參考:
Boundary Element Method

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1-D Boundary Elements

- 1-D boundary elements
  - To describe a boundary of a 2-D domain

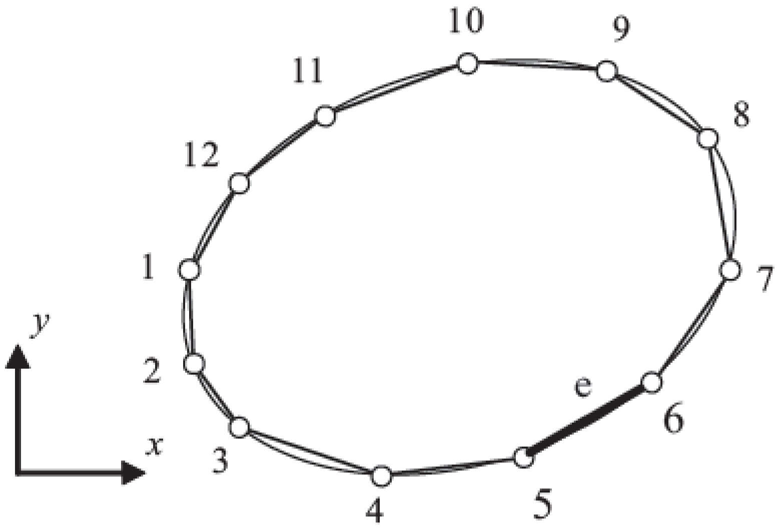


Figure from Beer

✓ Essentially same with 1-D finite elements except their usage

- Shape functions
  → Gaussian quadrature

$$\mathbf{x} = \sum_i N_i \mathbf{x}_i^e$$

2-D or 3-D Elements

- 2-D boundary elements
  - To describe boundaries of 3-D domains or to evaluate 2-D volume integrals
- 3-D cells
  - To evaluate 3-D volume integrals

Iso-parametric Elements

- Same shape functions for coordinates (element shape) and for physical quantity interpolation.
- The physical quantities can be discontinuous at element interfaces.

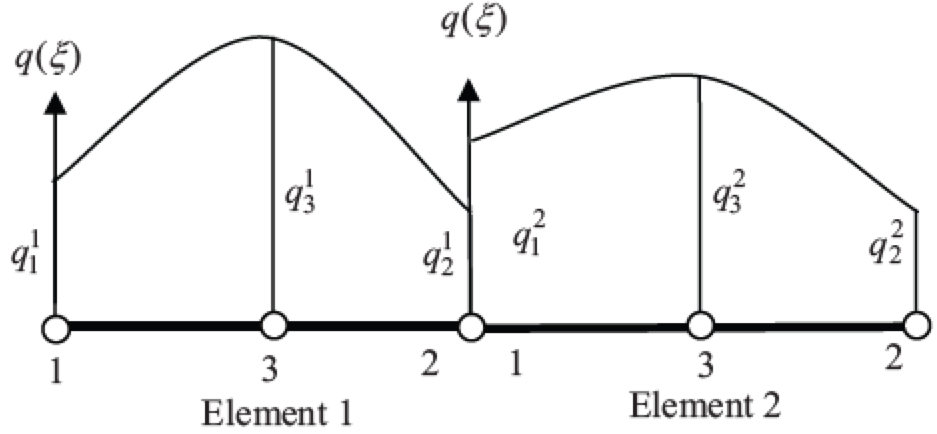
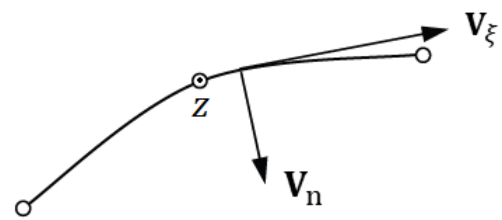


