Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) There exists a solution $y(t)$ of $y' = 2y - y^2$ satisfying $y(0) = y(1) = 1$.

b) The maximal solution of the initial value problem $y' = y^2 - t$, $y(0) = \frac{1}{2}$ exists at time $t = 2021$.

c) Every solution $y\colon (0,\infty) \to \mathbb{R}$ of $t^2y'' + 3t\,y' + 2y = 0$ has infinitely many zeros.

d) The initial value problem $(x^2+4)y'' + (x+4)y' - 4y = 0$, $y(1) = y'(1) = 1$ has a power series solution $y(x) = \sum_{n=0}^{\infty} a_n(x-1)^n$ which is defined at $x = 3$.

e) Suppose $\mathbf{A} \in \mathbb{R}^{2\times 2}$ satisfies $\mathbf{A}^3 = \mathbf{I}$ (the 2×2 identity matrix), but $\mathbf{A} \neq \mathbf{I}$. Then every solution $\mathbf{y}(t)$ of the linear system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ must satisfy $\lim\limits_{t\to\infty} \mathbf{y}(t) = (0,0)^{\mathsf{T}}$.

f) Suppose $f, g\colon (-1,1) \to \mathbb{R}$ are C^1-functions. Then the IVP $y' = f(t)g(y)$, $y(0) = 0$ has a solution $y(t)$ that is defined for all $t \in (-1,1)$.

Question 2 (ca. 10 marks)

Consider the differential equation

$$2x^2\,y'' + x(1-x)y' - 6\,y = 0. \tag{DE}$$

a) Verify that $x_0 = 0$ is a regular singular point of (DE).

b) Determine the general solution of (DE) on $(0,\infty)$.

c) Using the result of b), state the general solution of (DE) on $(-\infty, 0)$ and on $\mathbb{R}$.

Question 3 (ca. 7 marks)

Consider the ODE

$$y' = y^2 + \frac{5}{t}\,y + \frac{5}{t^2}, \qquad t > 0. \tag{R}$$

a) Show that there exists a solution $y_1(t)$ of the form $y_1(t) = c\,t^r$ with constants c, r.

b) Show that the substitution $y = y_1 + 1/z$ transforms (R) into a first-order linear ODE.

c) Using b), determine all maximal solutions of (R) and their domain.

Question 4 (ca. 6 marks)

For the matrix $\mathbf{A} = \begin{pmatrix} -8 & 0 & 5 & -2 \\ 5 & -1 & -4 & 1 \\ -10 & 0 & 7 & -2 \\ 0 & 0 & 3 & 2 \end{pmatrix}$ determine the general solution of the linear system $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

Question 5 (ca. 7 marks)

Consider the differential equation

$$x(3y^2-1)\,\mathrm{d}x + y\,\mathrm{d}y = 0. \tag{DF}$$

a) Determine the general solution of (DF) in implicit form.

b) Determine the maximal solution $y(x)$ satisfying $y(1)=\frac{1}{3}$ and its domain.
Hint: $\ln\left(\frac{3}{2}\right) \approx 0.4$

c) Is every point of $\mathbb{R}^2$ on a unique integral curve of (DF)?

Question 6 (ca. 8 marks)

a) Determine a real fundamental system of solutions of

$$y^{(5)} + 4\,y^{(4)} + 24\,y''' + 40\,y'' + 100\,y' = 0.$$

Hint: The characteristic polynomial is divisible by the square of a quadratic polynomial.

b) Determine the general real solution of

$$y^{(5)} + 4\,y^{(4)} + 24\,y''' + 40\,y'' + 100\,y' = 200\,t - \mathrm{e}^{-t}.$$

c) Find the Laplace transform $Y(s)$ of the solution of the ODE in b) with initial values $y(0)=y'(0)=y''(0)=y'''(0)=y^{(4)}(0)=0$.

