# Weakly Compressible SPH for Free Surface Flows

**Conference Paper** · January 2007

DOI: 10.1145/1272690.1272719 · Source: DBLP



**2 authors:**

Markus Becker
DHGS Deutsche Hochschule für Gesundheit und Sport
**16** PUBLICATIONS **855** CITATIONS
SEE PROFILE

Matthias Teschner
University of Freiburg
**136** PUBLICATIONS **6,404** CITATIONS
SEE PROFILE

Eurographics/ ACM SIGGRAPH Symposium on Computer Animation (2007), pp. 1–8
D. Metaxas and J. Popovic (Editors)

# Weakly compressible SPH for free surface flows

Markus Becker Matthias Teschner

University of Freiburg

**Abstract**

*We present a weakly compressible form of the Smoothed Particle Hydrodynamics method (SPH) for fluid flow based on the Tait equation. In contrast to commonly employed projection approaches that strictly enforce incompressibility, time-consuming solvers for the Poisson equation are avoided by allowing for small, user-defined density fluctuations. We also discuss an improved surface tension model that is particularly appropriate for single-phase free-surface flows. The proposed model is compared to existing models and experiments illustrate the accuracy of the approach for free surface flows. Combining the proposed methods, volume-preserving low-viscosity liquids can be efficiently simulated using SPH. The approach is appropriate for medium-scale and small-scale phenomena. Effects such as splashing and breaking waves are naturally handled.*

Categories and Subject Descriptors (according to ACM CCS): I.3.3 [Computer Graphics]: Three-Dimensional Graphics and Realism: Animation

**Keywords:** Physically-based simulation, Fluid dynamics, Smoothed Particle Hydrodynamics, Tait equation, volume preservation, surface tension

## 1. Introduction

Simulating turbulent liquids with breaking waves and splashes is among the most desired features in fluid animation. Lagrangian methods such as SPH are a promising way to capture such properties.

Particle-based approaches have been introduced by Miller and Pearcy [MP89]. While this early method is based on simple particle-particle forces, Stam and Fiume [SF95] have been the first to apply SPH in the simulation of gas and fire phenomena. Similarly, Takeshita et al. [TOT*03] have used particle methods for explosive flames. In addition to these compressible phenomena, SPH has also been used for the simulation of liquids [MCG03, KAG*05].

These approaches employ the ideal gas equation to relate pressure and density. This results in high compressibility and oscillations which is – in contrast to gases – undesired and visually displeasing. Enforcing incompressibility for particle methods is a challenging problem. Cummins and Rudman [CR99] propose a projection method for Lagrangian methods that is also used in Eulerian approaches. Similarly, Premoze et al. [PTB*03] have realized incompressibility for the Moving Particles Semi-implicit method (MPS). In this approach, a velocity estimate is projected onto a divergence free sub space based on a pressure correction. Therefore, the Poisson equation has to be solved. Although large time steps can be handled, the conjugate gradient solver used to solve the Poisson equation is time-consuming for large systems. Premoze notes 3 min for 100k particles, while compressible SPH implementations can process similar complexities in 5-10 s per time step.

In our SPH simulation, we propose to use the Tait equation with a high speed of sound, which results in a weakly compressible formulation with very low density fluctuations [Mon94]. The method avoids implausible simulation results due to compressibility as previously reported for SPH [KAG*05, BFMF06]. Although the approach requires smaller time steps compared to the Poisson method, the computation per time step is significantly faster and the implementation easily fits into the SPH framework. The computation time of the weakly compressible form corresponds to the time that is required for the compressible form based on the ideal gas equation.

In addition to volume preservation, surface tension plays an important role in realistic fluid animations. Since the handling of small details is mainly governed by surface tension, it is commonly considered in Lagrangian approaches

for free surface fluids, e. g. [KAG*05, MCG03]. However, these methods have been originally designed for multi phase flows [Mor99, HA06]. While not assessable in highly dynamic settings, they cannot handle large curvatures correctly. This is due to the fact, that particles have too few neighbors in those regions. Further, [Mor99] requires to calculate the second derivative of a color field which is very sensitive to particle disorder and thus not adequate for turbulent settings.

We propose a surface tension model based on cohesion forces. The model is particularly appropriate for single-phase flows, but can also handle multiple phases. The computation of derivatives is avoided which improves the efficiency and the stability of the approach.

## 2. Related work

Inspired by early procedural approaches of Max [Max81] and Peachy [Pea86], fluid simulation in computer graphics is mainly comprised of Eulerian and Lagrangian approaches. In general, Eulerian approaches exhibit a number of useful features. To name just a few, computations are performed on a stationary grid and the time-consuming search for neighbors – as required in Lagrangian approaches – is avoided. Further, inflow and outflow can easily be handled by setting the appropriate boundary conditions.

In [KM90], Kass and Miller present a height-field model based on a linearization of the shallow water equation. In [FM96], Foster and Metaxas introduce a finite-difference solution for the full set of Navier-Stokes equations in three dimensions. A 2D height field and surface marker particles are used to track the free surface and a one-way coupling of the fluid and rigid objects is realized. In [FM97], the 2D marker particle method is extended to 3D and a locally adaptive damping method is presented to improve the stability. An unconditionally stable solver for the incompressible Navier-Stokes equations in 3D is presented by Stam [Sta99]. Foster and Fedkiw augment the method for liquids [FF01] using an improved surface tracking by combining level sets and surface marker particles to alleviate mass dissipation. This is known as the particle level set method [EFFM02]. Elcott et al. [ETK*07] provide a thorough overview of further extensions of [Sta99]. They use the same semi-Langragian advection strategy, but present a circulation-preserving integration scheme.