Figure from Beer

Differential Geometry

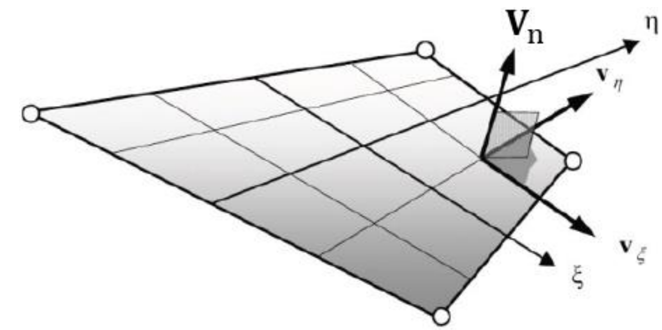
- Tangential vector

$$\mathbf{V}_\xi = \frac{\partial \mathbf{x}}{\partial \xi} = \sum_i \frac{\partial N_i}{\partial \xi} \mathbf{x}_i^e$$



- Normal vector
  - Line element

$$\mathbf{V}_\mathrm{n} = \mathbf{V}_\xi \times \mathbf{n}_z = \left(\frac{dy}{d\xi}, -\frac{dx}{d\xi}, 0\right)^T$$



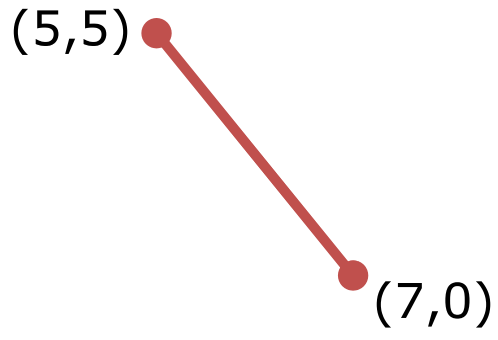
  - Surface element

$$\mathbf{V}_\mathrm{n} = \mathbf{V}_\xi \times \mathbf{V}_\eta = \left(\frac{\partial y}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial y}{\partial \eta}\frac{\partial z}{\partial \xi}, \frac{\partial z}{\partial \xi}\frac{\partial x}{\partial \eta} - \frac{\partial z}{\partial \eta}\frac{\partial x}{\partial \xi}, \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}\right)^T$$

❖ $V_\mathrm{n} = |\mathbf{V}_\mathrm{n}| = J$ (Jacobian)

Figures from Beer

Do It Yourself



- Imagine that you should apply BEM and take the above line segment as a (1-D) bar element. Find the relation between the actual coordinates and the natural coordinates of a point on this element.
- Calculate the Jacobian.

Singular Integrals

- Weakly singular integrals
  - Ex.) $\int_0^1 \ln|x-\alpha|\,dx$ where $0<\alpha<1$

$$= \lim_{\varepsilon\to 0}\int_0^{\alpha-\varepsilon} \ln|x-\alpha|\,dx + \lim_{\varepsilon\to 0}\int_{\alpha+\varepsilon}^{1} \ln|x-\alpha|\,dx$$

  - Integrands
    - 2-D: functions of order $\log r$
    - 3-D: functions of order $1/r$
  - These integrals can be handled by the Gaussian quadrature.

Singular Integrals

- Divergent or strongly singular integrals
  - Ex.) $\int_0^1 \frac{1}{x-\alpha} dx$ where $0 < \alpha < 1$

$$= \lim_{\varepsilon \to 0} \int_0^{\alpha-\varepsilon} \frac{1}{x-\alpha} dx + \lim_{\varepsilon \to 0} \int_{\alpha+\varepsilon}^1 \frac{1}{x-\alpha} dx$$

  - Integrands
    - 2-D: functions of order $1/r$
    - 3-D: functions of order $1/r^2$
  - These integrals are evaluated as *Cauchy principal values*.
    - Determination of Cauchy principal values is tricky. Sometimes they can be replaced by other equations.

