Finite Element Method 2

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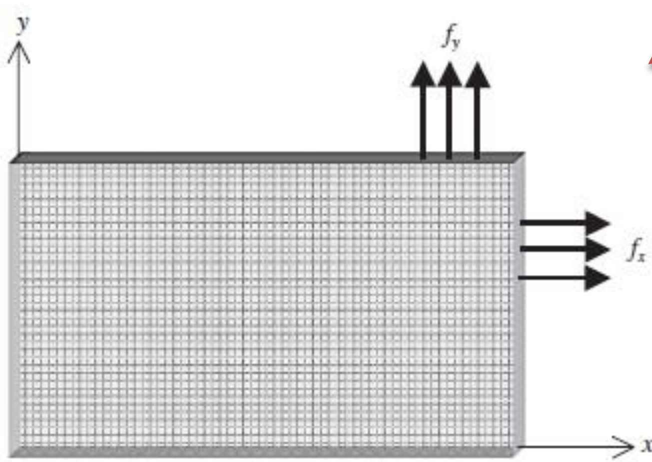
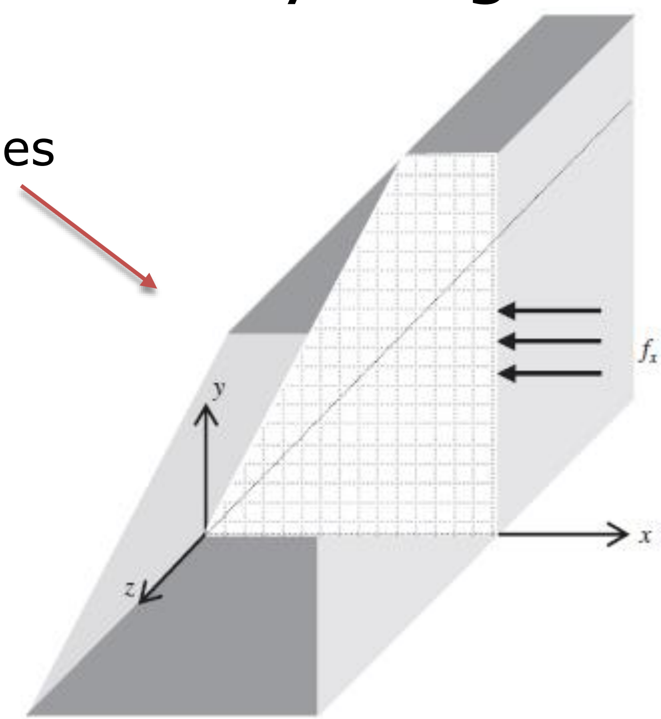
2-D Problems

- Case 1: Planes or sheets → negligible thickness
- Case 2: Huge thickness but everything is same in this direction

❖Mechanical problem examples





Figures from Liu & Quek

2-D Problems

- Case 3: Axisymmetric systems

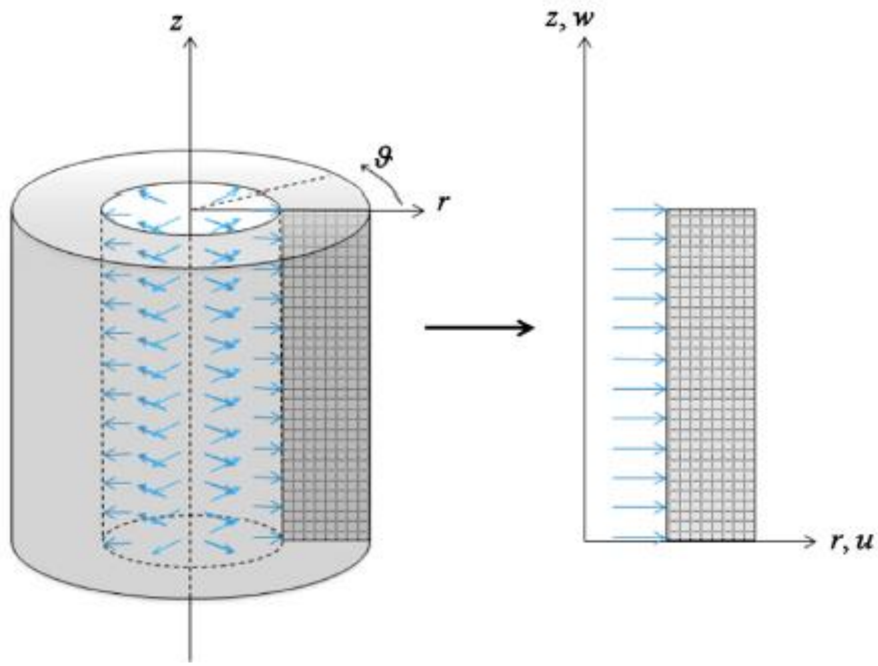


Figure from Liu & Quek

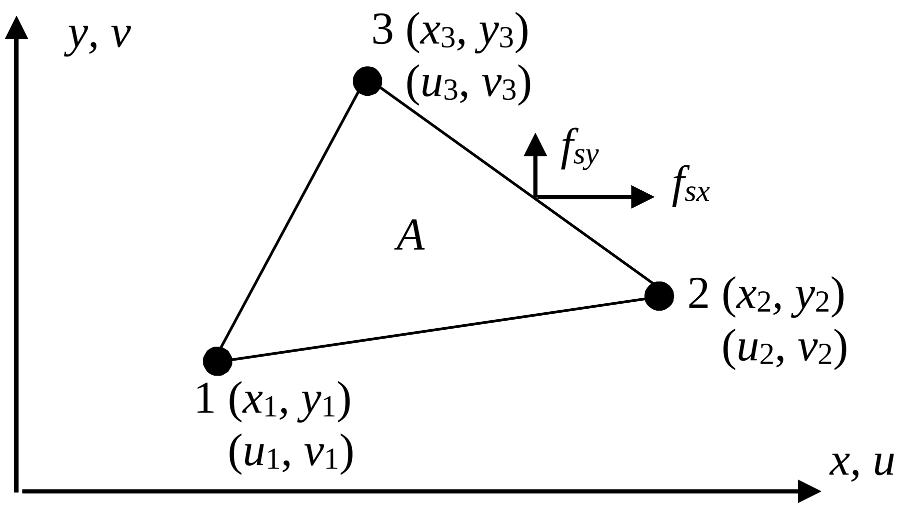
Elements for 2-D Problems

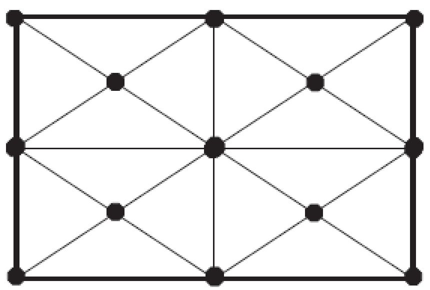
- Triangular elements
- Rectangular elements
- Quadrilateral elements

Linear Triangular Elements

– Liu & Quek pp. 164~172

- Less accurate than quadrilateral elements
- Suitable to complex geometries; used by most mesh generators





Figures from Liu & Quek

Linear Triangular Elements

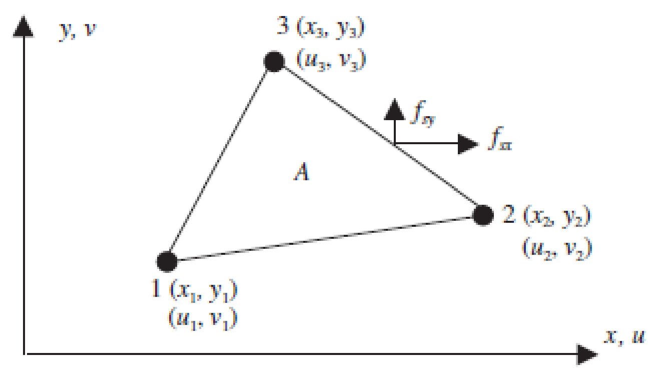
– Liu & Quek pp. 164~172

- Field variable interpolation

$$\mathbf{U}^h(x,y) = \mathbf{N}(x,y)\mathbf{d}_e$$

$$\text{where}\quad \mathbf{d}_e = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} u_1 \\ v_1 \end{matrix}}\right\} \text{displacements at node 1} \\ \left.\vphantom{\begin{matrix} u_2 \\ v_2 \end{matrix}}\right\} \text{displacements at node 2} \\ \left.\vphantom{\begin{matrix} u_3 \\ v_3 \end{matrix}}\right\} \text{displacements at node 3} \end{matrix}$$

$$\mathbf{N} = \left[\underbrace{\begin{matrix} N_1 & 0 \\ 0 & N_1 \end{matrix}}_{\text{Node 1}} \quad \underbrace{\begin{matrix} N_2 & 0 \\ 0 & N_2 \end{matrix}}_{\text{Node 2}} \quad \underbrace{\begin{matrix} N_3 & 0 \\ 0 & N_3 \end{matrix}}_{\text{Node 3}} \right]$$



Figures from Liu & Quek

Linear Triangular Elements

– Liu & Quek pp. 164~172

- Shape function construction
  - Let

$$N_i(x,y) = a_i + b_i x + c_i y, \qquad i = 1,2,3$$

  - Delta function property: $N_i(x_j, y_j) = \delta_{ij}$

$$N_1(x_1,y_1) = a_1 + b_1 x_1 + c_1 y_1 = 1$$
$$N_1(x_2,y_2) = a_1 + b_1 x_2 + c_1 y_2 = 0$$
$$N_1(x_3,y_3) = a_1 + b_1 x_3 + c_1 y_3 = 0$$

which gives

$$a_1 = \frac{x_2 y_3 - x_3 y_2}{(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x_1 + (x_3 - x_2)y_1}$$

Linear Triangular Elements

– Liu & Quek pp. 164~172

- Shape function construction

$$b_1 = \frac{y_2 - y_3}{(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x_1 + (x_3 - x_2)y_1}$$

$$c_1 = \frac{x_3 - x_2}{(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x_1 + (x_3 - x_2)y_1}$$

– Triangle area

$$A_e = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}[(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x_1 + (x_3 - x_2)y_1]$$

$$a_1 = \frac{x_2 y_3 - x_3 y_2}{2A_e}, \qquad b_1 = \frac{y_2 - y_3}{2A_e}, \qquad c_1 = \frac{x_3 - x_2}{2A_e}$$

$$N_1(x, y) = \frac{1}{2A_e}[(y_2 - y_3)(x - x_2) + (x_3 - x_2)(y - y_2)]$$

Linear Triangular Elements

– Liu & Quek pp. 164~172

- Shape function construction

$$N_1(x,y) = \frac{1}{2A_e}\left[(y_2 - y_3)(x - x_2) + (x_3 - x_2)(y - y_2)\right]$$

$$N_2(x,y) = \frac{1}{2A_e}\left[(y_3 - y_1)(x - x_3) + (x_1 - x_3)(y - y_3)\right]$$

$$N_3(x,y) = \frac{1}{2A_e}\left[(y_1 - y_2)(x - x_1) + (x_2 - x_1)(y - y_1)\right]$$

$$N_i(x,y) = a_i + b_i x + c_i y$$

$$a_i = \frac{x_j y_k - x_k y_j}{2A_e}, \qquad b_i = \frac{y_j - y_k}{2A_e}, \qquad c_i = \frac{x_k - x_j}{2A_e}$$

