

$$\langle H^* \boldsymbol{f}, g \rangle_{L^2} = \langle \boldsymbol{f}, Hg \rangle_{\ell^2} = \sum_{k=1}^{\infty} \lambda_k g_k \langle \boldsymbol{f}, Se_k \rangle_{\ell^2} = \sum_{k=1} \lambda_k g_k \sum_{i=1}^{N_d} f_i e_k(x_i),$$