Basic Ideas of BEM

- Use functions satisfying the PDE to approximate the solution
- Then actually, only boundary conditions are needed to be approximated
- Iso-parametric elements + numerical integration

Pros & Cons

- Pros
  - Efficiency ← relatively small number of nodes
  - Semi-analytical → good accuracy
  - Visualization-friendly (ex.: exact contours)
  - Good for extreme systems
    - Ex.) concentrated stress, infinite domain

- Cons
  - Inefficiency ← handling BEM matrices
  - Fundamental solution is required → Rarely applicable to inhomogeneous or non-linear systems

Comparison to FEM

BEM

- Discretizing boundary
- Small input data
- Small full matrices
- Better for infinite or semi-infinite domains
- Fundamental solution: required
- Hardly applicable to inhomogeneous or non-linear PDEs

FEM

- Discretizing entire domain
- Rare singularities
- Large sparse matrix
- Better for finite domains
- Fundamental solution: not required
- Applicable to any PDE systems

Application

- Solid Mechanics
  - Fracture, contact
- Acoustics
- Soil-structure, mining & tunnelling
- Fluid mechanics, solid-fluid interaction
- Electromagnetics

Laplace Equation

- Green's second identity

$$\int_D (\phi\Delta\psi - \psi\Delta\phi)\, dV = \oint_{\partial D} \left(\phi \frac{\partial \psi}{\partial \mathbf{n}} - \psi \frac{\partial \phi}{\partial \mathbf{n}}\right) dS$$

  - $\mathbf{n}$: outward surface normal vector

- Integral equation for Laplace equation

$$\int_D (\varphi(\mathbf{r})\Delta\psi(\mathbf{p},\mathbf{r}) - \psi(\mathbf{p},\mathbf{r})\Delta\varphi(\mathbf{r}))\, dV_{\mathbf{r}}$$

$$= \oint_{\partial D} \left(\varphi(\mathbf{q}) \frac{\partial \psi(\mathbf{p},\mathbf{q})}{\partial \mathbf{n}} - \psi(\mathbf{p},\mathbf{q}) \frac{\partial \varphi(\mathbf{q})}{\partial \mathbf{n}}\right) dS_{\mathbf{q}}$$

$$\mathbf{p} \in D \cup \partial D; \mathbf{q} \in \partial D; \mathbf{r} \in D$$

$$\Delta\varphi(\mathbf{r}) = 0 \qquad \Delta\psi(\mathbf{p},\mathbf{r}) = -\delta(\mathbf{p} - \mathbf{r})$$

ψ: auxiliary function

Laplace Equation

- Green function
  - Solution of $\mathrm{L}\psi(\mathbf{p},\mathbf{r}) = \delta(\mathbf{p}-\mathbf{r})$
    - L: linear differential operator
    - It is difficult to find ψ if L is non-linear, anisotropic, or inhomogeneous.
  - Green functions for $\Delta\psi(\mathbf{p},\mathbf{r}) = -\delta(\mathbf{p}-\mathbf{r})$
    - 2-D: $\psi(\mathbf{p},\mathbf{r}) = -\frac{1}{2\pi}\ln|\mathbf{p}-\mathbf{r}|$
    - 3-D: $\psi(\mathbf{p},\mathbf{r}) = \frac{1}{4\pi|\mathbf{p}-\mathbf{r}|}$

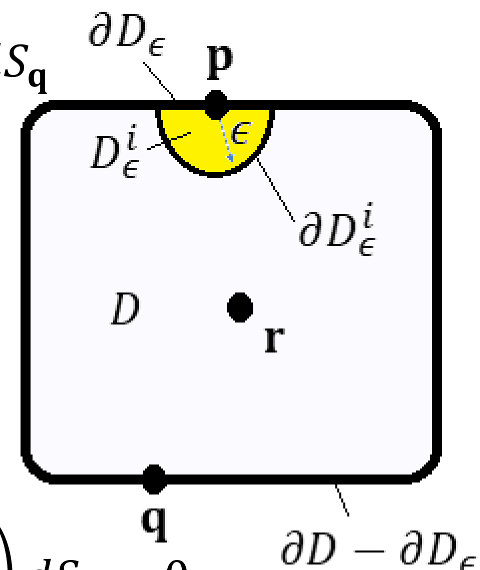
Laplace Equation

- Boundary integral equation

$$-C(\mathbf{p})\varphi(\mathbf{p}) = \lim_{\epsilon\to 0}\int_{\partial D-\partial D_\epsilon}\left(\varphi(\mathbf{q})\frac{\partial\psi(\mathbf{p},\mathbf{q})}{\partial\mathbf{n}} - \psi(\mathbf{p},\mathbf{q})\frac{\partial\varphi(\mathbf{q})}{\partial\mathbf{n}}\right)dS_{\mathbf{q}}$$

