### 3. Iterative Methods

Consider the following $N \times N$ cyclic tri-diagonal matrix:

$$\mathbb{T}_\alpha = \begin{pmatrix} \alpha & -1 & 0 & -1 \\ -1 & \alpha & -1 & 0 \\ 0 & \ddots & \ddots & -1 \\ -1 & 0 & -1 & \alpha \end{pmatrix} \quad \text{with} \quad \alpha \in \mathbb{R}. \tag{3}$$

(a) Show that the following vectors (usually called discrete Fourier modes)

$$\mathbf{v}_m = \begin{bmatrix} v_{m,0} \\ v_{m,1} \\ \vdots \\ v_{m,N-1} \end{bmatrix}, \quad \text{where } v_{m,n} = \exp\left(-i\frac{2\pi mn}{N}\right), \quad n, m = 0, \dots, N-1 \tag{4}$$

are eigenvectors of $\mathbb{T}_\alpha$ with eigenvalues given by

$$\lambda_m = \alpha - 2\cos\left(\frac{2\pi m}{N}\right). \tag{5}$$

$$T_\alpha \cdot \vec{V_m} = T_\alpha \cdot \begin{bmatrix} V_{m,0} \\ \vdots \\ V_{m,N-1} \end{bmatrix} \quad \text{where } V_{m,n} = \exp\left(-i\frac{2\pi mn}{N}\right)$$

Since $V_{m,N+n} = V_{m,n}$, $T_\alpha \vec{V_m}$ can derive the below equation

If $i \neq 0$, $i \neq N-1$

$$-V_{m,i-1} + \alpha V_{m,i} - V_{m,i+1} = \left(\alpha - \exp\left(-i\frac{2\pi m}{N}\right) - \exp\left(+i\frac{2\pi m}{N}\right)\right) V_{m,i}$$
$$= \left(\alpha - 2\cos\frac{2\pi m}{N}\right) V_{m,i}$$

If $i = 0$

$$\alpha V_{m,0} - V_{m,1} - V_{m,N-1} = \left(\alpha - \exp\left(-i\frac{2\pi m}{N}\right) - \underbrace{\exp\left(-i\frac{2\pi m}{N}(N-1)\right)}_{= \exp\left(i\frac{2\pi m}{N}\right)}\right) V_{m,0}$$
$$= \left(\alpha - 2\cos\frac{2\pi m}{N}\right) V_{m,0}$$

If $i = N-1$

$$\alpha V_{m,N-1} - V_{m,N-2} - V_{m,0} = \left(\alpha - \exp\left(+i\frac{2\pi}{N}m\right) - \underbrace{\exp\left(i\frac{2\pi m}{N}(N-1)\right)}_{\exp\left(-i\frac{2\pi}{N}m\right)}\right) V_{m,N-1}$$
$$= \left(\alpha - 2\cos\frac{2\pi m}{N}\right) V_{m,N-1}$$

Therefore, $$T_\alpha \vec{V_m} = \left(\alpha - 2\cos\frac{2\pi m}{N}\right)\begin{bmatrix} V_{m,0} \\ \vdots \\ V_{m,N-1} \end{bmatrix} = \left(\alpha - 2\cos\frac{2\pi m}{N}\right)\vec{V_m}$$

Thus $\vec{V_m}$ is eigenvector of $T_\alpha$ where $\lambda_m = \alpha - 2\cos\frac{2\pi m}{N}$

(b) Under what condition is the matrix $\mathbb{T}_\alpha$ positive definite?

To be positive definite, $T_\alpha$ should be symmetric and

$T_\alpha \succeq 0$ s.t. $\forall x \in \mathbb{R}^n$, $x^T T_\alpha x > 0$

Since $T_\alpha$ is circulant-tri-diagonal matrix of $\mathbb{R}^{n \times n}$, $T_\alpha$ is symmetric.