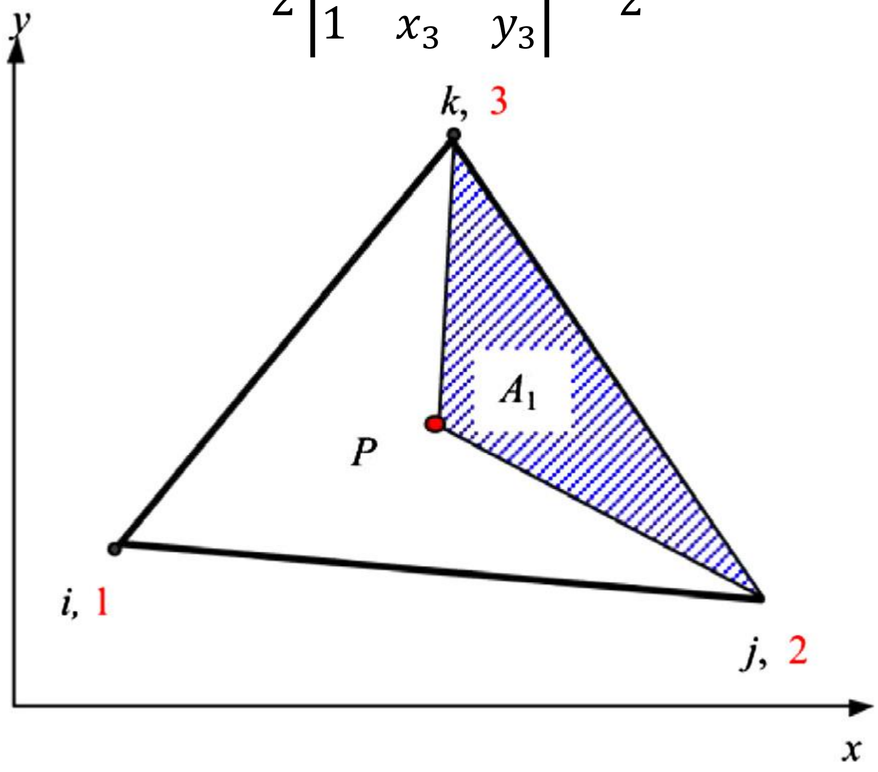
– i, j, k: cyclic permutation

Linear Triangular Elements

– Liu & Quek pp. 164~172

- Area coordinates

$$A_1 = \frac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$



$$L_1 := \frac{A_1}{A_e}$$

$$L_2 := \frac{A_2}{A_e}$$

$$L_3 := \frac{A_3}{A_e}$$

$$L_i(\mathbf{x}_j) = \delta_{ij}$$

$$L_1 + L_2 + L_3 = \frac{A_1 + A_2 + A_3}{A_e} = 1$$

Partition of unity

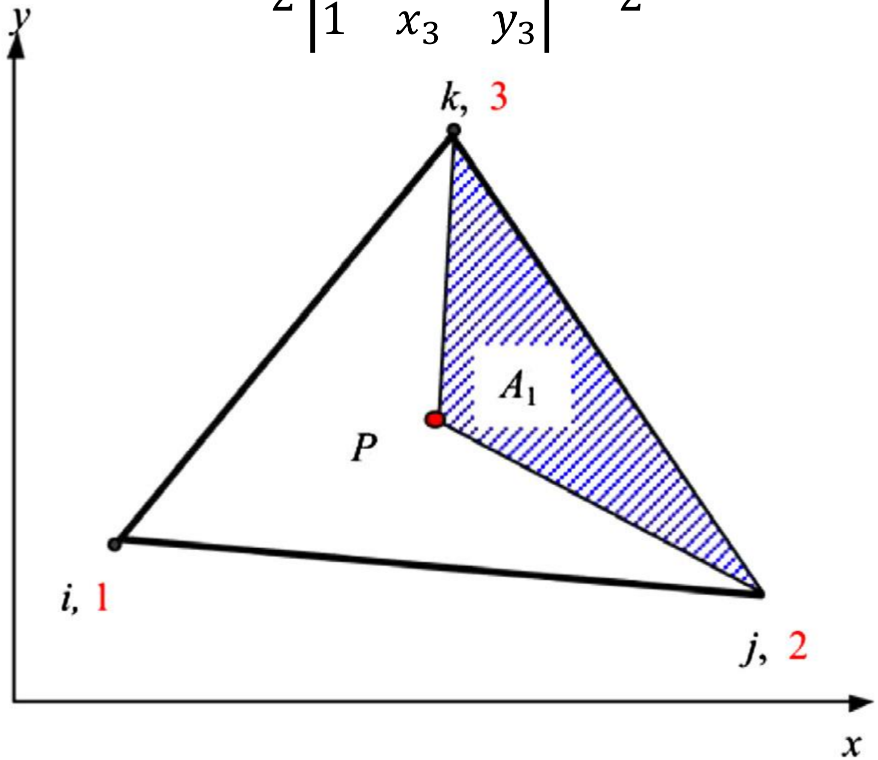
Figure from Liu & Quek

Linear Triangular Elements

– Liu & Quek pp. 164~172

- Area coordinates

$$A_1 = \frac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$



$$L_i(\mathbf{x}_j) = \delta_{ij}$$

$$L_1 + L_2 + L_3 = \frac{A_1 + A_2 + A_3}{A_e} = 1$$

$$L_1 = N_1$$
$$L_2 = N_2$$
$$L_3 = N_3$$

Figure from Liu & Quek

參考: Integration Rules

- Eisenberg and Malvern, Int. J. Numer. Methods. Eng. 7, 574 (1973)

- Area coordinates (for triangular elements)

$$\int_{A_e} L_1^m L_2^n L_3^p \, dA = \frac{m!\, n!\, p!}{(m+n+p+2)!} 2A_e$$

- Volume coordinates (for tetrahedron elements)

$$\int_{V_e} L_1^m L_2^n L_3^p L_4^q \, dV = \frac{m!\, n!\, p!\, q!}{(m+n+p+q+3)!} 6V_e$$

❖ $L_i = \frac{d_{P-jkl}}{d_{i-jkl}}$

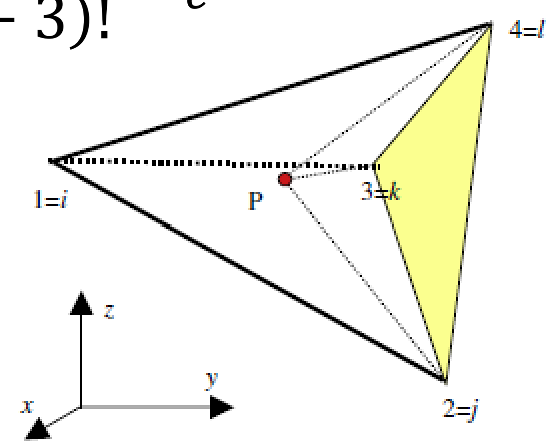


Figure from Liu & Quek

Linear Rectangular Elements

– Liu & Quek pp. 176~182

- Shape function construction
  - Field variable interpolation

$$\mathbf{U}^h(x,y) = \mathbf{N}(x,y)\mathbf{d}_e$$

$$\mathbf{d}_e = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ u_2 \\ u_3 \\ u_3 \\ u_4 \\ u_4 \end{Bmatrix} \begin{matrix} \} \text{ displacements at node 1} \\ \\ \} \text{ displacements at node 2} \\ \\ \} \text{ displacements at node 3} \\ \\ \} \text{ displacements at node 4} \\ \end{matrix}$$

$$\mathbf{N} = \begin{bmatrix} \underbrace{\begin{matrix} N_1 & 0 \\ 0 & N_1 \end{matrix}}_{\text{Node 1}} & \underbrace{\begin{matrix} N_2 & 0 \\ 0 & N_2 \end{matrix}}_{\text{Node 2}} & \underbrace{\begin{matrix} N_3 & 0 \\ 0 & N_3 \end{matrix}}_{\text{Node 3}} & \underbrace{\begin{matrix} N_4 & 0 \\ 0 & N_4 \end{matrix}}_{\text{Node 4}} \end{bmatrix}$$

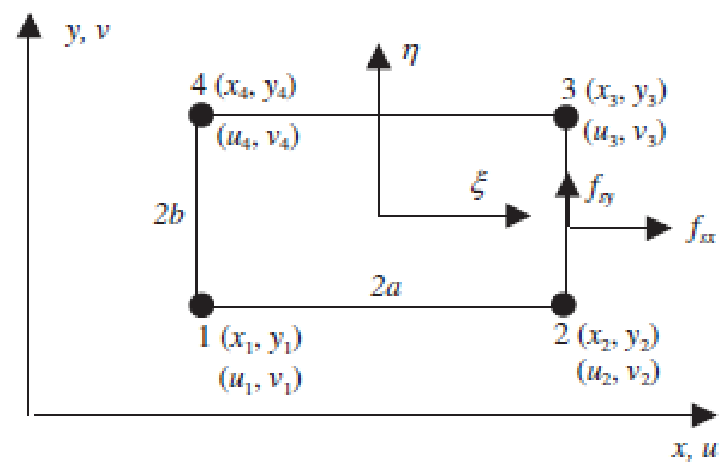


Figure from Liu & Quek

Linear Rectangular Elements

– Liu & Quek pp. 176~182

- Shape function construction
  - Natural coordinates

$$\xi = \frac{1}{a}\left(x - \frac{x_1 + x_2}{2}\right), \qquad \eta = \frac{1}{b}\left(y - \frac{y_1 + y_2}{2}\right)$$