$$+\lim_{\epsilon\to 0}\int_{\partial D_\epsilon^i}\left(\varphi(\mathbf{q})\frac{\partial\psi(\mathbf{p},\mathbf{q})}{\partial\mathbf{n}} - \psi(\mathbf{p},\mathbf{q})\frac{\partial\varphi(\mathbf{q})}{\partial\mathbf{n}}\right)dS_{\mathbf{q}}$$

$$\mathbf{p}\in D\cup\partial D_\epsilon;\, \mathbf{q}\in\partial D-\partial D_\epsilon \text{ or } \partial D_\epsilon^i$$



$$C(\mathbf{p})\varphi(\mathbf{p}) + \lim_{\epsilon\to 0}\int_{\partial D-\partial D_\epsilon}\left(\varphi(\mathbf{q})\frac{\partial\psi(\mathbf{p},\mathbf{q})}{\partial\mathbf{n}} - \psi(\mathbf{p},\mathbf{q})\frac{\partial\varphi(\mathbf{q})}{\partial\mathbf{n}}\right)dS_{\mathbf{q}} = 0$$

$$\mathbf{p}\in D\cup\partial D_\epsilon;\, \mathbf{q}\in\partial D$$

Geometric coefficient:
½ for smooth boundary
1 for inner domain

Laplace Equation

- Simple discretization

$$\varphi_k = \varphi(\mathbf{p}_k)$$

$$\sum_{k=1}^{N} H(j,k)\varphi_k = \sum_{k=1}^{N} G(j,k)\gamma_k$$

$$H(j,k) = \oint_{\partial D_k} \frac{\partial \psi}{\partial \mathbf{n}}\left(\mathbf{p}_j, \mathbf{q}\right) dS_{\mathbf{q}} + C\left(\mathbf{p}_j\right)\,\delta_{j,k}$$

$$G(j,k) = \oint_{\partial D_k} \psi\left(\mathbf{p}_j, \mathbf{q}\right) dS_{\mathbf{q}}\,, \qquad \gamma_k = \frac{\partial \varphi}{\partial \mathbf{n}}\left(\mathbf{p}_k\right)$$

→ $\mathbf{Hh} = \mathbf{Gg}$ form

Laplace Equation

- Discretization with shape functions

$$\varphi(\mathbf{q}) \to \sum N_k(\mathbf{q})\varphi_{\boldsymbol{k}}^{\boldsymbol{e}}, \qquad \frac{\partial\varphi(\mathbf{q})}{\partial\mathbf{n}} \to \sum N_k(\mathbf{q})\gamma_{\boldsymbol{k}}^{\boldsymbol{e}}$$

$$\lim_{\epsilon\to 0}\int_{\partial D-\partial D_\epsilon} \quad \to \sum_e \int_{\partial D_e}$$

$$\Rightarrow \quad C(\mathbf{p}_j)\varphi(\mathbf{p}_j) + \sum_e \sum_k \varphi_{\boldsymbol{k}}^{\boldsymbol{e}} \int_{\partial D_e} N_k(\mathbf{q}) \frac{\partial\psi}{\partial\mathbf{n}}(\mathbf{p}_j,\mathbf{q})\,dS_{\mathbf{q}}$$

$$-\sum_e \sum_k \gamma_{\boldsymbol{k}}^{\boldsymbol{e}} \int_{\partial D_e} N_k(\mathbf{q})\psi(\mathbf{p}_j,\mathbf{q})\,dS_{\mathbf{q}} = 0$$

