Deduction

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January 8, 2021

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Following the same notation as in HPIPM document, the cost function of optimal control function is:

$$\sum_{k=0}^{N-1} \frac{1}{2} \begin{bmatrix} \Delta u_k \\ \Delta x_k \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} R_k & S_k & r_k \\ S_k^{\mathrm{T}} & Q_k & q_k \\ r_k^{\mathrm{T}} & q_k^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \Delta u_k \\ \Delta x_k \\ 1 \end{bmatrix} + \frac{1}{2} \Delta x_N^{\mathrm{T}} Q_N \Delta x_N + \Delta x_N^{\mathrm{T}} q_N \tag{1}$$

which is equivalent to

$$\sum_{k=0}^{N-1} \frac{1}{2} \begin{bmatrix} \Delta u_k \\ \Delta x_k \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} R_k & S_k \\ S_k^{\mathrm{T}} & Q_k \end{bmatrix} \begin{bmatrix} \Delta u_k \\ \Delta x_k \end{bmatrix} + \begin{bmatrix} \Delta u_k \\ \Delta x_k \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} r_k \\ q_k \end{bmatrix} + \frac{1}{2} \Delta x_N^{\mathrm{T}} Q_N \Delta x_N + \Delta x_N^{\mathrm{T}} q_N \tag{2}$$

The dynamic constraints are

$$x_{k+1} = f(x_k, u_k) \tag{3a}$$

$$\Delta x_{k+1} = A_k \Delta x_k + B_k \Delta u_k + b_k \tag{3b}$$

$$A_k = \frac{\partial f}{\partial x_k}(x_k, u_k) \quad B_k = \frac{\partial f}{\partial u_k}(x_k, u_k) \quad b_k = f(x_k, u_k) - x_{k+1} \tag{3c}$$

The equality constraints are

$$g(x_k, u_k) = 0 \quad g \in \mathbb{R}^p \tag{4a}$$

$$C_k \Delta x_k + D_k \Delta u_k + e_k = 0 \tag{4b}$$

$$C_k = \frac{\partial g}{\partial x_k}(x_k, u_k) \quad C_k \in \mathbb{R}^{p \times n} \quad D_k = \frac{\partial g}{\partial u_k}(x_k, u_k) \quad D_k \in \mathbb{R}^{p \times m} \quad e_k = g(x_k, u_k) \tag{4c}$$

Now do the QR decomposition on the equality constraints

$$D_k \Delta u_k = -C_k \Delta x_k - e_k \tag{5a}$$

$$D_k^{\mathrm{T}} = \begin{bmatrix} Q_k^1 & Q_k^2 \end{bmatrix} \begin{bmatrix} R_k^1 \\ 0 \end{bmatrix} \quad Q_k^1 \in \mathbb{R}^{m \times p} \quad Q_k^2 \in \mathbb{R}^{m \times (m-p)} \quad R_k^1 \in \mathbb{R}^{p \times p} \tag{5b}$$

$$\text{The upper indices of } Q_k^1,\ Q_k^2 \text{ and } R_k^1 \text{ matrices do not mean power!} \tag{5c}$$

$$\begin{aligned} \Delta u_k &= Q_k^2 \Delta \tilde{u}_k + Q_k^1 (R_k^1)^{-\mathrm{T}} (-C_k \Delta x_k - e_k) && (5d) \\ &= Q_k^2 \Delta \tilde{u}_k - Q_k^1 (R_k^1)^{-\mathrm{T}} (C_k \Delta x_k + e_k) \quad \Delta \tilde{u}_k \in \mathbb{R}^{(m-p) \times 1} && (5e) \end{aligned}$$

The new variable $\tilde{\Delta u_k}$ is not constrained anymore. Now plug the new Δu_k into the cost terms. Begin with the first order terms