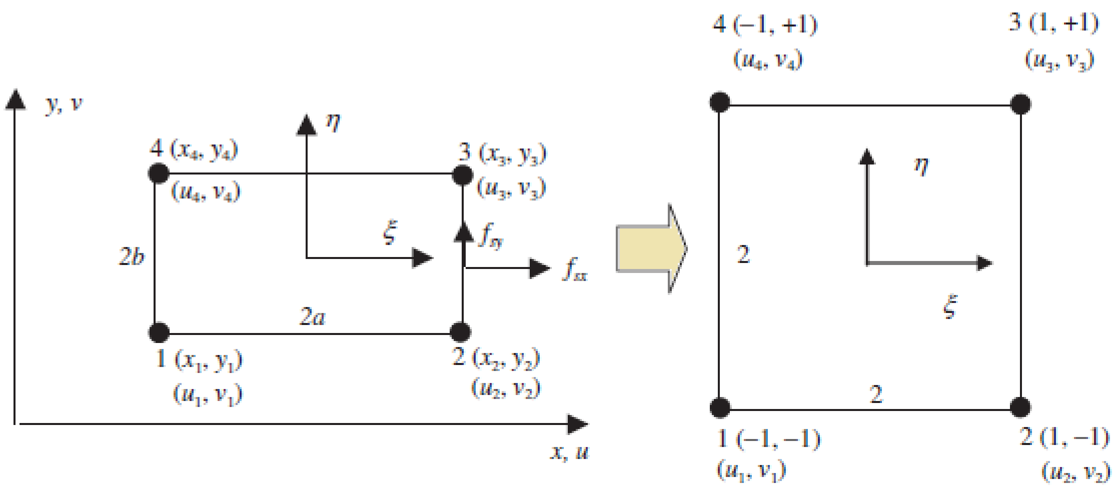


Figure from Liu & Quek

Linear Rectangular Elements

– Liu & Quek pp. 176~182

- Shape function construction

$$\left.\begin{aligned} N_1 &= \tfrac{1}{4}(1-\xi)(1-\eta) \\ N_2 &= \tfrac{1}{4}(1+\xi)(1-\eta) \\ N_3 &= \tfrac{1}{4}(1+\xi)(1+\eta) \\ N_4 &= \tfrac{1}{4}(1-\xi)(1+\eta) \end{aligned}\right\} N_j = \tfrac{1}{4}(1+\xi_j\xi)(1+\eta_j\eta)$$

– Partition of unity

$$\sum_{i=1}^{4} N_i = N_1 + N_2 + N_3 + N_4$$

$$= \frac{1}{4}[(1-\xi)(1-\eta) + (1+\xi)(1-\eta) + (1+\xi)(1+\eta) + (1-\xi)(1+\eta)]$$

$$= \frac{1}{4}[2(1-\xi) + 2(1+\xi)] = 1$$

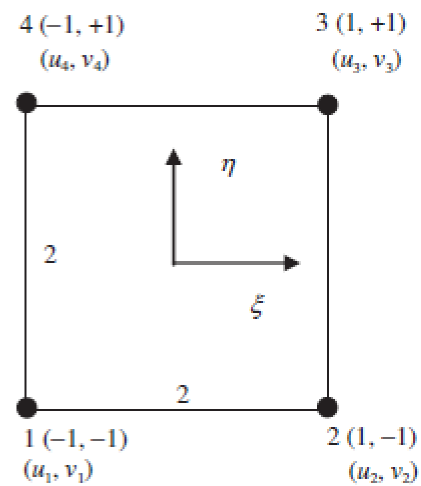


Figure from Liu & Quek

Linear Rectangular Elements

– Liu & Quek pp. 176~182

- Shape function construction
  - Delta function property

$$N_3|_{\text{at node 1}} = \frac{1}{4}(1+\xi)(1+\eta)|_{\substack{\xi=-1\\ \eta=-1}} = 0$$

$$N_3|_{\text{at node 2}} = \frac{1}{4}(1+\xi)(1+\eta)|_{\substack{\xi=1\\ \eta=-1}} = 0$$

$$N_3|_{\text{at node 3}} = \frac{1}{4}(1+\xi)(1+\eta)|_{\substack{\xi=1\\ \eta=1}} = 1$$

$$N_3|_{\text{at node 4}} = \frac{1}{4}(1+\xi)(1+\eta)|_{\substack{\xi=-1\\ \eta=1}} = 0$$

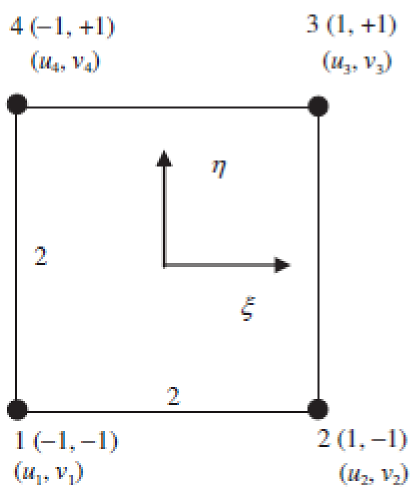


Figure from Liu & Quek

Linear Rectangular Elements

– Liu & Quek pp. 176~182

- Gaussian Quadrature

In 1 direction: $$I = \int_{-1}^{+1} f(\xi)\,\mathrm{d}\xi = \sum_{j=1}^{m} w_j f(\xi_j)$$

  - m gauss points gives exact solution of polynomial integrand of $n = 2m - 1$

In 2 directions:

$$I = \int_{-1}^{+1}\int_{-1}^{+1} f(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta = \sum_{i=1}^{n_x}\sum_{j=1}^{n_y} w_i w_j f(\xi_i,\eta_j)$$

Linear Rectangular Elements

- Liu & Quek pp. 176~182

- Gaussian Quadrature

m	ξ_j	w_j	Accuracy n
1	0	2	1
2	-1/√3, 1/√3	1, 1	3
3	-√0.6, 0, √0.6	5/9, 8/9, 5/9	5
4	-0.861136, -0.339981, 0.339981, 0.861136	0.347855, 0.652145, 0.652145, 0.347855	7
5	-0.906180, -0.538469, 0, 0.538469, 0.906180	0.236927, 0.478629, 0.568889, 0.478629, 0.236927	9

Linear Rectangular Elements

– Liu & Quek pp. 176~182

- Gaussian Quadrature

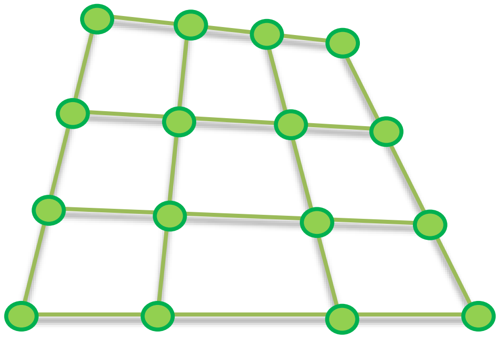
$$\int_{-1}^{+1}\int_{-1}^{+1} N_i N_j \,\mathrm{d}\xi \mathrm{d}\eta$$
$$= \frac{1}{16}\int_{-1}^{+1}(1+\xi_i\xi)(1+\xi_j\xi)\,\mathrm{d}\xi \int_{-1}^{+1}(1+\eta_i\eta)(1+\eta_j\eta)\,\mathrm{d}\eta$$
$$= \frac{1}{4}(1+\frac{1}{3}\xi_i\xi_j)(1+\frac{1}{3}\eta_i\eta_j)$$

– Ex.) $\int_{-1}^{+1}\int_{-1}^{+1} N_3 N_3 \,\mathrm{d}\xi \mathrm{d}\eta = \frac{4}{9}$

Linear Quadrilateral Elements

– Liu & Quek pp. 183~188

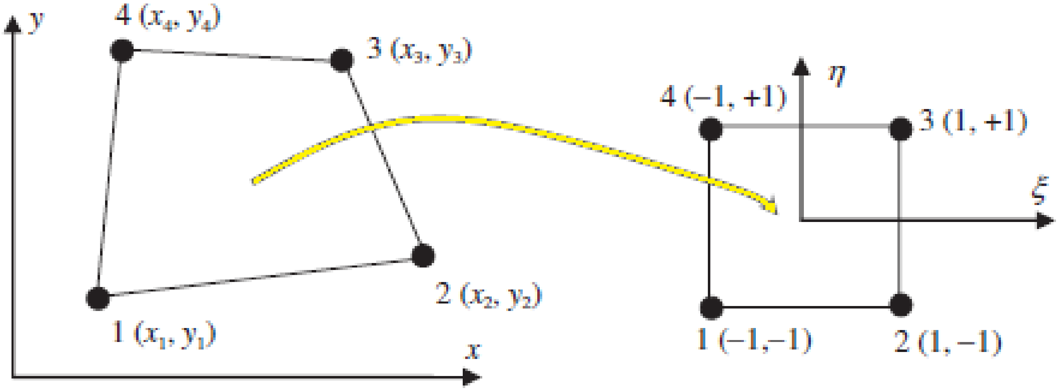
- Rectangular elements have limited application
- Quadrilateral elements with unparallel edges are more useful
- Irregular shape requires coordinate mapping before using Gauss integration



Linear Quadrilateral Elements

– Liu & Quek pp. 183~188

- Coordinate mapping



Physical coordinates   Natural coordinates

$$\mathbf{U}^h(\xi,\eta) = \mathbf{N}(\xi,\eta)\mathbf{d}_e$$ (Interpolation of displacements)

$$\mathbf{X}(\xi,\eta) = \mathbf{N}(\xi,\eta)\mathbf{x}_e$$ (Interpolation of coordinates)

Figure from Liu & Quek

Linear Quadrilateral Elements

– Liu & Quek pp. 183~188

- Coordinate mapping

$$\mathbf{X}(\xi,\eta) = \mathbf{N}(\xi,\eta)\mathbf{x}_e$$

where $\mathbf{X} = \begin{Bmatrix} x \\ y \end{Bmatrix}$,

$$\mathbf{x}_e = \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} x \\ y \end{matrix}}\right\} \text{coordinates at node 1} \\ \left.\vphantom{\begin{matrix} x \\ y \end{matrix}}\right\} \text{coordinates at node 2} \\ \left.\vphantom{\begin{matrix} x \\ y \end{matrix}}\right\} \text{coordinates at node 3} \\ \left.\vphantom{\begin{matrix} x \\ y \end{matrix}}\right\} \text{coordinates at node 4} \end{matrix}$$

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$
$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$
$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

$$\left\{ \begin{aligned} x(\xi,\eta) &= \sum_{i=1}^{4} N_i(\xi,\eta)x_i \\ y(\xi,\eta) &= \sum_{i=1}^{4} N_i(\xi,\eta)y_i \end{aligned} \right.$$

Linear Quadrilateral Elements

– Liu & Quek pp. 183~188

- Coordinate mapping

Substitute $\xi = 1$ into $x = \sum_{i=1}^{4} N_i(\xi,\eta)x_i$

$$x = \frac{1}{2}(1-\eta)x_2 + \frac{1}{2}(1+\eta)x_3 \qquad \text{or} \qquad x = \frac{1}{2}(x_2 + x_3) + \frac{1}{2}\eta(x_3 - x_2)$$

$$y = \frac{1}{2}(1-\eta)y_2 + \frac{1}{2}(1+\eta)y_3 \qquad \qquad y = \frac{1}{2}(y_2 + y_3) + \frac{1}{2}\eta(y_3 - y_2)$$