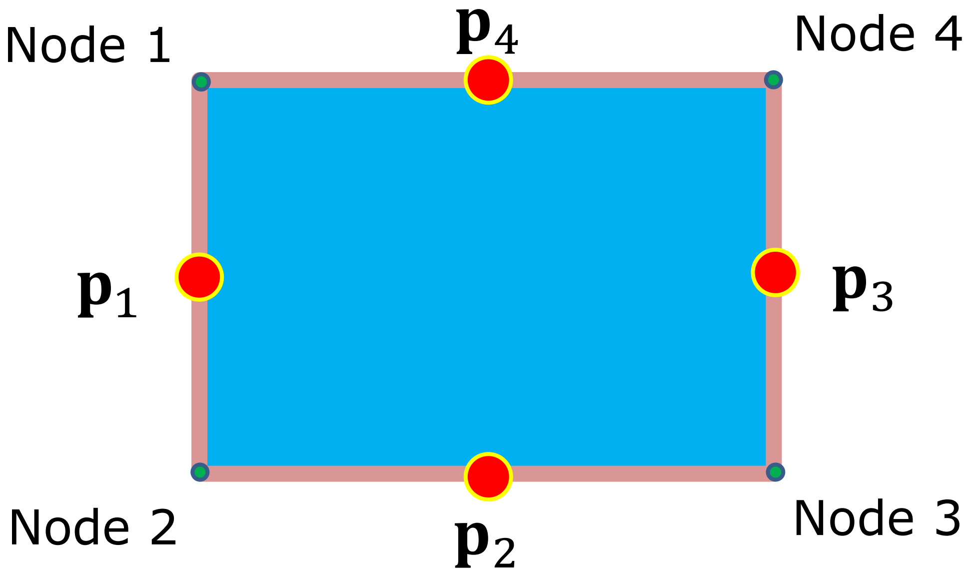
✓ $\varphi(\mathbf{p}_j) = \sum N_k(\mathbf{p}_j)\varphi_{\boldsymbol{k}}^{\boldsymbol{e}}$. $\varphi(\mathbf{p}_j) = \varphi_{\boldsymbol{k}}^{\boldsymbol{e}}$ if $\mathbf{p}_j = \mathbf{p}_k$ (kth node)

✓ $\mathbf{p}_j$'s are called collocation points.

Discretization Example

- 2-D rectangle
  - Index j: collocation points ($\mathbf{p}_1 \sim \mathbf{p}_4$)
  - Index k: nodes

✓ # of nodes
= # of shape functions



Numerical Integration

- Gaussian quadrature
  - Usually integration of polynomials.
  - But in BEM, you must integrate $\log r$, r^{-1}, r^{-2}, or r^{-3} terms. → often approximated as high order polynomials
  - From the formula of Strout and Secrest (Gaussian Quadrature Formulas, 1966), the upper bound of the integration error for r^{-1} is

$$\varepsilon \leq \frac{4}{(4r/L)^{2N}}$$

    - L: element length, N: number of Gauss points.
    - Note that $r = \left|\mathbf{p}_j - \mathbf{q}\right|$.

Numerical Integration

- Gaussian quadrature
  - The required number of Gauss points depends on R/L where R is the distance between $\mathbf{p}_j$ and its closest element boundary point when $\mathbf{p}_j$ is out of the element.

❖ **Minimum R/L for integration order 4 & 5**

N	R/L		
	$O(r^{-1})$	$O(r^{-2})$	$O(r^{-3})$
3	1.4025	2.3187	3.4170
4	0.6735	0.9709	1.2908

➢ U. Eberwien, C. Duenser, & W. Moser, *Engineering Analysis with Boundary Elements* **29**, 447 (2005).

Numerical Integration

- Gaussian quadrature
  - If the integrand has singularity of $\log r$, you should evaluate the integration analytically or use the Gauss-Laguerre type quadrature.

$$\int_0^1 f(\bar{\xi}) \log \frac{1}{\bar{\xi}} d\bar{\xi} \rightarrow \sum w_i f(\bar{\xi}_i)$$

  - Note: $0 < \bar{\xi} < 1$
    - Cf.) $-1 < \xi < 1$ in the Gauss-Legendre type

Numerical Integration

- If $\mathbf{p}_j$ is not located in the integration region, the integrals can be evaluated by using Gaussian quadrature.
  - If $\mathbf{p}_j$ is close to the integration region, subdivision is required to increase R/L. (This is not increasing the number of elements. Don't confuse.)

