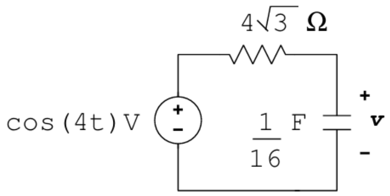
Homework 5

Due: Wednesday March 26, 2025, 11:59 pm

1. Consider the following circuit with $v\left(0^-\right) = 1\,V$. For $t > 0$, obtain:



   (a) the zero-state voltage across the capacitor's terminals, $v_{\mathrm{ZS}}(t)$,
   (b) the zero-input voltage across the capacitor's terminals, $v_{\mathrm{ZI}}(t)$,
   (c) the transient voltage across the capacitor's terminals, $v_{\mathrm{tr}}(t)$,
   (d) the steady state voltage across the capacitor's terminals, $v_{\mathrm{ss}}(t)$, and
   (e) the total voltage across the capacitor's terminals, $v(t)$.

Solution:

First we are going to find the ODE that rules the circuit. Applying KVL gives

$$\cos(4t) = v + v_R = v + R \times C\frac{dv}{dt}$$

which yields to the following ODE:

$$\cos(4t) = v + \frac{4\sqrt{3}}{16} \times \frac{dv}{dt},.$$

To solve this ODE we start from the particular solution

$$v_p(t) = A\cos(4t) + B\sin(4t).$$

Taking the derivative yields

$$\frac{dv_p(t)}{dt} = -4A\sin(4t) + 4B\cos(4t).$$

Substituting $v(t)$ and $\frac{dv(t)}{dt}$ into the ODE, we obtain

$$\cos(4t) = A\cos(4t) + B\sin(4t) + \frac{\sqrt{3}}{4} \times [-4A\sin(4t) + 4B\cos(4t)]$$

This identity imposes the two constraints

$$A + B\sqrt{3} = 1 \qquad \text{and} \qquad B - A\sqrt{3} = 0,$$

yielding

$$A = \frac{1}{4} \qquad \text{and} \qquad B = \frac{\sqrt{3}}{4}.$$

Therefore, the particular solution is

$$v_p(t) = \frac{1}{4}\cos(4t) + \frac{\sqrt{3}}{4}\sin(4t)\ V.$$

The homogeneous solution for the RC circuit has the form $v_h(t) = K_1 e^{-t/\tau}$, with $\tau = RC = \frac{\sqrt{3}}{4}$ s. Adding the homogeneous and the particular solutions yields to the general solution:

$$v(t) = v_h(t) + v_p(t) = K_1 e^{-t/\tau} + \frac{1}{4}\cos(4t) + \frac{\sqrt{3}}{4}\sin(4t)\ \text{V}.$$

(a) From the general solution we can calculate the zero-state response, i.e. the solution that satisfies $v(0^-) = 0$:

$$0 = K_1 + \frac{1}{4} \Rightarrow K_1 = -\frac{1}{4}.$$

Therefore the zero-state solution is

$$v_{\text{ZS}}(t) = -\frac{1}{4}e^{-\frac{4t}{\sqrt{3}}} + \frac{1}{4}\cos(4t) + \frac{\sqrt{3}}{4}\sin(4t)\ \text{V}.$$

(b) The zero-input response can be obtained applying the initial condition to the homogeneous solution, i.e.

$$v_h\left(0^-\right) = 1\,\text{V} = K_1 e^0 \Rightarrow K_1 = 1\,\text{V}.$$

Therefore the zero-input response is

$$v_{ZI}(t) = e^{-\frac{4t}{\sqrt{3}}}\ \text{V}.$$

(c) From the total response we notice that the transient response is

$$v_{\text{tr}}(t) = \frac{3}{4}e^{-\frac{4t}{\sqrt{3}}}\ \text{V},$$

(d) since that is the component of $v(t)$ that vanishes as $t \to \infty$. The remainder is the steady state response, i.e.

$$v_{\text{ss}}(t) = \frac{1}{4}\cos(4t) + \frac{\sqrt{3}}{4}\sin(4t)\ \text{V}.$$

(e) The total voltage across the capacitor is the sum of the zero-input and the zero-state responses. Also, it is the sum of transient response and steady state response:

$$v(t) = v_{\text{ZI}}(t) + v_{\text{ZS}}(t) = v_{tr}(t) + v_{\text{ss}}(t) = \frac{3}{4}e^{-\frac{4t}{\sqrt{3}}} + \frac{1}{4}\cos(4t) + \frac{\sqrt{3}}{4}\sin(4t)\ \text{V}.$$

2. A second-order linear system is described by

$$\frac{d^2v}{dt^2} + 3\frac{dv}{dt} + 2v(t) = \cos(2t).$$

Confirm that the transient function

$$v_h(t) = Ae^{-t} + Be^{-2t}$$

is homogeneous solution of the ODE and that its particular solution can be expressed as

$$v_p(t) = H\cos(2t + \theta).$$

Determine the values of H and θ. Hint: See Example 3.20 in Section 3.4.3 of the textbook.

