数学分析（甲）II（H）2021 - 2022 春夏期末答案

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一、一致收敛定义

对于函数列 $\{f_n(x)\}, \forall \varepsilon > 0, \exists N \in \mathbb{N},$ 其取值与 x 无关, 当 $n > N$ 时, $\forall x \in D,$ 有 $|f_n(x) - f(x)| < \varepsilon,$ 则称 $\{f_n(x)\}$ 在 D 上一致收敛于 $f(x),$ 记作 $f_n(x) \xrightarrow{D} f(x).$

$f_n(x) = \dfrac{\sin nx}{n^2}, f(x) = 0, \forall \varepsilon > 0, \exists N = \left[\varepsilon^{-\frac{1}{2}}\right],$ 当 $n > N$ 时, $\forall x \in \mathbb{R},$

$$|f_n(x) - f(x)| = |\frac{\sin nx}{n^2} - 0| \leqslant \frac{1}{n^2} < \frac{1}{N^2} = \varepsilon.$$

故 $\{\dfrac{\sin nx}{n^2}\}$ 在 $\mathbb{R}$ 上一致收敛于 $0.$

二、可偏导与可微

$f(x,y) = \begin{cases} \dfrac{\sin(xy)}{\sqrt{x^2 + y^2}}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}$, 则

$$\lim_{\substack{x \to 0 \\ y \to 0}} f(x,y) = \lim_{\rho \to 0} \frac{\rho^2 |\sin\theta\cos\theta|}{\rho} = \lim_{\rho \to 0} \rho|\sin\theta\cos\theta| = 0 = f(0,0),$$

所以 $f(x,y)$ 在 $(0,0)$ 处连续. 而

$$\lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x} = 0, \quad \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y} = 0,$$

故 $f(x,y)$ 在 $(0,0)$ 处可偏导. 但是

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{f(x,y) - (f'(0,0)x + f'(0,0)y)}{\sqrt{x^2 + y^2}} = \lim_{\rho \to 0} \frac{\rho^2 |\sin\theta\cos\theta|}{\rho^2} = |\sin\theta\cos\theta|,$$

取值与 θ 有关, 故 $f(x,y)$ 在 $(0,0)$ 处不可微.

三、隐函数存在定理与隐函数求导

Warning: 图灵回忆卷此处回忆有误，应为 $e^{x+y+1} + x^2 y = e$ 在 $0,0$ 的某邻域内唯一确定 y 关于 x 的函数

记 $F(x,y) = e^{x+y+1} + x^2 y - e, F(0,0) = 0.$ $\exists \delta > 0,$ 使得 $F(x,y)$ 在 $U(O,\delta)$ 内连续, 且 $F'_y(x,y) = e^{x+y+1} + x^2$ 在上述 $U(O,\delta)$ 内连续, 并成立 $F'_y(x,y) > 0,$ 故由隐函数存在定理, $F(x,y)$ 在 $U(O,\delta)$ 内可唯一确定 y 关于 x 的函数 $y = f(x),$ 且 $F(x,f(x)) = 0.$

等式 $e^{x+y+1} + x^2y - e = 0$ 两边同时对 x 求导可得

$$e^{x+y+1}(1 + \frac{\mathrm{d}y}{\mathrm{d}x}) + 2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

$$(e^{x+y+1} + x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = -e^{x+y+1} - 2xy,$$

故 $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{(0,0)} = -1$. 继续对 x 求导可得

$$(e^{x+y+1} + x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (e^{x+y+1}(1 + \frac{\mathrm{d}y}{\mathrm{d}x}) + 2x)\frac{\mathrm{d}y}{\mathrm{d}x} = -(e^{x+y+1}(1 + \frac{\mathrm{d}y}{\mathrm{d}x}) + 2(y + x\frac{\mathrm{d}y}{\mathrm{d}x})),$$

代入 $(0,0)$ 可得 $\left.\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{(0,0)} = 0$.

四、多元函数积分计算

1.

$$I = \int_0^{2\pi} \mathrm{d}\theta \int_0^\pi \mathrm{d}\varphi \int_0^R \rho^2\cos^2\varphi \cdot \rho \cdot \rho^2\sin\varphi\mathrm{d}\rho$$

$$= 2\pi \int_0^\pi \sin\varphi\cos^2\varphi\mathrm{d}\varphi \int_0^R \rho^5\mathrm{d}\rho$$

$$= -2\pi \times \frac{1}{3}\cos^3\varphi\Big|_0^\pi \times \frac{1}{6}\rho^6\Big|_0^R = \frac{2}{9}\pi R^6.$$

