

$$\langle H^*f, g \rangle_{L^2} = \langle f, Hg \rangle_{\ell^2} = \sum_{k=1}^{\infty} \lambda_k g_k \langle f, S e_k \rangle_{\ell^2} = \sum_{k=1}^{\infty} \lambda_k g_k \sum_{i=1}^{N_d} f_i e_k(x_i),$$