Week 4
Graphs Basics

INFSCI 2591

Spring 2023

Learning Objective

- As you have noticed in the textbook, many real-world problems are modeled as graphs. Since the textbook has limited information on graphs, the topic of this week is graph theory and basics of graphs is the main learning objective
- The primary purposes for this week's learning objective are
  - Review some of the basics of graphs discussed in the textbook
  - Provide additional graph details
- Read all the slides in this file as a general information and read further details, as needed, from other textbooks on algorithms, textbooks on graph theory, and other resources including online resources

Graph Definition

$$G = (V, E)$$

u and v: end nodes of a link

V set of nodes (vertices)

$$(u, v) \in E$$

E set of links (edges)

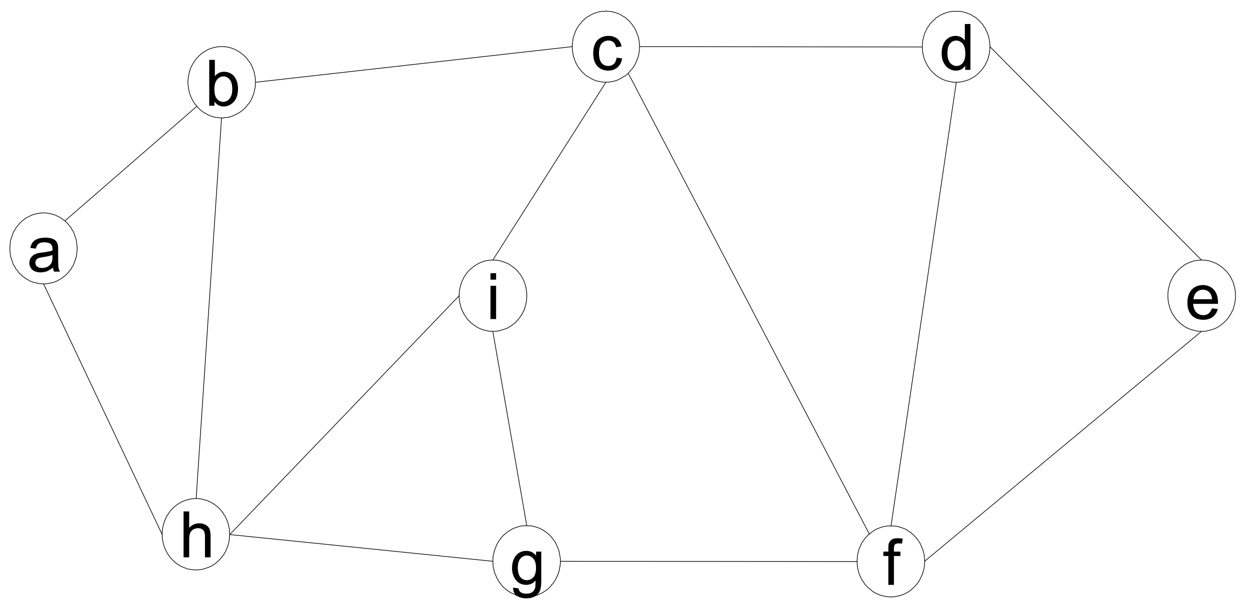
A Graph

- A graph can have
  - One or many vertices but no or a few edges
- There cannot be a graph with edges but no vertices
- In other words:
  - V can not be empty but E can be empty

Types of Graphs

- Undirected
  - Edges have no directions
    - All are two-way edges
- Directed (digraph)
  - Edges have directions
    - At least one, among all edges in the graph, is a one-way edge

