第三章习题

8. 解：(1) $P(X_0=0)=\frac{1}{3}$

$P(X_0=0, X_2=0)=\frac{1}{3}\times(\frac{1}{3}\times\frac{1}{3}+\frac{2}{3}\times\frac{2}{3}+0)$

$=\frac{5}{27}$

$\therefore P(X_2=0|X_0=0)=\frac{P(X_0=0, X_2=0)}{P(X_0=0)}=\frac{5}{9}$

$$P^2=\begin{bmatrix}\frac{5}{9} & \frac{2}{9} & \frac{2}{9}\\ \frac{2}{9} & \frac{7}{9} & 0\\ \frac{2}{3} & 0 & \frac{1}{3}\end{bmatrix}$$

有 $P(X_2=0)=\frac{1}{3}\times\frac{5}{9}+\frac{1}{3}\times\frac{2}{9}+\frac{1}{3}\times\frac{2}{3}=\frac{13}{27}$

$\therefore P(X_0=0|X_2=0)=\frac{P(X_2=0, X_0=0)}{P(X_2=0)}=\frac{5}{13}$

(2) $P(X_1=0)=\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{2}{3}=\frac{1}{3}$

$P(X_1=0, X_3=0, X_4=1, X_0=1)$

$$= \frac{1}{3}\times\frac{5}{9}\times\frac{2}{3}\times\frac{7}{9} = \frac{70}{729}$$

(3) 状态图如下

$$\begin{tikzcd}
0 \arrow[loop left, "\frac{1}{3}"] \arrow[r, bend left, "\frac{2}{3}"] & 1 \arrow[l, bend left, "\frac{2}{3}"] \arrow[r, bend left, "\frac{1}{3}"] & 2 \arrow[l, bend left, "1"]
\end{tikzcd}$$

有 $f_{11}^{(1)} = 0$

$$f_{11}^{(2)} = \frac{2}{3}\times\frac{2}{3}+\frac{1}{3}\times 1 = \frac{7}{9}$$

$n\geq 3$时 $f_{11}^{(n)} = \frac{2}{3}\times\frac{2}{3}\times\left(\frac{1}{3}\right)^{n-2} = 4\left(\frac{1}{3}\right)^{n}$

有 $f_{11} = \frac{7}{9}+\sum_{n=3}^{+\infty}4\left(\frac{1}{3}\right)^{n}$

$$= \frac{7}{9}+\frac{\frac{4}{27}}{1-\frac{1}{3}} = 1$$

$$\mu_1 = \frac{14}{9}+\sum_{n=3}^{+\infty}4n\left(\frac{1}{3}\right)^{n} = \frac{14}{9}+\frac{7}{9} = \frac{7}{3}$$

10、解：

(1) 先计算 P^2

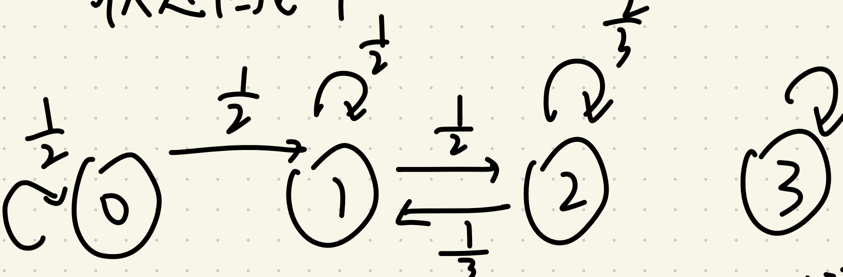
有 $P^2=\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{5}{12} & \frac{7}{12} & 0 \\ 0 & \frac{7}{18} & \frac{11}{18} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

故 $P(X_1=1, X_3=2)=\left(\frac{1}{3}\times\frac{1}{2}+\frac{1}{3}\times\frac{1}{2}\right)\times\frac{7}{12}$

$=\frac{1}{3}\times\frac{7}{12}=\frac{7}{36}$

$P(X_2=1)=\frac{1}{3}\times\frac{1}{2}+\frac{1}{3}\times\frac{5}{12}=\frac{11}{36}$

