7. An Inequality

1. Let $P(n)$ be " $(1+x)^n \geq 1+nx$ $(x \in \mathbb{R}, x>-1)$ ".
   We'll show $P(n)$ is true for all integers $n \geq 0$ by induction

2. Base Case ($n=0$): $(1+x)^0 = 1 \geq 1 = 1+0\cdot x$ So $P(0)$ is true

3. Induction Hypothesis: Suppose that $P(k)$ is true for some arbitrary integer $k \geq 0$

4. Induction Step Goal: show $P(k+1)$ i.e. show $(1+x)^{k+1} \geq 1+(k+1)x$

   by IH, we have $(1+x)^k \geq 1+kx$

   by $x \in \mathbb{R}$, $x>-1$ we have $1+x>0$, So we can multiply it on both sides of inequation

   $$(1+x)(1+x)^k \geq (1+kx)(1+x)$$

   now ① $(1+x)(1+x)^k = (1+x)^{k+1}$

   ② $(1+kx)(1+x) = 1+kx+x+kx^2 = 1+(k+1)x+kx^2$

   So $(1+x)^{k+1} \geq 1+(k+1)x+kx^2$

   $\because$ $x \in \mathbb{R}$ and $x>-1$ $k \geq 0$

   $\therefore$ $x^2 \geq 0$ $kx^2 \geq 0$

   $\therefore$ $(1+x)^{k+1} \geq 1+(k+1)x$ which is exactly $P(k+1)$

5. Thus $P(n)$ is true for all $n \in \mathbb{N}$. by induction

$\therefore$ for all $n \in \mathbb{N}$ $x \in \mathbb{R}$ $x>-1$

$$(1+x)^n \geq 1+nx$$