Math 286 Differential Equations Plus (Prof. Honold) **Midterm Exam** 2022/04/29

Name: ____________________ Student ID: ____________________ Group A

For each of the following problems, find the correct answer (tick as appropriate!). No justifications are required. Each problem has exactly one correct solution, which is worth 1 mark. Incorrect solutions (including no answer, multiple answers, or unreadable answers) will be assigned 0 marks; there are no penalties.

1. Which of the following ODE's has distinct solutions $y_1, y_2 \colon [-1,1] \to \mathbb{R}$ satisfying $y_1(0) = y_2(0)$?

   ☐ $y' = \sqrt{y^2+1}$ ☐ $t^2 y'' = y$ ☐ $y' = \sqrt{|t|}\,y$ ■ $(y')^3 = y$ ☐ $y' = t\,|y|$

2. The ODE $(y^2+y)\,\mathrm{d}x - x\,\mathrm{d}y$ has the integrating factor

   ☐ 0 ☐ x^{-1} ☐ y^{-1} ☐ x^{-2} ■ y^{-2}

3. For the solution $y(t)$ of the IVP $y' = y^4 - 4y^2,\ y(4.29) = 1$ the limit $\lim_{t\to+\infty} y(t)$ equals

   ☐ $-\infty$ ☐ -2 ■ 0 ☐ 2 ☐ $+\infty$

4. For the solution $y(t)$ of the IVP $y' = y \ln t,\ y(1) = 1$ the value $y(\mathrm{e})$ is equal to

   ☐ e^{-2} ☐ e^{-1} ☐ 1 ■ e ☐ e^2

5. For the solution $y\colon (0,\infty) \to \mathbb{R}$ of the IVP $t^2 y'' - t\,y' + y = 1,\ y(1) = y'(1) = 0$ the value $y(\mathrm{e})$ is equal to

   ☐ $\ln 4$ ■ 1 ☐ -1 ☐ $1+\ln 4$ ☐ $-1+\ln 4$

6. The power series $z + \frac{1}{2}z^2 + \frac{1}{4}z^4 + \frac{1}{8}z^8 + \frac{1}{16}z^{16} + \cdots$ has radius of convergence

   ☐ 0 ■ 1 ☐ $\sqrt{2}$ ☐ 2 ☐ ∞

7. The largest integer s such that $f_s(x) = \sum_{n=1}^{\infty} \frac{\cos(n^s x)}{n^3}$ is differentiable on $\mathbb{R}$ is equal to

   ☐ 0 ■ 1 ☐ 2 ☐ 3 ☐ 4

8. For which choice of $f_n(x)$ does the function series $\sum_{n=1}^{\infty} f_n$ converge uniformly on $\mathbb{R}$?

   ☐ $f_n(x) = \sqrt[n]{x^2}/n^4$ ☐ $f_n(x) = n/(x^4+n^2)$ ☐ $f_n(x) = \sqrt{x^2+n}/n^4$
   ■ $f_n(x) = n/(x^2+n^4)$ ☐ $f_n(x) = (-1)^n \ln(x^2+n)/n^4$

9. If $y(t)$ solves $y' = \dfrac{y+2t}{y+t}$ then $z(t) = y(t)/t$ solves

   ■ $z' = \dfrac{2-z^2}{t(z+1)}$ ☐ $z' = \dfrac{z^2-2}{t^2(z+1)}$ ☐ $z' = \dfrac{z+2}{z+1}$ ☐ $z' = \dfrac{z+2}{t(z+1)}$
   ☐ $z' = \dfrac{2-z^2}{z+1}$

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10. The sequence $\phi_0, \phi_1, \phi_2, \ldots$ of Picard-Lindelöf iterates for the IVP $y' = y + t,\ y(0) = -1$ has $\phi_2(t)$ equal to

☐ $\frac{1}{6}t^3$ ☐ $-1 - t - \frac{1}{2}t^2$ ■ $-1 - t + \frac{1}{6}t^3$ ☐ $-1 - t - \frac{1}{2}t^2 + \frac{1}{6}t^3$ ☐ $-1 - t$

11. $y'' + y = \cos t$ has a particular solution $y_p(t)$ of the form

☐ ct ☐ $c\cos t$ ☐ $c\sin t$ ☐ $ct\cos t$ ■ $ct\sin t$

with a constant c.

