參考: Lattice Boltzmann Method

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Modeling Scales

- Microscopic
  - Small particles or individuals.
- Macroscopic
  - Averaged quantities of groups (ex.: particle group). Approximation to continuum.
- **Mesoscopic**
  - Large particles or small clusters. Ensembles. Kinetic type. Often stochastic.
- Multi-scale: Combination of different scales

Kinetic Theory

- Object of interest
  - Particle distribution function (PDF)
    - Phase space probability density
- BBGKY hierarchy
      - Bogoliubov, Born, Green, Kirkwood, and Yvon
  - A set of equations for many particles
    - Each equation describes k-particle probability density function (k = 1, 2, … , N)
  - Boltzmann equation
    - Truncation at the first or the second equation

Boltzmann Equation

- Boltzmann equation
  - Considering 1-particle probability density function $f(\mathbf{x}, \mathbf{v}, t)$

    $$\partial_t f = (\partial_t f)_{\text{ext. force}} + (\partial_t f)_{\text{diffusion}} + (\partial_t f)_{\text{collision}}$$

    $$\partial_t f + (\mathbf{v}\cdot\triangledown)f + \mathbf{F}\cdot \partial f/\partial \mathbf{p} = \Omega(f, f)$$
  - $\mathbf{F}$: external force field
  - $\Omega(f, f)$: collision effect term

$$\Omega(f,f) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \sigma(\omega)|\mathbf{w}-\mathbf{v}|\,(f(\mathbf{v}')f(\mathbf{w}') - f(\mathbf{v})f(\mathbf{w}))\,d\omega d\mathbf{w}$$

Boltzmann Equation

- Boltzmann equation
  - $\Omega(f, f)$: collision effect term (also called collision operator)

$$\Omega(f,f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \sigma(\omega)|\mathbf{w} - \mathbf{v}| \left(f(\mathbf{v}')f(\mathbf{w}') - f(\mathbf{v})f(\mathbf{w}) \right) d\omega d\mathbf{w}$$

    - Collision of particles with velocity **v** and particles with velocity **w**
    - $\sigma(\omega)$: scattering cross section
    - ω : scattering angle

Boltzmann Equation

- Detailed balance
  - At equilibrium, (net gain) = (net loss)
    - $\Omega(f, f) = 0$
  - ◆ $f_0(\mathbf{v}')\, f_0(\mathbf{w}') = f_0(\mathbf{v})\, f_0(\mathbf{w})$ for gas model (no **F** case)
    - $f_0(\mathbf{v})$: a one-particle probability density function at equilibrium
- Maxwell-Boltzmann distribution

  $$f_0(\mathbf{v}) = n\,(m\beta/2\pi)^{3/2} \exp(-m\beta|\mathbf{v}-\mathbf{v}^0|^2/2) \text{ where } \beta = 1/(k_B T)$$

    - $n = 1$ after normalization

  k_B: Boltzmann constant

  - The equilibrium state for Boltzmann equation without external forces
    - Accurate for rarefied gases

Boltzmann Equation

- Conservation laws
  - Average value of A: $\langle A\rangle = \int Af\, d\mathbf{v} \,/\, \int f\, d\mathbf{v}$
  - Mass conservation

    $$\partial_t \rho + \nabla\cdot(\rho\mathbf{u}) = 0 \qquad (\rho = mn,\ n = \int f\, d\mathbf{v}\ ,\ \mathbf{u} = \langle\mathbf{v}\rangle)$$

  - Momentum conservation

    $$\partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\cdot\tau = n\mathbf{F} \qquad \left(\tau = \frac{1}{m}\int \mathbf{v}'\otimes\mathbf{v}' f\, d\mathbf{v}'\right)$$

    - τ: stress by fluctuation

$$\mathbf{v}\otimes\mathbf{w} = \begin{bmatrix} v_1w_1 & v_1w_2 & \cdots & v_1w_m \\ v_2w_1 & v_2w_2 & \cdots & v_2w_m \\ \vdots & \vdots & \ddots & \vdots \\ v_nw_1 & v_nw_2 & \cdots & v_nw_m \end{bmatrix}$$