状态图如下



可知1,2为互达等价类，而0不封闭，仅有0能到达0

故 $P(X_{10}=0)=\frac{1}{3}\times\left(\frac{1}{2}\right)^{10}=\frac{1}{3\cdot 2^{10}}$

(2) 0不封闭，故为暂留达

1、2为互达等价类，对1，有

$$f_{11}^{(1)}=\frac{1}{2} \quad f_{11}^{(2)}=\frac{1}{6} \quad f_{11}^{(3)}=\frac{1}{9}$$

故 $d(1)=1$，知1为非周期正常返态。

由互达等价类，2也为非周期正常返态，

将1、2拿出，有 $P=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{3} & \frac{2}{3}\end{bmatrix}$

有 $\begin{cases}\pi_1+\pi_2=1\\ \pi_1=\frac{1}{2}\pi_1+\frac{1}{3}\pi_2\end{cases}$

解得 $\begin{cases}\pi_1=\frac{2}{5}\\ \pi_2=\frac{3}{5}\end{cases}$

故 $\mu_1=\frac{5}{2}$ $\quad \mu_2=\frac{5}{3}$

3为吸收态，故为非周期正常返态，$\mu_3=1$

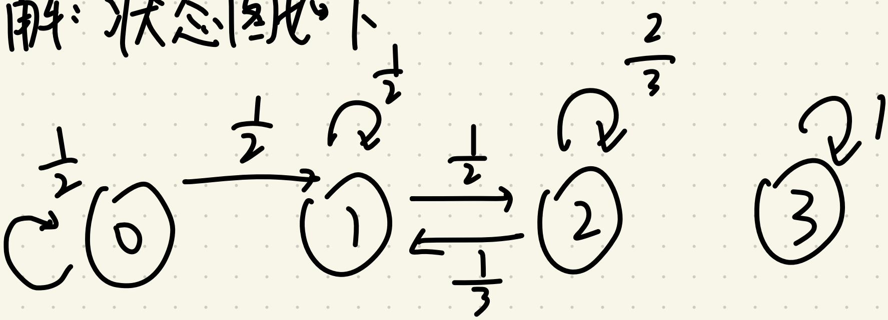
12、设平稳分布为 (π_0,π_1,π_2)

有 $\begin{cases} \pi_0+\pi_1+\pi_2=1 \\ \pi_2=\frac{1}{3}\pi_1 \\ \pi_1=\frac{2}{3}\pi_0+\pi_2 \end{cases}$

解得 $\begin{cases} \pi_0=\frac{3}{7} \\ \pi_1=\frac{3}{7} \\ \pi_2=\frac{1}{7} \end{cases}$

故平稳分布为 $\left(\frac{3}{7}, \frac{3}{7}, \frac{1}{7}\right)$

16、解：状态图如下



① $i=0$ 时，0为暂留态，因此 $\lim_{n\to\infty} P(X_n=0)=0$

② $i=3$ 时，3为非周期常返态，若初始为3，则必为3；若初始不为3，则无论如何皆无法进入3，故 $\lim_{n\to\infty} P(X_n=3)=\frac{1}{3}$

③ $i=1$时，此时非周期正常返，

有 $\lim\limits_{n\to\infty} P(X_n=1)=\lim\limits_{n\to\infty}\sum\limits_{i=1}^{3} P(X_n=1|X_0=i)P(X_0=i)$

$$=\frac{1}{3}\lim_{n\to\infty}P(X_n=1|X_0=0)+\frac{1}{3}\lim_{n\to\infty}P(X_n=1|X_0=1)$$

又由全概率公式求得 $h_{01}=h_{11}$

由10题知 $\pi_1=\frac{2}{5}$　$\pi_3=\frac{3}{5}$

故 $\lim\limits_{n\to\infty}P(X_0=0, X_n=1)=\pi_1=\frac{2}{5}$

$\lim\limits_{n\to\infty}P(X_0=1, X_n=1)=\pi_1=\frac{2}{5}$

$\therefore \lim\limits_{n\to\infty}P(X_n=1)=\frac{1}{3}\times\frac{2}{5}+\frac{1}{3}\times\frac{2}{5}=\frac{4}{15}$

