3. Happily Ever After

according to definition of set S. we can rewrite it as

$S: \{5, 9, x : x = 5m+9n$ for any integer $m>0, n\geq 0$ or $m\geq 0\ n>0\}$

let integer a be in this set S so $a = 5p+9q$ for some integer $p>0\ q\geq 0$ or $p\geq 0\ q>0$

1. let $P(n)$ be "$n \in S$" we'll prove $P(n)$ is true for all integer $n \geq a$

2. Base Case: $P(a)$ $a \in S$ by definition so $P(a)$ is true

3. Inductive Hypothesis: assume that for some arbitrary $k \geq a+4$, $P(j)$ is true for all integer j from $a+4$ to k

$k \leq a+4$

$$\boxed{\begin{aligned} a+1 &= 5p+9q+1 = 5(p+2)+9(q-1) \\ a+2 &= 5p+9q+2 = 5(p+4)+9(q-2) \quad \to q-2\geq 0 \\ a+3 &= 5p+9q+3 = 5(p-3)+9(q+2) \quad \to p-3\geq 0 \quad p\geq 3\ q\geq 2 \\ a+4 &= 5p+9q+4 = 5(p-1)+9(q+1) \\ &\text{So let } p=3,\ q=2 \text{ then } a = 3\times 5+2\times 9 = 33 \\ &\text{and all these base cases are true} \end{aligned}}$$

4. Inductive step

goal: to show $P(k+1)$ is always true. which means $k+1 \in S$

by IH and base case. if $a \leq j \leq k$, $j \in \mathbb{Z}$ then $j \in S$

$(k+1)-5 = k-4 \geq a$ and $a \leq k-4 \leq k$ $k-4 \in \mathbb{Z}$ $\therefore k-4 \in S$

$\therefore k-4 = 5r+9s$ for some integer r, s ($r>0\ s\geq 0$ or $r\geq 0\ s>0$)

$k+1 = k-4+5 = 5r+9s+5 = 5(r+1)+9s$ $\therefore k+1 \in S$

$\therefore P(k+1)$ is true

5. conclusion. therefore by strong induction, there is an integer a (a can be 33) such that for every integer $n \geq a$, $n \in S$