$S(\beta) = \Delta_{N+1} S_0(\beta_0) \Delta_{N+1} \doteq \beta(\sigma_N \dots \sigma_2 \sigma_1^3 \sigma_2 \dots \sigma_N)$