5. Palindromes

$(a+b) \bmod m = ((a \bmod m)+(b \bmod m)) \bmod m$
$ab \bmod m = (a \bmod m \times b \bmod m) \bmod m$ def of congruency

pre: Consider $a\in\mathbb{Z}$ $b\in\mathbb{Z}$ $m\in\mathbb{Z}$ $m>0$

$\therefore a \equiv a \bmod m \pmod m$ then
$b \equiv b \bmod m \pmod m$
(by def of mod)

$a+b \equiv (a \bmod m)+(b \bmod m) \pmod m$ addition Property
$ab \equiv (a \bmod m)(b \bmod m) \pmod m$ multiplication Property

suppose an integer P is palindromic and has 2m-bits (even) ($m\in\mathbb{Z}$ $m>0$)
so it is $(x_{2m-1}x_{2m-2}\cdots x_1x_0)_{10}$ and $x_{2m-1}=x_0$, $x_{2m-2}=x_1$, $\cdots$ $x_{m-1}=x_m$

$\therefore P=\sum_{i=0}^{m} x_i(10^i+10^{2m-1-i})$ and now we want to get $P \bmod 11$

$P \bmod 11 = \left(\sum_{i=0}^{m} x_i(10^i+10^{2m-1-i})\right) \bmod 11$

$\equiv \left(\sum_{i=0}^{m} (x_i(10^i+10^{2m-1-i}) \bmod 11)\right) \bmod 11$ (additivity of congruency)

$\equiv \left(\sum_{i=0}^{m} (x_i \bmod 11)((10^i+10^{2m-1-i}) \bmod 11)\right) \bmod 11$ (multiplicity)

$\equiv \left(\sum_{i=0}^{m} ((x_i \bmod 11)((10^i \bmod 11)+(10^{2m-1-i} \bmod 11)) \bmod 11, \bmod 11)\right) \bmod 11$ (additivity)

given: $10 \equiv -1 \pmod{11}$
$10^i \equiv (-1)^i \pmod{11}$ multiplicity for $i\in\mathbb{Z}$ $i\geq 0$

① when i is odd
then $10^i \bmod 11 = (-1)^i \bmod 11$
$= -1 \bmod 11$
$= 10$ (congruency)

② when i is even
then $10^i \bmod 11 = (-1)^i \bmod 11$
$= 1 \bmod 11$
$= 1$ (congruency)

$m\in\mathbb{Z}$
2m is always even
2m-1 is always odd

so if i is even $10^i \bmod 11 = 10$
2m-1-i is odd $10^{2m-1-i} \bmod 11 = 1$
and if i is odd $10^i \bmod 11 = 1$
then 2m-1-i is even $10^{2m-1-i} \bmod 11 = 10$.

$\therefore P \bmod 11 = \left(\sum_{i=0}^{m} ((x_i \bmod 11)((1+10) \bmod 11)\right) \bmod 11$

$= \left(\sum_{i=0}^{m} ((x_i \bmod 11)(11 \bmod 11))\right) \bmod 11$

$= \left(\sum_{i=0}^{m} (x_i \bmod 11)\times 0\right) \bmod 11$

$= 0 \bmod 11 = 0$

$\therefore 11 \mid P$ (divisibility, def of mod)

conclusion: every palindromic integer with an even number of digits is divisible by 11.