Eliminating η, $y = \dfrac{(y_3 - y_2)}{(x_3 - x_2)}\left\{x - \dfrac{1}{2}(x_2 + x_3)\right\} + \dfrac{1}{2}(y_2 + y_3)$

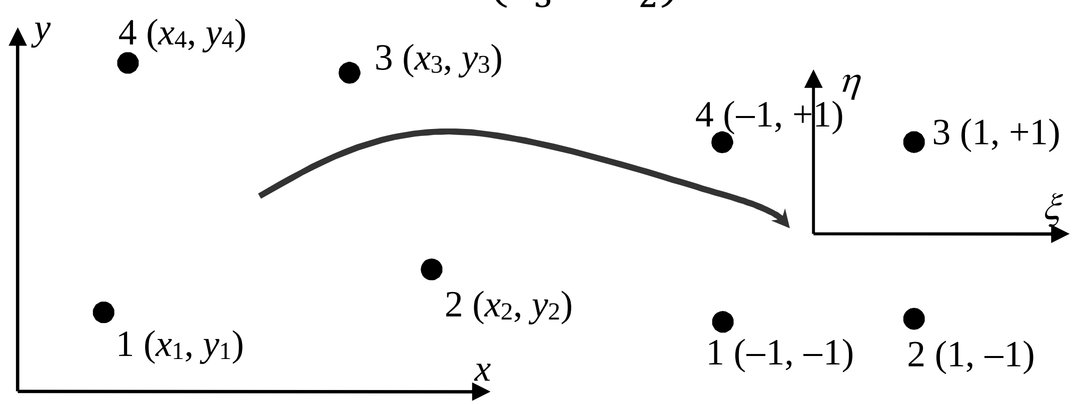


Figure from Liu & Quek

Linear Quadrilateral Elements

– Liu & Quek pp. 183~188

- Integration
  - Jacobian

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} \qquad \because \mathbf{X}(\xi,\eta) = \mathbf{N}(\xi,\eta)\mathbf{x}_e$$

$$\mathrm{d}x\mathrm{d}y = \det|\mathbf{J}|\,\mathrm{d}\xi\mathrm{d}\eta$$

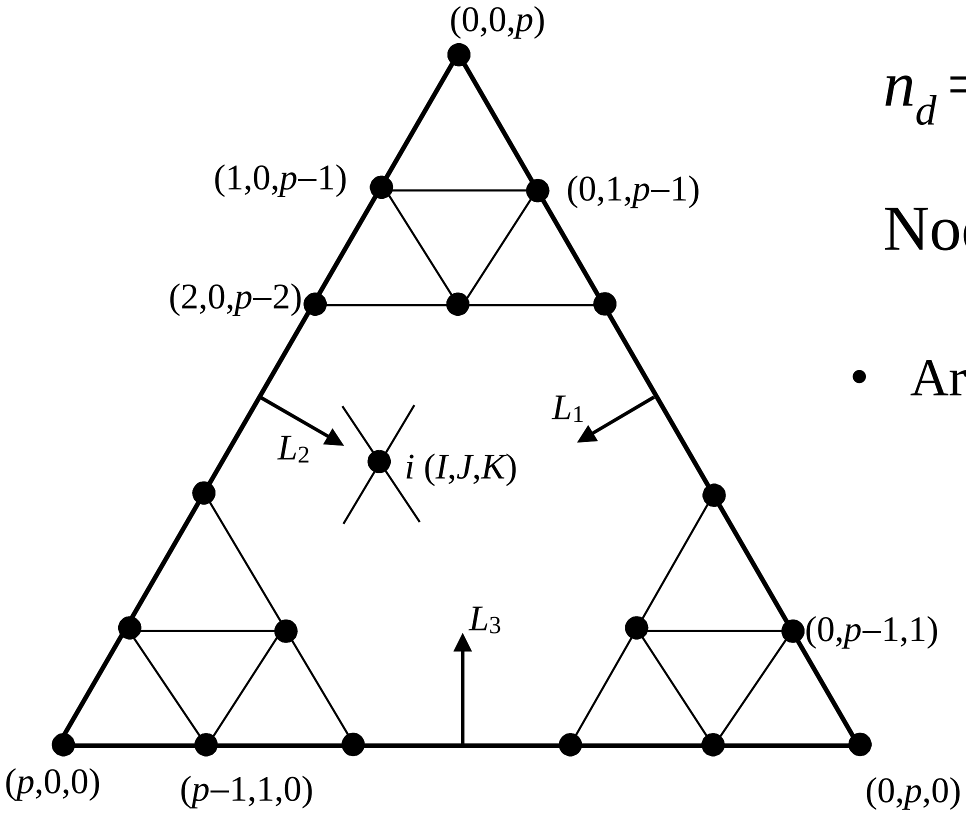
Linear Quadrilateral Elements

– Liu & Quek pp. 183~188

- Shape functions used for interpolating the coordinates are the same as the shape functions used for interpolation of the displacement field. Therefore, the element is called an *isoparametric element*.
- Note that the shape functions for coordinate interpolation and displacement interpolation do not have to be the same.

Higher Order Triangular Elements

– Liu & Quek pp. 191~194



$n_d = (p+1)(p+2)/2$

Node i, $\quad I + J + K = p$

- Argyris *et al.*, Aeronaut. J. 72, 618 (1968) :

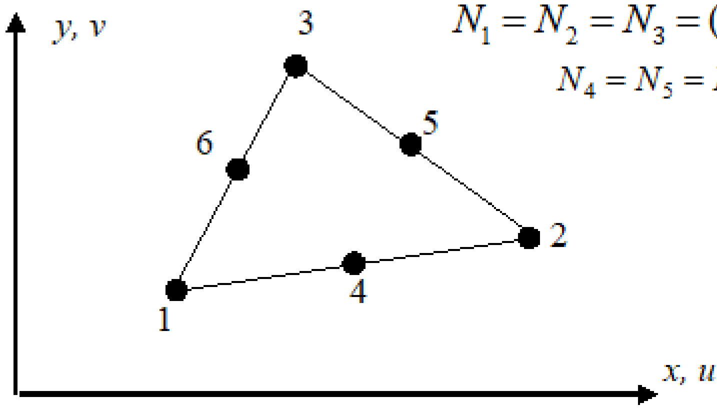
$$N_i = l_I^I(L_1) l_J^J(L_2) l_K^K(L_3)$$

$$l_\beta^\beta(L_\alpha) = \frac{(L_\alpha - L_{\alpha_0})(L_\alpha - L_{\alpha_1}) \cdots (L_\alpha - L_{\alpha_{(\beta-1)}})}{(L_{\alpha_I} - L_{\alpha_0})(L_{\alpha_I} - L_{\alpha_1}) \cdots (L_{\alpha_I} - L_{\alpha_{(\beta-1)}})}$$

Figure from Liu & Quek

Higher Order Triangular Elements

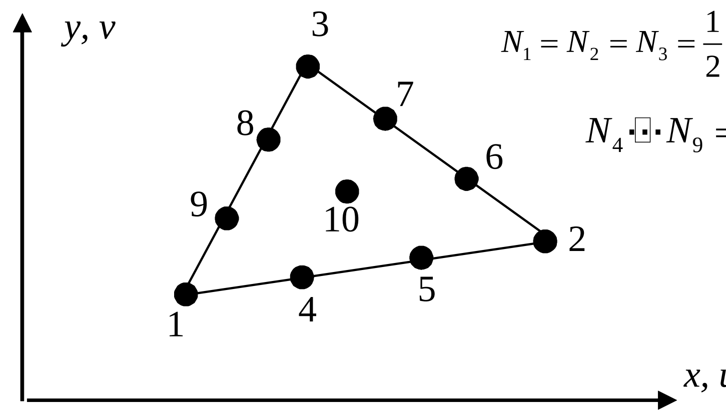
– Liu & Quek pp. 191~194



$$N_1 = N_2 = N_3 = (2L_1 - 1)L_1$$

$$N_4 = N_5 = N_6 = 4L_1L_2$$

Quadratic element



$$N_1 = N_2 = N_3 = \frac{1}{2}(3L_1 - 1)(3L_1 - 2)L_1$$

$$N_4 \cdots N_9 = \frac{9}{2}L_1L_2(3L_1 - 1)$$

$$N_{10} = 27L_1L_2L_3$$

Cubic element

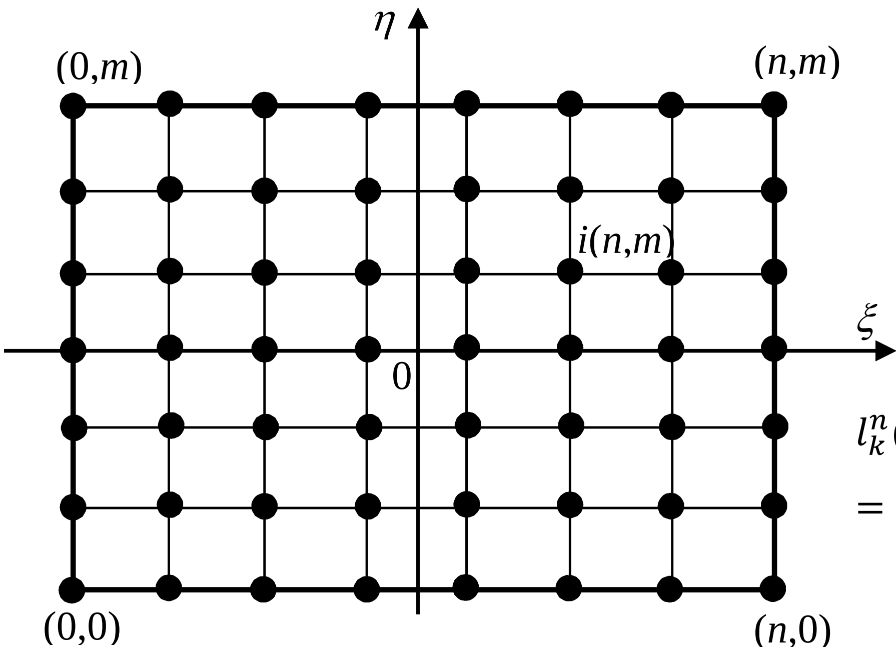
Figures from Liu & Quek

Higher Order Rectangular Elements

– Liu & Quek pp. 195~200

- Lagrange type

➢ Ref.) Zienkiewicz & Taylor



$$N_i = N_I^{1D} N_J^{1D} = l_I^n(\xi) l_J^m(\eta)$$

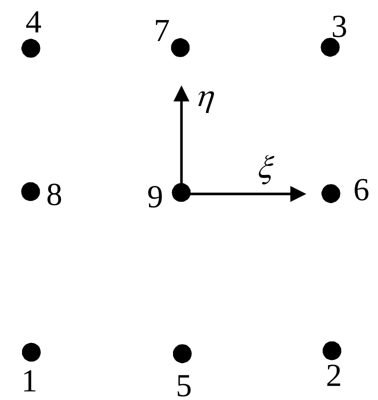
$$l_k^n(\xi) = \frac{(\xi - \xi_0)(\xi - \xi_1)\cdots(\xi - \xi_{k-1})(\xi - \xi_{k+1})\cdots(\xi - \xi_n)}{(\xi_k - \xi_0)(\xi_k - \xi_1)\cdots(\xi_k - \xi_{k-1})(\xi_k - \xi_{k+1})\cdots(\xi_k - \xi_n)}$$