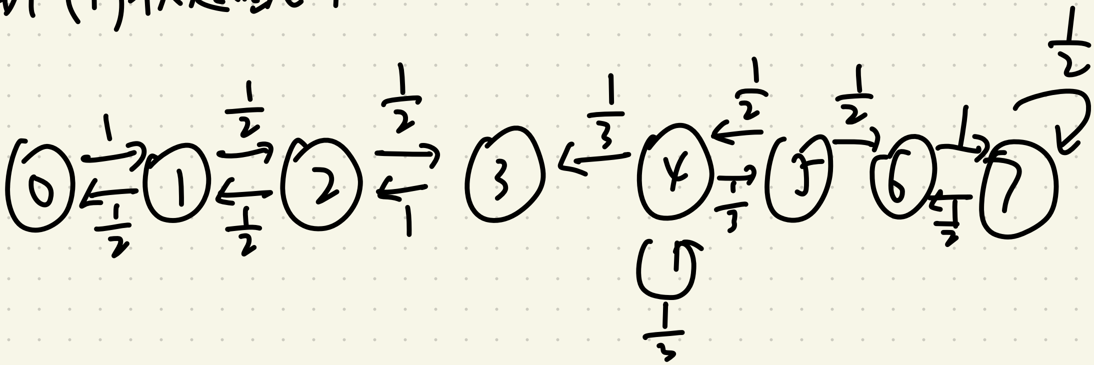
④ $i=2$时，此时非周期正常返，有 $h_{02}=h_{12}$

有 $\lim\limits_{n\to\infty}P(X_n=2)=\lim\limits_{n\to\infty}\sum\limits_{i=1}^{3}P(X_n=2|X_0=i)P(X_0=i)$

$$=\frac{1}{3}\lim_{n\to\infty}P(X_n=1|X_0=0)+\frac{1}{3}\lim_{n\to\infty}P(X_n=1|X_0=1)$$

$$=\frac{1}{3}\times\frac{3}{5}+\frac{1}{3}\times\frac{3}{5}=\frac{2}{5}$$

17、解：(1) 状态图如下



故知 $\{0,1,2,3\}$ $\{4,5\}$ $\{6,7\}$ 为互达等价类

显然 $\{0,1,2,3\}$ $\{6,7\}$ 为闭等价类，一旦进入就不会再出来，$\{4,5\}$ 不是闭等价类

(2) ① $\{0,1,2,3\}$ 为互达等价类，考察0，由于闭等价类，且 $\{0,1,2,3\}$ 有限，故必为正常返

有 $f_{00}^{(1)}=0$　$f_{00}^{(2)}=\frac{1}{2}$　$f_{00}^{(3)}=0$　$f_{00}^{(4)}=\left(\frac{1}{2}\right)^3$

… 显然若从0出发回到0必经过偶数步，最小公约数为 $d(0)=2$，故 $d(0)=d(1)=d(2)=d(3)=2$

② $\{4,5\}$ 不是闭的等价类，因此为暂留

③ 对 6、7，考察 7，以为正常返。

$f_{77}^{(1)}=\frac{1}{2}$ $f_{77}^{(2)}=\frac{1}{2}$

$n\geq 3$ 时，有 $f_{77}^{(n)}=0$

此时 $d(7)=1$，故 6、7 为非周期正常返

下分别对 $\{0,1,2,3\}$ 及 $\{6,7\}$ 求稳定分布

对 $\{0,1,2,3\}$，有

$$(\pi_0\ \pi_1\ \pi_2\ \pi_3)=(\pi_0\ \pi_1\ \pi_2\ \pi_3)\begin{pmatrix}0&1&0&0\\ \frac{1}{2}&0&\frac{1}{2}&0\\ 0&\frac{1}{2}&0&\frac{1}{2}\\ 0&0&1&0\end{pmatrix}$$

有 $\begin{cases}\pi_0+\pi_1+\pi_2+\pi_3=1\\ \pi_0=\frac{1}{2}\pi_1\\ \pi_3=\frac{1}{2}\pi_2\\ \pi_1=\pi_0+\frac{1}{2}\pi_2\end{cases} \Rightarrow \begin{cases}\pi_0=\frac{1}{6}\\ \pi_1=\frac{1}{3}\\ \pi_2=\frac{1}{3}\\ \pi_3=\frac{1}{6}\end{cases}$

