Supplementary Material

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Contents

1 Summary

This document contains derivations and proofs that supplement the paper ”Certifiably Optimal Mutual Localization with Anonymous Bearing Measurements” [YW]. We will provide some basic mathematical equations in Sec.(2). In Sec.(3), we provide derivation of loop error for one-observed-robot case. In Sec.(4), we provide derivation of loop error for multiple-observed-robot case. In Sec.(5), we provide detail constructions for all involved quadratic constraints. Furthermore, we provide proof of two lemmas in Sec.(6).

2 Basic Mathematical Equations

We make heavy use of matrix tricks in the problem formulation. Summarize some important mathematical equations as follows:

$$\text{vec}(AXB) = (B^T \otimes A)\text{vec}(X) \tag{1}$$
$$Xb = \text{vec}(IXb) = (b^T \otimes I)\text{vec}(X) \tag{2}$$
$$\text{tr}(ABC) = \text{tr}(BCA) = \text{tr}(CAB) \tag{3}$$

3 Derivation of Loop Error for One-Observed-Robot Case

In this section, we provide detail derivation of $e^{AB}_{j_1 j_2}$ in Sec.III-A of our paper.

$$
\begin{aligned}
e^{AB}_{j_1 j_2} &= s_{AB} R_{AB} \widehat{t}_{B_{j_1} B_{j_2}} + R_{AB} \widehat{R}_{B_{j_1} B_{j_2}} {}^{B}\bar{P} - (\widehat{t}_{A_{j_1} A_{j_2}} + R_{A_1 A_{j_2}} b_{j_2} D_{j_2} - R_{A_1 A_{j_1}} b_{j_1} D_{j_1}) \\
&\qquad (R_{AB} R_{B_{j_1} B_{j_2}} {}^{B}\bar{P} = ({}^{B}\bar{P}^T \otimes R_{AB}) \text{vec}(R_{B_{j_1} B_{j_2}})) \text{ according to Equ.(1).} \\
&= s_{AB} R_{AB} \widehat{t}_{B_{j_1} B_{j_2}} + ({}^{B}\bar{P}^T \otimes R_{AB}) \text{vec}(\widehat{R}_{B_{j_1} B_{j_2}}) - (\widehat{t}_{A_{j_1} A_{j_2}} + R_{A_1 A_{j_2}} b_{j_2} D_{j_2} - R_{A_1 A_{j_1}} b_{j_1} D_{j_1}) \\
&\qquad ({}^{B}\bar{P}^T \otimes R_{AB}) \text{vec}(R_{B_{j_1} B_{j_2}})) = (\text{vec}(R_{B_{j_1} B_{j_2}})^T \otimes I) \text{vec}({}^{B}\bar{P}^T \otimes R_{AB})) \text{ acoording to Equ.(2).} \\
&= s_{AB} R_{AB} \widehat{t}_{B_{j_1} B_{j_2}} + (\text{vec}(\widehat{R}_{B_{j_1} B_{j_2}})^T \otimes I) \text{vec}({}^{B}\bar{P}^T \otimes R_{AB})) - (\widehat{t}_{A_{j_1} A_{j_2}} + R_{A_1 A_{j_2}} b_{j_2} D_{j_2} - R_{A_1 A_{j_1}} b_{j_1} D_{j_1}) \\
&\qquad s_{AB} R_{AB} \widehat{t}_{B_{j_1} B_{j_2}} = (\widehat{t}^T_{B_{j_1} B_{j_2}} \otimes I) \text{vec}(s R_{AB}) \text{ according to Equ.(2).} \\
&= (\widehat{t}^T_{B_{j_1} B_{j_2}} \otimes I) \text{vec}(s_{AB} R_{AB}) + (\text{vec}(\widehat{R}_{B_{j_1} B_{j_2}})^T \otimes I) \text{vec}({}^{B}\bar{P}^T \otimes R_{AB})) - (\widehat{t}_{A_{j_1} A_{j_2}} + R_{A_1 A_{j_2}} b_{j_2} D_{j_2} - R_{A_1 A_{j_1}} b_{j_1} D_{j_1}) \\
&\qquad \text{More compact formulation in linear form.} \\
&= [\, \widehat{t}^T_{B_{j_1} B_{j_2}} \otimes I, \text{vec}(\widehat{R}_{B_{j_1} B_{j_2}})^T \otimes I, -\widehat{t}_{A_{j_1} A_{j_2}}, R_{A_1 A_{j_1}} b_{j_1}, -R_{A_1 A_{j_2}} b_{j_2} \,] x^{AB}_{j_1 j_2},
\end{aligned}
\tag{4}
$$