Other authors have introduced extensions to handle multi phase flow involving two or more fluids [LSSF06], two-way coupling with moving rigid solids [CMT04], deformable solids [CGFO06], and local refinement [LGF04]. O'Brien and Hodgins [OH95] enhance grid-based approaches by using a particle system to model droplets and Mihalef et al. [MMS04] address breaking waves for Eulerian fluid simulation. Recently, the Lattice Boltzmann method has been presented in fluid simulation [TIR06, ZQFK07]. In this alternative grid-based approach, particle distributions are evolved on a finite grid using collision and propagation rules.

The investigation of Lagrangian approaches in computer graphics is – at least to some extend – motivated by interactive applications. Lagrangian approaches are comparatively efficient in terms of memory and computation time. They are flexible for time-varying simulation domains and they are easy to implement. Based on the above-mentioned fundamental SPH approaches [SF95, MCG03, KAG*05], various extensions have been presented. SPH applications for fluids range from the simulation of rivers [KW06], multi phase fluid simulations [MSKG05] and high-viscous fluids [CBP05, SAC*99] to fluid control [TKPR06]. Desbrun and Cani [DC96] present an elastic SPH approach for deformable solids. Hadap and Magnenat-Thalmann [HMT01] use SPH for hair dynamics. In [MST*04], two-way coupling of SPH-based fluids and deformable solids is realized. Solenthaler et al. [SSP07] present a unified SPH approach for liquids, elastic and rigid objects. SPH has also been applied in virtual surgery, e. g. [Mül04].

Bell et al. [BYM05] present a particle method for granular materials based on Molecular Dynamics. Zhu and Bridson [ZB05] extend the Particle-In-Cell method to simulate water and granular materials.

For a good introduction into various fluid simulation techniques in graphics we refer to the excellent SIGGRAPH course notes of Bridson, Fedkiw and Müller-Fischer [BFMF06]. A thorough and comprehensive introduction of SPH is given in [Mon05].

## 3. SPH fluid model

In SPH, particles are used to interpolate a continuous function $A(\mathbf{x})$ at a position $\mathbf{x}$. The contributing particles, relevant for a position, are determined by a kernel function $W$ of finite support associated with each particle. For a fluid, the interpolation is given as

$$A(\mathbf{x}) = \sum_j m_j \frac{A_j}{\rho_j} W(\mathbf{x} - \mathbf{x}_j, h). \tag{1}$$

The value $m_j$ denotes the mass of a particle, $\rho_j$ denotes the density at the position of the particle and $h$ the particle radius. In our SPH implementation, we use a B-cubic spline interpolation kernel $W$ with a support of length $4h$. For a detailed discussion of kernel functions we refer to [GM82] and [MCG03].

### 3.1. The governing equations and their SPH formulation

In our approach, we solve the Euler equations, a simplification of the Navier-Stokes equations for inviscid flow. Momentum-conserving artificial viscosity is explicitly added to the model in order to improve the numerical stability of the method. In non-conservation form, the Euler equa-

tions are given as

$$\frac{d\rho}{dt} = -\rho\nabla\cdot\mathbf{v} \quad \text{(continuity equation)} \tag{2}$$

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla P + \mathbf{g} \quad \text{(momentum equation)} \tag{3}$$

with $\mathbf{v}$ denoting the velocity, $P$ the pressure and $\mathbf{g}$ external forces.

**Continuity equation.** The original SPH approach uses a summation formula to calculate the density:

$$\rho_a = \sum_b m_b W_{ab} \tag{4}$$

with $W_{ab} = W(\mathbf{x}_a - \mathbf{x}_b)$. Since the mass is carried by the particles, this form conserves mass exactly. However, due to the fact that surface particles in free surface scenarios have less neighbors, (4) computes an erroneous (lower) density near the surface for equally spaced particles.

Several approaches address this problem. For instance, [SDD06] uses an adaptive kernel length $h$ to enforce constant density at the surface and to model small scale effects at the surface. An alternative approach is introduced by Monaghan [Mon94] and used in [SAC*99] by solving the continuity equation, which results in the SPH form

$$\frac{d\rho_a}{dt} = \sum_b m_b \mathbf{v}_{ab}\nabla_a W_{ab} \tag{5}$$

with $\mathbf{v}_{ab} = \mathbf{v}_a - \mathbf{v}_b$. Each particle is initialized with a density $\rho_0$ and density changes are only due to relative motion of particles. The differential update (5) guarantees a correct density at free surfaces. Although initialization is easier using (5), the summation approach (4) is more stable and has been used in our experiments. We discuss an initialization procedure in Sec. 5.

**Momentum equation.** In SPH form, the momentum equation is written as

$$\frac{d\mathbf{v}_a}{dt} = -\sum_b m_b \left(\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2}\right)\nabla_a W_{ab} + \mathbf{g} \tag{6}$$

This symmetrized version conserves linear and angular momentum.

**Equation of state.** There exist various forms to relate pressure and density. If incompressibility is required, the Poisson equation $\nabla^2 P = \rho\frac{\nabla\mathbf{v}}{\Delta t}$ is commonly solved in Eulerian approaches, e. g. [FF01], and Lagrangian approaches [PTB*03]. However, solving the Poisson equation is time-consuming. In contrast, [MCG03] uses the ideal gas equation $P = k_p\rho$ or $P = k_p(\rho - \rho_0)$ with pressure constant $k_p$ for SPH which results in a rather high compressibility.

