1. Elliptic BVP

Let Ω be the planar domain $[0,1]^2$. Consider the boundary value problem (BVP)

$$\begin{cases} \nabla^2 u - u^4 = 0, & x \in \Omega, \\ u(x) = 1, & x \in \partial\Omega. \end{cases} \tag{1}$$

Ω is discretized into $N \times N$ cells. We approximate the Laplace operator using the classical 5-point second-order finite-difference approximation.

(a) Reformulate the boundary value problem (BVP) in equation (1) as a root-finding problem that can be solved using the Newton-Raphson method.

(a) Let $u \in \mathbb{R}^{N \times N}$, $u = \begin{bmatrix} u_{11} & \cdots & u_{1N} \\ \vdots & & \\ u_{N1} & \cdots & u_{NN} \end{bmatrix}$ and u_{1D} is

1-dimensional array of u with $u_{1D} \in \mathbb{R}^{N^2}$ such that

$u_{1D} = \begin{pmatrix} u_{11} \\ \vdots \\ u_{N1} \\ u_{12} \\ \vdots \\ u_{N2} \\ \vdots \end{pmatrix}$ then ∇^2 can be linearized as 2D Laplacian

operator which is denoted as $L \otimes I + I \otimes L$ where L

is 1-Dimensional discretized laplacian operator. $L, I \in \mathbb{R}^{N \times N}$

Let $\nabla^2 = \tilde{L} = L \otimes I + I \otimes L$

Then, $g(u_{1D}) := \tilde{L} u_{1D} - u_{1D}^4$, which is nonlinear function.

This is equivalent to root-finding problem where

$g(x) = 0$ with $x \in \mathbb{R}^m$ $(m = N^2)$

we can apply Newton-Raphson method to find the solution x^*.

$g(x_{t+1}) = g(x_t) + \frac{\partial g}{\partial x_t} \cdot \delta x_t = g(x_t) + J_g(x_t) \cdot (x_{t+1} - x_t) = 0$

$\Rightarrow \quad x_{t+1} = x_t - J_g^{-1} \cdot g(x_t)$ then we can find the solution !! :)

2. Beam-warming Scheme

Derive the modified equation for the Beam-warming scheme presented in the lecture on "Hyperbolic PDEs":

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+a\frac{3u_i^n-4u_{i-1}^n+u_{i-2}^n}{2h}=\frac{a^2\Delta t}{2}\frac{u_i^n-2u_{i-1}^n+u_{i-2}^n}{h^2}. \qquad (2)$$

Starting from

$$\frac{\partial}{\partial t}u + a\frac{\partial u}{\partial x} = 0 \quad \text{(Burger's equation)}$$

Let $\frac{\partial}{\partial t}u = \frac{1}{\Delta t}(u_i^{n+1}-u_i^n)$

$$u_i^{n+1} = u_i^n + \Delta t\frac{\partial u}{\partial t} + \frac{1}{2}\Delta t^2\frac{\partial^2 u}{\partial t^2} + \cdots$$

$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$ thus

$$u_i^{n+1} \doteq u_i^n - a\Delta t\frac{\partial u}{\partial x} + \frac{1}{2}a^2\Delta t^2\frac{\partial^2 u}{\partial x^2}$$

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} + a\frac{\partial u}{\partial x} = \frac{1}{2}a^2\Delta t\frac{\partial^2 u}{\partial x^2}$$

Use $\frac{\partial u}{\partial x} \doteq \frac{3u_i^n - 4u_{i-1}^n + u_{i-2}^n}{2h}$ and $\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1}^n - 2u_i^n + u_{i-1}^n}{h^2}$

$$\Rightarrow \frac{1}{\Delta t}(u_i^{n+1}-u_i^n) + a\frac{3u_i^n-4u_{i-1}^n+u_{i-2}^n}{2h} = \frac{1}{2}a^2\Delta t\cdot\frac{u_{i-1}^n-2u_i^n+u_{i-1}^n}{h^2}$$

3. Advection

Consider a 2D advection equation for $u(t,x,y)$ given by

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0, \tag{3}$$

You are given the initial condition

$$u(0,x,y) = \exp\left(-\frac{(x-1/2)^2 + (y-1/2)^2}{(3/20)^2}\right) \tag{4}$$

and advection speeds $a = 1$ and $b = 2$.