where $x^{AB}_{j_1 j_2} = [\, \text{vec}(s_{AB} R_{AB})^T, \text{vec}({}^{B}\bar{P}^T \otimes R_{AB})^T, y, D_{j_1}, D_{j_2} \,]^T$ is unknown variable.

4 Derivation of Loop Error for Multiple-Observed-Robot Case

In this section, we provide detailed derivation of $c^{X}_{j_1 j_2}$ in Sec.III-B of our paper.
Firstly we define the below variables as shown in our paper,

$$ {}^{Y}\mathbb{P} \doteq [s_{AY}, {}^{Y}\bar{P}^T]^T \in \mathbb{R}^{4\times 1}, \tag{5} $$
$$ {}^{Y}\mathbb{P}_X \doteq \theta_{XY} {}^{Y}\mathbb{P} \in \mathbb{R}^{4\times 1}, \tag{6} $$
$$ r_{XY} \doteq \text{vec}({}^{Y}\mathbb{P}^T_X \otimes R_{AY}) \in \mathbb{R}^{36\times 1}, \tag{7} $$
$$ r_X \doteq \text{vstack}(\{r_{XY}\}^N_{Y=1}) \in \mathbb{R}^{36N\times 1}, \tag{8} $$
$$ D_X \doteq \text{vstack}(\{D^X_j\}_{j\in J}) \in \mathbb{R}^{n\times 1}, \tag{9} $$
$$ r \doteq \text{vstack}(\{r_X\}^N_{X=1}) \in \mathbb{R}^{36N^2\times 1}, \tag{10} $$
$$ D \doteq \text{vstack}(\{D_X\}^N_{X=1}) \in \mathbb{R}^{nN\times 1}, \tag{11} $$
$$ x \doteq [\, r^T,\ y,\ D^T \,]^T \in \mathbb{R}^{(36N^2+1+nN)\times 1}. \tag{12} $$

Note that after introducing additional constraint $y^2 = 1$, the marginalized QCQP Problem in Sec.III-C of our paper is homogeneous and both $\pm(r^T, 1)$ are solutions of it. There for we can assume that $y = 1$.

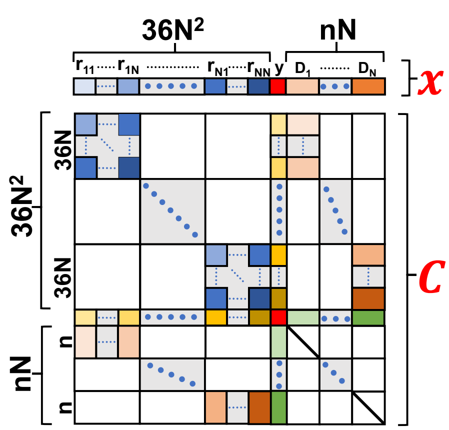

Figure 1: Structure of x and C.

Then utilizing the above variables, we derive loop error for multiple-observed-robot case as follow