Figure from Liu & Quek

Higher Order Rectangular Elements

– Liu & Quek pp. 195~200

- Lagrange type
  - Ex.) 9-node quadratic element



$$N_1 = N_1^{1D}(\xi)N_1^{1D}(\eta) = \frac{1}{4}\xi(1-\xi)\eta(1-\eta)$$

$$N_2 = N_2^{1D}(\xi)N_1^{1D}(\eta) = -\frac{1}{4}\xi(1+\xi)\eta(1-\eta)$$

$$N_3 = N_2^{1D}(\xi)N_2^{1D}(\eta) = \frac{1}{4}\xi(1+\xi)(1+\eta)\eta$$

$$N_4 = N_1^{1D}(\xi)N_2^{1D}(\eta) = -\frac{1}{4}\xi(1-\xi)(1+\eta)\eta$$

$$N_5 = N_3^{1D}(\xi)N_1^{1D}(\eta) = -\frac{1}{2}(1+\xi)(1-\xi)(1-\eta)\eta$$

$$N_6 = N_2^{1D}(\xi)N_3^{1D}(\eta) = \frac{1}{2}\xi(1+\xi)(1+\eta)(1-\eta)$$

$$N_7 = N_3^{1D}(\xi)N_2^{1D}(\eta) = \frac{1}{2}(1+\xi)(1-\xi)(1+\eta)\eta$$

$$N_8 = N_1^{1D}(\xi)N_1^{1D}(\eta) = -\frac{1}{2}\xi(1-\xi)(1-\eta)\eta$$

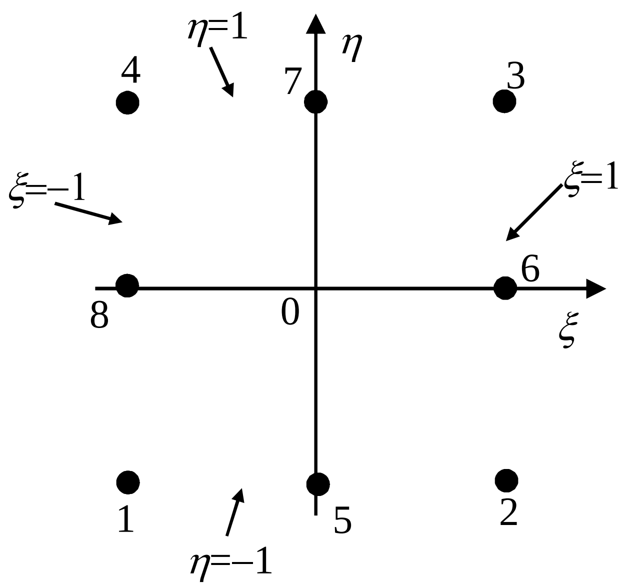
$$N_9 = N_3^{1D}(\xi)N_3^{1D}(\eta) = (1-\xi^2)(1-\eta^2)$$

Figure from Liu & Quek

Higher Order Rectangular Elements

– Liu & Quek pp. 195~200

- Serendipity type
  - 8-node quadratic element



$$N_j = \frac{1}{4}(1+\xi_j\xi)(1+\eta_j\eta)(\xi_j\xi+\eta_j\eta-1) \qquad j = 1, 2, 3, 4$$

$$N_j = \frac{1}{2}(1-\xi^2)(1+\eta_j\eta) \qquad j = 5,7$$

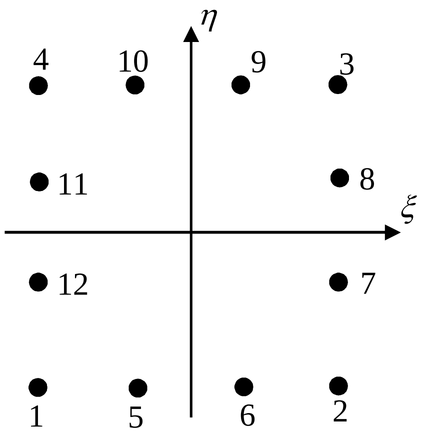
$$N_j = \frac{1}{2}(1+\xi_j\xi)(1-\eta^2) \qquad j = 6,8$$

Figure from Liu & Quek

Higher Order Rectangular Elements

– Liu & Quek pp. 195~200

- Serendipity type
  - 12–node quadratic element



$$N_j = \frac{1}{32}(1+\xi_j\xi)(1+\eta_j\eta)(9\xi^2+9\eta^2-10)$$

for corner nodes $j = 1, 2, 3, 4$

$$N_j = \frac{9}{32}(1+\xi_j\xi)(1-\eta^2)(1+9\eta_j\eta)$$

for side nodes $j = 7, 8, 11, 12$ where $\xi_j = \pm 1$ and $\eta_j = \pm\frac{1}{3}$

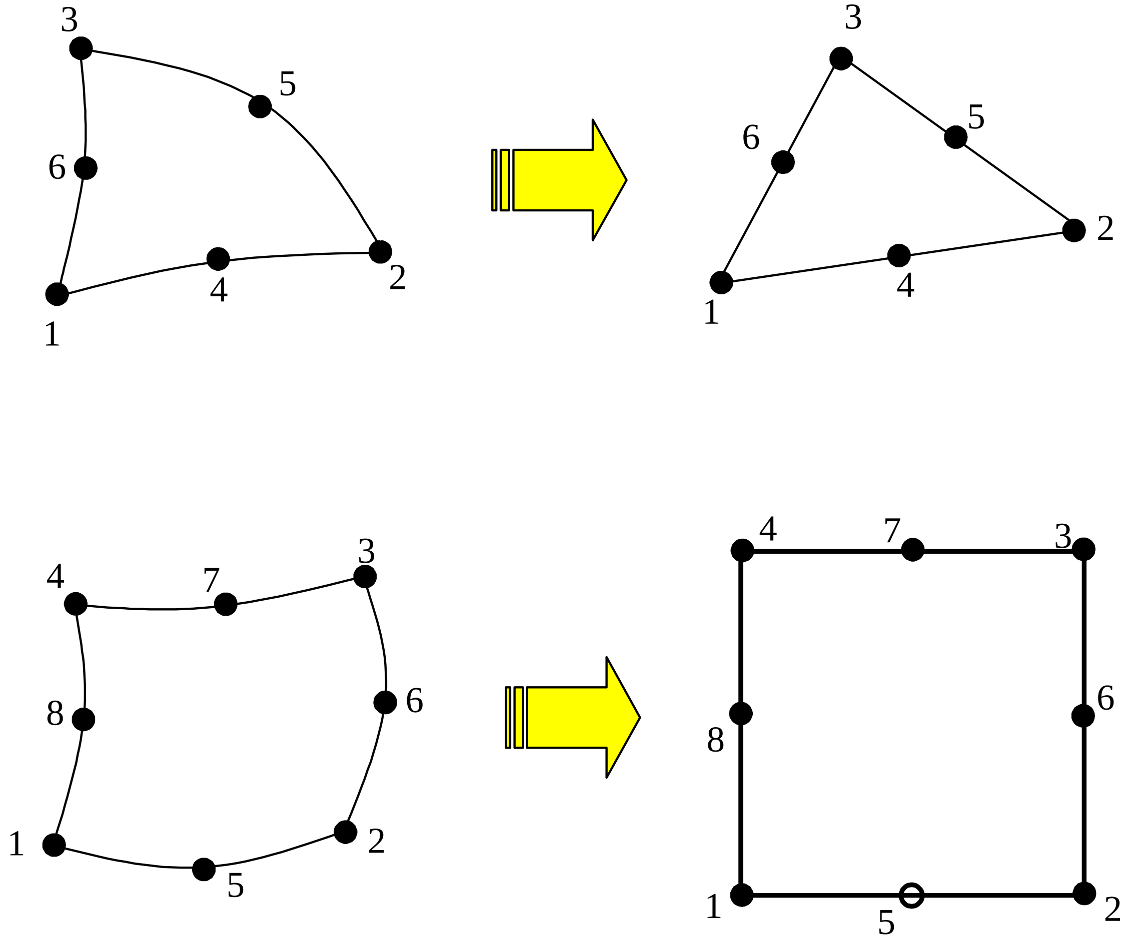
$$N_j = \frac{9}{32}(1+\eta_j\eta)(1-\xi^2)(1+9\xi_j\xi)$$

for side nodes $j = 5, 6, 9, 10$ where $\xi_j = \pm\frac{1}{3}$ and $\eta_j = \pm 1$

Figure from Liu & Quek

參考: Elements with Curved Edges

– Liu & Quek pp. 200~201



Figures from Liu & Quek

參考: Gaussian Quadrature

– Liu & Quek p. 201

- Using a smaller number of Gauss points tends to counteract the ***over-stiff behavior*** associated with the displacement-based FEM.
- Displacement in an element is assumed using shape functions. This implies that the deformation of the element is somehow prescribed in a fashion of the shape function. This prescription gives a constraint to the element. The so-constrained element behaves stiffer than it should. It is often observed that higher order elements are usually softer than lower order ones. This is because using higher order elements gives fewer constraint to the elements.
- Two Gauss points for linear elements, and two or three points for quadratic elements in each direction should be sufficient for most cases.