Solutions

1 a) False. Since $y' = y(2-y)$ is positive if $0 < y < 2$, and any solution starting in the strip $0 < y < 2$ is confined to this strip (e.g., because the strip is bounded by the constant solutions $y(t) \equiv 0$ and $y(t) \equiv 2$), the solution with $y(0) = 1$ must be strictly increasing and hence satisfy $y(1) > 1$. $\boxed{2}$

Alternatively one can argue that this ODE is of the form $y' = ay^2 + by + c$ and the corresponding canonical form, viz. $z' = -z^2 + 1$, is the same as for the Logistic Equation (cf. our discussion in the lecture and H16 of HW2). Hence solutions starting between the two equilibrium solutions must be monotonically increasing.

b) True. For $t \geq 0$ we have $-t \leq y^2 - t \leq y^2$, so that the solution $\phi_1(t)$ of $y' = y^2 \wedge y(0) = \frac{1}{2}$ is an upper bound for $y(t)$ and the solution $\phi_2(t)$ of $y' = -t \wedge y(0) = \frac{1}{2}$ is a lower bound for $y(t)$, Solving the two auxiliary IVP's gives $\phi_1(t) = 1/(2-t)$, $\phi_2(t) = (1-t^2)/2$. Hence $y(t)$ is defined at least on $[0, 2)$. $\boxed{1}$

Since $y(1) \leq \phi_1(1) = 1$ and $y^2 - t \leq y^2 - 1$ for $t \geq 1$, the solution $\phi_3(t)$ of $y' = y^2 - 1 \wedge y(1) = 1$ is an upper bound for $y(t)$. Solving the auxilieary IVP gives $\phi_3(t) \equiv 1$. Since $\phi_2(t)$ and $\phi_3(t)$ are defined on $[1, \infty)$, the same must be true of $y(t)$. $\boxed{+1}$

c) True. This is an Euler equation with parameters $\alpha = 3$, $\beta = 2$, indicial equation $r^2 + (\alpha - 1)r + \beta = r^2 + 2r + 2 = (r + 1 + \mathrm{i})(r + 1 - \mathrm{i}) = 0$, complex fundamental system $t^{-1\pm\mathrm{i}}$, and general real solution $y(t) = t^{-1}\,(c_1 \cos \ln t + c_2 \sin \ln t)$ on $(0, \infty)$. $\boxed{1}$

Since $c_1 \cos x + c_2 \sin x = A \sin(x - \alpha)$ for some $A \geq 0$, $\alpha \in [0, 2\pi)$, we can write the solution in the form $y(t) = A\,t^{-1} \sin(\ln t - \alpha)$ and conclude that $t_k = \mathrm{e}^{k\pi+\alpha}$, $k = 0, 1, 2, \dots$, are solutions. $\boxed{1}$

d) True. The explicit form of this homogeneous linear 2nd-order ODE is

$$y'' + \frac{x+4}{x^2+4}\,y' - \frac{4}{x^2+4}\,y = 0.$$

$x_0 = 1$ is an ordinary point and the coefficient functions $p(x) = \frac{x+4}{x^2+4}$, $q(x) = -\frac{4}{x^2+4}$ are analytic (even rational) in the disk $|z - 1| < \sqrt{5}$ (the disk with center 1 that has the singularities $\pm 2\mathrm{i}$ of $p(x)$ and $q(x)$ on its boundary). Hence (referring to a theorem proved in the lecture) there exists a power series solution $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$ of the IVP with radius of convergence $\geq \sqrt{5}$. Since $\sqrt{5} > 2$, this solution is defined at $x = 3$. $\boxed{2}$

e) True. If $\mathbf{v} \in \mathbb{C}^2$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda \in \mathbb{C}$, we have $\mathbf{v} = \mathbf{A}^3\mathbf{v} = \lambda^3\mathbf{v}$, and hence $\lambda^3 = 1$. Thus $\lambda \in \left\{1, \frac{-1+\mathrm{i}\sqrt{3}}{2}, \frac{-1-\mathrm{i}\sqrt{3}}{2},\right\}$. If $\mathbf{A}$ has an eigenvalue $\neq 1$, the eigenvalues must be $\lambda_{1/2} = \frac{-1\pm\mathrm{i}\sqrt{3}}{2}$ (since complex eigenvalues of real matrices occur in conjugate pairs). Since their real part is $-\frac{1}{2} < 0$, the matrix $\mathbf{A}$ is asymptotically stable. If $\lambda_1 = \lambda_2 = 1$ then $(\mathbf{A} - \mathbf{I})^2 = \mathbf{0}$ (by the Cayley-Hamilton Theorem), which gives $\mathbf{A}^2 = 2\mathbf{A} - \mathbf{I}$, $\mathbf{I} = \mathbf{A}^3 = 2\mathbf{A}^2 - \mathbf{A} = 3\mathbf{A} - 2\mathbf{I}$, and hence $\mathbf{A} = \mathbf{I}$; contradiction. $\boxed{+2}$

f) False. Separable ODE's $y' = f(t)g(y)$ may have maximal solutions with strictly smaller domain than $f(t)$. $\boxed{1}$