$$
\begin{aligned}
e^X_{j_1 j_2} &= \sum_{Y=1}^{N} \theta_{XY}(R_{AY} R_{Y_{j_1} Y_{j_2}}{}^{Y}\bar{P} + s_{AY} R_{AY} \widehat{t}_{Y_{j_1} Y_{j_2}}) - (\widehat{t}_{A_{j_1} A_{j_2}} + R_{A_1 A_{j_2}} b^X_{j_2} D^X_{j_2} - R_{A_1 A_{j_1}} b^X_{j_1} D^X_{j_1}) \\
&\quad \text{According to loop error for one-observed-robot case in Sec.(3).} \\
&= \sum_{Y=1}^{N} [\, \widehat{t}^T_{Y_{j_1} Y_{j_2}} \otimes I, \mathrm{vec}(R_{Y_{j_1} Y_{j_2}})^T \otimes I \,] r_{XY} - \widehat{t}_{A_{j_1} A_{j_2}} - (R_{A_1 A_{j_2}} b^X_{j_2} D^X_{j_2} - R_{A_1 A_{j_1}} b^X_{j_1} D^X_{j_1}) \\
&\quad \text{Fill zero to } c^X_D \text{ where corresponds to other distance varible } D^X_j. \\
&= \sum_{Y=1}^{N} \underbrace{[\, \widehat{t}^T_{Y_{j_1} Y_{j_2}} \otimes I, \mathrm{vec}(R_{Y_{j_1} Y_{j_2}})^T \otimes I \,]}_{:= c^{XY}_r \in R^{3\times 36}} r_{XY} - \widehat{t}_{A_{j_1} A_{j_2}} + \underbrace{[\, 0_{3\times 1} ..., R_{A_1 A_{j_1}} b^X_{j_1}, -R_{A_1 A_{j_2}} b^X_{j_2}, ... 0_{3\times 1} \,]}_{:= c^X_D \in R^{3\times n}} D_X \\
&\quad \text{Construct } c^{XY}_r \text{ for each observed robot (from 1 to N), and fill them into } c^X_r. \\
&= \underbrace{[\, c^{X1}_r, ..., c^{XY}_r, ..., c^{XN}_r \,]}_{:= c^X_r \in R^{3\times 36N}} r_X - \widehat{t}_{A_{j_1} A_{j_2}} + [\, 0_{3\times n}, ..., c^X_D, ..., 0_{3\times n} \,][\, D^T_1, ..., D^T_X, ... D^T_N \,]^T \\
&\quad \text{Fill zero to } c_r \text{ where correspondence to other } r_X \text{ variables.} \\
&= \underbrace{[\, 0_{3\times 36N}, ..., c^X_r, ..., 0_{3\times 36N} \,]}_{:= c_r \in R^{3\times (36N^2)}}[\, r^T_1, ... r^T_X, ..., r^T_N \,]^T - \widehat{t}_{A_{j_1} A_{j_2}} + \underbrace{[\, 0_{3\times n}, ..., c^X_D, ..., 0_{3\times n} \,]}_{:= c_D \in R^{3\times (Nn)}}[\, D^T_1, ..., D^T_X, ... D^T_N \,]^T \\
&\quad \text{More compact formulation in linear form.} \\
&= \underbrace{[\, c_r, -\widehat{t}_{A_{j_1} A_{j_2}}, c_D \,]}_{c^X_{j_1 j_2} \in R^{3\times (36N^2+1+Nn)}} x
\end{aligned}
\tag{13}
$$

Then, we can use $c^X_{j_1 j_2}$ to formulate mutual localization problem with anonymous measurements as a non-convex optimization and obtain a cost matrix C

$$
C \doteq \sum_{\substack{X\in[1,N] \\ \{j_1, j_2\}\in J}} (c^X_{j_1 j_2})^T w^X_{j_1 j_2} c^X_{j_1 j_2}
\tag{14}
$$

We divide C into 9 parts to present the structure of C and demonstrate C in Fig.(1).

$$
C = \left(\begin{array}{c|c|c} C_{1,1} & C_{1,2} & C_{1,3} \\ \hline C_{2,1} & C_{2,2} & C_{2,3} \\ \hline C_{3,1} & C_{3,2} & C_{3,3} \end{array}\right) = \left(\begin{array}{cccc|c|cccc}
E & 0 & \cdots & 0 & e & F_1 & 0 & \cdots & 0 \\
0 & E & \cdots & 0 & e & 0 & F_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & E & e & 0 & 0 & \cdots & F_N \\ \hline
e^T & e^T & \cdots & e^T & c & f^T_1 & \cdots & f^T_{N-1} & f^T_N \\ \hline
F^T_1 & 0 & \cdots & 0 & f_1 & G_1 & 0 & \cdots & 0 \\
0 & F^T_2 & \cdots & 0 & f_2 & 0 & G_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & F^T_N & f_N & 0 & 0 & \cdots & G_N
\end{array}\right)
$$

where $E \in R^{36N\times 36N}$, $F_i \in R^{36N\times n}$, $G \in R^{n\times n}$, $e \in R^{36N\times 1}$, $f_i \in R^{n\times 1}$ and c is a scalar.

