參考: Spectral Element Methods

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Spectral Element Method

- The Spectral Element Method has two meanings.

1. **Another class of highly accurate finite element method developed by Patera in 1984**

2. **Finite element method on frequency-domains via Fourier transform**

In the First Meaning ……

- Locating internal nodes at roots of some orthogonal polynomials gives the highest interpolation accuracy.

- The method increases accuracy by using high degree piecewise polynomial basis functions
  - Lagrange polynomials
  - Lobatto polynomials
  - Chebyshev polynomials

In the First Meaning ……

- Advantage
  - High accuracy with fewer degrees of freedom
  - Low computational cost
- Disadvantage
  - Difficult to handle complex geometries
- Application
  - Fluid dynamics
  - Seismology
  - ……

In the First Meaning ……

- Legendre-Gauss-Lobatto grid
  - LGL nodes ξ_k^N of order N: $N+1$ roots of $(1-x^2)L_N'(x)$ where $L_N(x)$ is the Legendre polynomial of degree N
  - Basis function

$$h_k(\xi) = \frac{(\xi^2 - 1)P_N'(\xi)}{N(N+1)P_N(\xi_k)(\xi - \xi_k)}$$

    - P_N: Legendre polynomial of degree N

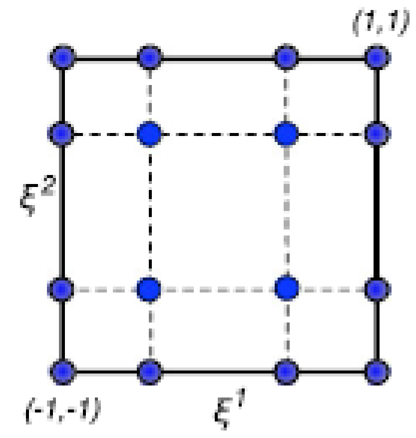


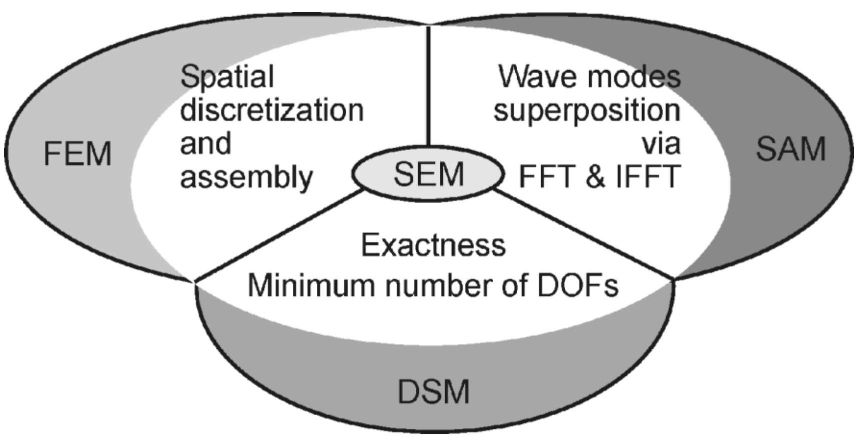
Figure from Lauritzen *et al.*

Discrete Fourier Transform

- Also known as Finite Fourier Transform or Discrete time Fourier Series
  - Continuous time or space → discrete time or space → Fourier transform → discrete frequency or wavenumber
  - $-\infty$ to ∞ → finite domains
- Fast Fourier Transform (FFT): fast version of discrete Fourier transform algorithm

In the Second Meaning ……

- Time-dependent PDEs → frequency-dependent PDEs
  - Shape functions: complex exponentials instead of polynomials
  - Limit: finite windows of frequencies



Spectral Analysis Method

Dynamic Stiffness Method: related to shape functions

Figure from Lee

In the Second Meaning ……

- Advantages
  - You can get frequency-response functions (good for inverse problems).
  - High accuracy and efficiency
    - Especially for infinite domains or frequency-response.
  - Directly compatible with digitized data
  - No shear locking problems
- Disadvantages
  - Exact SEM models are unavailable for 2-D or 3-D.
  - Less efficient to non-linear systems due to FFT
  - Inverse FFT may give less accurate solutions for time-domain.