Boltzmann Equation

- Conservation laws
  - Average value of A: $\langle A \rangle = \int Af\, d\mathbf{v} \,/\, \int f\, d\mathbf{v}$
  - Mass conservation

    $$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \qquad (\rho = mn,\ n = \int f\, d\mathbf{v}\ ,\ \mathbf{u} = \langle \mathbf{v} \rangle)$$
  - Momentum conservation

    $$\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \cdot \tau = n\mathbf{F} \qquad \left(\boldsymbol{\tau} = \frac{1}{m} \int \mathbf{v}' \otimes \mathbf{v}' f d\mathbf{v}' \right)$$
    - τ: stress by fluctuation
  - Energy conservation

    $$\partial_t E + \nabla \cdot (E\mathbf{u} + \tau \cdot \mathbf{u}) + \nabla \cdot \mathbf{q} = n\mathbf{F} \cdot \mathbf{u} \quad \left(\mathbf{q} = \frac{1}{m} \int \frac{|\mathbf{v}'|^2}{2} \mathbf{v}' f d\mathbf{v}' \right)$$
    - $E = \rho|\mathbf{u}|^2/2$ + (thermal E), $\mathbf{q}$: heat flux

Boltzmann Equation

- Knudsen number (Kn)

$$\mathrm{Kn} = \lambda / L$$

  - λ : mean free path (typical order: μm or nm)
  - L : system size (typical order: meter)
  - – Dimensionless Boltzmann equation

$$\partial_t f + (\mathbf{v} \cdot \bigtriangledown) f + \mathbf{F} \cdot \partial f / \partial \mathbf{p} = \Omega(f, f) / \mathrm{Kn}$$

- Local equilibrium approximation
  - – Assuming $\mathbf{F} = 0$,

$$\textit{If}\ \ \mathrm{Kn} \ll 1 \rightarrow \Omega(f, f) = 0 \rightarrow \text{ M.-B. Distribution}$$

BGK Approximation

– Bhatnagar, Gross, and Krook

- In most cases, the collision effect results a distribution not far from the equilibrium distribution.

$$\partial_t f + (\mathbf{v} \cdot \bigtriangledown) f + \mathbf{F} \cdot \partial f / \partial \mathbf{p} = (f_0 - f)/\tau$$

– τ: relaxation time
  - Elementary time of collisions
  - Related to transport coefficient or diffusion coefficient

Lattice Boltzmann Method

- Computational fluid dynamics method

- Discrete-velocity version of approximated Boltzmann equation
  - When we ignore the external force term,

$$\partial_t f_i + (\mathbf{v}_i \cdot \nabla) f_i = \Omega(f_i, f_i) \qquad (i\text{: velocity index})$$

$$\Rightarrow f_i(\mathbf{x} + \mathbf{v}_i \Delta t, t + \Delta t) = f_i(\mathbf{x}, t) + \Delta t \cdot \Omega[f_i(\mathbf{x}, t), f_i(\mathbf{x}, t)]$$

- Propagation and collision of fictitious particles (quasiparticles)
  - Mesoscopic viewpoint

Advantages of LBM

- Easy to include a particular molecular interaction
- Easy to be parallelized.
  - Massive-parallelism such as GPUs and supercomputers
- Suitable to mass-conserving flows in porus media
- Moving boundaries can be implemented.
- Multiphase and multicomponent methods are available.
- Various couplings between different flows are available.
  - Wave with sound, heat-transfer or chemical reactions
- Thermal fluctuations can be applied.
- Appropriate for simulating mesoscopic physics

Disadvantages of LBM

- Memory-intensive
- Time-dependent, even for steady flows
- Spurious currents near fluid-fluid interfaces, as in other lattice-based methods
- The range of viscosities and densities are limited in multiphase and multicomponent simulations
- Energy-conserving thermal simulations are not straightforward
- Inappropriate for long-range propagation of sound at real viscosity
- Inappropriate for strongly compressible flows

Lattice Boltzmann Models

- Lattice Boltzmann equation

  $$f_i(\mathbf{x} + \mathbf{v}_i\,\Delta t,\ t + \Delta t) = f_i(\mathbf{x},\ t) + \Delta t\cdot\Omega[f_i(\mathbf{x},\ t),\ f_i(\mathbf{x},\ t)]$$

  – Discretization: time & velocity

- The BGK approximation is usually used in LBM.

  $$f_i(\mathbf{x} + \mathbf{v}_i\,\Delta t,\ t + \Delta t) = f_i(\mathbf{x},\ t) + [f_{0,i}(\mathbf{x},\ t) - f_i(\mathbf{x},\ t)]\cdot\Delta t/\tau$$