5 Quadratic Constraints in QCQP Problem Formulation

In this section we will provide detailed construction of Q_i for all involved constraints. Firstly, we present all necessary auxiliary variables.

1. **Lifted Rotation Variable:** ℓ

$$
\ell_Y \doteq \mathrm{vec}({}^{Y}\mathbb{P}^T \otimes R_{AY}) \in \mathbb{R}^{36\times 1},
\tag{15}
$$

$$
\ell \doteq \mathrm{vstack}(\{\ell_Y\}^N_{Y=1}) \in \mathbb{R}^{36N\times 1}.
\tag{16}
$$

2. **Binary Variable:** φ_θ

$$\varphi_\theta^X \doteq \text{vstack}(\{\theta_{XY}\}_{Y=1}^N) \in \mathbb{R}^{N\times 1}, \tag{17}$$

$$\varphi_\theta \doteq \text{vstack}(\{\varphi_\theta^X\}_{X=1}^N) \in \mathbb{R}^{N^2\times 1}. \tag{18}$$

3. **Scale and Inner-Bias Variable:** φ_h

$$\varphi_h \doteq \text{vstack}(\{{}^X\mathbb{P}\}_{X=1}^N) \in \mathbb{R}^{4N\times 1}. \tag{19}$$

4. **Lifted Scale and Inner-Bias Variable:** φ_μ

$$\varphi_\mu^X \doteq \text{vstack}(\{{}^Y\mathbb{P}_X\}_{Y=1}^N) \in \mathbb{R}^{4N\times 1}, \tag{20}$$

$$\varphi_\mu \doteq \text{vstack}(\{\varphi_\mu^X\}_{X=1}^N) \in \mathbb{R}^{4N^2\times 1}. \tag{21}$$

We denote $\mathrm{i_y}$ and $\mathrm{i}_{\theta_\mathrm{XY}}$ as the index of variable representing y and θ_{XY} in f. Then we summarize all involved constrains in our paper.

5.1 Binary Constraint

For each θ_{XY}, $\theta_{XY} \in \{0,1\} \Longrightarrow \theta_{XY}^2 - y\theta_{XY} = 0$, we have

$$\forall\, \theta_{XY}, Q_{\mathrm{i,j}}^{\text{Binary}} = \begin{cases} 1 & \mathrm{i} = \mathrm{i}_{\theta_\mathrm{XY}},\ \mathrm{j} = \mathrm{i}_{\theta_\mathrm{XY}} \\ -1 & \mathrm{i} = \mathrm{i_y},\ \mathrm{j} = \mathrm{i}_{\theta_\mathrm{XY}} \\ 0 & \text{others} \end{cases}$$

The Number of this type of Q_i is N^2.

5.2 Correspondence Constraint

For $\sum_X \theta_{XY} = 1, \forall\, Y \in [1,N] \Longrightarrow \sum_X y\theta_{XY} = y^2$, we have

$$Q_{\mathrm{i,j}}^{\text{Corres}} = \begin{cases} 1, & \mathrm{i} = \mathrm{i_y}, \mathrm{j} = \mathrm{i}_{\theta_\mathrm{XY}} \\ -1, & \mathrm{i} = \mathrm{i_y}, \mathrm{j} = \mathrm{i_y} \\ 0, & \text{others} \end{cases}, \forall\, Y \in [1,N]$$

And for $\sum_Y \theta_{XY} = 1, \forall\, X \in [1,N] \Longrightarrow \sum_Y y\theta_{XY} = y^2$, we have

$$Q_{\mathrm{i,j}}^{\text{Corres}} = \begin{cases} 1 & \mathrm{i} = \mathrm{i_y}, \mathrm{j} = \mathrm{i}_{\theta_\mathrm{XY}} \\ -1 & \mathrm{i} = \mathrm{i_y}, \mathrm{j} = \mathrm{i_y} \\ 0 & \text{others} \end{cases}, \forall\, X \in [1,N]$$

The number of above two types of Q_i is $2N$.

