4. putting car into reverse is rac

1. Let $P(y)$ be " $(x \cdot y)^R = y^R \cdot x^R$ " for all $x \in \Sigma^*$ "

   We'll prove $P(y)$ for all $y \in \Sigma^*$ by Structural induction

2. Base case $y = \varepsilon$ For any $x \in \Sigma^*$ $(x \cdot \varepsilon)^R = x^R$ (Concatenation)

   $\varepsilon^R = \varepsilon$ (reversal) $= \varepsilon^R \cdot x^R$ (concatenation)

   $\therefore P(\varepsilon)$ is true

3. Inductive Hypothesis: Assume that $P(w)$ is true for some arbitrary $w \in \Sigma^*$

4. Inductive step: Goal: show that $P(wa)$ is true for every $a \in \Sigma$

   let $a \in \Sigma$, $x \in \Sigma^*$ then $(x \cdot wa)^R = ((x \cdot w)a)^R$ (concatenation)

   $= a(x \cdot w)^R$ (reversal)

   $= a\, w^R \cdot x^R$ (by IH)

   $= (wa)^R \cdot x^R$ (reversal)

   which is exactly $P(wa)$

   $\therefore P(wa)$ is true

5. by Induction, $(x \cdot y)^R = y^R \cdot x^R$ for all $x, y \in \Sigma^*$

5. Binary Strings

(a) Basis step: $\varepsilon \in S$

Recursive step: If $x \in S$, then $0x \in S$ $x0 \in S$ $10x \in S$ $x10 \in S$

Exclusion Rule: Every Element in S follows from the basis step and a finite number of recursive steps

every occurrence of 1 is immediatly followed by 0, so when we want to add 1, we have to add 0 following that 1. so we can add "10" but not "1". 0 has no limit so we can add as many as we want. and ε is also in S because no "1" is in ε.

(b) Basis step: $\varepsilon \in S$

Recursive step: If $x \in S$ then $x0 \in S$ $0x \in S$ $x11 \in S$ $11x \in S$ $1x1 \in S$

Exclusion Rule. Every element in S follows from the basis step and a finite numbers of recursive steps

0 has no limits. but 1 must have ourrence of even times. so every time we want to add "1", we have to add two "1"s together: all at front, all at back, or one at front one at back

ε suffices because 0 "1"s is in ε and 0 is even also.

(c) Basis step: $\varepsilon \in S$

Recursive steps: If $x \in S$ then $10x \in S$ $01x \in S$ $x01 \in S$ $x10 \in S$ $1x0 \in S$ $0x1 \in S$

Exclusion Rule. Every element in S follows from the basis step and a finite number of recursive steps.

"1" and "0" must have same number, so we have to add them together and both add only once. position is not limited.

ε suffices because 0 "1"s and 0 "0"s are in ε

6. Proving BST insertion works!

1. Let $P(T)$ be " for all $b \in Z$. $x \in Z$, if less(b,T) $b > x$, then less(b, insert(x, T))" we'll prove $P(T)$ is true for all Trees by structural induction.

2. Base Case: $T = Nil$, assume that less(b, T) $\equiv$ true, $b > x$ $b \in Z$ $x \in Z$ for some arbitrary integers

less(b, insert(x,T)) = less(b, insert(x, Nil))

"insert" function = less(b, Tree(x, Nil, Nil)) (insert function)

"less" program = $x < b$ $\wedge$ less(b, Nil) $\wedge$ less(b, Nil)

given and "less program" = $T \wedge T \wedge T$

= T (Idempotency)

$\therefore$ For Nil Tree, if less(b,T) and $b > x$ $b \in Z$ $x \in Z$ then less(b, insert(x,T)) (direct proof rule)

$\therefore$ P(Nil) is true

3. Induction Hypothesis. assume that for some tree L and R P(L) and P(R) is true then by recursive step in definition of binary Tree for some integer k Tree(k, L, R) is also a tree

4. Induction Step: goal: to show P(Tree(k, L, R)) is true

assume that less(b, Tree(k, L, R)) is true and $b > x$ b and x are arbitrary integers

$\therefore$ less(b, Tree(k, L, R)) (assumption)

$\therefore$ $k < b$ $\wedge$ less(b, L) $\wedge$ less(b, R) $\equiv$ T ("less" program)

$\therefore$ $k < b$, less(b, L), less(b, R) (Elim $\wedge$)

$P(L)$: $((less(b, L) \wedge b>x) \rightarrow less(b, insert(x, L))) \equiv T$ (by IH)
$P(R)$: $((less(b, R) \wedge b>x) \rightarrow less(b, insert(x, R))) \equiv T$

$less(b, L) \equiv T$ $\quad less(b, R) \equiv T$ (proved) $\quad b>x$ (given)

$\therefore$ $less(b, insert(x, L)) \equiv T$ $\quad less(b, insert(x, R)) \equiv T$ (Modus Ponens)

$less(b, insert(x, Tree(k, L, R)))$ $\qquad k<b$ (proved)

① if $x<k$

then $= less(b, Tree(k, insert(x, L), R))$ (def of insert)

$= k<b \wedge less(b, insert(x, L)) \wedge less(b, R)$ (def of less)

$= T \wedge T \wedge T$ (all proved)

$= T$ (idempotency)

② if $x \geq k$

then $= less(b, Tree(x, L, insert(x, R)))$ (def of insert)

$= k<b \wedge less(b, L) \wedge less(b, insert(x, R))$ (def of less)

$= T \wedge T \wedge T$ (all proved) $= T$ (idempotency)

$\therefore$ $less(b, insert(x, Tree(k, L, R)))$ is true for all $x \in \mathbb{Z}$

$\therefore$ by direct proof rule

$(less(b, Tree(k, L, R)) \wedge b>x) \rightarrow less(b, insert(x, Tree(k, L, R)))$

$\therefore$ $P(Tree(k, L, R))$ is true

5. Conclusion: $P(T)$ is true for all Tree by structural induction

$\therefore$ for all $b \in \mathbb{Z}$ $x \in \mathbb{Z}$ and all trees T

if $less(b, T)$ and $b>x$ then $less(b, insert(x, T))$