12. Maximal solutions of $y' = y^5 + y$ satisfying $y(0) = 1$ are defined on an interval of the form

☐ (a,b) ☐ $[a,b]$ ☐ $(a,+\infty)$ ■ $(-\infty,b)$ ☐ $(-\infty,+\infty)$

with $a, b \in \mathbb{R}$.

13. For $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ and $t \in \mathbb{R}$, the matrix $\mathrm{e}^{\mathbf{A}t}$ is equal to

■ $\begin{pmatrix} (1-t)\mathrm{e}^t & -t\,\mathrm{e}^t \\ t\,\mathrm{e}^t & (1+t)\mathrm{e}^t \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & \mathrm{e}^t \\ \mathrm{e}^{-t} & \mathrm{e}^{2t} \end{pmatrix}$ ☐ $\begin{pmatrix} (1+t)\mathrm{e}^t & t\,\mathrm{e}^t \\ -t\,\mathrm{e}^t & (1-t)\mathrm{e}^t \end{pmatrix}$

☐ $\begin{pmatrix} (1-t)\mathrm{e}^t & t\,\mathrm{e}^t \\ -t\,\mathrm{e}^t & (1+t)\mathrm{e}^t \end{pmatrix}$ ☐ $\begin{pmatrix} (1+t)\mathrm{e}^t & -t\,\mathrm{e}^t \\ t\,\mathrm{e}^t & (1-t)\mathrm{e}^t \end{pmatrix}$

14. The matrix norm of $\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$ (subordinate to the Euclidean length on $\mathbb{R}^2$) is contained in the interval

☐ $[1,2]$ ☐ $(2,3]$ ■ $(3,4]$ ☐ $(4,5]$ ☐ $(5,6]$

15. A map $T\colon M \to M$ satisfying $|T(x) - T(y)| \leq \frac{2021}{2022}|x-y|$ for all $x, y \in M$ must have a fixed point if M is equal to

☐ $(0,+\infty)$ ☐ $[0,1)$ ■ $[0,+\infty)$ ☐ $(0,1)$ ☐ $\mathbb{Q}$

All intervals are in $\mathbb{R}$, and $\mathbb{Q}$ is the field of rational numbers.

16. This midterm exam was

■ too easy ■ too difficult ■ too long ■ too short ■ just appropriate

Time allowed: 50 min CLOSED BOOK ***Good luck!***

Notes

Notes have only be written for Group A. In those places where Group B differs from Group A, the difference is indicated briefly at the end of the note. Group C is identical to Group A, except for the order of the questions.

1 This question was probably the most tricky.
Like $y' = \sqrt{|y|}$, which we have discussed in the lecture, $(y')^3 = y$ has nonzero solutions of the form $y(t) = ct^r$. Putting this „Ansatz" into the ODE, one finds $r = 3/2$, $c = \pm\frac{2\sqrt{2}}{3\sqrt{3}}$, $y(t) = \pm\frac{2\sqrt{2}}{3\sqrt{3}}t^{3/2}$, and $y'(t) = \pm\frac{\sqrt{2}}{\sqrt{3}}t^{1/2}$. These solutions are defined on $[0,\infty)$, and differentiable at $t = 0$ with one-sided derivative zero. Hence they can be glued with the all-zero solution on $(-\infty,0]$ to form a solution on $\mathbb{R}$. Thus we obtain at least three distinct solutions on $[-1,1]$ satisfying $y(0) = 0$. (In fact there are more, since solutions can be shifted horizontally to yield further solutions. Unlike for $y' = \sqrt{|y|}$, however, there are no solutions flowing into the all-zero solution, and hence for every $y_0 \neq 0$ there is a unique solution $y(t)$ on $\mathbb{R}$ or on $[-1,1]$ satisfying $y(0) = y_0$.)
For the ODE's $y' = \sqrt{y^2+1}$, $y' = \sqrt{|t|}\,y$, $y' = t\,|y|$, which satisfy the assumptions of the Existence and Uniqueness Theorem, the statement is false. For the 2nd-order ODE $t^2y'' = y$ it is false as well, since this is an Euler equation with indicial equation $r^2 - r - 1 = 0$, whose general solution on $(0,\infty)$ is $y(t) = c_1t^{r_1} + c_2t^{r_2}$ with $r_{1/2} = \frac{1\pm\sqrt{5}}{2}$. Since $r_2 < 0 < r_1 < 2$, the 2nd-order derivative of any nonzero solution blows up near $t = 0$, and hence no nonzero solution can exist on $[-1,1]$.