(a) Show that the analytical solution is

$$u(t,x,y) = u(0, x-at, y-bt). \tag{5}$$

(a) $\frac{\partial u}{\partial t} + a\cdot\frac{\partial u}{\partial x} + b\cdot\frac{\partial u}{\partial y} = 0$ let $\eta = x - at$ $\xi = y - bt$

and $u = u(\eta, \xi)$ then

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} = -a\frac{\partial u}{\partial \eta} - b\frac{\partial u}{\partial \xi}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \eta} \quad \Rightarrow \quad \frac{\partial}{\partial t}u = -a\frac{\partial u}{\partial x} - b\frac{\partial u}{\partial \xi}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} = \frac{\partial u}{\partial \xi}$$

Therefore $u = u(\eta, \xi)$ is a solution of $\frac{\partial}{\partial t}u + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0$

and $u(\eta, \xi) = u(0, x-at, y-bt)$ $\therefore$

(b) We will now derive a 2D finite-volume numerical scheme based on this exact solution called the Corner Transport Upstream (CTU) method.

(1) For 1D advection, defining $\mu \equiv a\Delta t/\Delta x$, we can reformulate the first-order upwind scheme using the exact solution for piecewise constant initial conditions, projected back to the initial mesh via conservative integration. Show geometrically (using a simple diagram) that projecting back the exact solution gives you the numerical solution

$$u_i^{n+1} = (1-\mu)u_i^n + \mu u_{i-1}^n \tag{6}$$

(2) Using the same geometrical strategy in 2D, derive the projected values of the exact solution back to the initial grid as a function of $\mu \equiv a\Delta t/\Delta x$ and $\nu \equiv b\Delta t/\Delta y$. This should give you the fully discrete form of the CTU scheme.

(3) Derive the modified equation of this scheme and analyze the leading-order error term.

(1) Let $\frac{\partial u}{\partial t} + a\cdot\frac{\partial u}{\partial x} = 0$ (1D advection problem)

and $\mu := a\Delta t/\Delta x$. This equation can be reformulated as

1st order upwind scheme by forward euler scheme.

$$\frac{\partial a}{\partial t} \simeq (a_i^{n+1} - a_i^n)/\Delta t \qquad \frac{\partial a}{\partial x} \simeq \frac{a_i^n - a_{i-1}^n}{\Delta x}$$

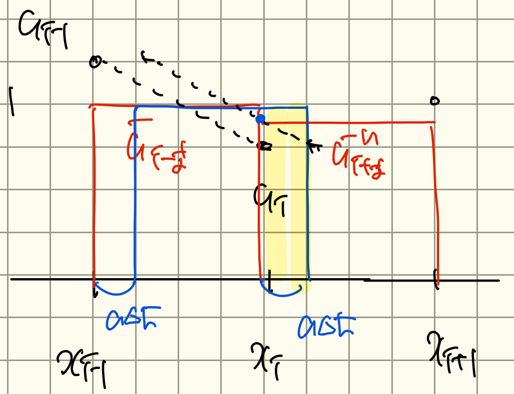
Then
$$\frac{\partial a}{\partial t} + a \cdot \frac{\partial a}{\partial x} \simeq \frac{1}{\Delta t}(a_i^{n+1} - a_i^n) + a \cdot \frac{1}{\Delta x}(a_i^n - a_{i-1}^n) = 0$$

$$a_i^{n+1} = a_i^n - a\frac{\Delta t}{\Delta x} a_i^n + a\frac{\Delta t}{\Delta x} a_{i-1}^n$$

$$a_i^{n+1} = (1-\mu)a_i^n + \mu a_{i-1}^n$$

This can be shown by geometrical interpretation.