This can happen even for autonomous ODE's, and we can take $y' = 2(y+1)^2$, i.e., $f(t) = 1$, $g(y) = 2(y+1)^2$ as counterexample. The general solution of this ODE is $y(t) = \frac{1}{C-2t} - 1$, $C \in \mathbb{R}$, and $y(0) = 0$ gives $C = 1$. But $y(t) = \frac{1}{1-2t} - 1$ is not defined for $t \in \left[\frac{1}{2}, 1\right)$. $\boxed{+2}$

Remarks:

a) The fact that any solution starting in the strip $0 < y < 2$ is confined to this strip is implicit in the theorem on using the phaseline to determine the asymptotic behavior of solutions of autonomous 1st-order ODE's (and was proved in its proof). Thus arguing with the phaseline was accepted as justification for this fact. But you had to indicate somehow that solutions with $0 < y(t) < 2$ are strictly increasing ($0 < y(t) \leq 1$ is not enough!).

As some students noted, one can also argue with the Mean Value Theorem of Calculus I: $y(0) = y(1) = 1$ implies $y'(t) = 0$ for some $t \in (0,1)$, which contradicts $y'(t) = y(t)\bigl(2 - y(t)\bigr) > 0$; see above.

b) Many students observed that $\phi_1(t) = 1/(2-t)$ is an upper bound for $y(t)$ where it is defined ($0 \leq t < 2$), but then falsely concluded that $y(t)$ blows up at (or before) $t = 2$ like $\phi_1(t)$ (0.5 marks). The correct statement of an upper/lower bound for $y(t)$ and its consequences for the domain of $y(t)$ was honored by 1 mark each. Thus, for example, the lower bound $y' \geq y^2 - 2021$, together with the fact that the solution of the IVP $y' = y^2 - 2021 \wedge y(0) = 1/2$ is defined for all $t \in \mathbb{R}$, gives that $y(t)$ for $t \leq 2021$ cannot escape to $-\infty$. Very few students (if any) solved this problem completely.

c) 1.5 marks were assigned for the correct real solution $y(t) = t^{-1}\left(c_1 \cos \ln t + c_2 \sin \ln t\right)$, $c_1, c_2 \in \mathbb{R}$, of the Euler equation, if the required additional argument proving infinitely many zeros was missing. For complex solutions the assertion is not true (look at $y(t) = t^{-1\pm i}$).

d) Some students didn't notice that the question is about power series solutions centered at $x_0 = 1$. For $x_0 = 0$ the answer is supposedly "False", since in this case the radius of convergence is 2 and 3 has distance $3 > 2$ to the origin.

e) It seems that no student provided a complete answer to this question. Many students noted that the asymptotic stability of $\mathbf{A}$ depends on the eigenvalues, but were either unable to determine the eigenvalues or made the unjustified assumption that $\mathbf{A}$ describes a plane rotation by 120°, which of course has the said eigenvalues. (One can show that $\mathbf{A}$ must be similar to a rotation matrix, but it need not be itself a rotation matrix.) One student reduced the proof of asymptotic stability of $\mathbf{A}$ to $\mathbf{A}^2 + \mathbf{A} + \mathbf{I} = \mathbf{0}$ (which is correct), but then claimed without proof that this equation follows from $\mathbf{A}^3 = \mathbf{I}$ and $\mathbf{A} \neq \mathbf{I}$. However, since matrix multiplication isn't cancellative, one cannot conclude this from $\mathbf{A}^3 - \mathbf{I} = (\mathbf{A} - \mathbf{I})(\mathbf{A}^2 + \mathbf{A} + \mathbf{I})$.

f) Many students claimed that the answer is "True", although we have discussed extensively in the lecture that for general ODE's (and in particular for separable ODE's) the Existence Theorem holds only locally. Some students provided alternative counterexamples such as $y' = (y+2)^2 \wedge y(0) = 0$ (solution $y(t) = \frac{1}{1/2-t} - 2$, defined only for $t < 1/2$) or $y' = y^2 + a^2 \wedge y(0) = 0$ with $a > 0$ sufficiently small (solution $y(t) = a\tan(at)$, defined only for $-\pi/2a < t < \pi/2a$).