In the Second Meaning ……

- Frequency-dependent PDE example

$$a u_{xx} + b u_x = c u_{tt}$$

$$\rightarrow \; a \frac{d^2 \hat{u}_n}{dx^2} + b \frac{d\hat{u}_n}{dx} + c\omega_n^2 \hat{u}_n = 0$$

  - $u(x,t) = \sum_{n=0}^{N-1} \hat{u}_n(x, \omega_n)\, e^{i\omega_n t}$

  – Characteristic equation

$$\left(k^2 + i\frac{b}{a}k + \frac{c\omega_n^2}{a}\right) A_n = 0$$

  - A_n: unknown constant related to $\hat{u}_n$

  – Solution

$$\hat{u}_n = A_n e^{-ik_n x} + B_n e^{ik_n x} \;\text{ where }\; k_n = \omega_n \sqrt{\frac{c}{a}}$$

In the Second Meaning ……

- General procedure
  1. FFT: Time-dependent governing PDE → frequency-dependent PDE
  2. Solution preparation with dynamic shape functions $N(x, \omega)$
  3. Formulation of spectral element equation
  4. Assembly
  5. Imposing boundary conditions
  6. Eigen-solutions by eigenvalue problem solving or dynamic responses by IFFT

In the Second Meaning ……

- Spectral element example
  - Gopalakrishnan *et al*. pp. 51~52
  - Wave equation: an isotropic homogenous rod

$$u_{tt} = c^2 u_{xx}$$

    - $c^2 = E/\rho$ (ρ: rod density, E: Young's modulus)
    - Constitutive relation: $F(x,t) = AEu_x$ (A: cross-section)

$\rightarrow c^2 d^2\hat{u}_n/dx^2 + c\omega_n^2\hat{u}_n = 0 \rightarrow (-c^2k^2 + \omega_n^2)u_0 = 0$

  - Solution: $\hat{u}_n = C_1 e^{-ik_n x} + C_2 e^{ik_n x}$

$$\begin{pmatrix}\hat{u}_1\\ \hat{u}_2\end{pmatrix} = \mathbf{T}_1\begin{pmatrix}C_1\\ C_2\end{pmatrix}, \mathbf{T}_1 = \begin{pmatrix}e^{-ik_n x_1} & e^{ik_n x_1}\\ e^{-ik_n x_2} & e^{ik_n x_2}\end{pmatrix}$$

    - For an element of length L (nodes: x_1 & x_2)

In the Second Meaning ……

- Spectral element example
  - Gopalakrishnan *et al*. pp. 51~52
  - Nodal forces
    - $\hat{F}_1 = -\hat{F}(x_1, \omega_n)$, $\hat{F}_2 = \hat{F}(x_2, \omega_n)$

$$\begin{pmatrix} \hat{F}_1 \\ \hat{F}_2 \end{pmatrix} = \mathbf{T}_2 \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \mathbf{T_2} = AE(ik_n)\begin{pmatrix} e^{-ik_n x_1} & -e^{ik_n x_1} \\ -e^{-ik_n x_2} & e^{ik_n x_2} \end{pmatrix}$$

  - Force-displacement relation

$$\begin{pmatrix} \hat{F}_1 \\ \hat{F}_2 \end{pmatrix} = \mathbf{T}_2 \mathbf{T}_1^{-1} \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix}$$

➢ C_1 & C_2 can be determined by the boundary conditions of u & F.

References

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  - The first meaning of SEM

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- P.H. Lauritzen *et al*., J. Adv. Model. Earth Syst. 10, 1537 (2018).
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- S. Gopalakrishnan *et al*., "Spectral Finite Element Method"
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