$$\begin{aligned}
\begin{bmatrix}\Delta u_k \\ \Delta x_k\end{bmatrix}^{\mathrm{T}} \begin{bmatrix} r_k \\ q_k \end{bmatrix} &= \begin{bmatrix} Q_k^2 \tilde{\Delta u_k} - Q_k^1 (R_k^1)^{-\mathrm{T}} (C_k \Delta x_k + e_k) \\ \Delta x_k \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} r_k \\ q_k \end{bmatrix} &\quad (6a)\\
&= \left(\tilde{\Delta u_k}^{\mathrm{T}} (Q_k^2)^{\mathrm{T}} - \Delta x_k^{\mathrm{T}} \left(Q_k^1 (R_k^1)^{-\mathrm{T}} C_k \right)^{\mathrm{T}} - e_k^{\mathrm{T}} \left(Q_k^1 (R_k^1)^{-\mathrm{T}} \right)^{\mathrm{T}} \right) r_k + \Delta x_k^{\mathrm{T}} q_k &\quad (6b)\\
&= \begin{bmatrix}\tilde{\Delta u_k} \\ \Delta x_k\end{bmatrix}^{\mathrm{T}} \begin{bmatrix} (Q_k^2)^{\mathrm{T}} r_k \\ q_k - \left(Q_k^1 (R_k^1)^{-\mathrm{T}} C_k \right)^{\mathrm{T}} r_k \end{bmatrix} - e_k^{\mathrm{T}} \left(Q_k^1 (R_k^1)^{-\mathrm{T}} \right)^{\mathrm{T}} r_k &\quad (6c)\\
&= \begin{bmatrix}\tilde{\Delta u_k} \\ \Delta x_k\end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \tilde{r_k^1} \\ \tilde{q_k^1} \end{bmatrix} + C_k^1 &\quad (6d)
\end{aligned}$$

$$\begin{aligned}
\tilde{r_k^1} &= (Q_k^2)^{\mathrm{T}} r_k &\quad (6e)\\
\tilde{q_k^1} &= q_k - \left(Q_k^1 (R_k^1)^{-\mathrm{T}} C_k \right)^{\mathrm{T}} r_k &\quad (6f)\\
C_k^1 &= -e_k^{\mathrm{T}} \left(Q_k^1 (R_k^1)^{-\mathrm{T}} \right)^{\mathrm{T}} r_k &\quad (6g)
\end{aligned}$$

Then the second order term $\frac{1}{2}\Delta u_k^{\mathrm{T}} R_k \Delta u_k$

$$\begin{aligned}
&\frac{1}{2}\Delta u_k^{\mathrm{T}} R_k \Delta u_k \quad (7a)\\
&= \frac{1}{2}\tilde{\Delta u_k}^{\mathrm{T}} (Q_k^2)^{\mathrm{T}} R_k Q_k^2 \tilde{\Delta u_k} + \frac{1}{2}\Delta x_k^{\mathrm{T}} \left(Q_k^1 (R_k^1)^{-\mathrm{T}} C_k \right)^{\mathrm{T}} R_k \left(Q_k^1 (R_k^1)^{-\mathrm{T}} C_k \right) \Delta x_k + \frac{1}{2} e_k^{\mathrm{T}} \left(Q_k^1 (R_k^1)^{-\mathrm{T}} \right)^{\mathrm{T}} R_k \left(Q_k^1 (R_k^1)^{-\mathrm{T}} \right) e_k \quad (7b)\\
&- \tilde{\Delta u_k}^{\mathrm{T}} (Q_k^2)^{\mathrm{T}} R_k \left(Q_k^1 (R_k^1)^{-\mathrm{T}} C_k \right) \Delta x_k - \tilde{\Delta u_k}^{\mathrm{T}} (Q_k^2)^{\mathrm{T}} R_k \left(Q_k^1 (R_k^1)^{-\mathrm{T}} \right) e_k - \Delta x_k^{\mathrm{T}} \left(Q_k^1 (R_k^1)^{-\mathrm{T}} C_k \right)^{\mathrm{T}} R_k \left(Q_k^1 (R_k^1)^{-\mathrm{T}} \right) e_k \quad (7c)
\end{aligned}$$

Then the second order term $\Delta u_k^{\mathrm{T}} S_k \Delta x_k$

$$\Delta u_k^{\mathrm{T}} S_k \Delta x_k = \tilde{\Delta u_k}^{\mathrm{T}} (Q_k^2)^{\mathrm{T}} S_k \Delta x_k - \Delta x_k^{\mathrm{T}} \left(Q_k^1 (R_k^1)^{-\mathrm{T}} C_k \right)^{\mathrm{T}} S_k \Delta x_k - e_k^{\mathrm{T}} \left(Q_k^1 (R_k^1)^{-\mathrm{T}} \right)^{\mathrm{T}} S_k \Delta x_k \quad (8a)$$