$$\sum_1 = 8 + 5$$

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2 a) The explicit form of (DE) is

$$y'' + \frac{1-x}{2x}\,y' - \frac{3}{x^2}\,y = 0$$

Using the notation of the lecture/textbook, $p(x) = \frac{1-x}{2x} = \frac{1}{2}\,x^{-1} - \frac{1}{2}$ has a pole of order 1 at 0, and $q(x) = -\frac{3}{x^2}$ has a pole of order 2 at 0. This implies that 0 is a regular singular point of (DE). $\boxed{1}$

Alternatively, use that the limits defining p_0, q_0 below are finite.

b) From a) we have $p_0 = \lim_{x\to 0} x\,p(x) = 1/2$, $q_0 = \lim_{x\to 0} x^2\,q(x) = -3$.
$\Longrightarrow$ The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = r^2 - \frac{1}{2}\,r - 3 = (r-2)(r+3/2) = 0.$$

$\Longrightarrow$ The exponents at the singularity $x_0 = 0$ are $r_1 = 2$, $r_2 = -3/2$. Since $r_1 - r_2 \notin \mathbb{Z}$, there exist two fundamental solutions y_1, y_2 of the form

$$\begin{aligned} y_1(x) &= x^2 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+2}, \\ y_2(x) &= x^{-3/2} \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} b_n x^{n-3/2} \end{aligned} \qquad \boxed{1}$$

with normalization $a_0 = b_0 = 1$.

First we determine the analytic solution $y_1(x)$. We have

$$\begin{aligned} 0 &= 2x^2\,y_1'' + x(1-x)y_1' - 6\,y_1 \\ &= 2x^2 \sum_{n=0}^{\infty}(n+2)(n+1)a_n x^n + (x-x^2)\sum_{n=0}^{\infty}(n+2)a_n x^{n+1} - 6\sum_{n=0}^{\infty} a_n x^{n+2} \\ &= \sum_{n=0}^{\infty}\left[2(n+2)(n+1) + n + 2 - 6\right] a_n x^{n+2} - \sum_{n=0}^{\infty}(n+2)a_n x^{n+3} \\ &= \sum_{n=0}^{\infty}(2n^2+7n)a_n x^{n+2} - \sum_{n=1}^{\infty}(n+1)a_{n-1}x^{n+2} \\ &= \sum_{n=1}^{\infty}\left[n(2n+7)a_n - (n+1)a_{n-1}\right] x^{n+2}. \end{aligned}$$

Equating coefficients gives the recurrence relation

$$a_n = \frac{n+1}{n(2n+7)} a_{n-1} \quad \text{for } n = 1, 2, 3, \dots. \qquad \boxed{1}$$

and with $a_0 = 1$ further $a_n = \frac{(n+1)!}{n!\, 9\cdot 11\cdot 13\cdots(2n+7)} = \frac{n+1}{9\cdot 11\cdot 13\cdots(2n+7)}$.

$$\begin{aligned}\implies y_1(x) &= \sum_{n=0}^{\infty} \frac{n+1}{9\cdot 11\cdot 13\cdots(2n+7)}\, x^{n+2} \qquad \boxed{1\tfrac{1}{2}}\\ &= x^2 + \frac{2}{9}\, x^3 + \frac{3}{9\cdot 11}\, x^4 + \frac{4}{9\cdot 11\cdot 13}\, x^5 + \frac{5}{9\cdot 11\cdot 13\cdot 15}\, x^7 + \cdots\end{aligned}$$

For the determination of $y_2(x)$ we repeat the process with exponents decreased by 3.5 :

$$\begin{aligned}0 &= 2x^2\, y_2'' + x(1-x)y_2' - 6\, y_2\\ &= 2x^2 \sum_{n=0}^{\infty} \left(n-\tfrac{3}{2}\right)\left(n-\tfrac{5}{2}\right) b_n x^{n-7/2} + (x-x^2)\sum_{n=0}^{\infty}\left(n-\tfrac{3}{2}\right) b_n x^{n-5/2} - 6\sum_{n=0}^{\infty} b_n x^{n-3/2}\\ &= \sum_{n=0}^{\infty}\left[2\left(n-\tfrac{3}{2}\right)\left(n-\tfrac{5}{2}\right) + n - \tfrac{3}{2} - 6\right] b_n x^{n-3/2} - \sum_{n=0}^{\infty}\left(n-\tfrac{3}{2}\right) b_n x^{n-1/2}\\ &= \sum_{n=0}^{\infty}(2n^2-7n) b_n x^{n-3/2} - \sum_{n=1}^{\infty}\left(n-\tfrac{5}{2}\right) b_{n-1} x^{n-3/2}\\ &= \sum_{n=1}^{\infty}\left[n(2n-7)b_n - \left(n-\tfrac{5}{2}\right) b_{n-1}\right] x^{n-3/2}.\end{aligned}$$

