

数学分析（甲）II（H）2020 - 2021 春夏期末答案

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一、多元函数可微性

1. $z = f(x, y)$ 在 (x_0, y_0) 的某邻域内有定义, 若存在常数 A, B 对充分小的 $\Delta x, \Delta y$ 均有

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = A\Delta x + B\Delta y + o(\rho), \quad \rho \rightarrow 0.$$

其中 $\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$, 则称函数 $z = f(x, y)$ 在点 (x_0, y_0) 处可微.

2. $f(x, y) = (xy)^{\frac{5}{7}}$, $f'_x(0, 0) = 0$, $f'_y(0, 0) = 0$. 并且有

$$\begin{aligned} & \frac{|f(x, y) - f(0, 0) - f'_x(0, 0)x - f'_y(0, 0)y|}{\sqrt{x^2 + y^2}} \\ &= \frac{|(xy)^{\frac{5}{7}}|}{\sqrt{x^2 + y^2}} \leq \frac{|(xy)^{\frac{5}{7}}|}{\sqrt{2}|xy|^{\frac{1}{2}}} = \frac{1}{\sqrt{2}}|xy|^{\frac{3}{14}} \rightarrow 0, \quad (x, y) \rightarrow (0, 0). \end{aligned}$$

故 $f(x, y)$ 在 $(0, 0)$ 处可微.

二、反常积分与级数的敛散性

1. $\frac{1}{x}$ 在 $[1, +\infty)$ 上单调递减, 且 $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$. 而 $|\int_1^u \sin x dx| = |\cos u - \cos 1| \leq 2$, 由 Dirichlet 判别法知, 积分 $\int_1^{+\infty} \frac{\sin x}{x} dx$ 收敛.

而

$$\frac{|\sin x|}{x} \geq \frac{\sin^2 x}{x} = \frac{1 - \cos 2x}{2x}$$

且 $\int_1^{+\infty} \frac{dx}{2x}$ 发散, $\int_1^{+\infty} \frac{\cos 2x}{2x} dx$ 收敛, 由比较判别法知 $\int_1^{+\infty} \frac{|\sin x|}{x} dx$ 发散.

2. 否. 比如 $u_n = \frac{(-1)^n}{\sqrt{n}}, v_n = \frac{(-1)^n}{\sqrt{n} + (-1)^n}$, 有 $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{\sqrt{n} + (-1)^n}{\sqrt{n}} = 1$.

但是 $\sum_{n=1}^{+\infty} u_n$ 收敛 (Leibniz 判别法), 而 $\sum_{n=1}^{+\infty} v_n$ 中,

$$v_n = \frac{(-1)^n}{\sqrt{n} + (-1)^n} = \frac{(-1)^n(\sqrt{n} - (-1)^n)}{n - 1} = \frac{(-1)^n \sqrt{n}}{n - 1} - \frac{1}{n - 1},$$

其中 $\sum_{n=1}^{+\infty} \frac{(-1)^n \sqrt{n}}{n - 1}$ 收敛, $\sum_{n=1}^{+\infty} \frac{1}{n - 1}$ 发散, 故 $\sum_{n=1}^{+\infty} v_n$ 发散.

三、多元函数积分计算

1.

$$\begin{aligned}
 I &= \int_0^1 e^{2x} \ln(1 + e^{2x}) dx = \frac{1}{2} \int_2^{1+e^2} \ln t dt \\
 &= \frac{1}{2} t(\ln t - 1) \Big|_2^{1+e^2} = \frac{1}{2} [(1+e^2)(\ln(1+e^2) - 1) - 2(\ln 2 - 1)].
 \end{aligned}$$

2.

$$\begin{aligned}
 I &= \int_{-c}^0 z dz \iint_{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 - \frac{z^2}{c^2}} dx dy = \int_{-c}^0 \pi ab (1 - \frac{z^2}{c^2}) z dz = \pi ab \int_{-c}^0 (z - \frac{z^3}{c^2}) dz \\
 &= \pi ab (\frac{z^2}{2} - \frac{z^4}{4c^2}) \Big|_{-c}^0 = -\frac{\pi}{4} abc^2.
 \end{aligned}$$

3. $C: x^2 + 4y^2 = \delta^2$, 顺时针. $I = \int_{L+C} + \int_{C-}$. 设 C 所围区域为 D .