Note there is no change to the second order term $\frac{1}{2}\Delta x_k^{\mathrm{T}} Q_k \Delta x_k$. Combining all above:

$$\begin{aligned}
&\frac{1}{2}\Delta u_k^{\mathrm{T}} R_k \Delta u_k + \frac{1}{2}\Delta x_k^{\mathrm{T}} Q_k \Delta x_k + \Delta u_k^{\mathrm{T}} S_k \Delta x_k &\quad (9a)\\
&= \frac{1}{2}\tilde{\Delta u_k}^{\mathrm{T}} \tilde{R_k} \tilde{\Delta u_k} + \frac{1}{2}\Delta x_k^{\mathrm{T}} \tilde{Q_k} \Delta x_k + \tilde{\Delta u_k}^{\mathrm{T}} \tilde{S_k} \Delta x_k + \begin{bmatrix}\tilde{\Delta u_k} \\ \Delta x_k\end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \tilde{r_k^2} \\ \tilde{q_k^2} \end{bmatrix} + C_k^2 &\quad (9b)
\end{aligned}$$

$$\begin{aligned}
\tilde{R_k} &= (Q_k^2)^{\mathrm{T}} R_k Q_k^2 &\quad (9c)\\
\tilde{Q_k} &= Q_k - 2\left(Q_k^1 (R_k^1)^{-\mathrm{T}} C_k \right)^{\mathrm{T}} S_k + \left(Q_k^1 (R_k^1)^{-\mathrm{T}} C_k \right)^{\mathrm{T}} R_k \left(Q_k^1 (R_k^1)^{-\mathrm{T}} C_k \right) &\quad (9d)\\
\tilde{S_k} &= (Q_k^2)^{\mathrm{T}} S_k - (Q_k^2)^{\mathrm{T}} R_k \left(Q_k^1 (R_k^1)^{-\mathrm{T}} C_k \right) &\quad (9e)\\
\tilde{r_k^2} &= -(Q_k^2)^{\mathrm{T}} R_k \left(Q_k^1 (R_k^1)^{-\mathrm{T}} \right) e_k &\quad (9f)\\
\tilde{q_k^2} &= -S_k \left(Q_k^1 (R_k^1)^{-\mathrm{T}} \right) e_k - \left(Q_k^1 (R_k^1)^{-\mathrm{T}} C_k \right)^{\mathrm{T}} R_k \left(Q_k^1 (R_k^1)^{-\mathrm{T}} \right) e_k &\quad (9g)\\
C_k^2 &= \frac{1}{2} e_k^{\mathrm{T}} \left(Q_k^1 (R_k^1)^{-\mathrm{T}} \right)^{\mathrm{T}} R_k \left(Q_k^1 (R_k^1)^{-\mathrm{T}} \right) e_k &\quad (9h)
\end{aligned}$$

The new cost is now

$$\sum_{k=0}^{N-1} \left(\frac{1}{2} \begin{bmatrix}\tilde{\Delta u_k} \\ \Delta x_k\end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \tilde{R_k} & \tilde{Q_k} \\ \tilde{S_k^{\mathrm{T}}} & \tilde{Q_k} \end{bmatrix} \begin{bmatrix}\tilde{\Delta u_k} \\ \Delta x_k\end{bmatrix} + \begin{bmatrix}\tilde{\Delta u_k} \\ \Delta x_k\end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \tilde{r_k} \\ \tilde{q_k} \end{bmatrix} + C_k^1 + C_k^2 \right) + \frac{1}{2}\Delta x_N^{\mathrm{T}} Q_N \Delta x_N + \Delta x_N^{\mathrm{T}} q_N \quad (10a)$$

$$\begin{aligned}
\tilde{r_k} &= \tilde{r_k^1} + \tilde{r_k^2} &\quad (10b)\\
\tilde{q_k} &= \tilde{q_k^1} + \tilde{q_k^2} &\quad (10c)
\end{aligned}$$

