2D fluid beam couple by SPH and FEM

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1 system governing equation

In this couple system, the self-dependent variable are particles' velocity vector $v(x, y, t)$ and FEM node position vector $w(x, t)$. so system governing equations are:

$$\rho \dot{v} = \nabla \sigma + g + f^{s2f}$$
$$\frac{d^2}{dx^2}(EI\frac{d^2 w}{dx^2}) + \rho \ddot{w} = f^{f2s}$$

The two equations are coupled by f^{s2f} and f^{f2s}. as one FEM node correspond to multi fluid particles, while each fluid particles correspond to at most 2 FEM node, so these two forces are actually not equivalent.

Note in this simple case, all beam nodes play dual-roles as FEM node as well as interface nodes. One way is to set interface node - fluid particle Pair $< i, I >$, here i stands for fluid particle, and I stands for interface node, which obtain:

$$f^{s2f} = \sum_i repulse\_force(< i, I >)$$
$$f^{f2s} = \sum_I -repulse\_force(< i, I >)$$

2 weak form of structural governing equation

1. kinematic equation

   An accurate 2D kinemtic relationship is, details in [?, p. 215]

$$U = u - z \sin\theta \tag{1a}$$
$$W = w - y(1 - \cos\theta) \tag{1b}$$

   here, u, w are displacements of neutral axis, and θ is rotational angle of an arbitrary cross-section.

2. displacemnt-strain equation

   taking derivativs of kinematic equation, in terms of x, y respectively:

$$U_{,x} = u_{,x} - y\theta_{,x}\cos\theta \tag{2a}$$
$$U_{,y} = -\sin\theta \tag{2b}$$
$$V_{,x} = v_{,x} - y\theta_{,x}\sin\theta \tag{2c}$$
$$V_{,y} = -(1-\cos\theta) \tag{2d}$$

here we still adopt Euler-Boulluni Assumption, that the cross-section plane keep straight and vertical to neutral axis before and after deformation.

Giving Green-Lagrangian strain:

$$E_{xx} = U_{,x} + (U_{,x}^2 + V_{,x}^2)/2 = u_{,x} - y\theta_{,x}\cos\theta = u_{,x} + (u_{,x}^2 + v_{,x}^2)/2 + y^2\theta_{,x}^2/2 - y\theta_{,x} \tag{3a}$$
$$E_{xy} = U_{,x} + V_{,x} + U_{,x}U_{,y} + V_{,x}V_{,y} = 0 \tag{3b}$$
$$E_{yy} = V_{,y} + (U_{,y}^2 + V_{,y}^2)/2 = 0 \tag{3c}$$

3. virtual work and balance equation _ weak form

virtual work equation In finite deformation theory, Green-Lagrangian Strain E_{xx} is energy-conjugate couple correspond to Second Piola-Kirchhoff stress S_{xx}. so the internal force virtual work is

$$\delta W^{in} = \int (S_{xx}\delta E_{xx}^T)dx = \int (E_{xx}E\delta E_{xx}^T)dx$$

where E is the linear elastic constant constitution module.

$$\delta W^{out} = \int q\delta w dx$$

balance equation and weak form

Taken an infinite element dx, consider vertical diretion deflection and moment equation,respectively. we will obtian the balance equation as :

$$q(x,t)dx + \frac{\partial Q}{\partial x} = \rho\frac{\partial^2 w}{\partial t^2}$$

where $q(x,t)$ is the transverse distribute force, $Q(x,t)$ is the internal shear force.

$$Q(x,t) = \frac{\partial M}{\partial x} + N\frac{\partial w}{\partial x}$$

substituting this equation into the above one, we obtain

$$\rho\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}(EI\frac{\partial^2 w}{\partial x^2}) - N(x,t)\frac{\partial^2 w}{\partial x^2} = q(x,t)$$

where $N(x,t)$ is the normal force, and $M(x,t)$ is the moment.