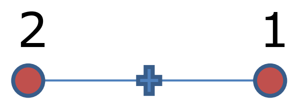
❖ Naturally, ($\mathbf{v}$ lattice unit) = ($\mathbf{x}$ lattice unit)/Δt

❖ Velocity lattice notation: D*d*Q*q*

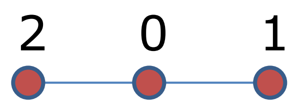
  – *d*: dimension, *q*: number of linkages

1-D Lattice Arrangements

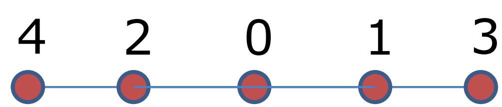
- D1Q2



  - Weighting factors: $w_1=1/2$, $w_2=1/2$.
- D1Q3



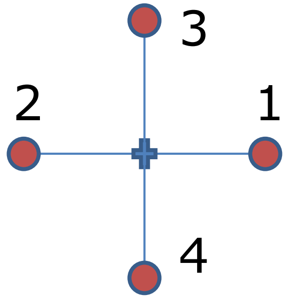
  - Weighting factors: $w_0=2/3$, $w_1=1/6$, $w_2=1/6$.
- D1Q5



  - Weighting factors: $w_0= 1/2$, $w_1=1/6$, $w_2=1/6$, $w_3=1/12$, $w_4=1/12$.

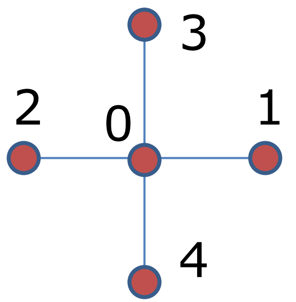
2-D Lattice Arrangements

- D2Q4



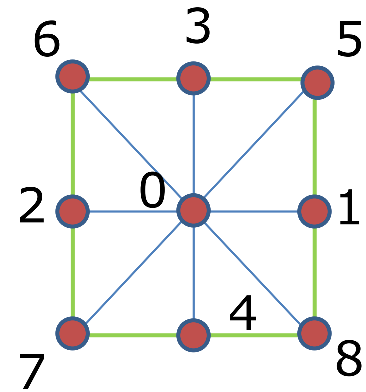
  - Weighting factors: $w_i=1/4$

- D2Q5



  - Weighting factors: $w_0= 1/3$, $w_1=1/6$, $w_2=1/6$, $w_3=1/6$, $w_4=1/6$.

- D2Q9

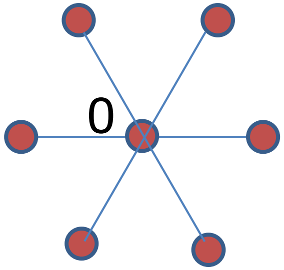


  - Weighting factors: $w_0= 4/9$, $w_1=1/9$, $w_2=1/9$, $w_3=1/9$, $w_4=1/9$, $w_5=1/36$, $w_6=1/36$, $w_7=1/36$, $w_8=1/36$

2-D Lattice Arrangements

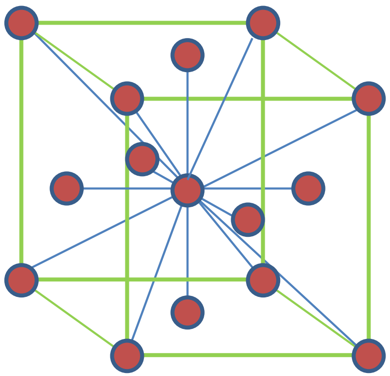
- D2Q7
  - Hexagonal grid
  - Weighting factors: $w_0 = 1/2$, $w_1 = 1/12$, ... , $w_6 = 1/12$,

$$c_s = \Delta x/(2\Delta t)$$



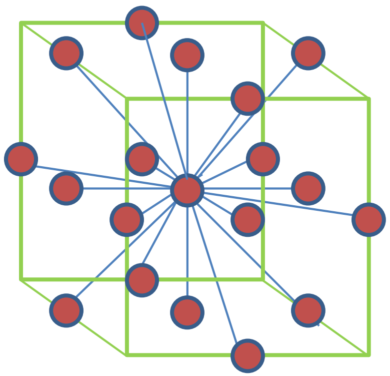
3-D Lattice Arrangements

- D3Q15



  - Weighting factors: w_0= 2/9, w_1=1/9, …… , w_6=1/9, w_7=1/72, …… , w_{14}=1/72.