2 Multiplying the ODE by y^{-2} gives $(1+1/y)\,\mathrm{d}x - (x/y^2)\,\mathrm{d}y = 0$ which of the form $P\,\mathrm{d}x + Q\,\mathrm{d}y$ with $P_y = -1/y^2 = Q_x$ and hence exact on $\mathbb{R}^2$. Answers B,C,D don't have this property. Answer A is also false: Zero is not considered as an integrating factor, since multiplication by zero renders the ODE useless.
In Group B the integrating factor is x^{-2}, which turns $y\,\mathrm{d}x - (x^2+x)\,\mathrm{d}y$ into the exact ODE $(y/x^2)\,\mathrm{d}x - (1+1/x)\,\mathrm{d}y = 0$ ($P_y = 1/x^2 = Q_x$).

3 The phase line can be used to answer this question. The ODE is of the form $y' = f(y)$ with $f(y) = y^4 - 4y^2 = y^2(y-2)(y+2)$, which is negative in the intervall determined by adjacent zeros of f into which the starting value $y_0 = 1$ falls, viz. $(0,2)$. Hence $y(t)$ tends to the left end point of this interval for $t \to +\infty$.

4 $dy/y = \ln t\,\mathrm{d}t \Longrightarrow \ln y = t\ln t - t + C \Longrightarrow y = \exp(t\ln t - t + C)$
The initial value $y(1) = 1$ gives $C = 1$, so that $y(t) = \exp(t\ln t - t + 1)$. It follows that $y(\mathrm{e}) = \exp(\mathrm{e} - \mathrm{e} + 1) = \mathrm{e}$.
In Group B the initial value $y(\mathrm{e}) = 1$ gives $C = 0$, i.e., $y(t) = \exp(t\ln t - t)$, and hence $y(1) = \exp(-1) = 1/\mathrm{e}$.
The ODE is not only separable but also first-order linear (homogeneous) and can be solved using the explicit solution formula for such ODE's.

5 This is an inhomogeneous Euler equation with $\alpha = -1$, $\beta = 1$, indicial equation $r^2 - 2r + 1 = (r-1)^2 = 0$, particular solution $y_p(t) \equiv 1$, and general solution $y(t) = 1 + c_1t + c_2t\ln t$ on $(0,\infty)$. The initial values yield $c_1 = -1$, $c_2 = 1$, so that the IVP is solved by $y(t) = 1 - t + t\ln t$, which has $y(\mathrm{e}) = 1 - \mathrm{e} + \mathrm{e} = 1$.
In Group B the value asked for is $y(2) = -1 + 2\ln 2 = -1 + \ln 4$.

6 The power series has coefficients

$$a_n = \begin{cases} 1/n & \text{if } n = 2^k \text{ for some } k \in \mathbb{N}_0, \\ 0 & \text{otherwise.} \end{cases}$$

The Cauchy-Hadamard formula gives $R = 1/L$ with

$$L = \limsup_{n\to\infty} \sqrt[n]{a_n} = \lim_{k\to\infty} \sqrt[2^k]{\frac{1}{2^k}} = \frac{1}{\lim_{k\to\infty} \sqrt[2^k]{2^k}} = 1,$$

since $\left(\sqrt[2^k]{2^k}\right)$ is a subsequence of $(\sqrt[n]{n})$, which converges to 1.
Alternatively, without using the Cauchy-Hadamard formula, one can observe that the series converges for $z = 1$ $(1 + 1/2 + 1/4 + 1/8 + \cdots = 2)$ and diverges for real $z > 1$ (since $z^n/n \to +\infty$ and hence also $z^{2^k}/2^k \to +\infty$ for such z).