Solution:

Substitute the transient function into the linear system.

$$\frac{d^2v_h}{dt^2} + 3\frac{dv_h}{dt} + 2v_h(t) = Ae^{-t} + 4Be^{-2t} - 3Ae^{-t} - 6Be^{-2t} + 2Ae^{-t} + 2Be^{-2t} = 0$$

Thus it is a homogeneous solution of the ODE.

We then substitute $v_p(t)$ to the left hand side of the linear system.

$$\begin{aligned}
\frac{d^2v_p}{dt^2} + 3\frac{dv_p}{dt} + 2v_p(t) &= -4H\cos(2t+\theta) - 6H\sin(2t+\theta) + 2H\cos(2t+\theta) \\
&= -2H\cos(2t+\theta) - 6H\sin(2t+\theta) \\
&= -H(e^{j(2t+\theta)} + e^{-j(2t+\theta)}) + j3H(e^{j(2t+\theta)} - e^{-j(2t+\theta)}) \\
&= (-H + j3H)e^{j(2t+\theta)} + (-H - j3H)e^{-j(2t+\theta)}
\end{aligned}$$

Note that Eq. should be equal to

$$\cos(2t) = \frac{1}{2}(e^{j2t} + e^{-j2t}),$$

which leads to

$$(-H + j3H)e^{j\theta} = \frac{1}{2}, \qquad (-H - j3H)e^{-j\theta} = \frac{1}{2}$$

We then have

$$\sqrt{(-H)^2 + (3H)^2} = \frac{1}{2}$$

Thus,

$$H = \frac{1}{2\sqrt{10}}.$$

$$\theta = -\left(\pi + \arctan\left(\frac{3H}{-H}\right)\right) = \arctan(3) - \pi.$$

3. Determine the phasor F of the following co-sinusoidal functions $f(t)$:

(a) $f(t) = -\sqrt{3}\cos(3t - \frac{\pi}{6})$,

(b) $f(t) = 4\sin(4t - \frac{\pi}{2})$.

Solution:

(a) $f(t) = -\sqrt{3}\cos(3t - \frac{\pi}{6}) = \sqrt{3}\cos\left(3t + \pi - \frac{\pi}{6}\right) = \sqrt{3}\cos\left(3t + \frac{5\pi}{6}\right)$. The magnitude of the cosine signal is $|F| = \sqrt{3}$. The phase shift is $\angle F = \frac{5\pi}{6}$ rad. Therefore the phasor is: $F = \sqrt{3}e^{j\frac{5\pi}{6}} = \sqrt{3}\angle 150^o$.

(b) $f(t) = 4\sin(4t - \frac{\pi}{2}) = 4\cos\left(\sqrt{3}t - \frac{\pi}{2} - \frac{\pi}{2} + 2\pi\right)$. In this case $|F| = 4$, $\angle F = \pi$ rad. Therefore the corresponding phasor is: $F = 4e^{j\pi} = 4\angle 180^o$.

4. Determine the phasor F of the following co-sinusoidal functions $f(t)$:

(a) $f(t) = 2\cos(2t + \frac{\pi}{3})$

(b) $f(t) = A\sin(\omega t)$

(c) $f(t) = -5\sin(\pi t)$

Solution:

(a) $|f(t)| = 2$, $\angle f(t) = \frac{\pi}{3}$. So $F = 2\angle\frac{\pi}{3}$.

(b) $f(t) = A\cos\left(\omega t - \frac{\pi}{2}\right)$. So $F = A\angle - \frac{\pi}{2}$.

(c) $f(t) = 5\cos\left(\pi t + \frac{\pi}{2}\right)$. So $F = 5\angle\frac{\pi}{2}$.

5. Use the phasor method to express the following signals $f(t)$ as single cosines:

(a) $f(t) = 3\cos(4t) - 4\sin(4t)$.

(b) $f(t) = 3\cos(3t) + 3\cos(3t - \pi/2)$.

Solution:

(a) $f(t) = -4\sin(4t) + 3\cos(4t) = 4\cos\left(4t + \frac{\pi}{2}\right) + 3\cos(4t)$. In phasor form $F = 4e^{+j\frac{\pi}{2}} + 3 = j4 + 3 = 5e^{j\left(\arctan\left(\frac{4}{3}\right)\right)}$, which yields the time-domain form:

$$f(t) = 5\cos\left(4t + \arctan\left(\frac{4}{3}\right)\right) \approx 5\cos(4t + 0.9273),$$

with amplitude 5 and phase shift $\arctan\left(\frac{4}{3}\right) \approx 0.9273\,\text{rad} = 53.1301^o$.

(b) $g(t) = 3\cos(3t) + 3\cos(3t - \pi/2)$. In phasor form $F = 3e^{-j\frac{\pi}{2}} + 3 = -j3 + 3 = 3\sqrt{2}e^{-j\frac{\pi}{4}}$, which yields

$$f(t) = 3\sqrt{2}\cos\left(3t - \frac{\pi}{4}\right),$$

with amplitude $3\sqrt{2}$ and phase shift $-\frac{\pi}{4}$ rad.