2D fluid beam couple by SPH and FEM

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1 system governing equation

In this couple system, the self-dependent variable are particles' velocity vector $v(x, y, t)$ and FEM node position vector $w(x, t)$. so system governing equations are:

$$\rho \dot{v} = \nabla \sigma + g + f^{s2f}$$

$$\rho \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}(EI \frac{\partial^2 w}{\partial x^2}) - N(x, t) \frac{\partial^2 w}{\partial x^2} = q(x, t)$$

where $N(x, t)$ is the normal force. defined as $N(x, t) = EA\epsilon(x, t)$
$q(x, t)$ is the average traction of nodal force from fluid to structural f^{f2s}.

$$\int_0^L q(x, t) dx = \sum_{i=1}^{np} f_i^{f2s} \tag{1a}$$

$$q_i = \frac{f_i + f_{i+1}}{2l_e}, \ 1 < i < np \tag{1b}$$

$$q_1 = (f_1 + f_2/2)/l_e \tag{1c}$$

$$q_{np} = (f_{np-1}/2 + f_{np})/l_e \tag{1d}$$

where $q(x, t)$ is the equivalent distributed traction, and np is the number of interface nodes. q_i stands for a constant traction on an element length.

The two equations are coupled by f^{s2f} and f^{f2s}. as one FEM node correspond to multi fluid particles, while each fluid particles correspond to at most 2 FEM node, so these two forces are actually not equivalent.

Note in this simple case, all beam nodes play dual-roles as FEM node as well as interface nodes. One way is to set interface node - fluid particle Pair $< i, I >$, here i stands for fluid particle, and I stands for interface node, which obtain:

$$f^{s2f} = \sum_i repulse\_force(< i, I >)$$

$$f^{f2s} = \sum_I -repulse\_force(< i, I >)$$

2 weak form of structural governing equation

1. kinematic equation

   An accurate 2D kinemtic relationship is, details in [**?**, p. 215]

$$U = u - z\sin\theta \tag{2a}$$
$$W = w - z(1-\cos\theta) \tag{2b}$$

here, u, w are displacements of neutral axis, and θ is rotational angle of an arbitrary cross-section.

2. displacemnt-strain equation

   taking derivativs of kinematic equation, in terms of x, y respectively:

$$U_{,x} = u_{,x} - z\theta_{,x}\cos\theta \tag{3a}$$
$$U_{,z} = -\sin\theta \tag{3b}$$
$$W_{,x} = w_{,x} - z\theta_{,x}\sin\theta \tag{3c}$$
$$W_{,z} = -(1-\cos\theta) \tag{3d}$$

   here we still adopt Euler-Boulluni Assumption, that the cross-section plane keep straight and vertical to neutral axis before and after deformation.

   Giving Green-Lagrangian strain:

$$E_{xx} = U_{,x} + (U_{,x}^2 + W_{,x}^2)/2 = u_{,x} - z\theta_{,x}\cos\theta = u_{,x} + (u_{,x}^2 + w_{,x}^2)/2 + z^2\theta_{,x}^2/2 - z\theta_{,x} \tag{4a}$$
$$E_{xz} = U_{,x} + W_{,x} + U_{,x}U_{,z} + W_{,x}W_{,z} = 0 \tag{4b}$$
$$E_{zz} = W_{,z} + (U_{,z}^2 + W_{,z}^2)/2 = 0 \tag{4c}$$

3. virtual work and weak form

   3.1 virtual work

   In finite deformation theory, Green-Lagrangian Strain E_{xx} is energy-conjugate couple correspond to Second Piola-Kirchhoff stress S_{xx}. so the internal force virtual work is

$$\delta W^{in} = \int (S_{xx}\delta E_{xx}^T)dx = \int (E_{xx}E\delta E_{xx}^T)dx$$

   where E is the linear elastic constant constitution module.

$$\delta W^{out} = \int -\rho\delta\ddot{w} + q\delta w dx$$

   3.2 weak form from balance equation

   Consider Green-Lagrangian Strain as before, and multiply δv at both side of balance equation,and integration on the whole beam, we obtain the weak form:

$$\int (EI\frac{d^2\delta v}{dx^2}\frac{d^2 w}{dx^2} + \rho\ddot{w}\delta v + N^L\frac{dw}{dx}\frac{d\delta v}{dx})dx = \int q\delta v dx - v(0)Q(0) + \frac{dv}{dx}|_{x=0}M(0) + v(L)Q(L) - \frac{dv}{dx}|_{x=L}M(L) + N^L$$

consider boundary condition: $\delta v(0) = \frac{\delta v}{dx} = M(L) = Q(L) = \frac{\partial w}{\partial x}|_{x=0} = 0$, so the weak form above is:

$$\int (EI\frac{d^2\delta v}{dx^2}\frac{d^2 w}{dx^2} + \rho\ddot{w}\delta v + N^L\frac{dw}{dx}\frac{d\delta v}{dx})dx = \int q\delta v dx + N^L\frac{dw}{dx}v_|x = L$$

4. matrix approximation

virtual work equation above is easy to implement based on Timoshenko Beam Assumption; and weak form from balance equation is based on Euler-Boulluni Beam Assumption. so we can use either linear interpolations for u, w, θ as TB or linear interpolation for u and Hermite interpolation for w as EB.

