$$\tilde{\sigma}^{-1}(\kappa_a^{\ell+1}) \notin V(\mathcal{P}_a^\ell) \cup V(\mathcal{P}_b^\ell),$$