Here we obtain the recurrence relation

$$b_n = \frac{n-5/2}{n(2n-7)}\, b_{n-1} = \frac{2n-5}{2n(2n-7)}\, b_{n-1} \quad \text{for } n = 1, 2, 3, \dots, \qquad \boxed{1}$$

and with $b_0 = 1$ further $b_n = \frac{(-3)(-1)\cdots(2n-5)}{2\cdot 4\cdots 2n(-5)(-3)\cdots(2n-7)} = \frac{2n-5}{2\cdot 4\cdots 2n(-5)}$.

$$\begin{aligned}y_2(x) &= -\frac{1}{5}\sum_{n=0}^{\infty} \frac{2n-5}{2\cdot 4\cdots 2n}\, x^{n-3/2} \qquad \boxed{1\tfrac{1}{2}}\\ &= x^{-3/2} + \frac{3}{5\cdot 2}\, x^{-1/2} + \frac{1}{5\cdot 2\cdot 4}\, x^{1/2} - \frac{1}{5\cdot 2\cdot 4\cdot 6}\, x^{3/2} - \frac{3}{5\cdot 2\cdot 4\cdot 6\cdot 8}\, x^{5/2} - \frac{5}{5\cdot 2\cdot 4\cdot 6\cdot 8\cdot 10}\, x^{7/2} - \cdots\end{aligned}$$

Alternative solution: We use the general recurrence relation for the rational functions $a_n(r)$, viz. $a_0(r) = 1$ and

$$a_n(r) = -\frac{1}{F(r+n)} \sum_{k=0}^{n-1} \left[(r+k)p_{n-k} + q_{n-k}\right] a_{n-1}(r) \quad \text{for } n \geq 1.$$

Since $F(r) = (r-2)(r+3/2)$, $p_1 = -1/2$, $p_2 = p_3 = \cdots = q_1 = q_2 = \cdots = 0$, we obtain

$$\begin{aligned}a_n(r) &= -\frac{1}{(r+n-2)(r+n+3/2)}\left[(r+n-1)(-1/2)\right] a_{n-1}(r)\\ &= \frac{r+n-1}{(r+n-2)(2r+2n+3)}\, a_{n-1}(r), \qquad n \geq 1.\end{aligned}$$

Thus the coefficients $a_n(2)$ of $y_1(x)$ satisfy the recurrence relation $a_n(2) = \frac{n+1}{n(2n+7)}\,a_{n-1}(2)$ (the same as for a_n above) and the coefficients $a_n(-3/2)$ of $y_2(x)$ satisfy the recurrence relation $a_n(-3/2) = \frac{n-5/2}{(n-7/2)2n}\,a_{n-1}(-3/2)$ (the same as for b_n above). The rest of the computation remains the same.

The general (real) solution on $(0,\infty)$ is then $y(x) = c_1\,y_1(x) + c_2\,y_2(x)$, $c_1, c_2 \in \mathbb{R}$. $\boxed{\frac{1}{2}}$

That solutions are defined on the whole of $(0,\infty)$, is guaranteed by the analyticity of $p(x)$, $q(x)$ in $\mathbb{C}\setminus\{0\}$, but follows also readily from the easily established fact that the radius of convergence of both power series is ∞. $\boxed{\frac{1}{2}}$

c) The solution on $(-\infty,0)$ is $y(x) = c_1\,y_1(x) + c_2\,y_2^-(x)$ with the same power series $y_1(x)$ as in b) and

$$y_2^-(x) = -\frac{1}{5\,|x|^{3/2}}\sum_{n=0}^{\infty}\frac{2n-5}{2\cdot 4\cdots 2n}\,x^n = -\frac{1}{5(-x)^{3/2}}\sum_{n=0}^{\infty}\frac{2n-5}{2\cdot 4\cdots 2n}\,x^n. \qquad \boxed{1}$$

(This is <u>not</u> the same as $y_2(-x)$, whose coefficients have an additional factor $(-1)^n$.)