5.3 Rotation Constraint

For rotation constraints, existing work [BKGJ18] provides similar derivations. Here, we firstly give out rotation constraints considering μ for completeness, then integrate them into constraint matrix. We denote

$$r \doteq \begin{bmatrix} \text{vec}(\mu R) \\ \mu \end{bmatrix}$$

.

5.3.1 Orthonormality of rotation columns

For $(\mu R)^T(\mu R) = \mu^2 I$, we have

$$\begin{aligned} C_{\text{Orth-col}} &: e_i^T((\mu R)^T(\mu R) - \mu^2 I_3)e_j \\ &= \text{tr}(e_i^T(\mu R)^T(\mu R)e_j) - \mu^2 e_i^T e_j \\ &= \text{tr}(e_{ij}^T(\mu R)^T(\mu R)) - \mu^2\delta_i j \\ &= \text{vec}(\mu R)^T(e_{ij}\otimes I_3)\text{vec}(\mu R) - \mu^2\delta_i j \\ &= r^T\begin{bmatrix} e_{ij}\otimes I_3 & 0 \\ 0 & -\delta_i j\end{bmatrix} r = 0 \end{aligned} \tag{22}$$

5.3.2 Orthonormality of rotation rows

For $(\mu R)(\mu R)^T = \mu^2 I.$, we have

$$\begin{aligned} C_{\text{Or-row}} &: e_i^T((\mu R)(\mu R)^T - \mu^2 I_3)e_j \\ &= \text{tr}(e_i^T(\mu R)(\mu R^T)e_j) - \mu^2 e_i^T e_j \\ &= \text{tr}((\mu R)^T e_{ji}(\mu R)) - \mu^2\delta_i j \\ &= \text{vec}((\mu R))^T(I_3\otimes e_{ji})\text{vec}((\mu R)) - \mu^2\delta_i j \\ &= r^T\begin{bmatrix} I_3\otimes e_{ji} & 0 \\ 0 & -\delta_i j\end{bmatrix} r = 0 \end{aligned} \tag{23}$$

5.3.3 Right-hand rule on rotation columns

For $(\mu R)^{(i)}\times(\mu R)^{(j)} = \mu(\mu R)^{(k)}.$, we have

$$\begin{aligned} C_{\text{hand}} &= e_a^T((\mu R)^{(i)}\times(\mu R)^{(j)} - (\mu R)^{(k)}) \\ &= e_a^T((\mu R e_i)\times(\mu R e_j) - (\mu R e_k)) \\ &= -e_i^T(\mu R)^T[e_a]_\times(\mu R)e_j - \mu e_a^T(\mu R)e_k \\ &= -\text{tr}(e_i^T(\mu R)^T[e_a]_\times(\mu R)e_j) - \mu\text{tr}(e_a^T(\mu R)e_k) \\ &= -\text{tr}(e_j e_i^T(\mu R)^T[e_a]_\times(\mu R)) - \mu\text{tr}(e_k e_a^T(\mu R)) \\ &= -\text{tr}(e_{ij}^T(\mu R)^T[e_a]_\times(\mu R)) - \mu\text{tr}((e_a e_k^T)^T(\mu R)) \\ &= -\text{vec}((\mu R))^T(e_{ij}\otimes[e_a]_\times)\text{vec}((\mu R)) - \mu\text{vec}(e_a e_k^T)^T\text{vec}((\mu R)) \\ &= -\text{vec}((\mu R))^T(e_{ij}\otimes[e_a]_\times)\text{vec}((\mu R)) - \mu(e_k\otimes e_a)\text{vec}((\mu R)) \\ &= r^T\begin{bmatrix} e_{ij}\otimes[e_a]_\times^T & (e_k\otimes e_a)^T \\ 0 & 0\end{bmatrix} r = 0 \end{aligned} \tag{24}$$

Then in actual constraint matrix construction, we will integrate matrix $C \in \{C_{\text{Orth-col}}, C_{\text{Or-row}}, C_{\text{hand}}\}$ into constraint matrix Q. Note that above rotation constraints involve both variable $\theta_{XY}h_Y\otimes R_{AY}$ and auxiliary variable $h_Y\otimes R_{AY}$. So take $\mathrm{i_r}$ is the first index of vec$(\theta_{XY}h_Y\otimes R_{AY})$ or vec$(h_Y\otimes R_{AY})$, then we have constraint matrix

$$Q^{\text{Rotation}}_{\text{i,j}} = \begin{cases} C_{\mathrm{i-i_r,j-i_r}} & \mathrm{i-i_r, j-i_r} \in [1,9] \\ C_{10,10} & \mathrm{i = i_y, j = i_y} \\ 0 & \text{others}\end{cases}$$