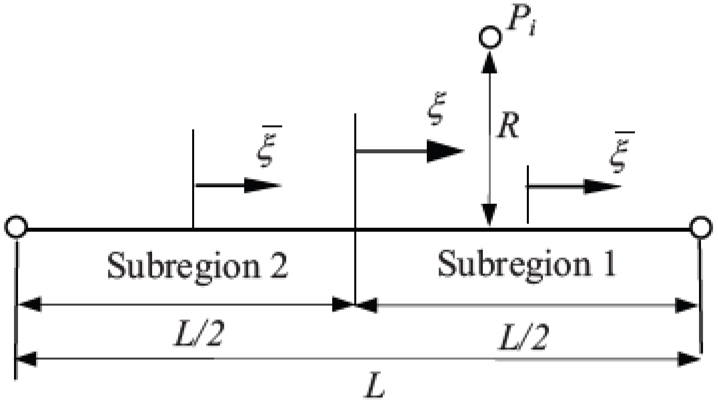
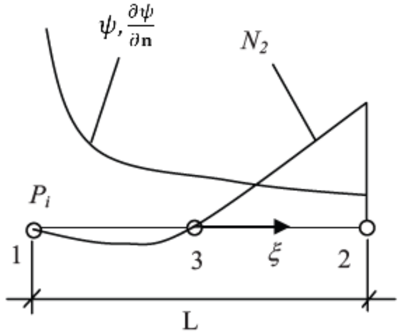
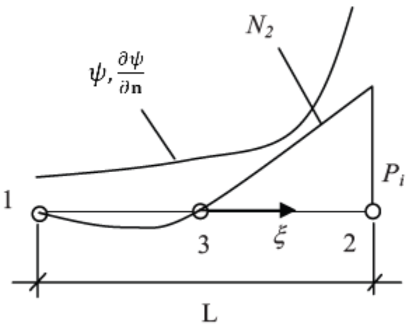


Figure from Beer

Numerical Integration

- If $\mathbf{p}_j$ is in the integration region, the functions ψ & $\frac{\partial \psi}{\partial \mathbf{n}}$ usually tend to diverge.
  - If the integral is weakly singular or N_k is zero at $\mathbf{p}_j$, still you can use Gaussian quadrature.
  - Otherwise, you can't choose but use techniques for strongly singular integrals.





Figures from Beer

BEM Procedure

1. Data input & initialization
2. Determination of connectivity of nodes & elements
3. Determination of element boundary conditions
4. Local element equation & integration
5. Assembly & solving
6. Postprocessing

Postprocessing

- Boundary results
  - From the result you've got $\varphi = \sum N_k \varphi_{\boldsymbol{k}}^{\boldsymbol{e}}$, you can get its derivatives along the boundary.
    - Remember the tangential vector $\mathbf{V}_\xi = \frac{\partial \mathbf{x}}{\partial \xi} = \sum_i \frac{\partial N_i}{\partial \xi} \mathbf{x}_i^e$

$$\frac{\partial \varphi}{\partial \bar{x}} = \sum_i \frac{\partial N_i}{\partial \xi} \frac{\partial \xi}{\partial \bar{x}} \varphi_{\boldsymbol{k}}^{\boldsymbol{e}}$$

    - $\frac{\partial \xi}{\partial \bar{x}}$: inverse Jacobian. For 2-D,

$$\frac{\partial \xi}{\partial \bar{x}} = \left(V_{\xi,x}^2 + V_{\xi,y}^2\right)^{-1/2}$$

