6. An Equality

1. Let $P(n)$ be " $\sum_{k=1}^{n} k2^k = (n-1)2^{n+1}+2$ ". We'll show $P(n)$ is true for all positive integer n by induction

2. Base Case ($n=1$): $1\times 2^1 = 2 = (1-1)2^{1+1}+2$ Therefore $P(1)$ is true

3. Induction Hypothesis: Suppose that $P(m)$ is true for some arbitrary positive integer $m \geq 1$

4. Induction Step:

Goal: show $P(m+1)$ i.e. show $\sum_{k=1}^{m+1} k2^k = (m+1-1)2^{m+1+1}+2$

$= m2^{m+2}+2$

$\sum_{k=1}^{m} k2^k = (m-1)2^{m+1}+2$ by IH

Adding $(m+1)2^{m+1}$ to both sides, we get:

$$\sum_{k=1}^{m} k2^k + (m+1)2^{m+1} = (m-1)2^{m+1}+2+(m+1)2^{m+1}$$

now ① $\sum_{k=1}^{m} k2^k + (m+1)2^{m+1} = \sum_{k=1}^{m+1} k2^k$

② $(m-1)2^{m+1}+2+(m+1)2^{m+1}$

$= (m-1+m+1)2^{m+1}+2$

$= 2m\cdot 2^{m+1}+2 = m\cdot 2^{m+2}+2$

So. we have

$\sum_{k=1}^{m+1} k2^k = m\cdot 2^{m+2}+2$ which is exactly $P(m+1)$

5. Thus $P(n)$ is true for all $n \in \mathbb{Z}$ $n>0$, by induction

$\therefore$ For all positive integer n

$$\sum_{k=1}^{n} k2^k = (n-1)2^{n+1}+2$$