5.4 Equality Relationship Constraint

As described in our paper, equality relationship constraints include

$$\underline{\theta_{XY}h_Y} = \underline{\theta_{XY}}\;\underline{h_Y}, \tag{25}$$

$$\underline{\text{vec}(\theta_{XY}h_Y R_{AY})} = \underline{\theta_{XY}}\;\underline{\text{vec}(h_Y R_{AY})}. \tag{26}$$

We denote $\mathrm{i_{h_Y}}$ and $\mathrm{i_{\theta_{XY}h_Y}}$ for index of h_Y and $\theta_{XY}h_Y$, and take $\mathrm{i_{hr}}$ as the first index of vec$(h_Y \otimes R_{AY})$. Then we construct constraint matrix for each equality relationship constraint.

1.For $\underline{\theta_{XY}h_Y} = \underline{\theta_{XY}}\ \underline{h_Y} \Longrightarrow \underline{\theta_{XY}h_Y}\ y = \underline{\theta_{XY}}\ \underline{h_Y}$, we have

$$Q^{\text{Equa(1)}}_{\mathrm{i,j}} = \begin{cases} 1 & \mathrm{i = i_{\theta_{XY}}, j = i_{h_Y}} \\ -1 & \mathrm{i = i_y, j = i_{\theta_{XY}h_Y}} \\ 0 & \text{others} \end{cases}$$

2.$\underline{\text{vec}(\theta_{XY}h_Y R_{AY})} = \underline{\theta_{XY}}\ \underline{\text{vec}(h_Y R_{AY})} \Longrightarrow \underline{\text{vec}(\theta_{XY}h_Y R_{AY})}^{(m)} y = \underline{\theta_{XY}}\ \underline{\text{vec}(h_Y R_{AY})}^{(m)}, m \in [1, 9]$.

$$Q^{\text{Equa(2)}}_{\mathrm{i,j}} = \begin{cases} 1 & \mathrm{i = i_{\theta_{XY}}, j = i_{hr} + m} \\ -1 & \mathrm{i = i_y, j = i_r + m} \\ 0 & \text{others} \end{cases}$$

6 Proof of Lemma

In this section, we aim to prove the two lemmas in Sec.IV-C of our paper.

6.1 Proof of Lemma 1

Lemma 6.1. *If $C \in \mathbb{R}^{n\times n}$ be positive semidefinite and $x^T Cx = 0$ for a vector x, then $Cx = 0$.*

Proof. Since C is a positive semidefinite, its eigenvalus are all non-negative. If the rank$(C) = r$, the eigenvalues can be list as $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r > 0 = \lambda_{r+1} = ... = \lambda_n$. Denote the i-th eigenvector as x_i. The eigenvectors are orthogonal to each other. Then we express $x = \sum_{i=1}^n a_i x_i$, and get that $Cx = C\sum_{i=1}^n a_i x_i = \sum_{i=1}^n a_i \lambda_i x_i$ and $x^T Cx = (\sum_{i=1}^n a_i x_i^T)(\sum_{i=1}^n a_i \lambda_i x_i) = \sum_{i=1}^n a_i^2 \lambda_i ||x_i||^2$. So if $x^T Cx = 0$, we have $a_i = 0$ for $i = 1, ..., r$. Thus $Cx = \sum_{i=1}^n a_i \lambda_i x_i = 0$. □

6.2 Proof of Lemma 2

Definition 1. *For QCQP Problem in our paper, the corank-one condition holds if the number of independent measurements n and the number of observed robots N, satisfy that $n \geq 18N + 2$, where independent measurements mean that $\{c^X_{j_1 j_2}\}$ are linearly independent vectors.*

Based in this condition, we have

Lemma 6.2. *Assume that the corank-one condition holds, then the cost matrix C is semidefinite and has corank one in noise-free cases. Furthermore, after Schur Compliment, $\bar{C}$ is also semidefinite and has corank one.*

Proof. Firstly, we prove that the $\mathrm{i_y}$-th column of C can be expressed as linear combination of other columns, which means that C is not full-rank, and at least have corank-one. Next we will prove that given $n >= 18N + 2$ independent measurements, the left columns are linear independent. Thus we obtain that $rank(C) = dim(C) - 1$,which means that C is corank-one.