$$N(x,t) = EA\epsilon(x,t)$$

Consider Green-Lagrangian Strain as before, and multiply δv at both side of balance equation,and integration on the whole beam, we obtain the weak form:

$$\int (EI\frac{d^2\delta v}{dx^2}\frac{d^2 w}{dx^2}+\rho\ddot{w}\delta v+N^L\frac{dw}{dx}\frac{d\delta v}{dx})dx = \int q\delta v dx - v(0)Q(0)+\frac{dv}{dx}|_{x=0}M(0)+v(L)Q(L)-\frac{dv}{dx}|_{x=L}M(L)+N^L$$

consider boundary condition, $\delta v(0) = \frac{\delta v}{dx} = M(L) = Q(L) = \frac{\partial w}{\partial x}|_{x=0} = 0$, so the weak form above is:

$$\int (EI\frac{d^2\delta v}{dx^2}\frac{d^2 w}{dx^2}+\rho\ddot{w}\delta v+N^L\frac{dw}{dx}\frac{d\delta v}{dx})dx = \int q\delta v dx + N^L\frac{dw}{dx}v_{x=L}$$

4. nodal force

   Beam nodes play dual-roles as FEM node as well as interface node, fluid-beam interaction is coupled by interface repulsive force f^{f2s}, and this force is only acted at node point, which will lead singularity in the weak form of structural equation. so a simpler equivalent is:

$$\int_0^L q(x,t)dx = \sum_{i=1}^{np} f_i^{f2s}$$

$$\sum_{i=1}^{np-1}\int_{x_{i-1}}^{x_i} q_i dx = f_1 + \sum_{i=2}^{np-1} f_i + f_{np}$$

   where $q(x,t)$ is the equivalent distributed traction, and np is the number of interface nodes. q_i stands for a constant traction on an element length.

   so $q_i = \frac{f_i+f_{i+1}}{2l_e}, 1 < i < np$, $q_1 = (f_1 + f_2/2)/l_e$, $q_{np} = (f_{np-1}/2 + f_{np})/l_e$

5. matrix approximation

3 SPH fluid equation

assuming weak-compressible, the mass conservation equation is

$$\frac{d\rho}{dt} = -\rho\nabla v$$

momentum equation is

$$\rho\frac{dv}{dt} = -\nabla P + \mu\Delta^2 v + g$$

To update pressure here, introduce $p-\rho$ relationship

$$p = c^2\rho$$

where $c^2 = 100u^2$ is virtual sound speed.

4 system approximation

for sph particle velocity vector:

$$\frac{du}{dt} = \sum_j m_i(\frac{\sigma_i}{\rho_i^2} + \frac{\sigma_j}{\rho_j^2})\nabla w_{ij} + \sum_j \frac{m_j}{\rho_j} f_j^{s2f} w_{ij}$$

Note to update the approximation fluid equation, we need calculate ρ_i, w_{ij} at first.
for beam deflection displacement:
as before, we adopt Hermite shape function $w = \sum_{k=1}^4 \Delta_k \theta_k$, then

$$\sum_{J=1}^{4} EI(\theta_I, \theta_J)\Delta_J - \rho\ddot{\Delta}_J l(\theta_I, \theta_J) = f_I^{f2s}$$

say $\sum_{J=1}^4 EI(\theta_I, \theta_J) = K_{IJ}$, $\rho l(\theta_I, \theta_J) = M_{IJ}$, $\mu\Delta u = L_i$, we can integratthe system equation in matrix form:

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \{ \begin{matrix} \ddot{\Delta}_J \\ \dot{v} \end{matrix} \} + \begin{bmatrix} K & 0 \\ 0 & L \end{bmatrix} \{ \begin{matrix} \Delta_J \\ v \end{matrix} \} = \{ \begin{matrix} f^{f2s} \\ f^{s2f} \end{matrix} \}$$

in which, Δ_J is the 4 DOF of FEM node, v is SPH velocity vector(2 dimension), L is Laplace Operator.

5 Reference