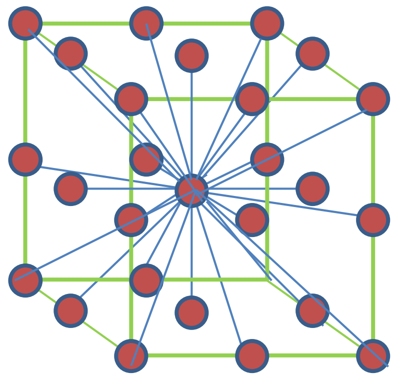
- D3Q19



  - Weighting factors: w_0= 1/3, w_1=1/18, …… , w_6=1/18, w_7=1/36, …… , w_{18}=1/36.

3-D Lattice Arrangements

- D3Q27
  - Weighting factors: $w_0 = 8/27$, $w_1=2/27$, …… , $w_6=2/27$, $w_7=1/216$, …… , $w_{14}=1/216$, $w_{15}=1/54$, …… , $w_{26}=1/54$.



Macroscopic Variables

$$f_i(\mathbf{x} + \mathbf{v}_i\,\Delta t,\, t + \Delta t) = f_i(\mathbf{x}, t) + [f_{0,i}(\mathbf{x}, t) - f_i(\mathbf{x}, t)]\cdot\Delta t/\tau$$

- Density $\rho(\mathbf{x}, t) = \Sigma\, f_i(\mathbf{x}, t)$
- Velocity $\mathbf{u}(\mathbf{x}, t) = \Sigma\, f_i(\mathbf{x}, t)\mathbf{v}_i \,/\, \rho(\mathbf{x}, t)$
- Kinematic shear viscosity (or diffusivity)

  $$\nu = c_s^{\,2}(\tau - \Delta t/2)$$

  – c_s: speed of sound (or an equivalent speed)
- Kinematic bulk viscosity $\nu_B = 2\nu/3$
- Viscous stress tensor

  $$\sigma_{\alpha\beta} \approx -\,[1 - \Delta t/(2\tau)]\sum_i \mathrm{v}_{i\alpha}\mathrm{v}_{i\beta}(f_i - f_{0,i})$$

Equilibrium Distribution Function

- Maxwell-Boltzmann distribution

$$f_0(\mathbf{v}) = (m\beta/2\pi)^{3/2} \exp(-m\beta|\mathbf{v}-\mathbf{v}^0|^2/2) \text{ where } \beta = k_B T$$

$$\rightarrow f_0(\mathbf{v}) = (m\beta/2\pi)^{3/2} \exp(-m\beta|\mathbf{v}-\mathbf{u}|^2/2) \text{ where } \beta = k_B T$$

This can be approximated by Hermite expansion.

- So $f_{0,i}$ can have a form of

$$f_{0,i} = \rho\, \mathrm{w_i}\, [A + B\, \mathbf{v}_i{\cdot}\mathbf{u} + C(\mathbf{v}_i{\cdot}\mathbf{u})^2 + D|\mathbf{u}|^2\,],\ \rho = \Sigma\, f_i(\mathbf{x}, t)$$

- For general fluids with constant density,

$$f_{0,i} = \rho_0\, \mathrm{w_i} \left[1 + \frac{\mathbf{v}_i{\cdot}\mathbf{u}}{c_s{}^2} + \frac{(\mathbf{v}_i{\cdot}\mathbf{u})^2}{2\, c_s{}^4} + \frac{|\mathbf{u}|^2}{2\, c_s{}^2} \right]$$

General LBM Algorithm

1. Initialization
   - ❖ Ex.) $f_{0,i}$ at prev. page & ρ = 1 & $\mathbf{u}$ = 0
2. Calculation: ρ & $\mathbf{u}$
3. Calculation: $f_{0,i}$ (and optionally, $\sigma_{\alpha\beta}$)
   - ✓ You can save ρ & $\mathbf{u}$ at this point.
4. Collision: $f_i(\mathbf{x}, t) + [f_{0,i}(\mathbf{x}, t) - f_i(\mathbf{x}, t)]\cdot\Delta t/\tau$
5. Streaming: $f_i(\mathbf{x} + \mathbf{v}_i \Delta t, t + \Delta t)$
   - & Applying boundary conditions
6. Time stepping: $t \rightarrow t + \Delta t$
7. Back to 2 or Stop