Since only lateral deflection is obvious,an even simpler implementation here is to let $u_{,x} = 0$, namely ignore axial displacement of neutral axis. And $z^2\theta_{,x}^2 = o(dx^2)$.

For large rotation, Timonshenko Theory is better, and it's easy to implement on both Total Langrange Method and Update Langrange Method. While Euler-Boulluni Theory is an good approximate, but strict to UL implementation.

in the following, we will introduce these two ways.

4.1 EU Beam element

Hermite Interpolation:

$$w(x,t) = w_1(t)\phi_1(x) + \theta_1(t)\phi_2(x) + w_2(t)\phi_3(x) + \theta_2(t)\phi_4(x)$$

Define master element on length l_e with coordinate ξ, ranging from $[-1, 1]$

$$\begin{aligned}
\phi_1(x) &= \frac{1}{4}(1-\xi)^2(2+\xi) \\
\phi_2(x) &= \frac{l_e}{8}(1-\xi)^2(1+\xi) \\
\phi_3(x) &= \frac{1}{4}(1+\xi)^2(2-\xi) \\
\phi_4(x) &= \frac{l_e}{8}(1+\xi)^2(\xi-1)
\end{aligned}$$

define $[w_1, \theta_1, w_2, \theta_2] = [\Delta_1, \Delta_2, \Delta_3, \Delta_4]$

from master element to physical element we need define an Jacobian Transformation, which is simple for 2-node beam element as $x = \frac{l_e}{2}(\xi + 1)$

$$J = \frac{\partial \xi}{\partial x} = 2/l_e$$

in an element, the following satisfy:

$$\sum_{i=1}^{4}\sum_{j=1}^{4}(\int_{-1}^{1}(EI\frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2}\Delta_j + \rho\frac{d^2\Delta_j}{dt^2}\phi_i\phi_j + N^L\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\Delta_j)dx) = \int_{-1^1}\phi_i q(x,t)dx$$

From local coordinate system to global coordiante system, we need define transformation matrix T

$$\{ \begin{matrix} w^l \\ \theta^l \end{matrix} \} = \begin{bmatrix} \cos\varphi & 0 \\ 0 & 1 \end{bmatrix} \{ \begin{matrix} w^g \\ \theta^g \end{matrix} \}$$

Define element stiff matrix, element mass matrix and element force as :

$$k^e = \sum_{i=1}^{4}\sum_{j=1}^{4}(\int_{x_i}^{x_{i+1}}(EI\frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2}\Delta_j + N^L\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\Delta_j)dx) \tag{5a}$$

$$m^e = \sum_{i=1}^{4}\sum_{j=1}^{4}(\int_{x_i}^{x_{i+1}}\rho\frac{d^2\Delta_j}{dt^2}\phi_i\phi_j dx \tag{5b}$$

$$f^e = \sum_{i=1}^{4}(\int_{x_i}^{x_{i+1}}\int_{x_i}^{x_{i+1}}\phi_i q(x,t)dx \tag{5c}$$

Assembly up, we obtain the whole structural system Matrix Format:

$$\sum_{ne} T^T m^e T\{\ddot{\Delta}\} + \sum_{ne} T^T k^e T\{\Delta\} = \sum_{ne} T^T f^e$$

where ne is the total number of elements, T is the transformation matrix.

Define global Mass Matrix, Stiff Matrix and Force Vector as

$$M = \sum_{ne} T^T m^e T$$

$$K = \sum_{ne} T^T k^e T$$

$$F = \sum_{ne} T^T f^e$$

4.2 Timosheko beam element Interpolate w, θ respectively, linear shape function is easy but not gurantee strain continuity at node, as $E_{xx} = w_{,x}^2/2 - z\theta_{,x}$, both w, θ here have first derivatives. So the right thing is to construct shape function at least second order. So both w, θ need use Hermite shape function.

$$w = \sum_{i=1}^{4}\Delta_i N_i$$

$$\theta = \sum_{j=1}^{4}\theta_j N_j$$

where N is shape function same as EB beam element.Taken into virtual work.

note that $\delta E_{xx} = w_{,x}\delta w_{,x} - z\delta\theta_{,x}$, for nonlinear term $w_{,x}^2$, decompose it as $w_{,x}^L w_{,x}$, here $w_{,x}^L$ is known from last step.

the element stiff matrix

$$(w^L_{,x})^2 \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial \phi_i}{\partial x} \end{bmatrix}^T \cdot \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial \phi_i}{\partial x} \end{bmatrix}$$

the element mass matrix and element force vector are same as EB beam element, and matrix assembly is same as EB.