Since $y_1(x)$ is analytic everywhere and $\lim_{x\downarrow 0} y_2(x) = \infty$, the general solution on $\mathbb{R}$ is $y(x) = c_1\,y_1(x)$, $c_1 \in \mathbb{R}$. $\boxed{1}$

Remarks: This question was generally answered well, except that most students missed the 0.5 marks assigned for the radius of convergence of the power series in b) and quite a few students asserted falsely that for the solution on $(-\infty,0)$ one only needs to replace x by $|x|$ in b) (that doesn't work for $y_2(x)$).

$$\sum_{2} = 10$$

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3 a) Substituting $y_1(t) = c\,t^r$ into the ODE gives

$$cr\,t^{r-1} = c^2t^{2r} + 5c\,t^{r-1} + 5\,t^{-2},$$

which holds if $r=-1$ and $-c = c^2+5c+5$, i.e., $c^2+6c+5=0$, which has solutions $c\in\{-1,-5\}$. Thus we can take $y_1(t) = -t^{-1}$ or $y_1(t) = -5\,t^{-1}$. $\boxed{1}$

b) Taking $y_1(t) = -t^{-1}$ in a), the substitution becomes $y = -t^{-1}+1/z$, $y' = 1/t^2 - z'/z^2$. Substituting this into (R) gives

$$\begin{aligned}
\frac{1}{t^2} - \frac{z'}{z^2} &= \left(-\frac{1}{t}+\frac{1}{z}\right)^2 + \frac{5}{t}\left(-\frac{1}{t}+\frac{1}{z}\right) + \frac{5}{t^2} = \frac{1}{t^2} - \frac{2}{tz} + \frac{1}{z^2} - \frac{5}{t^2} + \frac{5}{tz} + \frac{5}{t^2}\\
\iff\quad -\frac{z'}{z^2} &= \frac{3}{tz} + \frac{1}{z^2}\\
\iff\quad z' &= -\frac{3}{t}\,z - 1.
\end{aligned}$$

This is of the form $z' = a(t)z + b(t)$, hence first-order (inhomogeneous) linear. $\boxed{2}$

c) The general solution of $z' = (-3/t)z$ is

$$z(t) = c \exp \int -\frac{3}{t}\,\mathrm{d}t = \frac{c}{t^3}, \quad c \in \mathbb{R}. \qquad \boxed{1}$$

Variation of parameters then yields a particular solution z_p of $z' = (-3/t) - 1$:

$$z_p(t) = t^{-3} \int t^3(-1)\,\mathrm{d}t = -\frac{t}{4}. \qquad \boxed{1}$$

$\Longrightarrow$ The general solution of $z' = (-3/t)z - 1$ is

$$z(t) = -\frac{t}{4} + \frac{c}{t^3}, \quad c \in \mathbb{R}.$$

$\Longrightarrow$ The general solution of (R) is

$$y(t) = -\frac{1}{t} + \frac{1}{-t/4 + c/t^3} = -\frac{1}{t} + \frac{t^3}{c - t^4/4}, \quad c \in \mathbb{R} \cup \{\infty\}, \qquad \boxed{1}$$

where $c = \infty$ represent the solution y_1.

The maximal domain of $y(t)$ is $(0, \infty)$ for $c \in \{0, \infty\}$ and $c < 0$ ($c = 0$ corresponds to the 2nd solution $y(t) = -5\,t^{-1}$ discovered in a).) For $c > 0$ the expression for $y(t)$ defines two maximal solutions on $\big(0, \sqrt[4]{4c}\big)$ and $\big(\sqrt[4]{4c}, \infty\big)$. $\boxed{1}$

Remarks: This question was virtually the same as Q3 in the first sample final exam. Nevertheless, many students either missed the solution $y_1(t)$ obtained in a) when stating the general solution in c), or failed to provide the correct maximal domain of $y_c(t)$ (the solution with parameter c). Since solutions of ODE's are defined on intervals, it is not sufficient to exclude t with $c - t^4/4 = 0$. Rather this exclusion gives rise to two maximal solutions whose domains are intervals. Also quite a few students did b) and c) for both $y_1(t) = -t^{-1}$ and $-5\,t^{-1}$, which is unnecessary. The solution must be the same!