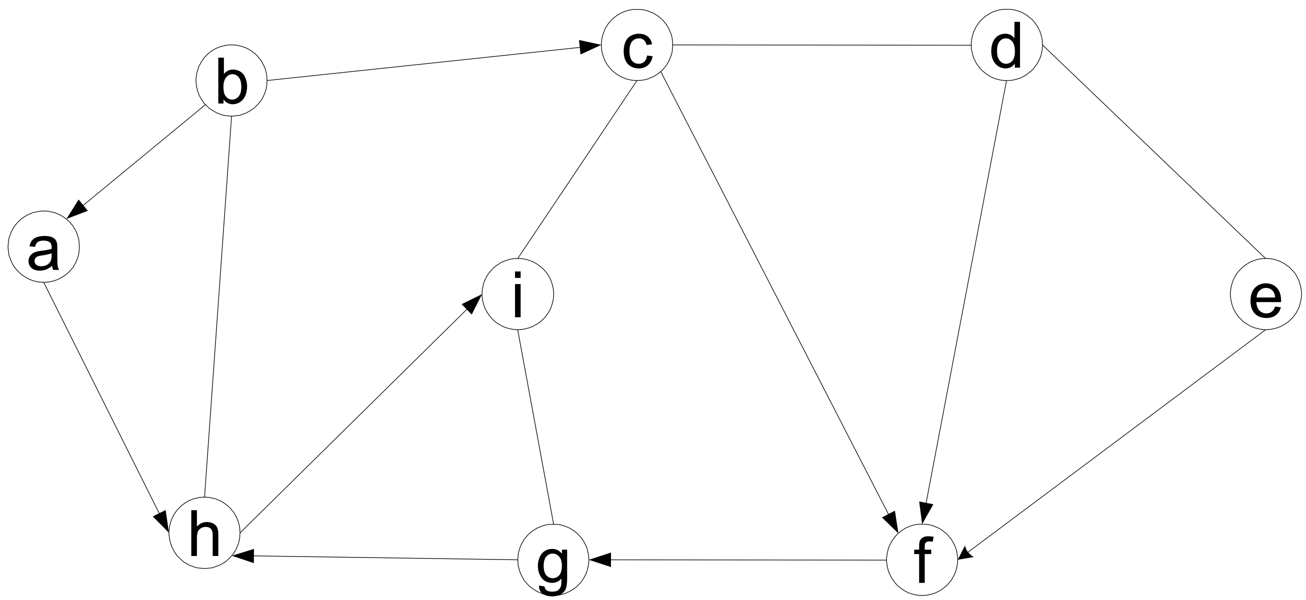
Example Undirected Graph



$$V = \{a, b, c, d, e, f, g, h, i\}$$

$$E = \{ab, ah, bc, bh, cd, ci, cf, de, df, ef, fg, gi, gh, ih\}$$

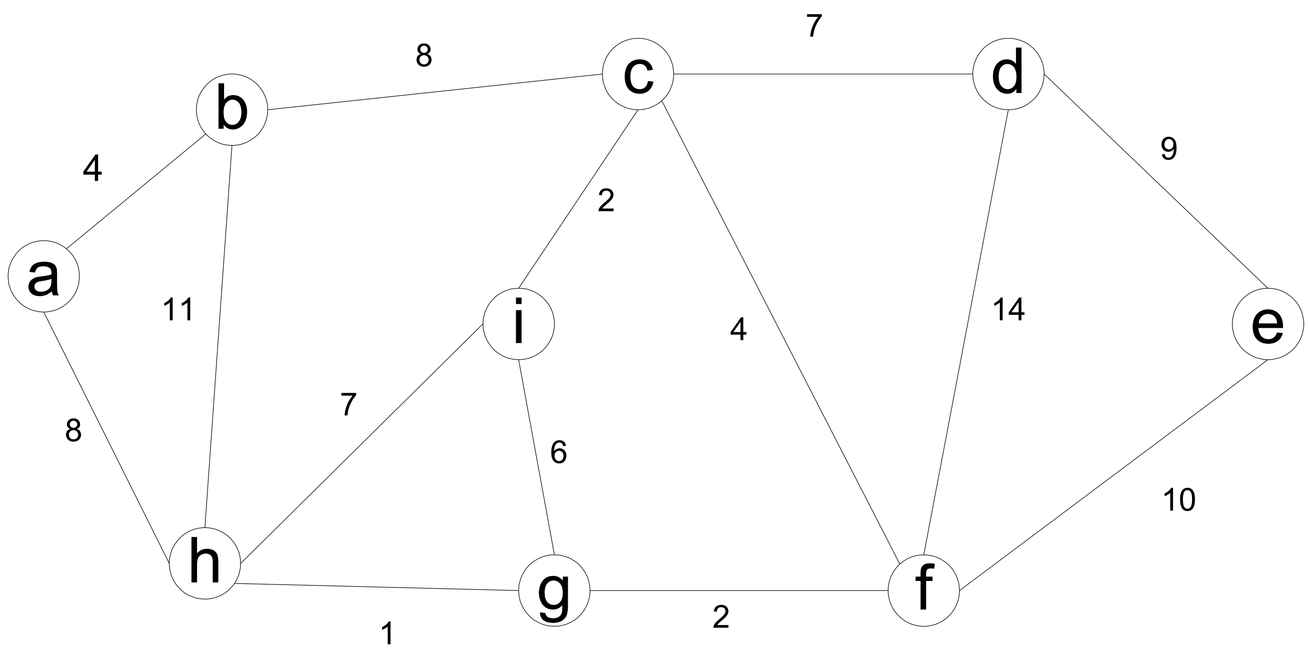
Example Directed Graph



$$V = \{a, b, c, d, e, f, g, h, i\}$$

$$E = \{\overrightarrow{ba}, \overrightarrow{ah}, bh, \overrightarrow{bc}, cd, \overrightarrow{cf}, ci, de, \overrightarrow{df}, \overrightarrow{ef}, \overrightarrow{fg}, gi, \overrightarrow{gh}, \overrightarrow{hi}\}$$