$T_\alpha \vec{v_m} = \lambda_m \vec{v_m} \Rightarrow [\vec{v_0} \cdots \vec{v_{n-1}}] \equiv P$

$T_\alpha P = P D(\lambda) \Rightarrow T_\alpha = P D(\lambda) P^T$

let any $x \in \mathbb{R}^n \Rightarrow x^T P D(\lambda) P^T x \geq 0$

$\Rightarrow y^T D(\lambda) y \geq 0$ for $\forall y \in \mathbb{R}^n$ then

$y = [y_1\ y_2 \cdots y_n]^T$ then, $\sum_i y_i^2 \lambda_i \geq 0$

we can think that $y = e_1 \cdots e_n$, and for each case,

$\lambda_i > 0 \Rightarrow \lambda_i = \alpha - 2\cos\left(\frac{2\pi i}{n}\right) > 0$

$\therefore \forall i \in \{0 \cdots n-1\}$ $\lambda_i = \alpha - 2\cos\left(\frac{2\pi i}{n}\right) > 0$

$\Rightarrow \alpha > \max\left\{2\cos\left(\frac{2\pi}{n} \cdot 0\right), \cdots, 2\cos\left(\frac{2\pi}{n}(n-1)\right)\right\} = 2$

$\therefore \alpha > 2 \leftarrow$ requirements for $T_\alpha$ to be positive definite

(c) In the lecture we define a general iterative method to solve a linear system $\mathbb{A}\mathbf{x} = \mathbf{b}$ by decomposing matrix $\mathbb{A}$ as $\mathbb{A} = \mathbb{M} - \mathbb{N}$. The iterative step can be written as

$$\mathbf{x}^{(k+1)} = \mathbb{M}^{-1}\mathbb{N}\mathbf{x}^{(k)} + \mathbb{M}^{-1}\mathbf{b}, \tag{6}$$

with iteration matrix $\mathbb{G} = \mathbb{M}^{-1}\mathbb{N}$. We have also defined the spectral radius $\rho(\mathbb{G})$, where $\rho(\mathbb{G}) < 1$ is a necessary condition for convergence. Compute the spectral radius of the Jacobi iteration scheme for matrix $\mathbb{T}_\alpha$.

Jacobi iteration scheme

$M = \mathrm{diag}(A)$ $\quad N = \mathrm{diag}(A) - A$ then

$M$ for $T_\alpha = \mathrm{diag}(T_\alpha) = \alpha I_N$ and $N = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & & & & \vdots \\ 0 & & \ddots & & & 0 \\ \vdots & & & & & 1 \\ 0 & & & & & \\ 1 & \cdots & & & 1 & 0 \end{bmatrix}$

Then $G = \frac{1}{\alpha} N = \begin{bmatrix} 0 & \frac{1}{\alpha} & \cdots & & 0 & \frac{1}{\alpha} \\ \frac{1}{\alpha} & 0 & \frac{1}{\alpha} & \cdots & 0 & 0 \\ \vdots & & \cdots & & 0 & \frac{1}{\alpha} \\ \frac{1}{\alpha} & 0 & \cdots & 0 & \frac{1}{\alpha} & 0 \end{bmatrix}$

$\rho := \max(\lambda)$ of $G$. To get $\lambda$,

We already know that

$$\begin{bmatrix} \alpha & -1 & 0 & \cdot & -1 \\ -1 & \alpha & -1 & \cdots & 0 \\ & & \vdots & & \end{bmatrix} V_m = \lambda_m V_m = \left(\alpha - 2\cos\frac{2\pi m}{N}\right) V_m$$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 0 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & 0 \\ & & \vdots & & \end{bmatrix} = \alpha G \quad \text{and} \quad \alpha G V_m = \left(-2\cos\frac{2\pi m}{N}\right) V_m$$

thus, $\quad G V_m = -\frac{2}{\alpha}\cos\frac{2\pi}{N} m \, V_m$

$\Rightarrow$ $V_m$ is also eigenvector of $G$ and $-\frac{2}{\alpha}\cos\frac{2\pi}{N} m$ is eigenvalue

$$\rho(G) := \max|\lambda| = \max\left|\frac{2}{\alpha}\cos\frac{2\pi}{N} m\right| = \frac{2}{\alpha}$$

The spectral radius $\rho(G) = \frac{2}{\alpha}$ $\therefore$