For reference, working with $y_1(t) = -5\,t^{-1}$ gives the linear ODE $z' = \frac{5}{t}\,z - 1$ in b), and $z_h(t) = c\,t^5$, $z_p(t) = t/4$, $z(t) = t/4 + c\,t^5$, $y(t) = -\frac{5}{t} + \frac{1}{t/4 + c\,t^5}$ in c).

$$\sum_{3} = 7$$

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4 The characteristic polynomial of $\mathbf{A}$ is

$$\chi_{\mathbf{A}}(X) = \begin{vmatrix} X+8 & 0 & -5 & 2 \\ -5 & X+1 & 4 & -1 \\ 10 & 0 & X-7 & 2 \\ 0 & 0 & -3 & X-2 \end{vmatrix} = (X+1) \begin{vmatrix} X+8 & -5 & 2 \\ 10 & X-7 & 2 \\ 0 & -3 & X-2 \end{vmatrix}$$

$$= (X+1) \begin{vmatrix} X+8 & -5 & 2 \\ 2-X & X-2 & 0 \\ 0 & -3 & X-2 \end{vmatrix} = (X+1)(X-2) \begin{vmatrix} X+8 & -5 & 2 \\ -1 & 1 & 0 \\ 0 & -3 & X-2 \end{vmatrix}$$

$$= (X+1)(X-2) \begin{vmatrix} X+3 & -5 & 2 \\ 0 & 1 & 0 \\ -3 & -3 & X-2 \end{vmatrix} = (X+1)(X-2) \begin{vmatrix} X+3 & 2 \\ -3 & X-2 \end{vmatrix}$$

$$= (X+1)(X-2)\,[(X+3)(X-2) - (-3)2]$$
$$= X(X+1)^2(X-2).$$

$\Longrightarrow$ The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 0$ with algebraic multiplicity 1, $\lambda_2 = -1$ with algebraic multiplicity 2, $\lambda_3 = 2$ with algebraic multiplicity 1. $\boxed{2}$

$\underline{\lambda_1 = 0:}$

$$\mathbf{A}-0\,\mathbf{I} = \begin{pmatrix} -8 & 0 & 5 & -2 \\ 5 & -1 & -4 & 1 \\ -10 & 0 & 7 & -2 \\ 0 & 0 & 3 & 2 \end{pmatrix} \to \begin{pmatrix} -8 & 0 & 5 & -2 \\ 5 & -1 & -4 & 1 \\ -2 & 0 & 2 & 0 \\ -8 & 0 & 8 & 0 \end{pmatrix} \to \begin{pmatrix} -8 & 0 & 5 & -2 \\ 5 & -1 & -4 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The latter is in "permuted" echelon form with x_1 as a free variable, say. Setting $x_1 = 1$ gives $x_3 = 1$, $x_4 = -3/2$, $x_2 = -1/2$.

$\Longrightarrow$ The eigenspace corresponding to $\lambda_1 = 0$ is generated by $\mathbf{v}_1 = (2, -1, 2, -3)^\mathsf{T}$.

$\underline{\lambda_2 = -1:}$

$$\mathbf{A}+\mathbf{I} = \begin{pmatrix} -7 & 0 & 5 & -2 \\ 5 & 0 & -4 & 1 \\ -10 & 0 & 8 & -2 \\ 0 & 0 & 3 & 3 \end{pmatrix} \to \begin{pmatrix} 3 & 0 & -3 & 0 \\ 5 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 \end{pmatrix} \to \begin{pmatrix} 3 & 0 & -3 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Longrightarrow$ The eigenspace corresponding to $\lambda_2 = -1$ is generated by $\mathbf{v}_2 = (0, 1, 0, 0)^\mathsf{T}$ and $\mathbf{v}_3 = (-1, 0, -1, 1)^\mathsf{T}$.

$\underline{\lambda_3 = 2:}$

$$\mathbf{A}-2\,\mathbf{I} = \begin{pmatrix} -10 & 0 & 5 & -2 \\ 5 & -3 & -4 & 1 \\ -10 & 0 & 5 & -2 \\ 0 & 0 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 5 & -3 & -4 & 1 \\ 0 & -6 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Setting $x_4 = 1$ gives $x_3 = 0$, $x_2 = 0$, $x_1 = -1/5$.

$\Longrightarrow$ The eigenspace corresponding to $\lambda_3 = 2$ is generated by $\mathbf{v}_4 = (-1, 0, 0, 5)^\mathsf{T}$.

Since eigenvectors corresponding to different eigenvalues are linearly independent, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ form a basis of $\mathbb{R}^4$ (and $\mathbf{A}$ is diagonalizable).