Poisson's Equation

- Strong form

$$-\triangle u = f \qquad (u = 0 \text{ at } \partial\Omega)$$

- Weak form

$$-\int_\Omega v \triangle u \; d\mathbf{x} = \int_\Omega v\, f \; d\mathbf{x}$$

  - Integration by parts gives

$$\int_\Omega \nabla v \bullet \nabla u \; d\mathbf{x} = \int_\Omega v\, f \; d\mathbf{x}$$

Galerkin Form

- Under a homogeneous Dirichlet boundary condition ($u = 0$ at $\partial\Omega$),

$$\int_\Omega \nabla v_h \bullet \nabla u_h \, d\mathbf{x} = \int_\Omega v_h f \, d\mathbf{x}$$

where $u_h \in H_0^1(\Omega)$, $v_h \in H_0^1(\Omega)$ and $f \in L_2(\Omega)$

- Then

$u_h = \sum c_i N_i$ & $v_h = \sum d_i N_i$

Galerkin Form

$$\sum_{ij} c_i d_j \int_\Omega \nabla N_i \bullet \nabla N_j \, d\mathbf{x} = \sum_j d_j \int_\Omega N_j \, f \, d\mathbf{x}$$

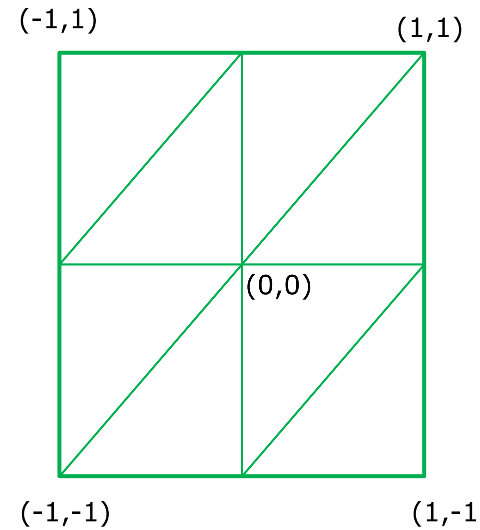
- $\mathbf{v}^T \mathbf{A} \mathbf{u} = \mathbf{v}^T \mathbf{f}$ holds for every d_j

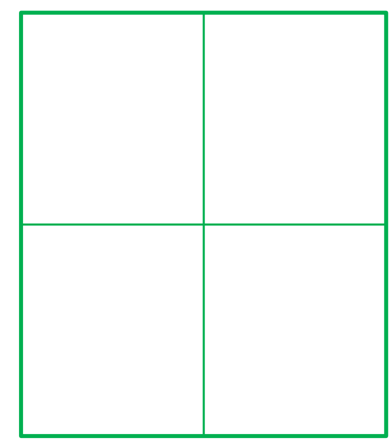
- $-\triangle u = f$ → $\mathbf{A}\mathbf{u} = \mathbf{f}$

$$\mathrm{A}_{ij} = \int_\Omega \nabla N_i \bullet \nabla N_j \, d\mathbf{x} \qquad f_j = \int_\Omega N_j \, f \, d\mathbf{x}$$

Do It Yourself

- This square domain is meshed with linear triangular elements (or rectangular elements).
- You should solve a Poisson equation $-\triangle u = 1$ with $u = 0$ at $\partial\Omega$.
- After assembly in a global equation, applying the boundary condition leaves only one matrix element related to the center node. Compute it. Find u at the center node.





Poisson's Equation

- General boundary conditions

  $u = g$ at $\partial\Omega_D$ & $\alpha u + \frac{\partial u}{\partial \mathbf{n}} = h$ at $\partial\Omega_R$

  – Weak form

$$-\int_\Omega v \triangle u \; d\mathbf{x} = \int_\Omega v f \; d\mathbf{x}$$

$$\Rightarrow \int_\Omega \nabla v \bullet \nabla u \; d\mathbf{x} = \int_\Omega v f \; d\mathbf{x} + \int_{\partial\Omega_R} v \frac{\partial u}{\partial \mathbf{n}} d\mathbf{x}$$

$$= \int_\Omega v f \; d\mathbf{x} + \int_{\partial\Omega_R} v(h - \alpha u) d\mathbf{x}$$

Galerkin Form

- General boundary conditions

$$u = g \text{ at } \partial\Omega_D \text{ \& } \alpha u + \frac{\partial u}{\partial \mathbf{n}} = h \text{ at } \partial\Omega_R$$

$$\sum_{ij} c_i d_j \int_\Omega \nabla N_i \bullet \nabla N_j \, d\mathbf{x} + \sum_{kj} g(\mathbf{x}_k) d_j \int_\Omega \nabla N_k \bullet \nabla N_j \, d\mathbf{x}$$

$$= \sum_j d_j \int_\Omega N_j \, f \, d\mathbf{x} + \sum_j d_j \int_{\partial\Omega_R} N_j \, h \, d\mathbf{x}$$

$$- \alpha \sum_{ij} c_i d_j \int_{\partial\Omega_R} N_i N_j \, d\mathbf{x}$$

$$(\mathbf{x}_i, \mathbf{x}_j \in \Omega - \Omega_D, \mathbf{x}_k \in \partial\Omega_D)$$

Galerkin Form

- General boundary conditions

$-\triangle u = f$, $u = g$ at $\partial\Omega_D$ & $\alpha u + \frac{\partial u}{\partial \mathbf{n}} = h$ at $\partial\Omega_R$

→ $\mathbf{Au} = \mathbf{b}$

$$\mathrm{A}_{ij} = \int_\Omega \triangledown N_i \bullet \triangledown N_j \, d\mathbf{x} + \alpha \int_{\partial\Omega_R} N_i N_j \, d\mathbf{x}$$

$$b_j = \int_\Omega N_j f \, d\mathbf{x} + \int_{\partial\Omega_R} N_j h \, d\mathbf{x}$$

$$- \sum_k g(\mathbf{x}_k) \int_\Omega \triangledown N_k \bullet \triangledown N_j \, d\mathbf{x}$$

$(\mathbf{x}_i, \mathbf{x}_j \in \Omega - \Omega_D, \mathbf{x}_k \in \partial\Omega_D)$

參考: Element Distortion

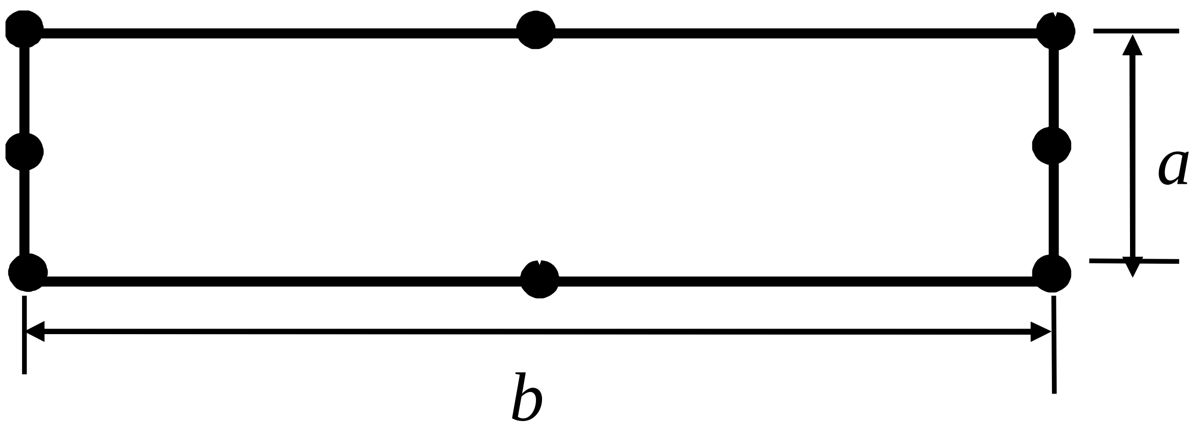
– Liu & Quek pp. 307~309

- Use of distorted elements in irregular and complex geometry is common but there are some limits to the distortion.
- The distortions are measured against the basic shape of the element
  - Square $\Rightarrow$ Quadrilateral elements
  - Isosceles triangle $\Rightarrow$ Triangle elements
  - Cube $\Rightarrow$ Hexahedron elements
  - Isosceles tetrahedron $\Rightarrow$ Tetrahedron elements

參考: Element Distortion

- Liu & Quek pp. 307~309

- *Aspect ratio distortion*



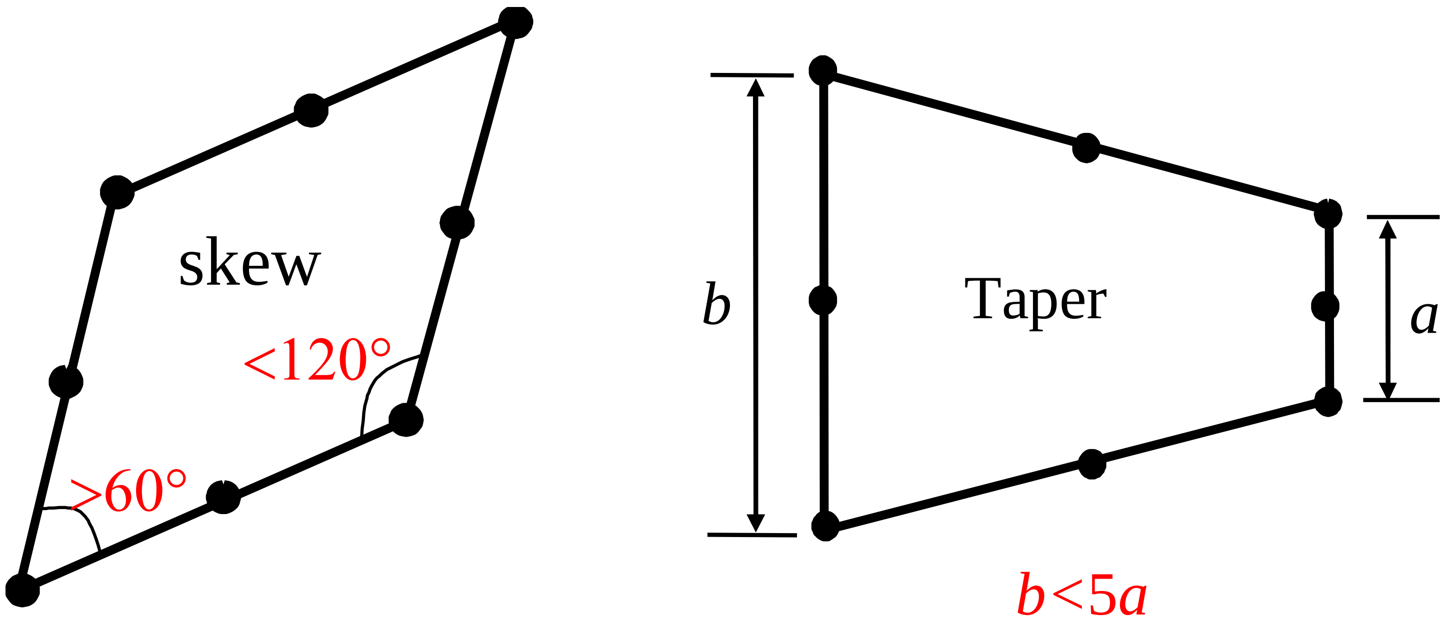
Rule of thumb: $\dfrac{b}{a} \leq \begin{cases} 3 & \text{Stress analysis} \\ 10 & \text{Displacement analysis} \end{cases}$