Memory and Loops in LBM

- Memory usage
  - ρ: $N_x \times N_y \times N_z$
  - u_x, u_y, u_z: $N_x \times N_y \times N_z$
  - f_i : $N_x \times N_y \times N_z \times q$ (q: number of $\mathbf{v}_i$)
    - You need $f_{i,old}$ & $f_{i,new}$, or you should arrange updating carefully with or without some buffer memory (some functions such as `circshift` in Matlab, `numpy.roll` in Python, and `CSHIFT` in Fortran may be helpful).
- Loops
  - ρ: for each position, summation over *i*
  - u_x, u_y, u_z: for each position, summation over *i*
  - f_i : for each position

Dimensionless Variables

$$f_i(\mathbf{x} + \mathbf{v}_i \Delta t, t + \Delta t) = f_i(\mathbf{x}, t) + [f_{0,i}(\mathbf{x}, t) - f_i(\mathbf{x}, t)] \cdot \Delta t/\tau$$

- Let’s assume your system has the characteristic length l, velocity V, and density ρ_0,
- $t \rightarrow tV/l$, $\mathbf{x} \rightarrow \mathbf{x}/l$, $\mathbf{v} \rightarrow \mathbf{v}/V$, $\rho \rightarrow \rho/\rho_0$
- $f \rightarrow fV^d/\rho_0$ (d : space dimension)

BGK Stability

- Sufficient condition

  $f_{0,i} \geq 0$ for all i and $\tau/\Delta t > 1/2$

- Optimal condition for $\tau/\Delta t \geq 1$

  $f_{0,0} > 0$ and $\tau/\Delta t \geq 1$

  $$\Rightarrow \quad |\mathbf{u}| < \sqrt{\frac{2}{3}}\frac{\Delta x}{\Delta t}$$

- If unstable,
  - Adjust Δt and Δx, or
  - Try MRT(Multi-Relaxation-Time) or TRT(Two-Relaxation-Time) instead of BGK approx.

Initial Conditions

- For a steady-state solution,

  $f_i = f_{0,i}$ & $\rho = 1$ & $\mathbf{u} = 0$

- For a time-dependent solution,

  $f_i = f_{0,i} + f_{\mathrm{n},i}$ (non-equilibrium part)

  – Consistent initialization
    - Ref.) R. Mei *et al*., Comput. Fluids **35**, 855 (2006).
    - Through iteration by the LB algorithm,
    - Initial velocity → f_i → ρ → f_i → ρ → ...... → converged
    - The end step has to be propagation rather than collision.

LBM for Diffusion Equation

- 1-D diffusion equation $\phi_t = \mu\phi_{xx}$ [+ S]
  - Diffusivity $\mu = c_s^2(\tau - \Delta t/2)$ or $\rho_0 c_s^2(\tau - \Delta t/2)$
  - In LBM, $c_s = \Delta x/(\Delta t\cdot\sqrt{c})$
    - c: 3 for most $\mathbf{v}_i$ lattices except D1Q2, D2Q4, & D2Q7
- Equilibrium distribution function
  - At equilibrium, $\mathbf{u}$ = 0.
  - Thus, $f_{0,i} = \rho\, \mathrm{w_i}$

    (Note: usually $\phi = \rho$ in diffusion equation)
- With source term

$$f_i(\mathbf{x} + \mathbf{v}_i \Delta t, t + \Delta t) = f_i(\mathbf{x}, t) + [f_{0,i}(\mathbf{x}, t) - f_i(\mathbf{x}, t)]\cdot\Delta t/\tau + \Delta t \mathrm{w_i} S$$

LBM for Diffusion Equation

- Boundary conditions (for $T_t = \mu T_{xx}$)
  - Constant temperature (Dirichlet B. C.)
    - $\Sigma f_i(\mathbf{x} = \mathbf{x}_0, t) = T_0 \rightarrow f_i(\mathbf{x} = \mathbf{x}_0) + f_j(\mathbf{x} = \mathbf{x}_0) = w_i T_0 + w_j T_0$
  - Adiabatic boundary (insulator, Neumann B. C.)
    - Using forward or backward finite differences
    - $[T(\mathbf{x} = \mathbf{x}_0+\Delta x) - T(\mathbf{x} = \mathbf{x}_0)]/\Delta x = 0 \rightarrow f_i(\mathbf{x} = \mathbf{x}_0) = f_i(\mathbf{x} = \mathbf{x}_0+\Delta x)$
  - Constant flux boundary
    - $T_t = S$ at the boundary
    - $T(\mathbf{x} = \mathbf{x}_0+\Delta x) - T(\mathbf{x} = \mathbf{x}_0) = \Delta x S$