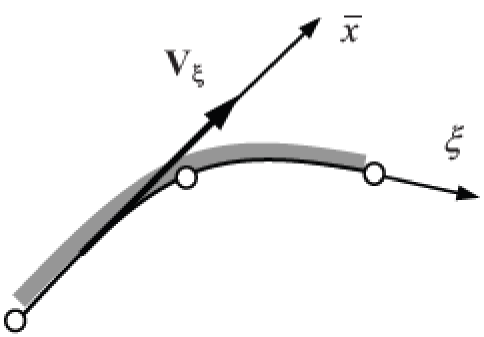


Figure from Beer

Postprocessing

- Internal domain results
  - Remember

$$\varphi(\mathbf{p}) = \sum_e \sum_k \gamma_{\boldsymbol{k}}^{\boldsymbol{e}} \int_{\partial D_e} N_k(\mathbf{q}) \psi(\mathbf{p},\mathbf{q}) dS_{\mathbf{q}} - \sum_e \sum_k \varphi_{\boldsymbol{k}}^{\boldsymbol{e}} \int_{\partial D_e} N_k(\mathbf{q}) \frac{\partial \psi}{\partial \mathbf{n}} (\mathbf{p},\mathbf{q}) dS_{\mathbf{q}}$$

  - Derivatives at $\mathbf{p}$
    - X-direction, for example,

$$\frac{\partial \varphi}{\partial x}(\mathbf{p}) = \sum_e \sum_k \gamma_{\boldsymbol{k}}^{\boldsymbol{e}} \int_{\partial D_e} N_k(\mathbf{q}) \frac{\partial \psi}{\partial x} (\mathbf{p},\mathbf{q}) dS_{\mathbf{q}} - \sum_e \sum_k \varphi_{\boldsymbol{k}}^{\boldsymbol{e}} \int_{\partial D_e} N_k(\mathbf{q}) \frac{\partial}{\partial x} \frac{\partial \psi}{\partial \mathbf{n}} (\mathbf{p},\mathbf{q}) dS_{\mathbf{q}}$$

Corners & Edges

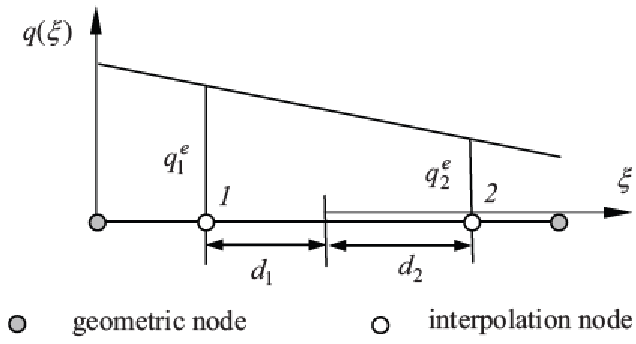
$$C(\mathbf{p})\varphi(\mathbf{p}) = \sum_e \sum_k \gamma_{\boldsymbol{k}}^{\boldsymbol{e}} \int_{\partial D_e} N_k(\mathbf{q})\psi(\mathbf{p},\mathbf{q})dS_{\mathbf{q}} - \sum_e \sum_k \varphi_{\boldsymbol{k}}^{\boldsymbol{e}} \int_{\partial D_e} N_k(\mathbf{q}) \frac{\partial \psi}{\partial \mathbf{n}}(\mathbf{p},\mathbf{q})dS_{\mathbf{q}}$$

- $C(\mathbf{p}) = 1 - \frac{\alpha}{2\pi}$ for sharp corners at $\mathbf{p}$
  - where α is the interior angle of the corner

- Discontinuous Elements
  - All interpolation nodes are located inside the element.
  - This type of elements are used to avoid indefinite $\frac{\partial \psi}{\partial \mathbf{n}}$ values.

Corners & Edges

- Discontinuous Elements
  - 1-D linear shape functions



$$N_1(\xi) = \frac{d_1-\xi}{d_1+d_2}, N_2(\xi) = \frac{d_2-\xi}{d_1+d_2}$$

- $\xi \neq \pm 1$ at interpolation nodes

  - Integration
    - Same as for continuous elements if $\mathbf{p}_j$ is not located at one of interpolation nodes
    - You should divide the integration region at $\mathbf{p}_j$ if $\mathbf{p}_j$ is located at an interpolation node.

Figure from Beer

References

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