In contrast to both approaches, we use the Tait equation [Mon94, Bat67] that enforces very low density variations and is efficient to compute. Neglecting atmospheric pressure that can simply be added as a constant pressure term, Tait's equation has the form

$$P = B\left(\left(\frac{\rho}{\rho_0}\right)^\gamma - 1\right) \tag{7}$$

with $\gamma = 7$. The pressure constant $B$ governs the relative density fluctuation $\frac{|\Delta\rho|}{\rho_0}$ with $\Delta\rho = \rho - \rho_0$. To determine values for $B$, we consider that compressibility effects scale with $O(M^2)$ with $M$ denoting the Mach number of the flow. This results in the following relation

$$\frac{|\Delta\rho|}{\rho_0} \sim \frac{|\mathbf{v}_f|^2}{c_s^2} \tag{8}$$

with $\mathbf{v}_f$ denoting the speed of flow and $c_s$ denoting the speed of sound in the fluid. Thus, we can ensure that $\frac{|\Delta\rho|}{\rho_0} \sim \eta$ if the speed of sound is assumed to be large enough such that $\frac{\mathbf{v}_f}{c_s} < \sqrt{\eta}$. Typically, $\eta$ is set to 0.01, i. e. allowing density variations of the order of 1%. To enforce this condition, $B$ is chosen as

$$B = \frac{\rho_0 c_s^2}{\gamma} \tag{9}$$

*Example.* We have a collapsing column of water with initial density $\rho_0 = 1000\frac{kg}{m^3}$, $\eta = 0.01$, height $H = 4m$, gravity $9.81\frac{m}{s^2}$, and particle radius $h = 0.1m$. Using the estimation $\mathbf{v}_f \sim \sqrt{2gH}$ for the maximum velocity, we get $c_s \approx 88.5\frac{m}{s}$ and $B \approx 1119kPa$.

Assuming a high speed of sound to keep density fluctuations low, Tait's equation requires smaller time steps compared to incompressible particle approaches [PTB*03]. In the above-mentioned example, we have used a time step of $\Delta t = 4.52\cdot 10^{-4}$s. However, each time step needs considerably less computation time since we avoid to solve the Poisson equation. For 100k particles, one time step can be computed twenty times faster compared to MPS. Experiments indicate that the Tait equation can be used to efficiently enforce volume preservation (see Sec. 5).

### 3.2. Viscosity

In order to improve the numerical stability and to allow for shock phenomena, artificial viscosity is employed [Mon05]. It is given as

$$\frac{d\mathbf{v}_a}{dt} = \begin{cases} -\sum_b m_b \Pi_{ab}\nabla_a W_{ab} & \mathbf{v}_{ab}^T\mathbf{x}_{ab} < 0 \\ 0 & \mathbf{v}_{ab}^T\mathbf{x}_{ab} \geq 0 \end{cases} \tag{10}$$

with $\mathbf{v}_{ab}^T\mathbf{x}_{ab} > 0$ being equivalent to $\nabla\cdot\mathbf{v} > 0$. $\Pi_{ab}$ is given as

$$\Pi_{ab} = -\nu\left(\frac{\mathbf{v}_{ab}^T\mathbf{x}_{ab}}{|\mathbf{x}_{ab}|^2 + \varepsilon h^2}\right) \tag{11}$$

with the viscous term $\nu = \frac{2\alpha h c_s}{\rho_a + \rho_b}$ and the viscosity constant $\alpha$. $\alpha$ is in between 0.08 and 0.5 for our experiments. The term $\varepsilon h^2$ is introduced to avoid singularities for $|\mathbf{x}_{ab}| = 0$,

with $\varepsilon = 0.01$. Linear and angular momentum are conserved in (11). $\Pi_{ab}$ is Galilean invariant and vanishes for rigid body rotations [Mon05].

### 3.3. Surface Tension

In this section, we briefly introduce existing surface tension models followed by the presentation of our new approach.

Current surface tension models are commonly based on a color value $c_a$ assigned to a particle $a$ [Mor99, MCG03, KAG*05]. All particles of a phase have the same color value and color values are interpolated according to

$$c = \sum_b \frac{m_b c_b}{\rho_b} W_{ab} \tag{12}$$

The surface normal is calculated as $\mathbf{n} = \nabla c$ and tension forces corresponding to a surface tension $\kappa$ are calculated as

$$\mathbf{f}^{tension} = -\kappa \nabla^2 c \frac{\mathbf{n}}{|\mathbf{n}|} \tag{13}$$

Due to the normalization, normals are only considered if their length exceed a certain threshold [MCG03]. This model has several drawbacks. Calculating the curvature $-\nabla^2 c$ is error-prone close to the surface where particles have only a few neighbors. Further, calculating the second derivative is sensitive to particle disorder.

Hu and Adams [HA06] propose a modified version of [Mor99]. The express the surface tension as the divergence of the stress tensor $\Pi$ which is uniquely defined by the color field $c$:

$$\Pi = \kappa \frac{1}{|\nabla c|} \left( \frac{1}{3} \mathbf{I} |\nabla c|^2 - \nabla c \nabla c^T \right) \tag{14}$$

This expression results in the acceleration equation

$$\frac{d\mathbf{v}_a}{dt} = \frac{1}{m_a} \sum_b \nabla W_{ab} \left( \frac{\Pi_a}{\sigma_a^2} + \frac{\Pi_b}{\sigma_b^2} \right) \tag{15}$$

with $\sigma_a = \sum_b W(r_a - r_b)$. Hu and Adams solve the above-mentioned issues of [Mor99], since they do not calculate the curvature explicitly and the tensor naturally tends to zero for $|\nabla c| \to 0$.

