3 Solve it

(c) Prove by contradiction → for some integer x

Suppose $51x \equiv 2 \pmod{141}$, then by definition of congruency, $141 \mid (51x-2)$, and by def of divisibility, $\exists k \in \mathbb{Z}$ that enables $51x-2 = 141k$. $\gcd(141, 51) = 3$, which means the equation can be change to be

$\frac{51}{3}x - \frac{2}{3} = \frac{141}{3}k \Rightarrow 17x - \frac{2}{3} = 47k$, since $k \in \mathbb{Z}$

$47k \in \mathbb{Z}$ but $\frac{2}{3}$ is not integer, which means x is not a integer, contradicts our condition ($x \in \mathbb{Z}$)

so $51x \equiv 2 \pmod{141}$ for some integer x cannot be true, $\therefore 51x \equiv 2 \pmod{141}$ has no solution

(d) $10x \equiv 70 \pmod{135}$ using Property $ca \equiv cb \pmod{cm} \iff a \equiv b \pmod{m}$

$c = 5$ then $2x \equiv 14 \pmod{27}$

$\gcd(27, 2) = \gcd(12, 27 \bmod 2)$

$= \gcd(2, 1) = \gcd(1, 2 \bmod 1)$

$= \gcd(1, 0) = 1$

$27 = 13 \times 2 + 1 \quad 1 = 27 - 13 \times 2$

$1 = 1 \times 27 - 13 \times 2$

$\therefore -13 \bmod 27 = 14$ is multiplicative inverse of 2 mod 27

$\therefore -14 \times 2 \equiv 1 \pmod{27}$

$2 \times 14 \times 14 = 1 \times 14 \pmod{27}$ (multiplicity)

$\therefore$ any $x \equiv 14 \times 14 \pmod{27}$

$\equiv 7 \pmod{27}$

$\therefore x = 7 + 27k$ for any integer k is solution

4. Modular Exponentiation Question

$3^1 \bmod 100 = 3 \bmod 100 = 3$

$3^2 \bmod 100 = 9 \bmod 100 = 9$

$3^4 \bmod 100 = (3^2 \bmod 100)^2 \bmod 100 \equiv 9^2 \bmod 100 = 81$

$3^8 \bmod 100 = (3^4 \bmod 100)^2 \bmod 100 = 81^2 \bmod 100 = 61$

$3^{16} \bmod 100 = (3^8 \bmod 100)^2 \bmod 100 = 61^2 \bmod 100 = 21$

$3^{32} \bmod 100 \equiv (3^{16} \bmod 100)^2 \bmod 100 = 21^2 \bmod 100 = 41$

$3^{66} \bmod 100 = (3^{32} \bmod 100)^2 \bmod 100 = 41^2 \bmod 100 = 81$

therefore

$3^{70} \bmod 100 = (3^{66} \bmod 100)(3^4 \bmod 100)(3^2 \bmod 100) \bmod 100$

$= (81 \times 81 \times 9) \bmod 100$

$= 49$

$\therefore 3^{70} \bmod 100 = 49$

8 multiplications are used