Figure from Liu & Quek

參考: Element Distortion

– Liu & Quek pp. 307~309

- *Angular distortion*



Figures from Liu & Quek

參考: Element Distortion

– Liu & Quek pp. 307~309

- *Curvature distortion*

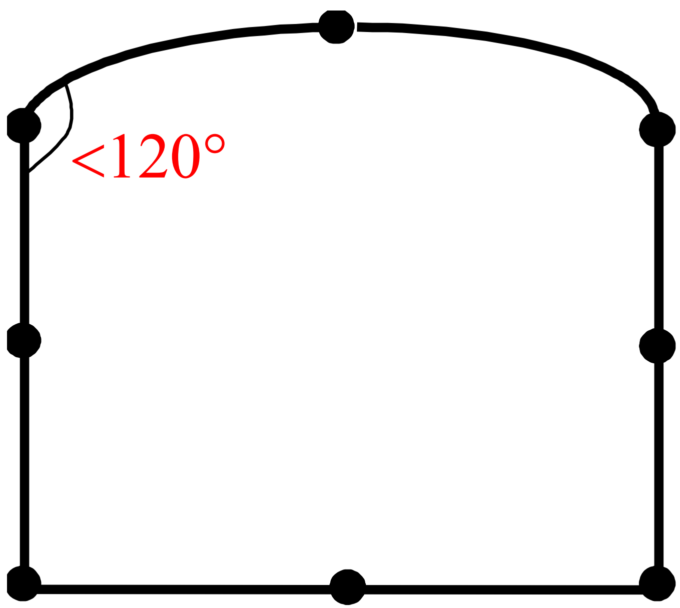


Figure from Liu & Quek

參考: Element Distortion

- Liu & Quek pp. 307~309

- *Volumetric distortion*
  - Area outside distorted element maps into an internal area – negative volume integration

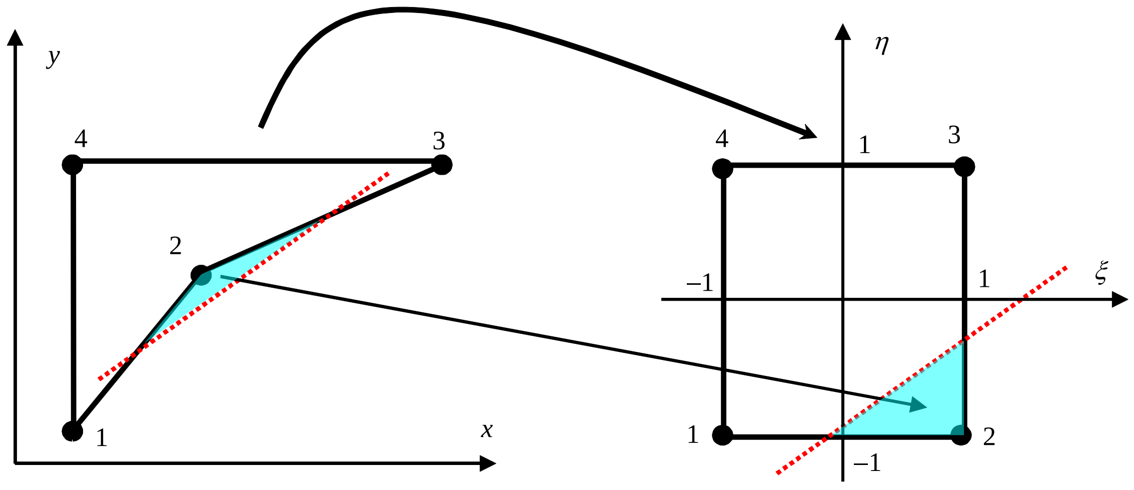
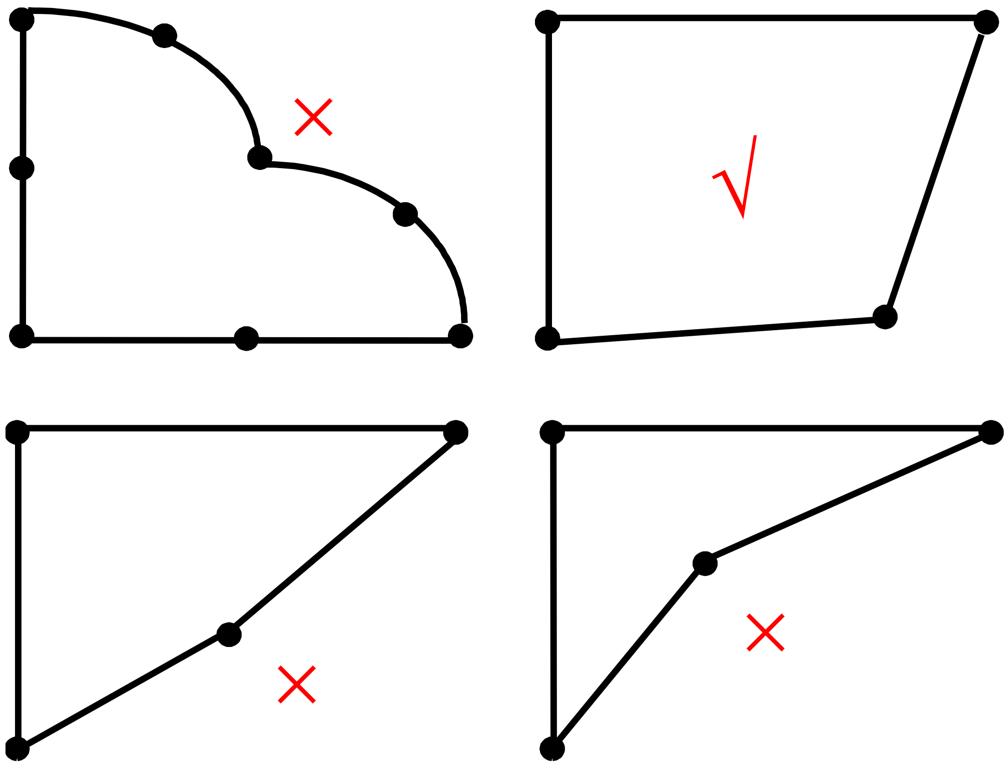


Figure from Liu & Quek

參考: Element Distortion

– Liu & Quek pp. 307~309

- *Volumetric distortion*

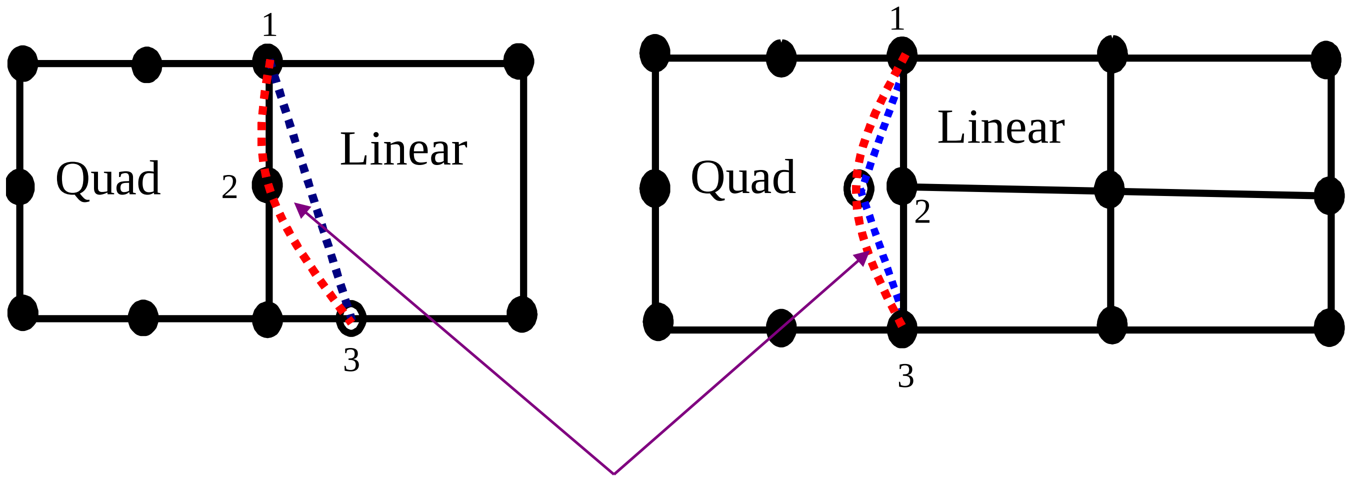


Figures from Liu & Quek

Mesh Compatibility

- Liu & Quek pp. 310~313

- Different order of elements



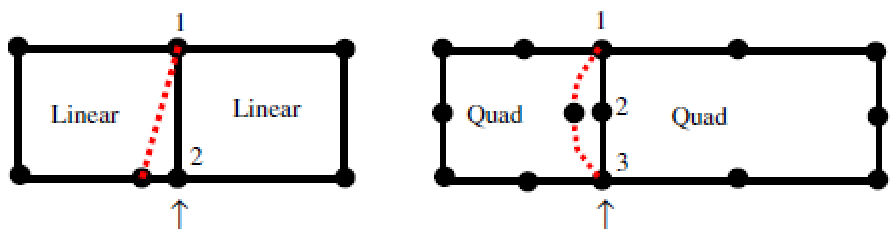
Crack like behavior – incorrect results

Figures from Liu & Quek

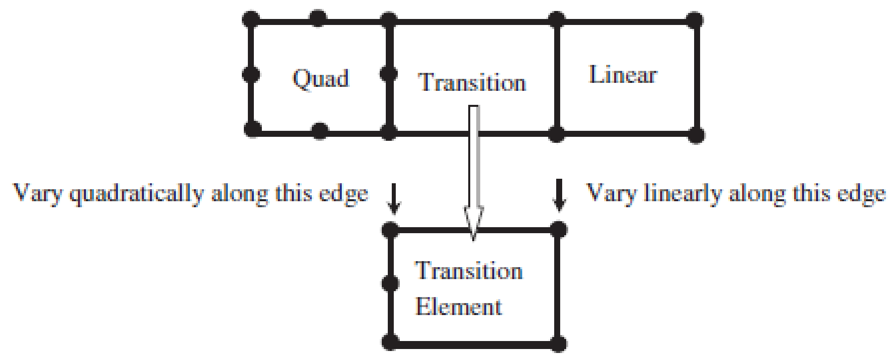
Mesh Compatibility

– Liu & Quek pp. 310~313

- Solutions for different order of elements
  - Use same type of elements throughout