$$\begin{aligned}
 \frac{\partial Q}{\partial x} &= \frac{x^2 + 4y^2 - 2x(x+4y)}{(x+4y^2)^2} = \frac{-x^2 - 8xy + 4y^2}{(x+4y^2)^2}, \\
 \frac{\partial P}{\partial y} &= \frac{-(x^2 + 4y^2) - 8y(x-y)}{(x+4y^2)^2} = \frac{-x^2 - 8xy + 4y^2}{(x+4y^2)^2}.
 \end{aligned}$$

故 $\int_{L+C} = 0$, 而

$$\int_{C-} = \frac{1}{\delta^2} \oint (x-y)dx + (x+4y)dy = \frac{2}{\delta^2} \iint_D dx dy = \frac{2}{\delta^2} \times \frac{\pi \delta^2}{2} = \pi.$$

所以 $I = \pi$.4. $I = \iint_S (x \cos \alpha + y \cos \beta + z \cos \gamma) dS$, 法向量为 $\vec{n} = (x, y, z)$, 则

$$\begin{aligned}
 \cos \alpha &= \frac{x}{\sqrt{x^2 + y^2 + z^2}} = x, \\
 \cos \beta &= \frac{y}{\sqrt{x^2 + y^2 + z^2}} = y, \\
 \cos \gamma &= \frac{z}{\sqrt{x^2 + y^2 + z^2}} = z.
 \end{aligned}$$

故

$$I = \iint_S (x^2 + y^2 + z^2) dS = \iint_S dS = \frac{1}{8} \times 4\pi 1^2 = \frac{\pi}{2}.$$

四、条件极值计算

目标函数为 $f(x, y, z) = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$, 约束条件为 $(x, y, z) \in Ax + By + Cz + D = 0$.
由拉格朗日乘数法, 设拉格朗日函数为

$$L(x, y, z, \lambda) = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 - \lambda(Ax + By + Cz + D).$$

则有

$$\begin{cases} L'_x = 2(x-x_0) - A\lambda = 0 \Rightarrow x = x_0 + \frac{A}{2}\lambda, \\ L'_y = 2(y-y_0) - B\lambda = 0 \Rightarrow y = y_0 + \frac{B}{2}\lambda, \\ L'_z = 2(z-z_0) - C\lambda = 0 \Rightarrow z = z_0 + \frac{C}{2}\lambda, \\ L'_\lambda = Ax + By + Cz + D = 0. \end{cases}$$

代入得

$$\frac{1}{2}(A^2 + B^2 + C^2)\lambda = -(Ax_0 + By_0 + Cz_0 + D),$$

$$\lambda = -\frac{2(Ax_0 + By_0 + Cz_0 + D)}{A^2 + B^2 + C^2}.$$

故

$$f_{min} = (Ax_0 + By_0 + Cz_0 + D)^2 \cdot \frac{A^2 + B^2 + C^2}{(A^2 + B^2 + C^2)^2} = \frac{(Ax_0 + By_0 + Cz_0 + D)^2}{A^2 + B^2 + C^2}.$$

$$d_{min} = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

五、一致收敛

1. $\forall \varepsilon > 0, \exists N \in \mathbb{N}^*$, 当 $n > N$ 时, $\forall x \in I, |f_n(x) - f(x)| < \varepsilon$, 则称函数列 $\{f_n(x)\}$ 在区间 I 上一致收敛于 $f(x)$.

2. $\forall [\alpha, \beta] \subset (a, b)$, $f(x)$ 有一阶连续导函数, 故 $f'(x)$ 在 $[\alpha, \beta]$ 上一致连续.

$\forall x \in (a, b), f_n(x) = n(f(x + \frac{1}{n}) - f(x))$, 由拉格朗日中值定理, 存在 $\theta_n \in (0, 1)$, 使得

$$f_n(x) = n \cdot \frac{1}{n} \cdot f'(x + \frac{\theta_n}{n}) = f'(x + \frac{\theta_n}{n}).$$

因为 $f'(x)$ 在 $[\alpha, \beta]$ 上一致连续, 故 $\forall \varepsilon > 0, \exists \delta > 0, \forall x', x'' \in [\alpha, \beta]$, 当 $|x' - x''| < \delta$ 时, $|f'(x') - f'(x'')| < \varepsilon$. 而 $\forall \varepsilon > 0, \exists N \in \mathbb{N}^*$, 当 $n > N$ 时, $\frac{\theta_n}{n} < \frac{1}{n} < \frac{1}{N}$.