Step 1: We firstly aim to prove that if $x^* = \{r_1^{*T}, ..., r_X^{*T}, ..., r_N^{*T}, y^*, D_1^{*T}, ..., D_X^{*T}, ..., D_N^{*T}\}$ is given, after defining the following variables

$$x^\circ \doteq \{r_1^{*T}, ..., r_X^{*T}, ..., r_N^{*T}, D_1^{*T}, ..., D_X^{*T}, ..., D_N^{*T}\}$$

$$C^\circ \doteq \begin{pmatrix} C_{1,1} & C_{1,3} \\ C_{2,1} & C_{2,3} \\ C_{3,1} & C_{3,3} \end{pmatrix}, q \doteq \begin{pmatrix} C_{1,2} \\ C_{2,2} \\ C_{3,2} \end{pmatrix}$$

then there will be $C^\circ x^\circ + q = 0$, which means C at least have corank-one.

Recall the construction of $c^X_{j_1 j_2}$ and C

$$c^X_{j_1 j_2} = [\ 0_{3\times 36N}, ..., c^X_r, ..., 0_{3\times 36N}, -\widehat{t}_{A_{j_1}A_{j_2}}, 0_{3\times n}, ..., c^X_D, ..., 0_{3\times n}\ ] \tag{27}$$

$$C = \sum_{\substack{X\in[1,N] \\ \{j_1,j_2\}\in J}} (c_{r,j_1 j_2}, -\widehat{t}_{A_{j_1}A_{j_2}}, c_{D,j_1 j_2})^T w^X_{j_1 j_2} (c_{r,j_1 j_2}, -\widehat{t}_{A_{j_1}A_{j_2}}, c_{D,j_1 j_2}) \tag{28}$$

So there are

$$C_{1,1} = \sum_{\substack{X \in [1,N] \\ \{j_1,j_2\} \in J}} c_{r,j_1 j_2}^T c_{r,j_1 j_2}, \quad C_{1,2} = \sum_{\substack{X \in [1,N] \\ \{j_1,j_2\} \in J}} -c_{r,j_1 j_2}^T \widehat{t}_{A_{j_1} A_{j_2}}, \quad C_{1,3} = \sum_{\substack{X \in [1,N] \\ \{j_1,j_2\} \in J}} c_{r,j_1 j_2}^T c_{D,j_1 j_2} \tag{29}$$

$$C_{2,1} = \sum_{\substack{X \in [1,N] \\ \{j_1,j_2\} \in J}} -\widehat{t}_{A_{j_1} A_{j_2}}^T c_{r,j_1 j_2}, \quad C_{2,2} = \sum_{\substack{X \in [1,N] \\ \{j_1,j_2\} \in J}} \widehat{t}_{A_{j_1} A_{j_2}}^T \widehat{t}_{A_{j_1} A_{j_2}}, \quad C_{2,3} = \sum_{\substack{X \in [1,N] \\ \{j_1,j_2\} \in J}} -\widehat{t}_{A_{j_1} A_{j_2}}^T \widehat{t}_{A_{j_1} A_{j_2}} \tag{30}$$

$$C_{3,1} = \sum_{\substack{X \in [1,N] \\ \{j_1,j_2\} \in J}} c_{D,j_1 j_2}^T c_{r,j_1 j_2}, \quad C_{3,2} = \sum_{\substack{X \in [1,N] \\ \{j_1,j_2\} \in J}} -c_{D,j_1 j_2}^T \widehat{t}_{A_{j_1} A_{j_2}}, \quad C_{3,3} = \sum_{\substack{X \in [1,N] \\ \{j_1,j_2\} \in J}} c_{D,j_1 j_2}^T c_{D,j_1 j_2} \tag{31}$$