7 The series $\sum_{n=1}^{\infty} \frac{\cos(n^s x)}{n^3}$ itself converges for all s uniformly on $\mathbb{R}$ (Weierstrass test with $M_n = 1/n^3$), so that $f_s(x)$ is well-defined. For $s = 1$ the series of derivatives is $\sum_{n=1}^{\infty} \frac{-\sin(nx)}{n^2}$, which converges uniformly on $\mathbb{R}$ (Weierstrass test with $M_n = 1/n^2$). Hence, according to the Differentiation Theorem $f_1(x)$ is differentiable on $\mathbb{R}$. For $s = 2$ the series of derivatives is $\sum_{n=1}^{\infty} \frac{-\sin(n^2 x)}{n}$, for which the Weierstrass test fails. Hence you can guess that $f_2(x)$ is not differentiable in every $x \in \mathbb{R}$ (unless your professor is a really bad guy), which is indeed the case. For $s \geq 3$ the summands of the series of derivatives are even bigger, and the same remains true.

8 For $f_n(x) = n/(x^2 + n^4)$ we have $|f_n(x)| = f_n(x) \leq n/n^4 = 1/n^3$ and can apply the Weierstrass test with $M_n = 1/n^3$ to conclude that $\sum_{n=1}^{\infty} f_n$ converges uniformly on $\mathbb{R}$. Answer B doesn't converge at all, since $n/(x^4 + n^2) \simeq 1/n$ for $n \to \infty$ and the harmonic series $\sum 1/n$ diverges. The remaining three series don't converge uniformly, since, e.g., given any n we can choose x such that $f_n(x) = \pm 1$. Uniform convergence of $\sum_{n=1}^{\infty} f_n$, however, requires in particular that $f_n \to 0$ uniformly.

9 $y' = \frac{y/t+2}{y/t+1} = \frac{z+2}{z+1} \Longrightarrow z' = \frac{y't-y}{t^2} = \frac{y'-z}{t} = \frac{1}{t}\left(\frac{z+2}{z+1} - z\right) = \frac{1}{t(z+1)}\big(z + 2 - z(z+1)\big) = \frac{2-z^2}{t(z+1)}$

10 $\phi_0(t) = -1$, $\phi_1(t) = -1 + \int_0^t \phi_0(s) + s\,\mathrm{d}s = -1 + \int_0^t -1 + s\,\mathrm{d}s = -1 - t + t^2/2$, $\phi_2(t) = -1 + \int_0^t \phi_1(s) + s\,\mathrm{d}s = -1 + \int_0^t -1 - s + s^2/2 + s\,\mathrm{d}s = -1 + \int_0^t -1 + s^2/2\,\mathrm{d}s = -1 - t + t^3/6$

11 The suggested way to solve this ODE is complexification: Solve $z'' + z = \mathrm{e}^{\mathrm{i}t}$ and obtain the solution as $y(t) = \mathrm{Re}\,z(t)$. Since $\mu = \mathrm{i}$ is a root of the characteristic polynomial, viz. $a(X) = X^2 + 1$, of multiplicity 1, the correct „Ansatz“ is $z(t) = ct\,\mathrm{e}^{\mathrm{i}t}$. Since $a(\mathrm{D})z = (D + \mathrm{i})(D - \mathrm{i})\left[ct\mathrm{e}^{\mathrm{i}t}\right] = c(D + \mathrm{i})\mathrm{e}^{\mathrm{i}t} = c(2\mathrm{i})\mathrm{e}^{\mathrm{i}t}$, we must take $c = 1/(2\mathrm{i}) = -\mathrm{i}/2$ and $y(t) = \mathrm{Re}\left(-\frac{1}{2}\mathrm{i}t\,\mathrm{e}^{\mathrm{i}t}\right) = \frac{1}{2}t\sin t$.
The question can also be answered by putting each of the five answers offered into the ODE. The only other reasonable candidate is Answer D, which turns out not to work.
In Group B the ODE is $y'' + y = \sin t$, which has the solution $y_p(t) = \mathrm{Im}\left(-\frac{1}{2}\mathrm{i}t\,\mathrm{e}^{\mathrm{i}t}\right) = -\frac{1}{2}t\cos t$.