2. 设 Σ 为曲线在平面 $x - y + z = 2$ 上围成的部分，取上侧. 则

$$I = \iint_\Sigma \begin{vmatrix} \mathrm{d}x\mathrm{d}y & \mathrm{d}y\mathrm{d}z & \mathrm{d}z\mathrm{d}x \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z - y & x - z & x - y \end{vmatrix}$$

$$= \iint_\Sigma (-1 + 1)\mathrm{d}x\mathrm{d}y + (1 - 1)\mathrm{d}y\mathrm{d}z + (1 + 1)\mathrm{d}z\mathrm{d}x$$

$$= 2\iint_\Sigma \mathrm{d}z\mathrm{d}x = 2\iint_\Sigma \mathrm{d}S\cos\beta = 2\iint_\Sigma \frac{\mathrm{d}x\mathrm{d}y}{\cos\gamma}\cos\beta = -2\iint_\Sigma \mathrm{d}x\mathrm{d}y = -2\pi$$

其中 $\vec{n_0} = (\cos\alpha, \cos\beta, \cos\gamma) = (\dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}})$ 为平面的单位法向量.

3. 添加 L: $y = 0$, x 从 π 到 0. $I = \int_{C+L} - \int_L = -\iint_D + \int_{L^-}$.
$\int_{L^-} = e^x(1 - \cos y)\mathrm{d}x - e^x(1 - \sin y)\mathrm{d}y = 0$. 故

$$I = -\iint_D \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ P & Q \end{vmatrix} = \iint_D (e^x(1 - \sin y) + e^x\sin y)\mathrm{d}x\mathrm{d}y$$

$$= \iint_D e^x\mathrm{d}x\mathrm{d}y = \int_0^\pi \mathrm{d}x \int_0^{\sin x} e^x\mathrm{d}y = \int_0^\pi e^x\sin x\mathrm{d}x.$$

进而运用分部积分

$$I = e^x\sin x\Big|_0^\pi - \int_0^\pi e^x\cos x\mathrm{d}x = -e^x\cos x\Big|_0^\pi - \int_0^\pi e^x\sin x\mathrm{d}x = (e^\pi + 1) - I,$$

$$I = \frac{e^\pi + 1}{2}.$$

4. 添加平面 $D = \begin{cases} x^2 + y^2 \leqslant 1, \\ z = 0 \end{cases}$ ，取下侧. $I = \iint_{\Sigma+D} - \iint_D = \iiint_\Omega + \iint_{D^-}$.

$$\iint_{D^-} = \iint_{D^-} \mathrm{d}x\mathrm{d}y = \pi$$

$$\iiint_\Omega = \iiint_\Omega 2y + 2z + (1 - 2y - 2z)\mathrm{d}V = \iiint_\Omega \mathrm{d}V = \frac{2}{3}\pi.$$

$$I = \frac{5}{3}\pi.$$

五、有条件极值

讨论点在内部还是在边缘. $D:\ x^2 + y^2 \leqslant 5$, $f(x,y) = xy + x - y$.

1. $(x,y) \in int\ D$. 令

$$\begin{cases} f'_x(x,y) = y + 1 = 0, \\ f'_y(x,y) = x - 1 = 0, \end{cases}$$

解得 $(x,y) = (1,-1)$, 设为点 P. 因为 $x_p^2 + y_p^2 = 2 < 5$, 故 $P \in int\ D$. 但 $A = f''_{xx}(P) = 0, C = f''_{yy}(P) = 0, B = f''_{xy}(P) = 1$, 有 $B^2 - AC = 1 > 0$, 所以 $f(P)$ 不是极值.

2. $(x,y) \in \partial D$. 利用拉格朗日乘数法, 设 $L(x,y,\lambda) = xy + x - y - \lambda(x^2 + y^2 - 5)$, 令

$$\begin{cases} L'_x(x,y,\lambda) = y + 1 - 2\lambda x = 0, \\ L'_y(x,y,\lambda) = x - 1 - 2\lambda y = 0, \\ L'_\lambda(x,y,\lambda) = x^2 + y^2 - 5 = 0, \end{cases}$$

联立 $\begin{cases} y + 1 - 2\lambda x = 0, \\ x - 1 - 2\lambda y = 0, \end{cases}$ 可得 $(1 - 2\lambda)(x + y) = 0$.

- 当 $\lambda = \dfrac{1}{2}$ 时, 解得 $(x,y) = (2,1)$ 或 $(x,y) = (-1,-2)$, 分别设为 Q_1, Q_2. 代入得 $f(Q_1) = f(Q_2) = 3$.

- 当 $x + y = 0$ 时, 代入解得 $(x,y) = (-\sqrt{5}, \sqrt{5})$ 或 $(x,y) = (\sqrt{5}, -\sqrt{5})$, 分别设为 Q_3, Q_4. 代入得 $f(Q_3) = f(Q_4) = -5 - 2\sqrt{5}$.