Although [Mor99] and [HA06] have been applied in free-surface problems, both approaches have originally been intended for multi-phase scenarios. Therefore, they do not compute correct surface tension forces in absence of a second phase. They lack an appropriate number of adjacent particles, particularly in surface areas with high curvature.

We propose a new surface tension model that relies on cohesion forces. As previous methods have adopted a macroscopic view of the model, our approach is inspired by a microscopic view since - on a molecular level - surface tension arises due to attractive forces between molecules. The approach is closely related to the model of [TM05]. In contrast to [TM05], we scale the attractive forces using the smoothing kernel $W$ as a weighting function:

$$\frac{d\mathbf{v}_a}{dt} = -\frac{\kappa}{m_a} \sum_b m_b W(\mathbf{x}_a - \mathbf{x}_b) \tag{16}$$

Our method does not rely on a second phase and is well-suited for free-surface problems with high curvatures. The attractive forces can be computed efficiently. The proposed method can easily be generalized to multi-phase problems by applying attractive forces only to particles of the same phase. In Sec. 5, we compare our approach with [HA06].

## 4. Visualization

Some visualization techniques for particle methods are based on an explicit surface reconstruction, e. g. [MCG03]. Using Marching Cubes [LC87], an iso-surface is reconstructed from a characteristic function. However, capturing small details requires a very fine or an adaptive discretization of the underlying space.

Particle level sets do not only use an iso-surface, but additionally introduce tracer particles which are advected with the velocity field [FF01, EFFM02]. They can handle small drops using the tracer particles. However, these drops do not necessarily capture the resolution of the simulation. For large deformations, tracer particles must be reseeded. In [KCC*06], tracer particles are advected with a Lagrangian approach.

We employ a ray tracing approach that is inspired by the surface splatting algorithms of Zwicker et al. [ZPvBG01] and Adams et al. [ALD06]. The visualization of small splashes and single drops is inspired by [KCC*06] and [TRS06]. The method guarantees that features down to the size of a single particle are captured appropriately. The normal of each pixel is calculated similar to [Mor99] as

$$\mathbf{n}_a = \sum_b \frac{m_b}{\rho_b} \nabla W(\mathbf{x}_a - \mathbf{x}_b) \tag{17}$$

where we sum over all particles hit by a ray. Visualizing 130k particles takes about 1-2 seconds on a Pentium 4 with 3.4 GHz.

## 5. Results

In this section, we illustrate some properties of our SPH implementation. We start with discussing the simulation time step, the boundary conditions, and the initialization of the approach. Afterwards, we illustrate the effect of using the Tait equation compared to the ideal gas equation and we discuss the properties of our surface tension model. Finally, we present an exemplary scenario with breaking waves.

If not stated otherwise, performance is measured using an Intel Pentium 4 with 3.4 GHz and 2 GB RAM. Neighborhood search is realized with uniform spatial subdivision.

Integration is performed with a Leap-frog scheme. All experiments are performed using the density of water $\rho_0 = 1000 \frac{kg}{m^3}$.

**Time step.** The appropriate choice of the time step is crucial for stability **and** convergence. In our case, the time step is derived from the Courant-Friedrichs-Lewy(CFL) condition, the viscous term, and the force terms [Mon92]:

$$\Delta t = \min\left(0.25 \cdot \min_a\left(\frac{h}{|\mathbf{f}_a|}\right), 0.4 \cdot \frac{h}{c_s \cdot (1+0.6\alpha)}\right) \quad (18)$$

with $\mathbf{f}_a$ denoting external forces. Although the time step could be larger with respect to numerical stability, we use the proposed time step to ensure convergence.

**Boundary conditions.** Boundary particles are inserted where fluid particles collide with a boundary. Following [Mon05], the force $\mathbf{f}_{ak}$ applied to a fluid particle $a$ that collides with a boundary particle $k$ is computed as:

$$\mathbf{f}_{ak} = \frac{m_k}{m_a + m_k} \Gamma(\mathbf{x}_a, \mathbf{x}_b) \frac{\mathbf{x}_a - \mathbf{x}_k}{|\mathbf{x}_a - \mathbf{x}_k|} \quad (19)$$

The function $\Gamma$ is defined as

$$\Gamma(\mathbf{x}_a, \mathbf{x}_b) = 0.02 \frac{c_s^2}{|\mathbf{x}_a - \mathbf{x}_b|} \begin{cases} \frac{2}{3} & 0 < q < \frac{2}{3} \\ (2q - \frac{3}{2}q^2) & \frac{2}{3} < q < 1 \\ \frac{1}{2}(2-q)^2 & 1 < q < 2 \\ 0 & \text{otherwise} \end{cases} \quad (20)$$

with $q = \frac{|\mathbf{x}_a - \mathbf{x}_b|}{h}$.

**Initialization.** In order to obtain a valid state of nearly constant density using the summation approach (4), we evolve the particles for one hundred steps with a heavily damped version of (6). In case of a simple initial geometry, this initialization can lead to small visual artifacts as can be seen in the accompanying video.

### 5.1. Volume preservation

We have performed two experiments in order to compare our implementation based on the Tait equation with the ideal gas equation. In both experiments, we measure the ratio of the maximum density and the initial density in order to assess the volume preservation.

In the first experiment, we simulate a corner-breaking dam scenario with 130k particles. The height of the dam is $H = 5$. For the Tait equation, the density fluctuation $\eta$ is set to 0.01. For the gas equation, the pressure constant $k_p$ is set to 300. The density fluctuation and pressure constant are minimized and maximized, respectively, in order to use the same time step in both approaches. In both versions, the particle radius and the initial spacing are 0.1, the viscous term $\alpha$ is 0.1, and the computation time is about 4.5 s per time step. The measured density ratio for the gas equation is 4.06, the density ratio for the Tait equation 1.01, which is illustrated in Fig. 1.

