Nonlinear Beam Element

Zhengjiang Li

1 Beam Kinematics Assumption

classic beam kinematics assumptions include, Euler-Bernoulli beam(EB) theory that neglects transverse shear strain; Timoshenko beam(TB) theory, that accounts for the transverse shear strain in a simpler way; and higher order beam theroy with additional terms into assumed displacement field.

in TB, the assumed displacement field is

$$u(x,z) = u_0(x) + z\phi_x(x)$$
$$v(x,z) = 0$$
$$w(x,z) = w_0(x)$$

in which u, v, w are displacements in longitudinal, lateral and transverse respectively. u_0, w_0 denote displacement of a point on mid-plane of an undeformed beam along axial(x) and transverse(z) directions

in EB, the assumed displacement field is

$$u(x,z) = u_0(x) - z\frac{dw_0}{dx}$$
$$v(x,z) = 0$$
$$w(x,z) = w_0(x)$$

which means that the plane sections perpendicular to the mid-plane of the beam before deformation remain plane, and rotate such that they remain perpendicular to hte mid-plane after deformation.

2 strain-displacement relations

$$\epsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) + \frac{1}{2}((\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j})$$

according to Kirchhoff's hypothesis(plane strain), ϵ_{zz}, ϵ_{xz}, ϵ_{yz} equal zero, and for a thin beam(ration of length and radius is larger than 10), ϵ_{yy}, ϵ_{xy} is zero. so the only nonzero strain is axial strain:

$$\epsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}[(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2 + (\frac{\partial w}{\partial x})^2$$

which gives,

$$\epsilon_{xx} = \epsilon_{xx}^0 + z\epsilon_{xx}^1$$

$$\epsilon_{xx}^0 = \frac{du_0}{dx} + \frac{1}{2}(\frac{dw_0}{dx})^2$$
$$\epsilon_{xx}^1 = -\frac{d^2 w_0}{dx^2}$$

From here, we can see there are 3 DOFs at each node $u, w, theta$. for EB theory, $theta = \frac{\partial w}{\partial x}$, so actually we need only 2 DOFs at each node. for Timonsenko theory, $u, w, theta$ are independent, os we need three DOFs at each node.

consider linear elastic of isotropic materials, $\sigma_{xx} = E\epsilon_{xx}$

3 weak form

consider virtual work principle and D'Alembert principle for dynamic problem, and only distributed pressure external force. we have:

$$\delta W_{ext} = \delta W_{int}$$
$$\delta W_{ext} = \int_0^L q\delta w dx$$
$$\delta W_{int} = \sigma_{xx}\delta\epsilon_{xx} + \rho A\ddot{w}\delta w$$

for EB theory, $\delta\theta = -\frac{\delta w}{\delta x}$ for Timonsenko theory, $\delta\theta$ and δw are independent.

$$\sigma_{xx}\delta\epsilon_{xx} = A\int(\delta\epsilon_{xx}^0 + z\delta\epsilon_{xx}^1)E_{11}(\epsilon_{xx}^0 + z\epsilon_{xx}^1)dx = A\int_0^L(E_{11}(\delta\epsilon_{xx}^0\epsilon_{xx}^0 + z(\delta\epsilon_{xx}^0\epsilon_{xx}^1 + \delta\epsilon_{xx}^1\epsilon_{xx}^0) + z^2(\delta\epsilon^1\epsilon^1)]dx$$

define C_1, C_2, C_3 as
$$(C_1, C_2, C_3) = \int E(1, z, z^2)dA$$

for $\delta\epsilon_{xx}^0 = \frac{d\delta u_0}{dx} + \frac{dw_0}{dx}\frac{ddeltaw_0}{dx}$, and $\delta\epsilon_{xx}^1 = -\frac{d^2\delta\epsilon_0}{dx^2}$

$$\begin{aligned}\delta w_{int} = \int_0^L (C_1[\frac{d\delta u_0}{dx}(\frac{du_0}{dx} + \frac{1}{2}(\frac{dw_0}{dx})^2) + \frac{d\delta w_0}{dx}\frac{dw_0}{dx}(\frac{du_0}{dx} + \frac{1}{2}(\frac{dw_0}{dx})^2)] \\ - C_2[\frac{d\delta u_0}{dx}(\frac{d^2 w_0}{dx^2}) + \frac{d\delta w_0}{dx}\frac{d\delta w_0}{dx}\frac{dw_0}{dx}(\frac{dw_0}{dx}d^2 w_0 dx^2) + \frac{d^2\delta w_0}{dx^2}(\frac{du_0}{dx} + \frac{1}{2}(\frac{dw_0}{dx})^2)] \\ + C_3\frac{d^2\delta w_0}{dx^2}(\frac{d^2 w_0}{dx^2})) + \rho A\ddot{w}\delta w \quad (1)\end{aligned}$$

4 Finite Element Formulation

for EB theory, approximation of u, w can be expressed as:

$$u(x,t) = u_1(t)\psi_1(x) + u_2(t)\psi_2(x)$$
$$w(x,t) = w_1(t)\phi_1(x) + theta_1(t)\phi_2(x) + w_2(t)\phi_3(x) + theta_2(t)\phi_4(x)$$

in which $\phi_i(t) = -\frac{dw_i(t)}{dx}$

$$\delta W_{in} = \int_0^L (C_1(\delta u_i \frac{d\psi_i}{dx}[\sum_{j=1}^{2} u_j \frac{d\psi_i}{dx} + \frac{1}{2}\frac{dw}{dx}\sum_{J=1}^{4}\Delta_J \frac{d\phi_J}{dx}] + \delta\Delta_I \frac{d\phi_I}{dx}\frac{dw}{dx}[\sum_{j=1}^{2} u_j \frac{d\psi_j}{dx}\frac{1}{2}\frac{dw}{dx}\sum_{J=1}^{4}\Delta_J \frac{d\phi_J}{dx}])$$

$$- C_2(\delta u_i \frac{d\psi}{dx}(\sum_{J=1}^{4}\Delta_J \frac{d^2\phi_J}{dx^2}) + \delta\Delta_I \frac{d\phi_I}{dx}(\frac{w}{dx}\sum_{J=1}^{4}\Delta_J \frac{d^2\phi_J}{dx^2}) + \delta\Delta_I \frac{d^2\phi_I}{dx^2}(\sum_{j=1}^{2} u_j \frac{d\psi_j}{dx} + \frac{1}{2}\frac{dw}{dx}\sum_{J=1}^{4}\Delta_J(\frac{d\phi_J}{dx})$$

$$+ C_3(\delta\Delta_I \frac{d^2\phi_I}{dx^2}\sum_{J=1}^{4}\Delta_J \frac{d^2\phi_J}{dx^2}) + \rho A \delta\Delta_i \phi_i \sum_{J=1}^{4}\phi_J \frac{d^2\Delta_J}{dt^2})dx \quad (2)$$

define $\delta U^T = [\delta u_1, \delta u_2, \delta w_1, \delta\theta_1, \delta w_2, \delta\theta_2]^T$; $U = [u_1, u_2, w_1, \theta_1, w_2, \theta_2]$;
we can summary the above equation as:

$$\delta W_{int} = \delta U^T[M]\{\ddot{U}\} + \delta U^T[K]\{U\}$$

similarly, external virtual work can be obtained as:

$$\delta W_{ext} = \sum_{J=1}^{4}\int_0^L q(x,t)\ddot{\Delta}_i \phi_i \phi_J \delta\Delta_J dx$$

in summary, we obtain

$$[M]\{\ddot{U}\} + [K]\{U\} = [F]$$