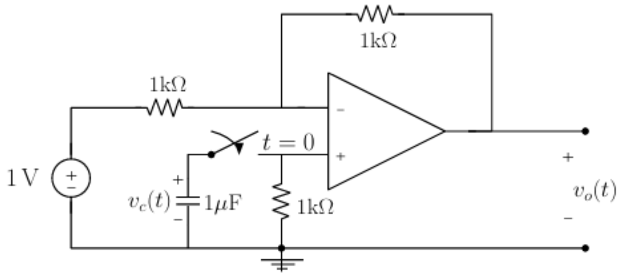
Homework 4

Due: Wednesday March 19, 2025, 11:59 pm

1. Assuming linear operation and making use of the ideal op-amp approximations, determine $v_o(t)$ at $t = 1\,\text{ms}$ in the following circuit. Consider $v_c(0) = -1\,\text{V}$.



Solution:

Assuming linear operation there is no current flowing into the op-amp through either terminals (+) nor (-). So we can isolate the RC circuit with time constant $\tau = RC = (1\text{k}\Omega)(1\mu\text{F}) = 1\text{ms}$, which has the general solution:

$$v_c(t) = K_1 e^{-t/\tau} + K_2.$$

When $t \to \infty$ all the charge in the capacitor should be gone, therefore

$$v_c(t \to \infty) = 0 = K_2$$

Applying initial condition yields

$$v_c(0^+) = v_c(0^-) = -1\,\text{V} = K_1 + K_2 \Rightarrow K_1 = -1\,\text{V}.$$

Therefore $v_c(t)$ is

$$v_C(t) = -e^{-\frac{t}{1\text{ms}}}\,\text{V},$$

for $t > 0$. Writing a KCL equation in the (-) terminal of the op-amp where$v_- \approx v_+ = v_c(t)$, we have

$$\frac{1\text{V} - v_C(t)}{1\text{k}\Omega} = \frac{v_C(t) - v_0(t)}{1\text{k}\Omega},$$

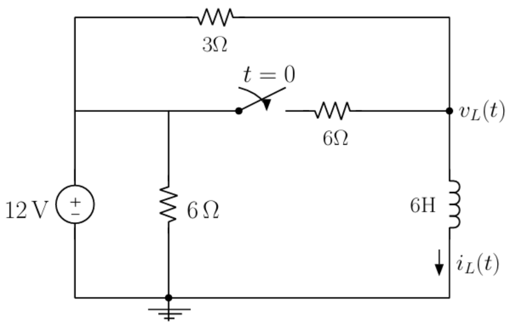
which simplifies to

$$\begin{aligned} v_0(t) &= 2v_C(t) - 1\,\text{V} \\ &= -2e^{-\frac{t}{1\text{ms}}} - 1\,\text{V}. \end{aligned}$$

Evaluating at $t = 1\text{ms}$,

$$\begin{aligned} v_0(t = 1\text{ms}) &= -2e^{-1} - 1\ \text{V} \\ &\approx -2 * 0.37 - 1 = -1.74\ \text{V}. \end{aligned}$$

2. The circuit shown below is in DC steady-state before the switch flips at $t = 0$. Find $v_L(0^-)$ and $i_L(0^-)$, as well as $i_L(t)$ and $v_L(t)$, for $t > 0$.

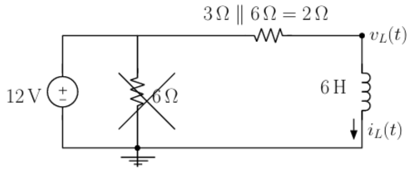


Solution:

In the DC steady-state, before closing the switch, the inductor acts like a short, then $v_L(0^-) = 0\text{V}$. Therefore by Ohm's Law we have the initial condition

$$i_L(0^-) = \frac{12\,\text{V}}{3\,\Omega} = 4\,\text{A}.$$

After closing the switch we have the following circuit



In this case we neglected the $6\,\Omega$ resistance, since it is in parallel with a voltage source. The general solution for the current in the inductor in an RL circuit with time constant $\tau = L/R = (6\,\text{H})/(2\,\Omega) = 3\,\text{s}$ is

$$i_L(t) = K_1 e^{-t/\tau} + K_2.$$

When $t \to \infty$ the inductor acts like a short, therefore, by Ohm's Law

$$i_L(t \to \infty) = \frac{12\,\text{V}}{2\,\Omega} = 6\,\text{A} = K_2$$

Applying the initial condition yields

$$i_L(0^+) = i_L(0^-) = 4\,\text{A} = K_1 + K_2 \Rightarrow K_1 = -2\,\text{A}.$$

Therefor $i_L(t)$ is

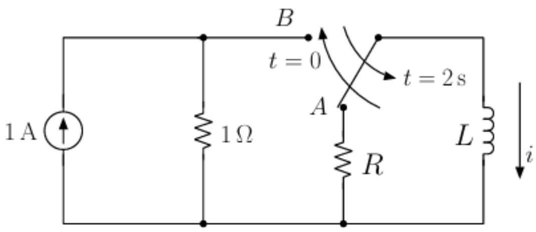
$$i_L(t) = 6 - 2e^{-\frac{t}{3\,\text{s}}}\,\text{A},$$

for $t > 0$.

Using the $v - i$ relation for the inductor $v_L = L\frac{d}{dt}i(t)$, we obtain

$$\begin{aligned} v_L(t) &= 6\,\text{H} \times \frac{d}{dt}(6 - 2e^{-\frac{t}{3}})\text{A} \\ &= 6\,\text{V} \times (\frac{2}{3}e^{-\frac{t}{3}}) \\ &= 4\,e^{-\frac{t}{3}}\,\text{V}. \end{aligned}$$

3. Assume the switch has been in position A for a long time. It moves to position B at time $t = 0\,\text{s}$, and back to position A at time $t = 2\,\text{s}$. Find values L and R such that $i(t) = 1 - e^{-1}\,\text{A}$ at $t = 2\,\text{s}$, and $i(t) = \left(1 - e^{-1}\right)e^{-1}\,\text{A}$ at $t = 6\,\text{s}$.



Solution:

Before $t = 0$, the circuit is in steady-state and without any sources there is no current flowing through the inductor, i.e.

$$i(0^-) = i(0^+) = 0\,\text{A}.$$

After the switch moves to position B we have an RL circuit with time constant $\tau = L/(1\,\Omega)$ and general equation for the current through the inductor

$$i(t) = K_1 e^{-t/\tau} + K_2.$$

If the switch would stay in position B for a long time, the inductor would become a short and the current flowing through it would be 1 A, which means

$$K_2 = 1\,\mathrm{A}.$$

Applying the initial condition we obtain K_1:

$$i(0^-) = i(0^+) = 0\,\mathrm{A} = K_1 + K_2 \Rightarrow K_1 = -1\,\mathrm{A}.$$

Therefore the current through the inductor in the interval $0 < t < 2\,\mathrm{s}$ is

$$i(t) = 1 - e^{-t/L}.$$

When the switch moves back to position A at $t = 2\,\mathrm{s}$, the current through the inductor remains

$$i(2^+) = i(2^-) = 1 - e^{-2/L}\,\mathrm{A}.$$

Now the general equation for the current has a new time constant: $\tau = L/R$ and starting at $t = 2\,\mathrm{s}$ is

$$i(t) = K_3 e^{-(t-2)/\tau} + K_4.$$

With no source in the new circuit we have $i(t \to \infty) = 0$ and consequently $K_4 = 0\,\mathrm{A}$. Applying the initial condition we can obtain K_3:

$$i(2^+) = i(2^-) = 1 - e^{-2/L}\,\mathrm{A} = K_3 e^{-(2-2)/\tau} \Rightarrow K_3 = 1 - e^{-2/L}\,\mathrm{A}.$$