In the second experiment, we simulate two breaking dams of height $H = 2$ with an overall number of 16k particles and viscosity $\alpha = 0.5$ colliding with each other in a rectangular basin. The pressure constant $k_p$ for the gas equation is set to 200, the density fluctuation parameter $\eta$ is set to 0.01. The maximum measured density ratio using the gas equation is 1.2. As determined by the parameters, the measured density ratio is 1.01 for the Tait equation.

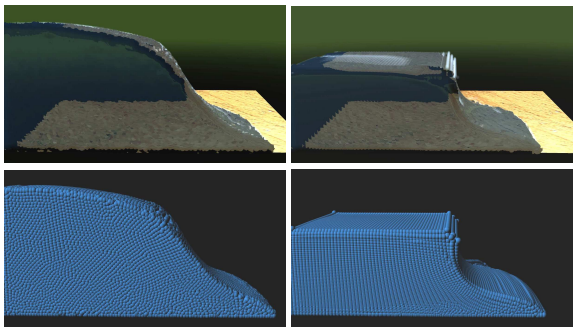

**Figure 1:** *Corner-breaking dam using the Tait equation (left) and the ideal gas equation (right). The gas equation leads to high compressibility.*

Both experiments have been initialized with the same density and the same mass per particle. Experiments indicate that for constant initial density and particle radius the density ratio for the gas equation increases with the volume of the simulated fluid and thereby with the number of particles. In contrast to the weakly compressible approach, considerable volume changes are observed for the gas equation. This is especially perceivable when the dams collapse. Using our weakly compressible approach, the density fluctuation can be kept constant for an increasing number of particles. In terms of volume preservation, the Tait equation is therefore appropriate for handling large particle systems.

### 5.2. Surface tension

In order to compare our surface tension approach with [HA06], we use a fluid cube with 27k particles that is supposed to deform to a drop in a zero-gravity scenario. For comparable results, surface tension is 0.8 for our approach and 30.6 for [HA06]. Alternative parameter settings mostly influence the rate of convergence to an equilibrium state. The qualitative result is not affected as long as the parameters stay within a reasonable range. Results can be seen in Fig. 2.

Since [HA06] has originally been intended for multi phase scenarios, it is less appropriate for high curvatures in free surface scenarios. In contrast to our approach, high curvatures of the initial state are basically preserved.

In another experiment, we investigate parameter settings for our surface tension model. Therefore, we simulate a corner-breaking dam with two different surface tensions.

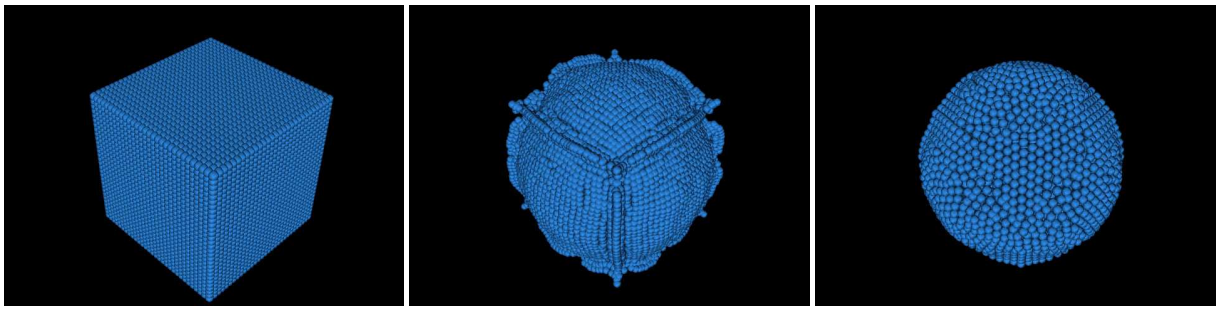

**Figure 2:** *The effect of surface tension. Initial state (left), resulting equilibrium using [HA06] (middle) and our surface tension approach (right).*

The surface tension 0.01 is close to water, the value 3.4 is the maximum stable value in this setting. As can be seen in Fig 3, choosing a larger surface tension value results in larger splashes of the fluid.

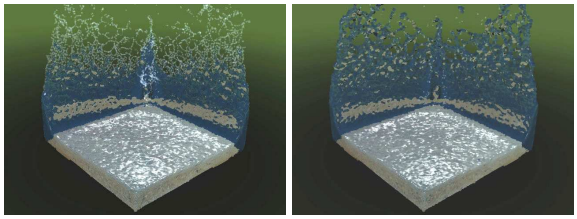

**Figure 3:** *Two corner-breaking dams using a surface tension of 0.01 (left) and 3.4 (right).*

### 5.3. Breaking waves

We have designed a scenario to illustrate the handling of breaking waves. We use 200k particles in a 26 x 5 basin with a ground plane of slope 12°. Radius and initial spacing are set to 0.1. The density fluctuation is set to 0.01. The viscosity term is $\alpha = 0.08$. To generate the waves, we use a planar wave generator. The computation time is 7.5 s per time step. As can be seen in Fig. 4, splashing and breaking waves can be modeled using our approach.

## 6. Conclusion

We have presented a weakly compressible SPH approach for free surface flows. By allowing for small user-defined density fluctuations, the Tait equation is an efficient alternative compared to approaches that strictly enforce incompressibility. In contrast to the gas equation, density fluctuations remain small even for a large number of particles. The volume preservation has been illustrated in various scenarios. Natural phenomena such as breaking waves in low viscous fluids with splashing drops can be simulated using our methods.