$$\tag{32}$$

We start from the equation

$$\begin{aligned}
& [\, c_{r,j_1 j_2}, -\widehat{t}_{A_{j_1} A_{j_2}}, c_{D,j_1 j_2} \,] x^* = 0 \\
\Longrightarrow & -\widehat{t}_{A_{j_1} A_{j_2}} = -[\, c_{r,j_1 j_2}, c_{D,j_1 j_2} \,] x^\circ \\
\Longrightarrow & -[\, c_{r,j_1 j_2}, -\widehat{t}_{A_{j_1} A_{j_2}}, c_{D,j_1 j_2} \,]^T \widehat{t}_{A_{j_1} A_{j_2}} \\
& = [\, c_{r,j_1 j_2}, -\widehat{t}_{A_{j_1} A_{j_2}}, c_{D,j_1 j_2} \,]^T [\, c_{r,j_1 j_2}, c_{D,j_1 j_2} \,](-x^\circ) \\
\Longrightarrow & -\sum_{\{j_1,j_2\} \in J} [\, c_{r,j_1 j_2}, -\widehat{t}_{A_{j_1} A_{j_2}}, c_{D,j_1 j_2} \,]^T \widehat{t}_{A_{j_1} A_{j_2}} \\
& = (\sum_{\{j_1,j_2\} \in J} [\, c_{r,j_1 j_2}, -\widehat{t}_{A_{j_1} A_{j_2}}, c_{D,j_1 j_2} \,]^T [\, c_{r,j_1 j_2}, c_{D,j_1 j_2} \,])(-x^\circ) \\
\Longrightarrow & \begin{pmatrix} C_{1,2} \\ C_{2,2} \\ C_{3,3} \end{pmatrix} = \begin{pmatrix} C_{1,1} & C_{1,3} \\ C_{2,1} & C_{2,3} \\ C_{3,1} & C_{3,3} \end{pmatrix} (-x^\circ) \\
\Longrightarrow & q = -C^\circ x^\circ
\end{aligned} \tag{33}$$

which means that C has at least corank-one.

Step2: We then aim to find that under what condition, $C_{!y} \doteq \begin{pmatrix} C_{1,1} & C_{1,3} \\ C_{3,1} & C_{3,3} \end{pmatrix}$ is full-rank. We observe that

$$C_{!y} = \sum_{\substack{X \in [1,N] \\ \{j_1,j_2\} \in J}} \sum_{i=1}^{3} [\, c_{r,j_1 j_2}, c_{D,j_1 j_2} \,]_i^T [\, c_{r,j_1 j_2}, c_{D,j_1 j_2} \,]_i$$

where $[\, c_{r,j_1 j_2}, c_{D,j_1 j_2} \,]_i$ denotes ith row of $[\, c_{r,j_1 j_2}, c_{D,j_1 j_2} \,]$. For simplicy, we can wirte $C_{!y} = \sum_{i=1}^{N*(n-1)*3} u_i u_i^T$, where $u_i \in R^{36N^2+N*n}$. So when $3*N*(n-1) >= 36N^2 + N*n$, which means $n >= 18N + 3/2, n >= 18N + 2$, and each u_i is linear independent, $C_{!y}$ is full rank. Finally, since C is symmetric, we can easily obtain that rank($C_{!y}$) = rank(C°), which means C° is also full rank.

Until now, we obtain that C has corank-one. Then we prove that $C_{3,3}$ is full rank. Same with above C full rank proof. $C_{3,3} = \sum_{\substack{X \in [1,N] \\ \{j_1,j_2\} \in J}} c_{D,j_1 j_2}^T c_{D,j_1 j_2} = \sum_{i=1}^{N*(n-1)*3} v_i v_i^T$, where $v_i \in R^{Nn}$, so if $n >= 18N + 2$, $C_{3,3}$ is full rank too. Then, after Schur Complement, $\widehat{C} = C/C_{3,3}$, rank$(\widehat{C})$ + rank$(C_{3,3})$ = rank(C). According to above proof, we known that rank$(C_{3,3})$ = dim$(C_{3,3})$, rank(C) = dim$(C) - 1$. Furthermore, based on the truth that dim$(C_{3,3})$ + dim$(\widehat{C})$ = dim(C), we can obtain that rank$(\widehat{C})$ = dim$(\widehat{C}) - 1$, which means $\widehat{C}$ has corank-one. □

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