Hence the equation for the current through the inductor for $t > 2\,\mathrm{s}$ is

$$i(t) = \left(1 - e^{-2/L}\right) e^{-\frac{(t-2)}{L/R}}.$$

The condition of the problem says that $i(t) = 1 - e^{-1}\,\mathrm{A}$ at $t = 2\,\mathrm{s}$. Therefore, the inductor should be

$$L = 2\,\mathrm{H}.$$

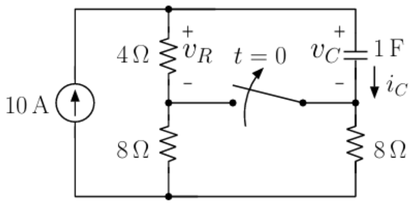
The second conditions tells us $i(t) = \left(1 - e^{-1}\right) e^{-1}\,\mathrm{A}$ at $t = 6\,\mathrm{s}$, which means

$$-\left.\frac{(t-2)}{L/R}\right|_{t=6\,\mathrm{s}} = -1.$$

For this to be true, the resistor R should be

$$R = \frac{L}{4} = \frac{2}{4} = 0.5\,\Omega$$

4. Assume that the switch has been closed for a long time and it opens at $t = 0\,\mathrm{s}$. Find and sketch $v_R(t)$, $v_C(t)$, and $i_C(t)$.



Solution:

In the steady-state, before the switch opens, i.e. $t < 0$, there is no current flowing through the fully charged capacitor

$$i_C(0^-) = 0.$$

And the voltage right before $t = 0$ can be obtained from Ohm's Law

$$v_R(0^-) = v_C(0^-) = v_C(0^+) = (10\,\mathrm{A})\,(4\,\Omega) = 40\,\mathrm{V}.$$

After the switch opens, we notice that the Thevenin resistor that the capacitor sees between its terminals is

$$R_T = 4 + 8 + 8 = 20\,\Omega.$$

And the general equation for the RC circuit with time constant $\tau = RC = 20\,\mathrm{s}$ becomes

$$v_C(t) = K_1 e^{-t/20} + K_2.$$

When $t \to \infty$, there will be no current flowing through the capacitor, and it will be fully charged with the same voltage across the resistors in series $4\,\Omega$ and $8\,\Omega$

$$v_C(t \to \infty) = (10\,\mathrm{A})\,(4\,\Omega + 8\,\Omega) = 120\,\mathrm{V},$$

which yields

$$K_2 = 120\,\mathrm{V}.$$

Finally applying the initial condition

$$v_C(0^+) = 40\,\mathrm{V} = K_1 + K_2,$$

we obtain

$$K_1 = -80\,\mathrm{V}.$$

Consequently the equation $v_C(t)$ for $t > 0$ becomes

$$v_C(t) = 120 - 80e^{-t/20}\,\mathrm{V}.$$

We can find $i_C(t)$ using the $v - i$ relation

$$i_C(t) = C\frac{dv_C}{dt} = 4e^{-t/20}\,\mathrm{A}.$$

From a KCL in the top node we can obtain

$$\frac{v_R(t)}{4\,\Omega} + i_C(t) = 10\,\mathrm{A},$$

which yields the voltage across the resistor

$$v_R(t) = 40 - 4i_C(t) = 40 - 16e^{-t/20}\,\mathrm{V}.$$

For sketching $v_R(t)$, $v_C(t)$, and $i_C(t)$, is useful to know some approximations of e^{-x}, e.g. $e^{-1} \approx 0.37$, and $e^{-5} \approx 0.01$, which has the following practical implications:

(a) After one time constant (τ), a capacitor has either discharged down to 37% of its initial charge (when discharging) or has reached 63% of its final charge (when charging).

(b) At five times the time constant (5τ) a capacitor is either almost fully charged (when charging) or almost completely discharged (when discharging).