Additionally, we have proposed a modified surface tension algorithm based on cohesion forces. In contrast to previous approaches, high curvatures in single-phase free-surface flows can be handled by our model.

Ongoing work focuses on surface tension for multiple phases and adaptive viscosity schemes, as e. g. proposed in [MM97].

## 7. Acknowledgments

We would like to thank Hendrik Tessendorf for his help with creating the videos.

## References

[ALD06] ADAMS B., LENAERTS T., DUTRE P.: *Particle Splatting: Interactive Rendering of Particle-Based Simulation Data*. Tech. Rep. CW 453, Katholieke Universiteit Leuven, 2006.

[Bat67] BATCHELOR G.: *An Introduction to Fluid Dynamics*. Cambridge University Press, 1967.

[BFMF06] BRIDSON R., FEDKIW R., MÜLLER-FISCHER M.: Fluid simulation: Siggraph 2006 course notes. In *SIGGRAPH '06: ACM SIGGRAPH 2006 Courses* (New York, NY, USA, 2006), ACM Press, pp. 1–87.

[BYM05] BELL N., YU Y., MUCHA P. J.: Particle-based simulation of granular materials. In *SCA '05: Proceedings of the 2005 ACM SIGGRAPH/Eurographics symposium on Computer animation* (New York, NY, USA, 2005), ACM Press, pp. 77–86.

[CBP05] CLAVET S., BEAUDOIN P., POULIN P.: Particle-based viscoelastic fluid simulation. In *SCA '05: Proceedings of the 2005 ACM SIGGRAPH/Eurographics symposium on Computer animation* (New York, NY, USA, 2005), ACM Press, pp. 219–228.

[CGFO06] CHENTANEZ N., GOKTEKIN T., FELDMAN B., O'BRIEN J.: Simultaneous coupling of fluids and deformable bodies. In *SCA '06: Proceedings of the 2006 ACM SIGGRAPH/Eurographics symposium on Computer animation* (Aire-la-Ville, Switzerland, Switzerland, 2006), Eurographics Association, pp. 83–89.

[CMT04] CARLSON M., MUCHA P., TURK G.: Rigid fluid: animating the interplay between rigid bodies and fluid. In *SIGGRAPH '04: ACM SIGGRAPH 2004 Papers* (New York, NY, USA, 2004), ACM Press, pp. 377–384.

[CR99] CUMMINS S., RUDMAN M.: An SPH Projection Method. *Journal of Computational Physics 152*, 2 (1999), 584–607.

[DC96] DESBRUN M., CANI M.-P.: Smoothed particles: A new paradigm for animating highly deformable bodies. *6th Eurographics Workshop on Computer Animation and Simulation '96* (1996), 61–76.

[EFFM02] ENRIGHT D., FEDKIW R., FERZIGER J., MITCHELL I.: A hybrid particle level set method for improved interface capturing. *Journal of Computational Physics 183*, 1 (2002), 83–116.

[ETK*07] ELCOTT S., TONG Y., KANSO E., SCHROEDER P., DESBRUN M.: Stable, circulation-preserving, simplicial fluids. *ACM Trans. Graph. 26*, 1 (2007), 4.

[FF01] FOSTER N., FEDKIW R.: Practical animation of liquids. In *SIGGRAPH '01: Proceedings of the 28th annual conference on Computer graphics and interactive techniques* (New York, NY, USA, 2001), ACM Press, pp. 23–30.

[FM96] FOSTER N., METAXAS D.: Realistic animation of liquids. In *GI '96: Proceedings of the conference on Graphics interface '96* (Toronto, Ont., Canada, 1996), Canadian Information Processing Society, pp. 204–212.

[FM97] FOSTER N., METAXAS D.: Controlling fluid animation. In *CGI '97: Proceedings of the 1997 Conference on Computer Graphics International* (Washington, DC, USA, 1997), IEEE Computer Society, pp. 178–188.

[GM82] GINGOLD R., MONAGHAN J.: Kernel estimates as a basis for general particle methods in hydrodynamics. *Journal of Computational Physics 46* (1982), 429–453.

[HA06] HU X., ADAMS N.: A multi-phase SPH method for macroscopic and mesoscopic flows. *Journal of Computational Physics 213*, 2 (2006), 844–861.

[HMT01] HADAP S., MAGNENAT-THALMANN N.: Modeling Dynamic Hair as a Continuum. *Computer Graphics Forum 20*, 3 (2001), 329–338.

[KAG*05] KEISER R., ADAMS B., GASSER D., BAZZI P., DUTRÉ P., GROSS M.: A Unified Lagrangian Approach to Solid-Fluid Animation. *Eurographics Symposium on Point-Based Graphics* (2005), 125–133.

[KCC*06] KIM J., CHA D., CHANG B., KOO B., IHM I.: Practical Animation of Turbulent Splashing Water. *Proceedings of the 2006 ACM SIGGRAPH/Eurographics symposium on Computer animation* (2006), 335–344.

[KM90] KASS M., MILLER G.: Rapid, stable fluid dynamics for computer graphics. *Proceedings of the 17th annual conference on Computer graphics and interactive techniques* (1990), 49–57.

[KW06] KIPFER P., WESTERMANN R.: Realistic and interactive simulation of rivers. *Proceedings of the 2006 conference on Graphics interface* (2006), 41–48.