FIG 1



$\frac{\partial a}{\partial t} + a \cdot \frac{\partial a}{\partial x} = 0$: moving flow with a velocity

If Δt time pass, then the amount of moving length is $a\Delta t$

Define
$$\bar{a}^n_{i+\frac{1}{2}} = \frac{1}{\Delta x}\int_{x_i}^{x_{i+1}} a(x,t)\,dx$$
then
$$\int \left(\frac{\partial a}{\partial t} + a\frac{\partial a}{\partial x}\right) dx\,dt = 0$$

$$\Rightarrow \int_{t_n}^{t_{n+1}}\int_{x_i}^{x_{i+1}} \left(\frac{\partial}{\partial t}a + a\frac{\partial a}{\partial x}\right) dx\,dt = \left(\bar{a}^{n+1}_{i+\frac{1}{2}} - \bar{a}^n_{i+\frac{1}{2}}\right)\Delta x + \Delta t\, a\left(\bar{a}^{n+\frac{1}{2}}_{i+1} - \bar{a}^{n+\frac{1}{2}}_{i}\right) = 0$$

$$\Rightarrow \bar{a}^{n+1}_{i+\frac{1}{2}} - \bar{a}^n_{i+\frac{1}{2}} + \Delta t\frac{a}{\Delta x}\cdot\left(\bar{a}^{n+\frac{1}{2}}_{i+1} - \bar{a}^{n+\frac{1}{2}}_{i}\right) = 0$$

Let $i \to i-1$ then this is same as our problem

$$\bar{a}^{n+1}_{i-\frac{1}{2}} - \bar{a}^n_{i-\frac{1}{2}} + \frac{a\Delta t}{\Delta x}\left(\bar{a}^{n+\frac{1}{2}}_{i} - \bar{a}^{n+\frac{1}{2}}_{i-1}\right) = 0$$

Since the amount of quantities pass at $x = x_i$ is $-a\Delta t\left(\bar{a}^{n+\frac{1}{2}}_{i} - \bar{a}^{n+\frac{1}{2}}_{i-1}\right)$

this is equal to the change of height $\bar{a}^n_{i-\frac{1}{2}}$ and the amount is $\left(\bar{a}^{n+1}_{i-\frac{1}{2}} - \bar{a}^n_{i-\frac{1}{2}}\right)\Delta x = -a\Delta t\left(\bar{a}^{n+\frac{1}{2}}_{i} - \bar{a}^{n+\frac{1}{2}}_{i-1}\right)$

$\Rightarrow$ Thus $(\bar{a}^{n+1}_{i-\frac12} - \bar{a}^{n}_{i-\frac12}) + \mu(\bar{a}_i^{n+\frac12} - \bar{a}_{i-1}^{n+\frac12}) = 0$ ∴

$\Rightarrow$ projected back then $(a_i^{n+1} - a_i^n) + \mu(a_i^n - a_{i-1}^n) = 0$ ∴ (equivalent)

(2) Let $\mu = a\Delta t/\Delta x$ $\nu = b\Delta t/\Delta y$ then

Define $$\bar{a}^{n}_{i+\frac12, j+\frac12} = \int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} a(t,x,y)\,dx\,dy \cdot \frac{1}{\Delta x \Delta y}$$

then, $$\frac{\partial a}{\partial t} + a\frac{\partial a}{\partial x} + b\frac{\partial a}{\partial y} = 0 \quad \times \int dx\,dy\,dt$$

$$\Rightarrow \Delta x\Delta y\,(\bar{a}^{n+1}_{i+\frac12, j+\frac12} - \bar{a}^{n}_{i+\frac12, j+\frac12}) + a(\bar{a}^{n+\frac12}_{i+1, j+\frac12} - \bar{a}^{n+\frac12}_{i, j+\frac12})\Delta y\Delta t$$

$$+ b\,\Delta x\Delta t\,(\bar{a}^{n+\frac12}_{i+\frac12, j+1} - \bar{a}^{n+\frac12}_{i+\frac12, j}) = 0$$