故 $f(x,y)$ 在 D 上的最大值为 3, 最小值为 $-5 - 2\sqrt{5}$.

（其实也可以三角函数代换来解，说明起来更充分些）

六、函数项级数的基本计算

设 $u = \dfrac{1}{3}x$, 则 $I = \sum_{n=0}^{+\infty} \dfrac{u^n}{n+1}$. $u = -1$ 时, I 收敛；$u = 1$ 时, I 发散. 故 I 的收敛域为 $[-3, 3)$, $r = 3$.

$$uI = \sum_{n=0}^{+\infty} \frac{u^{n+1}}{n+1}$$

$$(uI)' = \sum_{n=0}^{+\infty} u^n = \frac{1}{1-u}, \quad u \in [-1, 1)$$

$$uI = -\ln(1-u), \quad u \in [-1, 1)$$

$$I = -\frac{\ln(1-u)}{u} = -\frac{3\ln(1 - \frac{1}{3}x)}{x}, \quad x \in [-3, 3).$$

七、 Fourier 级数

进行周期延拓，其为偶函数，故 $b_n = 0$.

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}x(2\pi - x)\cos nx\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi}(\frac{\pi}{2}x\cos nx - \frac{1}{4}x^2\cos nx)\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{2\pi}x\cos nx\mathrm{d}x - \frac{1}{4\pi}\int_0^{2\pi}x^2\cos nx\mathrm{d}x = \frac{1}{2n}\int_0^{2\pi}x\mathrm{d}(\sin nx) - \frac{1}{4n\pi}\int_0^{2\pi}x^2\mathrm{d}(\sin nx)$$

$$= \frac{1}{2n}x\sin nx\Big|_0^{2\pi} - \frac{1}{2n}\int_0^{2\pi}\sin nx\mathrm{d}x - \frac{1}{4n\pi}x^2\sin nx\Big|_0^{2\pi} + \frac{1}{2n\pi}\int_0^{2\pi}x\sin nx\mathrm{d}x$$

$$= \frac{1}{2n\pi}\int_0^{2\pi}x\sin nx\mathrm{d}x = -\frac{1}{2n^2\pi}x\cos nx\Big|_0^{2\pi} + \frac{1}{2n^2\pi}\int_0^{2\pi}\cos nx\mathrm{d}x = -\frac{1}{n^2}, n\geqslant 1.$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}x(2\pi - x)\mathrm{d}x = \frac{1}{4\pi}(\pi x^2 - \frac{1}{3}x^3)\Big|_0^{2\pi} = \frac{1}{4\pi}\times\frac{4\pi^3}{3} = \frac{\pi^2}{3}.$$

所以

$$f(x) = \frac{\pi^2}{6} - \sum_{n=1}^{+\infty}\frac{\cos nx}{n^2}.$$

$$\frac{1}{4}x(2\pi - x) = \frac{\pi^2}{6} - \sum_{n=1}^{+\infty}\frac{\cos nx}{n^2}.$$

当 $x = 0$ 时，$\sum_{n=1}^{+\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

八、复杂函数列一致收敛的证明

$$F(x) = \lim_{n\to+\infty}f_n(x) = \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}f(x+\frac{k}{n}) = \int_0^1 f(x+t)\mathrm{d}t = \sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(x+t)\mathrm{d}t.$$

$$f_n(x) = \frac{1}{n}\sum_{k=0}^{n-1}f(x+\frac{k}{n}) = \sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(x+\frac{k}{n})\mathrm{d}t.$$

$$|f_n(x) - F(x)| = \Big|\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}}(f(x+\frac{k}{n}) - f(x+t))\mathrm{d}t\Big|.$$

$f(x)$ 在 $\mathbf{R}$ 上连续，则 $\forall[\alpha,\beta]\subset\mathbf{R}$, $f(x)$ 在其上一致连续. 即 $\forall\varepsilon > 0, \exists\delta > 0, x',x''\in[\alpha,\beta]$ 时，若 $|x' - x''| < \delta$, 则 $|f(x') - f(x'')| < \varepsilon$.

$\forall\varepsilon > 0, \exists N = \left[\frac{1}{\delta}\right] + 1$, 当 $n > N$ 时，$|(x+\frac{k}{n}) - (x+t)| \leqslant \frac{1}{n} < \delta$, 故

$$|f_n(x) - F(x)| \leqslant \sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}}|(f(x+\frac{k}{n}) - f(x+t))|\mathrm{d}t = \sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\varepsilon\mathrm{d}t = \varepsilon.$$

即有 $\{f_n(x)\}$ 在 $\mathbf{R}$ 上内闭一致收敛.