[LC87] LORENSEN W. E., CLINE H. E.: Marching cubes: A high resolution 3d surface construction algorithm. In *SIGGRAPH '87: Proceedings of the 14th annual conference on Computer graphics and interactive techniques* (New York, NY, USA, 1987), ACM Press, pp. 163–169.

[LGF04] LOSASSO F., GIBOU F., FEDKIW R.: Simulating water and smoke with an octree data structure. *International Conference on Computer Graphics and Interactive Techniques* (2004), 457–462.

[LSSF06] LOSASSO F., SHINAR T., SELLE A., FEDKIW R.: Multiple interacting liquids. In *SIGGRAPH '06: ACM SIGGRAPH 2006 Papers* (New York, NY, USA, 2006), ACM Press, pp. 812–819.

[Max81] MAX N.: Vectorized procedural models for natural terrain: Waves and islands in the sunset. In *SIGGRAPH '81: Proceedings of the 8th annual conference on Computer graphics and interactive techniques* (New York, NY, USA, 1981), ACM Press, pp. 317–324.

[MCG03] MÜLLER M., CHARYPAR D., GROSS M.: Particle-based fluid simulation for interactive applications. In *SCA '03: Proceedings of the 2003 ACM SIGGRAPH/Eurographics symposium on Computer animation* (Aire-la-Ville, Switzerland, Switzerland, 2003), Eurographics Association, pp. 154–159.

[MM97] MORRIS J., MONAGHAN J.: A Switch to Reduce SPH Viscosity. *Journal of Computational Physics 136*, 1 (1997), 41–50.

[MMS04] MIHALEF V., METAXAS D., SUSSMAN M.: Animation and control of breaking waves. In *SCA '04: Proceedings of the 2004 ACM SIGGRAPH/Eurographics symposium on Computer animation* (Aire-la-Ville, Switzerland, Switzerland, 2004), Eurographics Association, pp. 315–324.

[Mon92] MONAGHAN J.: Smoothed particle hydrodynamics. *Ann. Rev. Astron. Astrophys. 30* (1992), 543–574.

[Mon94] MONAGHAN J.: Simulating free surface flows with SPH. *Journal of Computational Physics 110*, 2 (1994), 399–406.

[Mon05] MONAGHAN J.: Smoothed particle hydrodynamics. *Reports on Progress in Physics 68*, 8 (2005), 1703–1759.

[Mor99] MORRIS J.: Simulating surface tension with smoothed particle hydrodynamics. *Int. J. Numer. Meth. Fluids 33* (1999), 333–353.

[MP89] MILLER G., PEARCE A.: Globular Dynamics: A Connected Particle System for Animating Viscous Fluids. *Computers and Graphics 13*, 3 (1989), 305–309.

[MSKG05] MÜLLER M., SOLENTHALER B., KEISER R., GROSS M.: Particle-based fluid-fluid interaction. *Proceedings of the 2005 ACM SIGGRAPH/Eurographics symposium on Computer animation* (2005), 237–244.

[MST*04] MÜLLER M., SCHIRM S., TESCHNER M., HEIDELBERGER B., GROSS M.: Interaction of fluids with deformable solids. *Computer Animation and Virtual Worlds 15*, 34 (2004), 159–171.

[Mül04] MÜLLER M.: Interactive blood simulation for virtual surgery based on smoothed particle hydrodynamics. *Technology and Health Care 12*, 1 (2004), 25–31.

[OH95] O'BRIEN J. F., HODGINS J. K.: Dynamic simulation of splashing fluids. In *CA '95: Proceedings of the Computer Animation* (Washington, DC, USA, 1995), IEEE Computer Society, p. 198.

[Pea86] PEACHEY D.: Modeling waves and surf. In *SIGGRAPH '86: Proceedings of the 13th annual conference on Computer*

*graphics and interactive techniques* (New York, NY, USA, 1986), ACM Press, pp. 65–74.

[PTB*03] PREMOZE S., TASDIZEN T., BIGLER J., LEFOHN A., WHITAKER R.: Particle-based simulation of fluids. *Computer Graphics Forum (Proc. of Eurographics) 22* (2003), 401–410.

[SAC*99] STORA D., AGLIATI P., CANI M.-P., NEYRET F., GASCUEL J.: Animating lava flows. *Graphics Interface 99* (1999), 203–210.

[SDD06] SIGALOTTI L., DAZA J., DONOSO A.: Modelling free surface flows with smoothed particle hydrodynamics. *Condensed Matter Physics 9*, 2(46) (2006), 359–366.

[SF95] STAM J., FIUME E.: Depicting fire and other gaseous phenomena using diffusion processes. In *SIGGRAPH '95: Proceedings of the 22nd annual conference on Computer graphics and interactive techniques* (New York, NY, USA, 1995), ACM Press, pp. 129–136.

[SSP07] SOLENTHALER B., SCHLÄFLI J., PAJAROLA R.: A unified particle model for fluid solid interactions: Research articles. *Comput. Animat. Virtual Worlds 18*, 1 (2007), 69–82.

[Sta99] STAM J.: Stable fluids. *Proceedings of the 26th annual conference on Computer graphics and interactive techniques* (1999), 121–128.

[TIR06] THÜREY N., IGLBERGER K., RÜDE U.: Free surface flows with moving and deforming objects with lbm. In *Vision, Modeling, Visualization VMV'06* (2006), pp. 193–200.

[TKPR06] THÜREY N., KEISER R., PAULY M., RÜDE U.: Detail-preserving fluid control. In *SCA '06: Proceedings of the 2006 ACM SIGGRAPH/Eurographics symposium on Computer animation* (Aire-la-Ville, Switzerland, Switzerland, 2006), Eurographics Association, pp. 7–12.