Weighted Graph



$$w(b,c) = 8 \qquad w(g,h) = 1 \qquad w(e,f) = 10$$

Graph Representation

- Adjacency matrix: 2D array
- Rows and columns: vertices
- Each array (matrix) element
  - Unweighted: a number (or symbol) showing whether the two corresponding vertices are connected or not
    - E.g., "1" for connected and "0" for not connected
  - Weighted: a value of interest related to an application
    - E.g., in a road network a weight could be distance on a road segment, in Facebook network a weight could be number of messages between two friends

Adjacency Matrix: Undirected Graph (symmetric)

	a	**b**	**c**	**d**	**e**	**f**	**g**	**h**	**i**
a	0	1	0	0	0	0	0	1	0
b	1	0	1	0	0	0	0	1	0
c	0	1	0	1	0	1	0	0	1
d	0	0	1	0	1	1	0	0	0
e	0	0	0	1	0	1	0	0	0
f	0	0	1	1	1	0	1	0	0
g	0	0	0	0	0	1	0	1	1
h	1	1	0	0	0	0	1	0	1
i	0	0	1	0	0	0	1	1	0

Adjacency Matrix: Directed Graph (non-symmetric)

	a	**b**	**c**	**d**	**e**	**f**	**g**	**h**	**i**
a	0	0	0	0	0	0	0	1	0
b	1	0	1	0	0	0	0	1	0
c	0	0	0	1	0	1	0	0	1
d	0	0	1	0	1	1	0	0	0
e	0	0	0	1	0	1	0	0	0
f	0	0	0	0	0	0	1	0	0
g	0	0	0	0	0	0	0	1	1
h	0	1	0	0	0	0	0	0	1
i	0	0	1	0	0	0	1	0	0

Adjacency Matrix: Undirected Weighted Graph

	a	**b**	**c**	**d**	**e**	**f**	**g**	**h**	**i**
a	0	4	0	0	0	0	0	8	0
b	4	0	8	0	0	0	0	11	0
c	0	8	0	7	0	4	0	0	2
d	0	0	7	0	9	14	0	0	0
e	0	0	0	9	0	10	0	0	0
f	0	0	4	14	10	0	2	0	0
g	0	0	0	0	0	2	0	1	6
h	8	11	0	0	0	0	1	0	7
i	0	0	2	0	0	0	6	7	0

Loop and Parallel Edge

- Loop
  - An edge with one vertex at both ends
- Parallel edge
  - Two edges with the same end vertices

Special Graphs

- Simple
  - A graph with neither loops or parallel edge
- Complete
  - A graph with n vertices in which there is an edge between every pair of distinct vertices
- Bipartite: a graph $G = (V, E)$ is bipartite if V_1 and V_2 are subsets of V where
  - Intersection of V_1 and V_2 is empty
  - Union of V_1 and V_2 is V
  - Each edge in E has one vertex in V_1 and one vertex in V_2

Subgraphs and Connected Graphs

- Subgraph
  - A graph G' whose vertex set is a subset of graph G
  - Adjacency matrix of G' is a subset of adjacency matrix of G
- Connected graph
  - There is a path from any vertex to any other vertex

Paths and Cycles

- Let v and w be vertices in a graph G
  - A simple path from v to w is a path from v to w with no repeated vertices
  - A cycle is a path of nonzero length from v to v with no repeated edges
  - A simple cycle is a cycle from v to v in which, except for the beginning and ending vertices that are both equal to v, there are no repeated vertices

Special Cycles

- Hamiltonian cycle
  - A cycle that passes through every vertex exactly once
- Euler cycle
  - A cycle that passes through every edge exactly once

Independent Sets and Cliques

- Independent set
  - A set of vertices in an undirected graph where no pair of vertices in the set are adjacent
- Clique
  - A set of vertices in an undirected graph where every pair of vertices in the set are adjacent

Graph Complement

- The complement of a simple graph G is the simple graph G' with the same vertices as G and an edge exists in G' if and only if it does not exist in G

Graph Isomorphism

- G1 and G2 are simple, undirected graphs
- A1 is an adjacency matrix for G1 corresponding to some ordering of G1's vertices
- A2 is adjacency matrix for G2 corresponding to some ordering of G's vertices
- A1=A2