$\therefore$ 有 $\mu_0=6$，$\mu_1=3$，$\mu_2=3$，$\mu_3=6$

对$\{6,7\}$，有$(\pi_6\ \ \pi_7)=(\pi_6\ \ \pi_7)\begin{bmatrix}0 & 1\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$

有$\begin{cases}\pi_6+\pi_7=1\\ \pi_6=\frac{1}{2}\pi_7\end{cases}\Rightarrow\begin{cases}\pi_6=\frac{1}{3}\\ \pi_7=\frac{2}{3}\end{cases}$

故$\mu_6=3$，$\mu_7=\frac{3}{2}$

(3) 4、5不是闭的非周期等价类，故4、5暂留

有$\lim\limits_{n\to\infty}p_{45}^{(n)}=0$

6、7为非周期正常返，为闭的等价类，

故$\lim\limits_{n\to\infty}p_{67}^{(n)}=\pi_7=\frac{2}{3}$

(4) ①对$i=0$，有

$\lim\limits_{n\to\infty}P(X_n=0)=\frac{1}{2}\lim\limits_{n\to\infty}P(X_n=0|X_0=3)+\frac{1}{2}\lim\limits_{n\to\infty}P(X_n=0|X_0=4)$

又$\lim\limits_{n\to\infty}P(X_n=0,X_0=3)=\pi_0=\frac{1}{6}$

故$\lim\limits_{n\to\infty}P(X_n=0)=\frac{1}{12}+\frac{1}{2}h_{40}$

又 $h_{4,0}=\sum_{i=1}^{3}\lim_{n\to\infty}P(X_n=0\mid X_1=i,X_0=4)P(X_1=i,X_0=4)$

$$=\frac{1}{3}h_{4,0}+\frac{1}{3}h_{3,0}+\frac{1}{3}h_{5,0}$$

$h_{5,0}=\frac{1}{2}h_{4,0}+\frac{1}{2}h_{6,0}=\frac{1}{2}h_{4,0}$

故知 $h_{4,0}=\frac{1}{9}$

有 $\lim_{n\to\infty}P(X_n=0)=\frac{1}{12}+\frac{1}{18}=\frac{5}{36}$

② 若 $i=1$，有

$\lim_{n\to\infty}P(X_n=1)=\frac{1}{2}\lim_{n\to\infty}P(X_n=1\mid X_0=3)+\frac{1}{2}\lim_{n\to\infty}P(X_n=1\mid X_0=4)$

$$=\frac{1}{6}+\frac{1}{2}h_{4,1}$$

又 $h_{4,1}=\frac{1}{3}h_{4,1}+\frac{1}{3}h_{3,1}+\frac{1}{3}h_{5,1}$

$h_{5,1}=\frac{1}{2}h_{4,1}+\frac{1}{2}h_{6,1}=\frac{1}{2}h_{4,1}$

故 $h_{4,1}=\frac{2}{9}$，$\lim_{n\to\infty}P(X_n=1)=\frac{5}{18}=\frac{10}{36}$

③ 若 $i=2$，分析可知

$\lim_{n\to\infty} P(X_n=2)=\lim_{n\to\infty}(X_n=1)=\frac{5}{18}=\frac{10}{36}$

④若 $i=3$，分析可知

$\lim_{n\to\infty} P(X_n=3)=\lim_{n\to\infty}(X_n=0)=\frac{5}{36}$

⑤若 $i=4$，此时4为暂留态，不是闭等价类，

有 $\lim_{n\to\infty} P(X_n=4)=0$

⑥若 $i=5$，此时5为暂留态，不属于闭等价类

有 $\lim_{n\to\infty} P(X_n=5)=0$

⑦若 $i=6$，有

$\lim_{n\to\infty} P(X_n=6)=\frac{1}{2}\lim_{n\to\infty} P(X_n=6 \mid X_0=4)$

$=\frac{1}{2}h_{4,6}$

又 $h_{4,6}=\frac{1}{3}h_{4,6}+\frac{1}{3}h_{5,6}$

$h_{5,6}=\frac{1}{2}h_{4,6}+\frac{1}{2}h_{6,6}=\frac{1}{2}h_{4,6}+\frac{1}{6}$

故 $h_{4,6}=\frac{1}{9}$

$\therefore \lim\limits_{n\to\infty} P(X_n=6) = \frac{1}{2}\times\frac{1}{9} = \frac{1}{18} = \frac{2}{36}$