  - Use transition elements



  - Use multipoint constraint equations

Figures from Liu & Quek

Mesh Compatibility

  - Liu & Quek pp. 310~313

- Straddling elements
  - Avoid straddling of elements in mesh

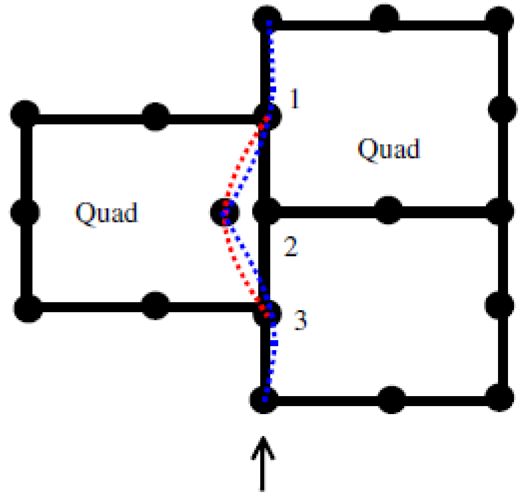


Figure from Liu & Quek

Enforcement of Mesh Compatibility

– Liu & Quek pp. 334~335

- Use lower order shape function to interpolate

$$d_x = 0.5(1-\eta)\ d_1 + 0.5(1+\eta)\ d_3$$
$$d_y = 0.5(1-\eta)\ d_4 + 0.5(1+\eta)\ d_6$$

- Substitute value of η at node 3

$$0.5\ d_1 - d_2 + 0.5\ d_3 = 0$$
$$0.5\ d_4 - d_5 + 0.5\ d_6 = 0$$

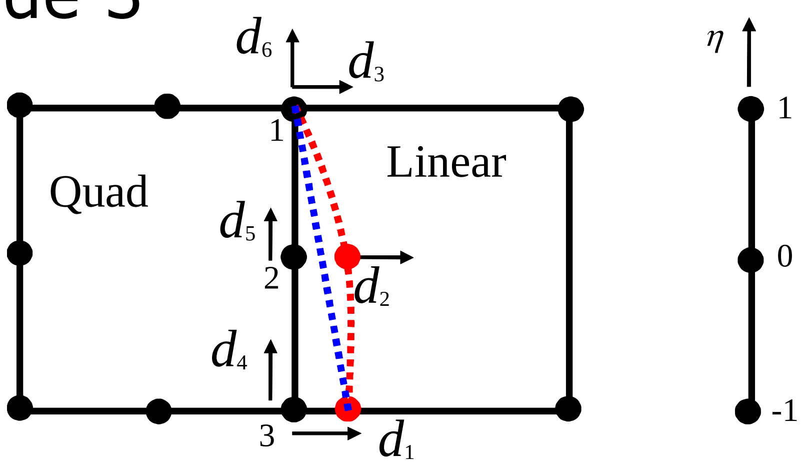


Figure from Liu & Quek

Enforcement of Mesh Compatibility

– Liu & Quek pp. 334~335

- Use shape function of longer element to interpolate

$d_x = -0.5\eta\,(1-\eta)\,d_1 + (1+\eta)(1-\eta)\,d_3 + 0.5\eta\,(1+\eta)\,d_5$

- Substituting the values of η for the two additional nodes

$d_2 = 0.25\times1.5\,d_1 + 1.5\times0.5\,d_3$
$\quad - 0.25\times0.5\,d_5$

$d_4 = -0.25\times0.5\,d_1 + 0.5\times1.5\,d_3$
$\quad + 0.25\times1.5\,d_5$

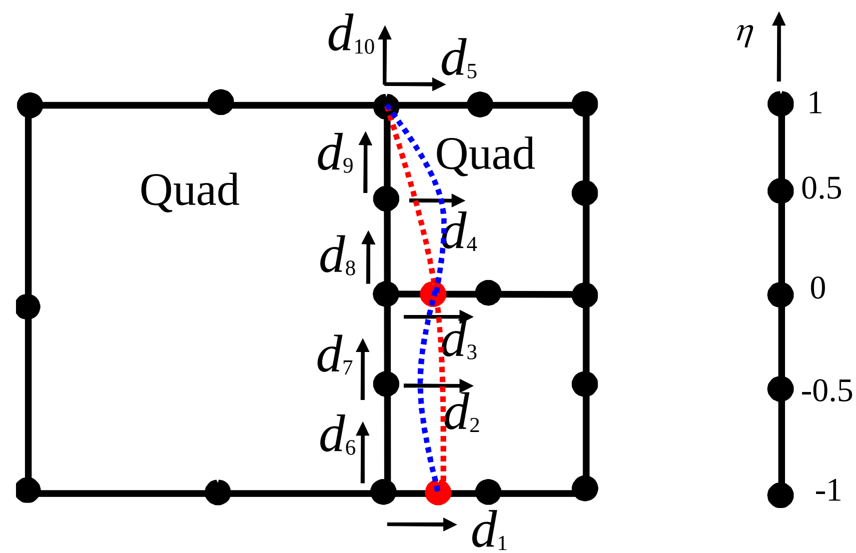


Figure from Liu & Quek

Enforcement of Mesh Compatibility

– Liu & Quek pp. 334~335

- In x direction,

$$0.375\, d_1 - d_2 + 0.75\, d_3 - 0.125\, d_5 = 0$$

$$-0.125\, d_1 + 0.75\, d_3 - d_4 + 0.375\, d_5 = 0$$

- In y direction,

$$0.375\, d_6 - d_7 + 0.75\, d_8 - 0.125\, d_{10} = 0$$

$$-0.125\, d_6 + 0.75\, d_8 - d_9 + 0.375\, d_{10} = 0$$

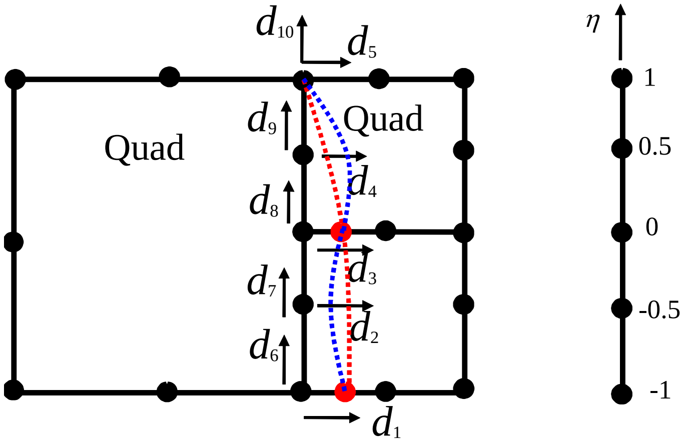


Figure from Liu & Quek

Evolution Problems

- For time dependent problems

  → FDM(time) + FEM(space)

  - Parabolic PDE example (Süli chapter 5)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \qquad x \in (0,1), t \in (0,T],$$
$$u(0,t) = u(1,t) = 0, \qquad t \in [0,T],$$
$$u(x,0) = u_0(x), \qquad x \in [0,1].$$

$$\Downarrow$$

$$\int_0^1 \frac{\partial}{\partial t} u_h(x,t) w_h(x)\,dx + \int_0^1 u_{h,x}(x,t) w_h'(x)\,dx = \int_0^1 f(x,t) w_h(x)\,dx,$$
$$w_h \in H_0^1(0,1)$$
$$\int_0^1 u_h(x,0) w_h(x)\,dx = \int_0^1 u_0(x) w_h(x)\,dx$$

Evolution Problems

- For time dependent problems
  → FDM(time) + FEM(space)
    - Parabolic PDE example (Süli chapter 5)
      – Forward Euler

$$\int_0^1 \frac{u_h^{m+1}(x) - u_h^m(x)}{\Delta t} w_h(x)\, dx + \int_0^1 (u_h^m)'(x) w_h'(x)\, dx = \int_0^1 f(x, m\Delta t) w_h(x)\, dx,$$

$$u_h^m(x) \equiv u_h(x, m\Delta t) \in H_0^1(0,1), \qquad w_h \in H_0^1(0,1).$$

$$\int_0^1 u_h^{m+1}(x) w_h(x)\, dx$$

$$= \int_0^1 u_h^m(x) w_h(x)\, dx - \Delta t \int_0^1 (u_h^m)'(x) w_h'(x)\, dx + \Delta t \int_0^1 f(x, m\Delta t) w_h(x)\, dx,$$

$$\int_0^1 u_h^0(x) w_h(x)\, dx = \int_0^1 u_0(x) w_h(x)\, dx$$

Evolution Problems

- For time dependent problems

  → FDM(time) + FEM(space)

  - Parabolic PDE example (Süli chapter 5)

    – Backward Euler

$$\int_0^1 u_h^{m+1}(x) w_h(x)\,dx + \Delta t \int_0^1 (u_h^{m+1})'(x) w_h'(x)\,dx$$
$$= \int_0^1 u_h^{m}(x) w_h(x)\,dx + \Delta t \int_0^1 f(x,[m+1]\Delta t) w_h(x)\,dx$$

    – Crank-Nicolson

$$\int_0^1 u_h^{m+1}(x) w_h(x)\,dx + \frac{\Delta t}{2} \int_0^1 (u_h^{m+1})'(x) w_h'(x)\,dx$$
$$= \int_0^1 u_h^{m}(x) w_h(x)\,dx - \frac{\Delta t}{2} \int_0^1 (u_h^{m})'(x) w_h'(x)\,dx + \Delta t \int_0^1 \frac{f(x,[m+1]\Delta t) + f(x,m\Delta t)}{2} w_h(x)\,dx$$

參考: Evolution Problems

- Free vibration
  - Liu & Quek 3.6 $\quad (\mathbf{AU} + \mathbf{B\ddot{U}} = 0)$
  - FEM matrix-vector eq.: $\mathbf{KD} + \mathbf{M\ddot{D}} = 0$
  - Eigenvalue analysis
    - Let $\mathbf{D} = \boldsymbol{\varphi} \exp(i\omega t)$
    - $\mathbf{KD} + \mathbf{M\ddot{D}} = 0 \rightarrow \left(\mathbf{K} - \omega^2\mathbf{M}\right)\boldsymbol{\varphi} = 0$ (eigenvalue equation)
    - Finding eigenvalues: $\det\left(\mathbf{K} - \omega^2\mathbf{M}\right) = \left|\mathbf{K} - \omega^2\mathbf{M}\right| = 0$
    - ω: vibration (angular) frequency ($= 2\pi f$)
    - Eigenvectors: vibration modes
  - There are many numerical methods to find eigenvalues and eigenvectors.

參考: Evolution Problems

- Transient response
  - General dynamic mechanical FEM matrix-vector equation: $\mathbf{KD} + \mathbf{C}\dot{\mathbf{D}} + \mathbf{M}\ddot{\mathbf{D}} = \mathbf{F}$
  - Special FDM
    - Explicit 2nd order method (central difference method: 2nd order version of the leapfrog method) or implicit 2nd order method (Newmark's method: Taylor series → quadrature of combination of $\ddot{\mathbf{D}}_t$ & $\ddot{\mathbf{D}}_{t+\Delta t}$ with parameters β & γ → implicit formula)
    - See the section 3.7 of Liu & Quek for more details.

Further Study

- About **linear tetrahedron** elements

References

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