3 SPH fluid equation

assuming weak-compressible, the mass conservation equation is

$$\frac{d\rho}{dt} = -\rho \nabla v$$

momentum equation is

$$\frac{dv}{dt} = -\frac{1}{\rho}\nabla P + \Gamma + g$$

where v is fluid particle velocity, and Γ refers to dissipative terms and $g = (0, 0, -9.81)m/s^{-2}$. Basically two ways here to implement:

1. artificial viscosity artificial viscosity proposed by Monaghan(1992), the equation above is:

$$\frac{dv_i}{dt} = -\sum_j m_j(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} + \Pi_{ij})\nabla_i W_{ij} + g$$

where viscosity term Π_{ij} is given by

$$\Pi_{ij} = \{ \begin{matrix} \frac{-\alpha \bar{c_{ij}} \mu_{ij}}{\rho_{ij}} \; v_{ij} \cdot r_{ij} < 0 \\ 0 \; v_{ij} \cdot r_{ij} > 0 \end{matrix}$$

where $\mu_{ij} = \frac{h v_{ij} r_{ij}}{r_{ij}^2 + \eta^2}$, $\eta^2 = 0.01h^2$, $\bar{c_{ij}} = \frac{1}{2}(c_i + c_j)$ is the mean sound speed. $\alpha = 0.3$, $c = 10$ for DamBreak case.

2. laminar viscosity proposed by Lo and Shao (2002)

$$\frac{dv_i}{dt} = -\sum_j m_j(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2})\nabla_i W_{ij} + g + \sum_j \frac{4\nu_0 r_{ij} \cdot \nabla_i w_{ij}}{(\rho_i + \rho_j)(r_{ij}^2 + \eta^2)} v_{ij}$$

$$\frac{dv_i}{dt} = -\sum_j m_j(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2})\nabla_i W_{ij} + g + \sum_j \frac{4\nu_0 r_{ij} \cdot \nabla_i w_{ij}}{(\rho_i + \rho_j)(r_{ij}^2 + \eta^2)} v_{ij}$$

where ν_0 is kinetic viscosity $10^{-6} m^2 s^{-1}$

choose Quadratic Kernel function

$$w(r_{ij}, h) = \alpha_D[\frac{3}{16}q^2 - \frac{3}{4}q + \frac{3}{4}] \; 0 \le q \le 2$$

the frist gradient is $\nabla_i w_{ij} = \alpha_D(\frac{3}{8}q - \frac{3}{4})$
where $q = r_{ij}/h$, $alpha_D = \frac{2}{\pi h^2}$ for 2D case.
For artificial viscosity, define

$$L = \sum_j \frac{m_j \alpha c_{ij}}{\rho_{ij}} \frac{h r_{ij}}{r_{ij}^2 + \eta^2} \nabla_i w_{ij}$$

when $v_{ij} \cdot r_{ij} < 0$ else $L = 0, v_{ij} \cdot r_{ij} > 0$
For laminar viscosity, define

$$L = \sum_j \frac{4\nu_0 r_{ij} \cdot \nabla_i w_{ij}}{(\rho_i + \rho_j)(r_{ij}^2 + \eta^2)}$$

and for both artificial viscosity and laminar viscosity, we have the term unrelated to velocity, defined as

$$RHS_0 = -\sum_j m_j (\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2}) \nabla_i W_{ij} + g$$

so sph governing equation in matrix format is $\ddot{v}_i - L v_{ij} + RHS_0$
To update density of fluid particles, we use

$$\frac{d\rho_i}{dt} = \sum_j m_j v_{ij} \nabla_i w_{ij}$$

and to update pressure we use Monaghan(1994) state equation as

$$P = B[(\frac{\rho}{\rho_0})^\kappa - 1]$$

where $\kappa = 7$, $B = c_0^2 \rho_0/\kappa$, $\rho_0 = 1000kgm^{-3}$
we can see here, the update of particles density and pressure are not directly related to the coupled system. so these two parameters are not solve simultaneous with particle velocities v in final.

4 coupled system approximation

here use EB beam element

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \{ \begin{matrix} \ddot{\Delta}_J \\ \dot{v} \end{matrix} \} + \begin{bmatrix} K & 0 \\ 0 & L \end{bmatrix} \{ \begin{matrix} \Delta_J \\ v \end{matrix} \} = \{ \begin{matrix} f^{f2s} \\ f^{s2f} + RHS_0 \end{matrix} \}$$

where Δ_J stands for displacement in structural, v stands for velocity of particles.

5 Reference