故取 $\delta = \frac{1}{N}$, $x' = x + \frac{\theta_n}{n}$, $x'' = x$, 有 $|x' - x''| < \delta$, 所以 $|f'(x + \frac{\theta_n}{n}) - f'(x)| < \varepsilon$.
所以 $\{f_n(x)\}$ 在 (a, b) 上内闭一致收敛于 f' .

六、Fourier 级数

1. 奇延拓后, $a_n = 0, n = 0, 1, 2, \dots$,

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{2}{\pi} \int_0^{\pi} (\pi - x) \sin nx dx \\ &= 2 \int_0^{\pi} \sin nx dx - \frac{2}{\pi} \int_0^{\pi} x \sin nx dx \\ &= -\frac{2}{n} \cos nx \Big|_0^{\pi} + \frac{2}{n\pi} x \cos nx \Big|_0^{\pi} - \frac{2}{n^2\pi} \int_0^{\pi} \cos nx dx = \frac{2}{n} \end{aligned}$$

故 $f(x)$ 的 Fourier 级数为 $\sum_{n=1}^{+\infty} \frac{2}{n} \sin nx$. 其在 $[-\pi, \pi]$ 上的取值为

$$\begin{cases} \pi - x, & 0 < x \leq \pi, \\ 0, & x = 0, \\ -\pi - x, & -\pi \leq x < 0. \end{cases}$$

2. 利用 Cauchy 收敛准则. $|b_n + \dots + b_{n+p}| = 2 \left| \frac{\sin nx}{n} + \dots + \frac{\sin(n+p)x}{n+p} \right|$.

取 $x = x_0 = \frac{\pi}{4n}$, $p = n$, $\varepsilon_0 = \frac{1}{\sqrt{2}}$, 有

$$|b_n + \dots + b_{n+p}| > 2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{n}{2n} = \varepsilon_0.$$

所以 f 的 Fourier 级数在 $(0, \pi)$ 上不一致收敛.

七、多元函数 Taylor 定理

1. $f(x, y) = f(x_0, y_0) + (\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})f(x_0, y_0) + \frac{1}{2!}(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})^2 f(x_0, y_0) + o(\rho^2)$, 其中 $\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$.

因为 $P_0(x_0, y_0)$ 是稳定点, 所以 $(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})f(x_0, y_0) = 0$. 又因为 Hesse 矩阵正定, 即有 $Q(\Delta x, \Delta y) = (\Delta x, \Delta y)H(P_0)(\Delta x, \Delta y)^T > 0$, 其中

$$H(P_0) = \begin{pmatrix} \frac{\partial^2}{\partial x^2} f & \frac{\partial^2}{\partial x \partial y} f \\ \frac{\partial^2}{\partial y \partial x} f & \frac{\partial^2}{\partial y^2} f \end{pmatrix}_{P_0}.$$

进而存在一不依赖于 $\Delta x, \Delta y$ 的常数 $q > 0$, 使得 $Q(\Delta x, \Delta y) \geq q((\Delta x)^2 + (\Delta y)^2)$.

所以 $\exists \delta > 0$, 当 $(x, y) \in U(P_0, \delta)$ 时, 有

$$f(x, y) - f(x_0, y_0) \geq ((\Delta x)^2 + (\Delta y)^2)(q + o(1)) > 0.$$

故 $f(x, y)$ 在 P_0 处取极小值.

2. 若 f 存在两个或以上的稳定点, 不妨取其中两个 $P_1(x_1, y_1), P_2(x_2, y_2)$, 由多元函数 Taylor 定理的 Lagrange 余项形式, 有

$$f(x, y) - f(x_1, y_1) = \frac{1}{2}(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})^2 f(x_1 + \theta_1 \Delta x, y_1 + \theta_1 \Delta y),$$

$$f(x, y) - f(x_2, y_2) = \frac{1}{2}(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})^2 f(x_2 + \theta_2 \Delta x, y_2 + \theta_2 \Delta y).$$

而因为 f 在每个点的 Hesse 矩阵都是正定的, 故 $\frac{1}{2}(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})^2 f(x_1 + \theta_1 \Delta x, y_1 + \theta_1 \Delta y) > 0$,

$\frac{1}{2}(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})^2 f(x_2 + \theta_2 \Delta x, y_2 + \theta_2 \Delta y) > 0$.

所以就会得出 $f(x_2, y_2) > f(x_1, y_1)$ 且 $f(x_1, y_1) > f(x_2, y_2)$ 的矛盾. 所以 f 至多有一个稳定点.