⑧ $i=7$，有

$$\lim_{n\to\infty} P(X_n=7) = \frac{1}{2}\lim_{n\to\infty} P(X_n=7 \mid X_0=4)$$
$$= \frac{1}{2}h_{4,7}$$

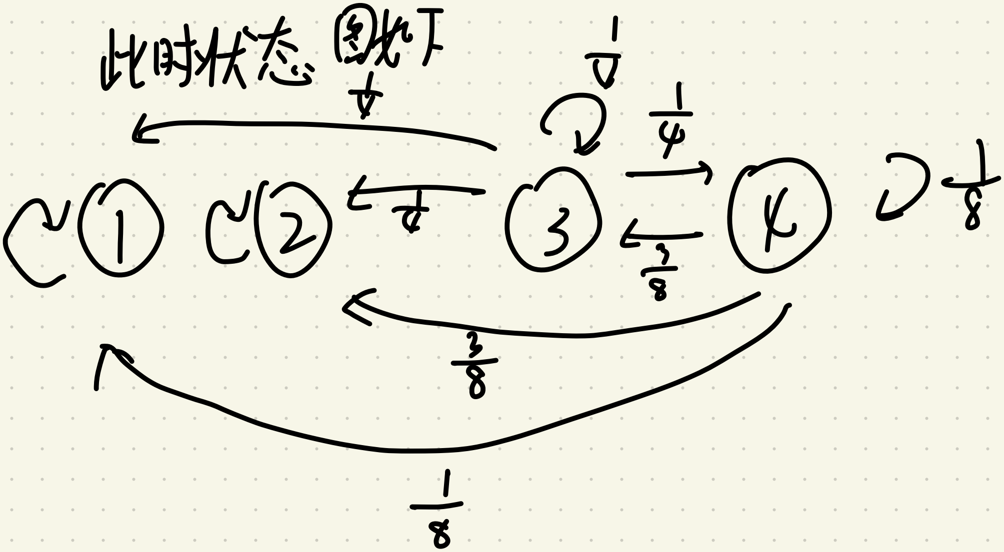
又 $h_{4,7} = \frac{1}{3}h_{4,7} + \frac{1}{3}h_{5,7}$

$h_{5,7} = \frac{1}{2}h_{4,7} + \frac{1}{2}h_{6,7} = \frac{1}{2}h_{4,7} + \frac{1}{3}$

$\therefore h_{4,7} = \frac{2}{9}$，$\lim\limits_{n\to\infty} P(X_n=7) = \frac{1}{9} = \frac{4}{36}$

19、解：若 $T_1 = \inf\{n \geq 0, X_n = 1\}$

此时状态图如下



此时有 $P(T_1<\infty \mid X_0=3)=\lim\limits_{n\to\infty}P(X_n=1\mid X_0=3)$

有 $\lim\limits_{n\to\infty}P(X_n=1\mid X_0=3)=\frac{1}{4}+\frac{1}{4}h_{3,1}+\frac{1}{4}h_{4,1}$

对 $h_{4,1}$，有 $h_{4,1}=\frac{1}{8}+\frac{3}{8}h_{3,1}+\frac{1}{8}h_{4,1}$

$\therefore$ 有 $\begin{cases} h_{3,1}=\frac{1}{4}+\frac{1}{4}h_{3,1}+\frac{1}{4}h_{4,1} \\ h_{4,1}=\frac{1}{8}+\frac{3}{8}h_{3,1}+\frac{1}{8}h_{4,1} \end{cases}$

解得 $\begin{cases} h_{3,1}=\frac{4}{9} \\ h_{4,1}=\frac{1}{3} \end{cases}$

故有 $\lim\limits_{n\to\infty}P(X_n=1\mid X_0=3)=\frac{1}{4}+\frac{1}{9}+\frac{1}{12}=\frac{4}{9}$

故 $P(T_1<\infty \mid X_0=3)=\frac{4}{9}$