12 Since maximal solutions of IVPs are unique, the statement should have read “The maximal solution …” rather than “Maximal solutions …”.
Solutions $y = y(t)$ satisfy $\int_1^y \frac{\mathrm{d}\eta}{\eta^5+\eta} = \int_0^t \mathrm{d}\tau = t$. We have $\lim_{y\downarrow 0} \int_1^y \frac{\mathrm{d}\eta}{\eta^5+\eta} = -\int_0^1 \frac{\mathrm{d}\eta}{\eta^5+\eta} = -\infty$ (since $\frac{1}{\eta^5+\eta} \simeq \frac{1}{\eta}$ for $\eta \downarrow 0$) and $b := \lim_{y\uparrow\infty} \int_1^y \frac{\mathrm{d}\eta}{\eta^5+\eta} = \int_1^\infty \frac{\mathrm{d}\eta}{\eta^5+\eta} < \infty$). This shows that the maximal solution is defined on an interval of the form $(-\infty, b)$.

13 The columns of $t \mapsto \mathrm{e}^{\mathbf{A}t}$ are solutions of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ and among the solutions uniquely determined by their initial values $\mathbf{y}(0) = (1,0)^\mathsf{T}$ resp. $(0,1)^\mathsf{T}$. The system $\mathbf{y}' = (y_1', y_2')^\mathsf{T} = \mathbf{A}\mathbf{y}$ is equivalent to $y_1' = -y_2$, $y_2' = y_1 + 2y_2$. The first equation is satisfied by the first column of Answer A, but not by the first column of any other answer. Thus the correct Answer must be A (provided Q12 is stated in accordance with the rules for the midterm).
The correct answer can also be written as $\mathrm{e}^t\left[\mathbf{I}_2 + t(\mathbf{A} - \mathbf{I}_2)\right]$, which can be seen directly when you know a little more theory about the matrix exponential function. We'll discuss this further in the lecture.
In Group B the matrix $\mathbf{A}$ is $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 2 \end{smallmatrix}\right)$, and the first equation of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ is $y_1' = y_2$. This is satisfied by the first column of Answer D, but not by the first column of any other answer.

14 For a matrix $\mathbf{A}$ of the form $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ (scaled orthogonal matrix) we have

$$\left|\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix}\right|^2 = (ax-by)^2+(bx+ay)^2 = (a^2+b^2)(x^2+y^2) = (a^2+b^2)\left|\begin{pmatrix} x \\ y \end{pmatrix}\right|^2.$$

It follows that $\|\mathbf{A}\| = \sqrt{a^2+b^2}$ for such matrices. In particular, $\left|\left(\begin{smallmatrix} 2 & -3 \\ 3 & 2 \end{smallmatrix}\right)\right| = \sqrt{2^2+3^2} = \sqrt{13} \in (3,4]$.
In Group B the correct answer is $\left|\left(\begin{smallmatrix} 3 & -4 \\ 4 & 3 \end{smallmatrix}\right)\right| = \sqrt{3^2+4^2} = 5 \in (4,5]$.

15 Banach's Fixed Point Theorem implies that the answer is True, provided M is a complete metric space w.r.t. the aboslute value, which for subsets of $\mathbb{R}$ is equivalent to "closed". The only closed set among the answers offered is $[0,+\infty)$. For the other answers offered, one can find a corresponding map T having no fixed point.

16 Here all answers (including multiple answers) were accepted, but at least one answer had to be present.