Now we plug the new variable $\Delta u_k = Q_k^2 \tilde{\Delta u_k} - Q_k^1 (R_k^1)^{-\mathrm{T}} (C_k \Delta x_k + e_k)$ into the dynamic equality constraints

$$\begin{aligned}
\Delta x_{k+1} &= A_k \Delta x_k + B_k \Delta u_k + b_k && \text{(11a)}\\
&= A_k \Delta x_k + B_k Q_k^2 \tilde{\Delta u_k} - B_k Q_k^1 (R_k^1)^{-\mathrm{T}} C_k \Delta x_k - B_k Q_k^1 (R_k^1)^{-\mathrm{T}} e_k + b_k && \text{(11b)}\\
&= \left(A_k - B_k Q_k^1 (R_k^1)^{-\mathrm{T}} C_k\right) \Delta x_k + B_k Q_k^2 \tilde{\Delta u_k} + b_k - B_k Q_k^1 (R_k^1)^{-\mathrm{T}} e_k && \text{(11c)}\\
&= \tilde{A_k} \Delta x_k + \tilde{B_k} \tilde{\Delta u_k} + \tilde{b_k} && \text{(11d)}
\end{aligned}$$

$$A_k = \frac{\partial f}{\partial x_k}(x_k, u_k) \quad B_k = \frac{\partial f}{\partial u_k}(x_k, u_k) \quad b_k = f(x_k, u_k) - x_{k+1} \tag{11e}$$

$$\tilde{A_k} = \left(A_k - B_k Q_k^1 (R_k^1)^{-\mathrm{T}} C_k\right) \quad \tilde{B_k} = B_k Q_k^2 \quad \tilde{b_k} = b_k - B_k Q_k^1 (R_k^1)^{-\mathrm{T}} e_k \tag{11f}$$

2 Integration method

The notation of continuous model is $\dot{x} = f_c(x, u, t)$. Different discrete models all use $x_{k+1} = f_d(x_k, u_k, t_k)$.

From the lecture notes on numerical optimization:

$$A_k = \frac{\partial f_d}{\partial x_k}(x_k, u_k, t_k) \tag{12}$$

$$B_k = \frac{\partial f_d}{\partial u_k}(x_k, u_k, t_k) \tag{13}$$

$$b_k = f_d(x_k, u_k, t_k) - x_{k+1} \tag{14}$$

2.1 One-step Euler

$$x_{k+1} = f_d(x_k, u_k, t_k) = x_k + f_c(x_k, u_k, t_k)\Delta t \tag{15}$$

$$A_k = I_d + \Delta t \frac{\partial f_c}{\partial x_k}(x_k, u_k, t_k) \tag{16}$$

$$B_k = \Delta t \frac{\partial f_c}{\partial u_k}(x_k, u_k, t_k) \tag{17}$$

$$b_k = x_k + f_c(x_k, u_k, t_k)\Delta t - x_{k+1} \tag{18}$$

2.2 RK4

$$x_{k+1} = f_d(x_k, u_k, t_k) = x_k + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{19}$$

$$k_1 = f_c(x_k, u_k, t_k) \tag{20}$$

$$k_2 = f_c(x_k + \frac{\Delta t}{2} k_1, u_k, t_k + \frac{\Delta t}{2}) \tag{21}$$

$$k_3 = f_c(x_k + \frac{\Delta t}{2} k_2, u_k, t_k + \frac{\Delta t}{2}) \tag{22}$$