    $\rightarrow f_i(\mathbf{x} = \mathbf{x}_0) = f_i(\mathbf{x} = \mathbf{x}_0+\Delta x) - \Delta x w_i S$

Do It Yourself

- Consider LBM for 1-D diffusion equation $T_t = \mu T_{xx}$ with D1Q2 lattice.
  - Set $\Delta t = \Delta x = 1$ and prepare f_i and T for your own space grid (set initial $f_i(x, t = 0) = 0$ & $T(x, t = 0) = 0$)
  - Set τ (> 0.5) and b. c. value T_0
  - Loop: $T(x, t) = \Sigma f_i(x, t)$, $f_{0,i} = T\,\mathrm{w_i}$, &
    $f_i(x + \mathrm{v}_i \Delta t, t + \Delta t) = f_i(x, t) + [f_{0,i}(x, t) - f_i(x, t)]\cdot\Delta t/\tau$
    - Applying B. C.: $f_1(x = 0, t) = T_0 - f_2(x = 0, t)$ & $f_2(x = L, t) = -f_1(x = L, t)$
  - Make and run your code. Check out $T(x, t)$

Boundary Conditions for Fluids

- Link-wise
  - Bounce-back method
    - Inverting f_i across the boundary wall. No-slip condition. Simple and stable. Moderate accuracy.

- Wet-node
  - Equilibrium scheme
    - ρ & $\mathbf{u}$ at the boundary by $f_{0,i}$. Simple and stable.
  - Non-equilibrium extrapolation method
    - Non-eq. contribution by nearby fluid. Moderate.
  - Non-equilibrium bounce-back method
    - Complex. Very accurate but less stable.

Boundary Conditions for Fluids

- Pressure B. C.
  - Periodic B. C.
    - For constant pressure, but periodic pressure gradient is also available if you follow the scheme of Zhang & Kwok (Phys. Rev. E, 2006)
    - You should check dependence and change of velocities
  - Pressure at inlet or outlet
    - Ghost boundary + $f_{0,i}$ by the desired pressure, or
    - Density at inlet or outlet from the desired pressure, and finding f_i and normal velocity from the non-equilibrium distribution functions
      - Zou & He (Phys. Fluids, 1997), Guo *et al.* (Phys. Rev. E, 2002)

LBM for Advection-Diffusion

- 1-D advection-diffusion eq. $\phi_t + u_0\phi_x = \mu\phi_{xx}$
  - Like LBM for diffusion, $\mu = c_s{}^2(\tau - \Delta t/2)$ or $\rho_0 c_s{}^2(\tau - \Delta t/2)$
- Equilibrium distribution function

  $f_{0,i} = \rho\, \mathrm{w_i}\, (1 + \mathbf{v}_i{\cdot}\mathbf{u}_0/c_s{}^2)$
- Source term: Same as for the diffusion eq.
- Boundary conditions
  - Periodic B. C.
  - Bounce-back wall
  - Neumann B. C.: setting $u_x = 0$ at the wall if x is the wall direction. (Remember $\rho\mathbf{u} = \Sigma\, f_i(\mathbf{x}, t)\mathbf{v}_i$)

LBM for Navier-Stokes Equation

- 2-D incompressible Navier-Stokes equation

$$\rho u_t + \rho(\mathbf{u}\cdot\triangledown)u = -p_x + \mu(u_{xx} + u_{yy})$$

$$\rho v_t + \rho(\mathbf{u}\cdot\triangledown)v = -p_y + \mu(v_{xx} + v_{yy})$$

  - With the continuity equation: $\rho(u_x + v_y) = 0$
  - where $\mathbf{u} := (u, v)$
  - Dynamic viscosity $\mu = \rho_0 c_s{}^2(\tau - \Delta t/2)$
  - Remember $c_s = \Delta x/(\Delta t\cdot\sqrt{c})$
- Reynolds number $Re = \rho_0 UL/\mu$
- Mach number $Ma = U/c_s$

