pts Perform a Monte Carlo simulation analysis and provide a 95% confidence interval for μ_z using $N = 1000$ samples (*Hint*: it should be close to π!).

(c) 2 pts Create a second process generator using the antithetic derived state variable Z_a:

$$Z_a = \frac{Z_1 + Z_2}{2}, \quad Z_1 = \begin{cases} 4 & \text{if } \sqrt{X^2 + Y^2} \leq 1 \\ 0 & \text{otherwise} \end{cases}, \quad Z_2 = \begin{cases} 4 & \text{if } \sqrt{(1-X)^2 + (1-Y)^2} \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Repeat the Monte Carlo simulation analysis and provide a 95% confidence interval for μ_z using $N = 1000$ samples (*Hint*: it should be close to π!).

3.2 Aircraft Range Estimation [9 points]

The Breguet range equation expresses aircraft range R (the distance it can fly without refueling) for steady, level flight as:

$$R = V \times \left(\frac{L}{D}\right) \times I_{sp} \times \ln\left(\frac{W_i}{W_f}\right)$$

where

$$\begin{aligned}
R &: \text{aircraft range [m]} \\
V &: \text{velocity of the aircraft [m/s]} \\
\frac{L}{D} &: \text{lift-to-drag ratio as a measure of aerodynamic efficiency [-]} \\
I_{sp} &: \text{specific impulse of the propulsion system as a measure of fuel efficiency [s]} \\
\frac{W_i}{W_f} &: \text{ratio of the initial (with fuel) to final (without fuel) weight of the aircraft [-]}
\end{aligned}$$

During design of a new aircraft, some parameters are uncertain but constrained by lower- and upper-bound requirements. A systems engineer provides the following estimates:

$$\begin{aligned}
V &= 255 \text{ [m/s]} \\
\frac{L}{D} &\sim \text{uniform}(a = 15, b = 18) \\
I_{sp} &= 5950 \text{ [s]} \\
\frac{W_i}{W_f} &\sim \text{ramp_up}(a = 1.3, b = 1.5)
\end{aligned}$$

where ramp_up(a, b) refers to a triangular distribution with lower bound a and mode/upper bound b.[1]

(a) 2 PTS Identify or implement process generators for the random variables $\frac{L}{D}$ and $\frac{W_i}{W_f}$.

(b) 2 PTS Implement a function to compute the derived state variable R as a function of known and sampled variables.

(c) 5 PTS Perform a Monte Carlo simulation analysis using $N = 1000$ samples and report:

   (i) Plot a histogram of the samples using appropriately-sized bins in kilometers [km]
   (ii) Sample mean ($\bar{R}$) in kilometers [km]
   (iii) 25th percentile sample in kilometers [km] (*Hint*: find the 250th smallest sample)
   (iv) Point estimate of probability of exceeding 9000 km range (*Hint:* find the number of samples with range above 9000 km divided by total number of samples.)

[1] See Chapter 3 of Farr's course notes (p. 3-50) for help on ramp-up process generators. Note the ramp-up cumulative distribution function (CDF) is given by:

$$F(x) = \left(\frac{x - a}{b - a}\right)^2, \quad a \leq x \leq b$$

3.3 Sven's Sweets [11 points]

Sven's Sweets sells gourmet cookies. His wholesale orders for special occasions must be placed in 10-box cases. The "Spooky Special" box costs Sven \$10 wholesale and sells for \$12 retail. Any boxes not yet sold by October 31 (Halloween) are discounted by 75% (to \$3) to sell quickly and clear inventory space. Sven estimates demand at full price as a triangular distribution with a minimum of 40 boxes, a mode of 70 boxes, and a maximum of 90 boxes.

(a) 1 PT Treat customer demand before Halloween (D) as a continuous random variable. (This is an elementary state variable.) Plot its probability density function.

(b) 1 PT What is the "design" variable or decision which Sven must make? (This is also an elementary state variable for the model.)

(c) 2 PTS What is the derived state variable which Sven seeks to maximize? Write it as a mathematical function in terms of the primary random variable in (a) and the design variable in (b). (*Hint:* define piece-wise to handle multiple conditions.)

(d) 2 PTS Implement a process generator to sample a value for (a). Use built-in process generators if desired.[2]

(e) 1 PT Implement a function to compute the derived state variable in (c) using a selected value for (b) and a sampled value for (a).

(f) 4 PTS Perform Monte Carlo simulation experiments for various values of (b) and make a recommendation to Sven supported by a table and/or plots to explain your analysis.

[2]See Chapter 3 of Farr's course notes (p. 3-51) for help on triangular process generators.