$$\Rightarrow \frac{1}{\Delta t}(\bar{a}^{n+1}_{i+\frac12, j+\frac12} - \bar{a}^{n}_{i+\frac12, j+\frac12}) + a\cdot\frac{1}{\Delta x}\underbrace{(\bar{a}^{n+\frac12}_{i+1, j+\frac12} - \bar{a}^{n+\frac12}_{i, j+\frac12})}_{①}$$

$$+ b\cdot\frac{1}{\Delta y}\underbrace{(\bar{a}^{n+\frac12}_{i+\frac12, j+1} - \bar{a}^{n+\frac12}_{i+\frac12, j})}_{②} = 0$$

$$\Rightarrow \bar{a}^{n+1}_{i+\frac12, j+\frac12} = \bar{a}^{n}_{i+\frac12, j+\frac12} - \mu\cdot① - \nu\cdot②$$

In our problem, $i, j \to i-1, j-1$ thus

$$\bar{a}^{n}_{i-\frac12, j-\frac12} = \bar{a}^{n}_{i-\frac12, j-\frac12} - \mu\cdot(\bar{a}^{n+\frac12}_{i, j-\frac12} - \bar{a}^{n+\frac12}_{i-1, j-\frac12}) - \nu\cdot(\bar{a}^{n+\frac12}_{i-\frac12, j} - \bar{a}^{n+\frac12}_{i-\frac12, j-1})$$

By projecting it back,

$$a^{n+1}_{i,j} = a^n_{i,j} - \mu(a^n_{i,j} - a^n_{i-1,j}) - \nu(a^n_{i,j} - a^n_{i,j-1})$$

$$a^{n+1}_{i,j} = (1-\mu-\nu)a^n_{i,j} + \mu a^n_{i-1,j} + \nu a^n_{i,j-1}$$

(a) $$u_{i,j}^{n+1} = u_{i,j}^{n} + \frac{\partial u}{\partial t}\cdot\Delta t + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t^2$$

$$u_{i-1,j}^{n} = u_{i,j}^{n} - \frac{\partial u}{\partial x}\cdot\Delta x + \frac{1}{2}\Delta x^2\frac{\partial^2 u}{\partial x^2} + \cdots$$

$$u_{i,j-1}^{n} = u_{i,j}^{n} - \frac{\partial u}{\partial y}\Delta y + \frac{1}{2}\Delta y^2\frac{\partial^2 u}{\partial y^2} + \cdots$$

$$\Rightarrow \quad \frac{\partial u}{\partial t}\Delta t + \frac{1}{2}\Delta t^2\frac{\partial^2 u}{\partial t^2} + \cancel{O(\Delta t^3)} = -\mu\Delta x\frac{\partial u}{\partial x} - \nu\Delta y\frac{\partial u}{\partial y}$$

$$+ \frac{1}{2}\mu\Delta x^2\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\nu\Delta y^2\frac{\partial^2 u}{\partial y^2} + \cancel{O(\Delta x^3)} + \cancel{O(\Delta y^3)}$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} + \mu\frac{\Delta x}{\Delta t}\frac{\partial u}{\partial x} + \nu\frac{\Delta y}{\Delta t}\frac{\partial u}{\partial y} = \frac{1}{2}\mu\frac{\Delta x^2}{\Delta t}\cdot\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\nu\frac{\Delta y^2}{\Delta t}\frac{\partial^2 u}{\partial y^2} - \frac{1}{2}\Delta t\frac{\partial^2 u}{\partial t^2}$$

$$* \quad \frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + b^2\frac{\partial^2 u}{\partial y^2} + 2ab\frac{\partial^2 u}{\partial x\partial y}$$

Finally,

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = \frac{1}{2}(a\Delta x - a^2\Delta t)\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}(b\Delta y - b^2\Delta t)\frac{\partial^2 u}{\partial y^2} - ab\Delta t\frac{\partial^2 u}{\partial x\partial y}$$

or

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = \frac{1}{2}a\Delta x(1-\mu)\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}b\Delta y(1-\nu)\frac{\partial^2 u}{\partial y^2} - ab\Delta t\frac{\partial^2 u}{\partial x\partial y}$$

The modified equation has additional diffusion terms which are proportional to Δx, Δy, and Δt with respectively.