U: characteristic velocity
L: characteristic length

LBM for Navier-Stokes Equation

- Equilibrium distribution function

$$f_{0,i} = \rho_0 \, \mathsf{w_i} \left[1 + \frac{\mathbf{v}_i \cdot \mathbf{u}}{c_\mathrm{s}^{\,2}} + \frac{(\mathbf{v}_i \cdot \mathbf{u})^2}{2\, c_\mathrm{s}^{\,4}} + \frac{|\mathbf{u}|^2}{2\, c_\mathrm{s}^{\,2}} \right]$$

  – If Ma^2 is not sufficiently small but $Ma^3 \ll 1$

$$f_{0,i} = \rho \, \mathsf{w_i} + \rho_0 \, \mathsf{w_i} \left[\frac{\mathbf{v}_i \cdot \mathbf{u}}{c_\mathrm{s}^{\,2}} + \frac{(\mathbf{v}_i \cdot \mathbf{u})^2}{2\, c_\mathrm{s}^{\,4}} + \frac{|\mathbf{u}|^2}{2\, c_\mathrm{s}^{\,2}} \right]$$

LBM for Navier-Stokes Equation

- Boundary conditions
  - Same as for the advection-diffusion eq., or
  - Dirichlet B. C.: fixing density/pressure at the boundary
    - You need to find f_i and normal velocity at the wall.
- Gravity: adjust the equilibrium velocity.
- Temperature
  - Compare the result with Maxwell-Boltzmann distribution, or
  - Use an equation of state.

Accuracy

- Spatial truncation errors
  - Dominated by O($[\tau - \Delta t/2]^2$) terms
- Modelling errors
  - Ground concept: LBM can be viewed as an approximation of Navier-Stokes equation.
    - See T. Krüger *et al*. section 4 or M. Junk, Numer. Methods Partial Differential Eq. 17: 383-402 (2001).
  - Space and time accuracy: second order
  - Stress tensor errors: O(u^3)
    - In case of standard lattice
    - This limits LBM to weakly compressible flows.
    - Insignificant if $Ma^2 \ll 1$
  - Compressibility errors: O(Ma^2)
    - If you reduce these and spatial truncation errors simultaneously, time accuracy reduces to first order.

MRT Collision Operator

- $f_i(\mathbf{x} + \mathbf{v}_i\,\Delta t,\, t + \Delta t) = f_i(\mathbf{x}, t) + \Delta t \cdot \Omega[f_i(\mathbf{x}, t), f_i(\mathbf{x}, t)]$

→ $f_i(\mathbf{x} + \mathbf{v}_i\,\Delta t,\, t + \Delta t) = f_i(\mathbf{x}, t) + \Delta t \cdot \mathbf{M}^{-1}\mathbf{S}[\mathbf{m}_0(\mathbf{x},t) - \mathbf{m}(\mathbf{x},t)]$

- For D*d*Q*q* velocity lattice,

  $m_j = \sum_{i=0}^{q-1} M_{ji} f_i$ for $j = 0, \ldots\ldots, q - 1$

  $m_{0,j} = \sum_{i=0}^{q-1} M_{ji} f_{0,i}$ for $j = 0, \ldots\ldots, q - 1$

  – See Mohamad or Krüger *et al*. for the values of M_{ji}

- $\mathbf{S}$: diagonal matrix of parameters.
  – If $\mathbf{S} = (1/\tau)\mathbf{I}$ → BGK approx.

TRT Collision Operator

- $f_i(\mathbf{x} + \mathbf{v}_i \Delta t, t + \Delta t) = f_i(\mathbf{x}, t) + [f_{0,i}^{+}(\mathbf{x}, t) - f_i^{+}(\mathbf{x}, t)] \cdot \Delta t/\tau^{+} + [f_{0,i}^{-}(\mathbf{x}, t) - f_i^{-}(\mathbf{x}, t)] \cdot \Delta t/\tau^{-}$
- Two relaxation time parameters
- $f_i^{+} = (f_i + f_{-i})/2$, $f_i^{-} = (f_i - f_{-i})/2$
  - where f_{-i} means that for the opposite velocity (*i.e.*, $-\mathbf{v}_i$)
- $f_i = f_i^{+} + f_i^{-}$, $f_{0,i} = f_{0,i}^{+} + f_{0,i}^{-}$
- Dynamic viscosity $\mu = \rho_0 c_s^2(\tau^{+} - \Delta t/2)$
  - while τ^{-} is a free parameter

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