$$k_4 = f_c(x_k + \Delta t k_3, u_k, t_k + \Delta t) \tag{23}$$

$$\tag{24}$$

$$
\begin{aligned}
A_k &= I_d + \frac{\Delta t}{6}(\frac{\partial k_1}{\partial x_k} + 2\frac{\partial k_2}{\partial x_k} + 2\frac{\partial k_3}{\partial x_k} + \frac{\partial k_4}{\partial x_k}) && (25)\\
\frac{\partial k_1}{\partial x_k} &= \frac{\partial f_c}{\partial x_k}(x_k, u_k, t_k) && (26)\\
\frac{\partial k_2}{\partial x_k} &= \frac{\partial f_c}{\partial (x_k + \frac{\Delta t}{2}k_1)}(x_k + \frac{\Delta t}{2}k_1, u_k, t_k + \frac{\Delta t}{2})\frac{\partial (x_k + \frac{\Delta t}{2}k_1)}{\partial x_k} && (27)\\
&= \frac{\partial f_c}{\partial (x_k + \frac{\Delta t}{2}k_1)}(x_k + \frac{\Delta t}{2}k_1, u_k, t_k + \frac{\Delta t}{2})(I_d + \frac{\Delta t}{2}\frac{\partial k_1}{\partial x_k}) && (28)\\
\frac{\partial k_3}{\partial x_k} &= \frac{\partial f_c}{\partial (x_k + \frac{\Delta t}{2}k_2)}(x_k + \frac{\Delta t}{2}k_2, u_k, t_k + \frac{\Delta t}{2})\frac{\partial (x_k + \frac{\Delta t}{2}k_2)}{\partial x_k} && (29)\\
&= \frac{\partial f_c}{\partial (x_k + \frac{\Delta t}{2}k_2)}(x_k + \frac{\Delta t}{2}k_2, u_k, t_k + \frac{\Delta t}{2})(I_d + \frac{\Delta t}{2}\frac{\partial k_2}{\partial x_k}) && (30)\\
\frac{\partial k_4}{\partial x_k} &= \frac{\partial f_c}{\partial (x_k + \Delta t k_3)}(x_k + \Delta t k_3, u_k, t_k + \Delta t)\frac{\partial (x_k + \Delta t k_3)}{\partial x_k} && (31)\\
&= \frac{\partial f_c}{\partial (x_k + \Delta t k_3)}(x_k + \Delta t k_3, u_k, t_k + \Delta t)(I_d + \Delta t\frac{\partial k_3}{\partial x_k}) && (32)
\end{aligned}
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\begin{aligned}
B_k &= \frac{\Delta t}{6}(\frac{\partial k_1}{\partial u_k} + 2\frac{\partial k_2}{\partial u_k} + 2\frac{\partial k_3}{\partial u_k} + \frac{\partial k_4}{\partial u_k}) && (33)\\
\frac{\partial k_1}{\partial u_k} &= \frac{\partial f_c}{\partial u_k}(x_k, u_k, t_k) && (34)\\
\frac{\partial k_2}{\partial u_k} &= \frac{\partial f_c}{\partial (x_k + \frac{\Delta t}{2}k_1)}(x_k + \frac{\Delta t}{2}k_1, u_k, t_k + \frac{\Delta t}{2})\frac{\partial (x_k + \frac{\Delta t}{2}k_1)}{\partial u_k} + \frac{\partial f_c}{\partial u_k}(x_k + \frac{\Delta t}{2}k_1, u_k, t_k + \frac{\Delta t}{2}) && (35)\\
&= \frac{\partial f_c}{\partial (x_k + \frac{\Delta t}{2}k_1)}(x_k + \frac{\Delta t}{2}k_1, u_k, t_k + \frac{\Delta t}{2})\frac{\Delta t}{2}\frac{\partial k_1}{\partial u_k} + \frac{\partial f_c}{\partial u_k}(x_k + \frac{\Delta t}{2}k_1, u_k, t_k + \frac{\Delta t}{2}) && (36)\\
\frac{\partial k_3}{\partial u_k} &= \frac{\partial f_c}{\partial (x_k + \frac{\Delta t}{2}k_2)}(x_k + \frac{\Delta t}{2}k_2, u_k, t_k + \frac{\Delta t}{2})\frac{\partial (x_k + \frac{\Delta t}{2}k_2)}{\partial u_k} + \frac{\partial f_c}{\partial u_k}(x_k + \frac{\Delta t}{2}k_2, u_k, t_k + \frac{\Delta t}{2}) && (37)\\
&= \frac{\partial f_c}{\partial (x_k + \frac{\Delta t}{2}k_2)}(x_k + \frac{\Delta t}{2}k_2, u_k, t_k + \frac{\Delta t}{2})\frac{\Delta t}{2}\frac{\partial k_2}{\partial u_k} + \frac{\partial f_c}{\partial u_k}(x_k + \frac{\Delta t}{2}k_2, u_k, t_k + \frac{\Delta t}{2}) && (38)\\
\frac{\partial k_4}{\partial u_k} &= \frac{\partial f_c}{\partial (x_k + \Delta t k_3)}(x_k + \Delta t k_3, u_k, t_k + \Delta t)\frac{\partial (x_k + \Delta t k_3)}{\partial u_k} + \frac{\partial f_c}{\partial u_k}(x_k + \Delta t k_3, u_k, t_k + \Delta t) && (39)\\
&= \frac{\partial f_c}{\partial (x_k + \Delta t k_3)}(x_k + \Delta t k_3, u_k, t_k + \Delta t)\Delta t\frac{\partial k_3}{\partial u_k} + \frac{\partial f_c}{\partial u_k}(x_k + \Delta t k_3, u_k, t_k + \Delta t) && (40)\\
& && (41)
\end{aligned}
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