[TM05] TARTAKOVSKY A., MEAKIN P.: Modeling of surface tension and contact angles with smoothed particle hydrodynamics. *Physical Review E 72*, 2 (2005), 26301.

[TOT*03] TAKESHITA D., OTA S., TAMURA M., FUJIMOTO T., MURAOKA K., CHIBA N.: Particle-based visual simulation of explosive flames. *Computer Graphics and Applications, 2003. Proceedings. 11th Pacific Conference on* (2003), 482–486.

[TRS06] THUEREY N., RUEDE U., STAMMINGER M.: Animation of Open Water Phenomena with Coupled Shallow Water and Free Surface Simulations. *Proceedings of the 2006 ACM SIGGRAPH/Eurographics symposium on Computer animation* (2006), 157–164.

[ZB05] ZHU Y., BRIDSON R.: Animating sand as a fluid. In *SIGGRAPH '05: ACM SIGGRAPH 2005 Papers* (New York, NY, USA, 2005), ACM Press, pp. 965–972.

[ZPvBG01] ZWICKER M., PFISTER H., VAN BAAR J., GROSS M.: Surface splatting. In *SIGGRAPH '01: Proceedings of the 28th annual conference on Computer graphics and interactive techniques* (New York, NY, USA, 2001), ACM Press, pp. 371–378.

[ZQFK07] ZHAO Y., QIU F., FAN Z., KAUFMAN A.: Flow Simulation with Locally-Refined LBM. In *Proc. of ACM SIGGRAPH Symposium on Interactive 3D Graphics and Games* (2007). to appear.

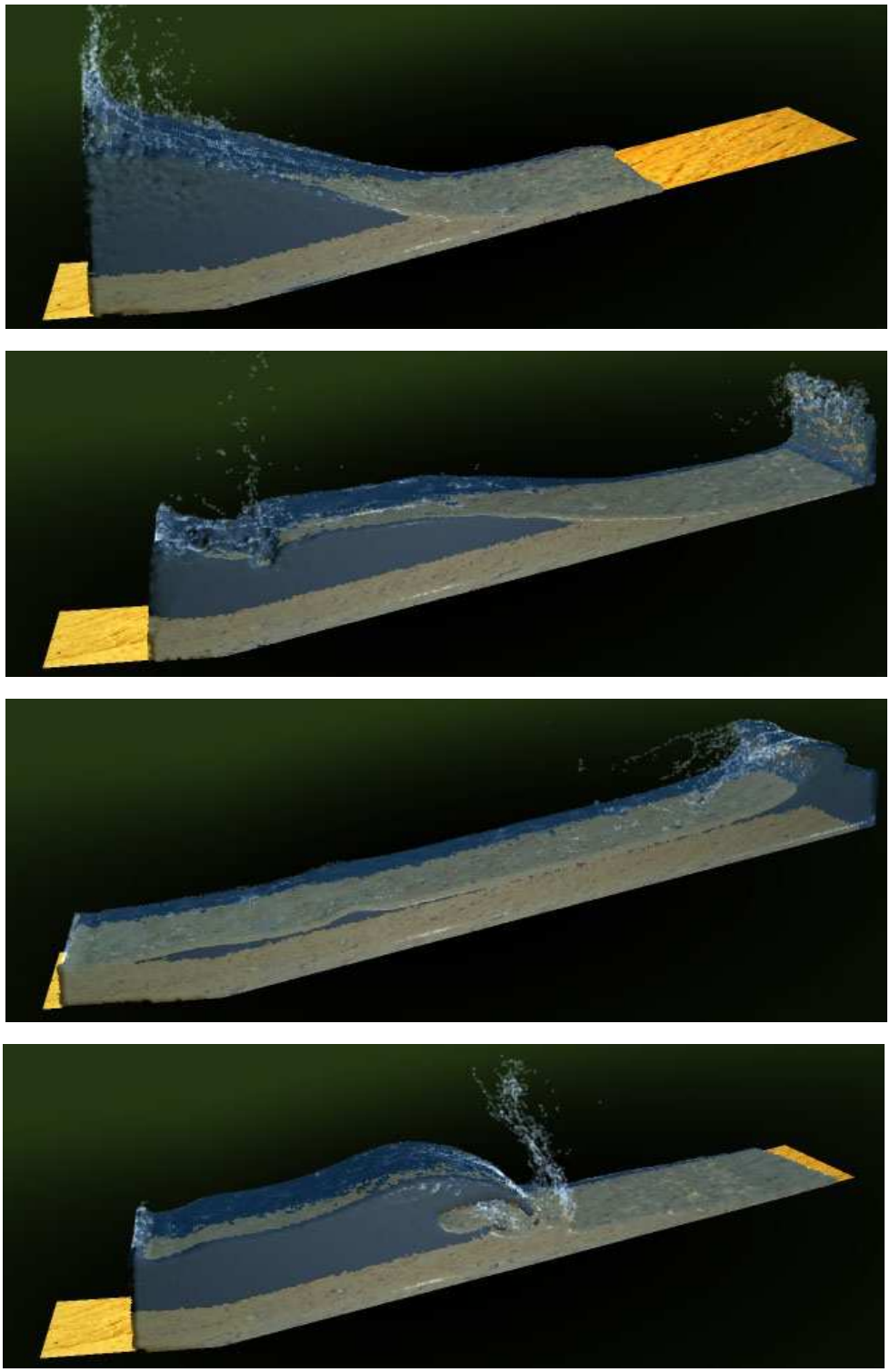

**Figure 4:** *A breaking wave is generated using a wave generator.*