$\Longrightarrow$ A fundamental system of solutions of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ is

$$\mathbf{y}_1(t) \equiv \begin{pmatrix} 2 \\ -1 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{y}_2(t) = \mathrm{e}^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{y}_3(t) = \mathrm{e}^{-t} \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{y}_4(t) = \mathrm{e}^{2t} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 5 \end{pmatrix},$$

$\boxed{4}$

and the general (real) solution is $\mathbf{y}(t) = c_1\,\mathbf{y}_1(t) + c_2\,\mathbf{y}_2(t) + c_3\,\mathbf{y}_3(t) + c_4\,\mathbf{y}_4(t)$, $c_1, c_2, c_3, c_4 \in \mathbb{R}$.

Remarks: Many students had difficulties with computing the characteristic polynomial $\chi_\mathbf{A}(X)$. Often Laplace expansion with respect to the first or last row was used, of which the latter is more efficient (and perhaps also more efficient than my solution). Doing this simultaneously for two rows, e.g., starting the expansion with $(X+8)(X+1)\left|\begin{smallmatrix} X-7 & 2 \\ -3 & X-2 \end{smallmatrix}\right|$ is dangerous because of the signs involved in Laplace expansion. At least this could explain, why so many students who expanded the determinant with respect to the first row ended up with the wrong sign for the last term, which in fact is $(-2)(-X-1)\left|\begin{smallmatrix} 10 & X-7 \\ 0 & -3 \end{smallmatrix}\right|$.

When computing eigenspaces (solution spaces of $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$) based on one or more false eigenvalues λ, the alarm bell should have rung if only the zero vector was obtained as solution. (Recompute the solution, and if that turns out be correct you know that λ is not an eigenvalue and you should then recompute the characteristic polynomial!) Nevertheless a few students stated the corresponding "solution" as $\mathbf{y}(t)=\mathrm{e}^{\lambda t}(0,0,0,0)^{\mathsf{T}}$.

A few students, having obtained the correct eigenspace for $\lambda=-1$ (i.e., two basis vectors $\mathbf{v}_1,\mathbf{v}_2$), stated the corresponding solutions as $t\mapsto \mathrm{e}^{-t}\mathbf{v}_1$ and $t\mapsto t\,\mathrm{e}^{-t}\mathbf{v}_2$. The 2nd "solution" is wrong: Solutions of $\mathbf{y}'=\mathbf{A}\mathbf{y}$ with diagonalizable $\mathbf{A}$ involve only pure exponentials $\mathrm{e}^{\lambda t}$ and, since distinct nonzero exponential polynomials are linearly independent, cannot be stated in this form.

Merely stating 4 solutions $\mathbf{y}_i(t)$ without mentioning at least that these form a fundamental system of solutions, or a basis of the solution space, wasn't accepted as satisfactory answer (-0.5 marks).

$$\sum_4 = 6$$

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5 a) Dividing (DF) by $3y^2-1$ gives the exact (even separable) equation

$$x\,\mathrm{d}x+\frac{y}{3y^2-1}\,\mathrm{d}y=0.$$

A function F with $\mathrm{d}F=x\,\mathrm{d}x+\frac{y}{3y^2-1}\,\mathrm{d}y$ is $F(x,y)=\frac{1}{2}x^2+\frac{1}{6}\ln|3y^2-1|$, and hence the general solution of (DF) in implicit form is

$$\frac{1}{2}x^2+\frac{1}{6}\ln\left|3y^2-1\right|=C,\quad C\in\mathbb{R}. \qquad \boxed{2}$$

This must be complemented by the horizontal lines $y=\pm 1/\sqrt{3}$, which have been lost when dividing by $3y^2-1$. Since $y=$ const. implies $\mathrm{d}y=0$, these are indeed solutions (even explicit solutions $y(x)\equiv\pm 1/\sqrt{3}$). $\boxed{1}$

b) $y(1)=1/3$ requires $C=\frac{1}{2}+\frac{1}{6}\ln\frac{2}{3}$. Then we solve the corresponding contour equation for y:

$$\begin{aligned}
\frac{1}{6}\ln\left|3y^2-1\right| &= \frac{1}{2}(1-x^2)+\frac{1}{6}\ln\frac{2}{3}\\
\ln\left|3y^2-1\right| &= 3(1-x^2)+\ln\frac{2}{3}\\
\left|3y^2-1\right| &= \frac{2}{3}\,\mathrm{e}^{3(1-x^2)}\\
3y^2-1 &= -\frac{2}{3}\,\mathrm{e}^{3(1-x^2)} && \left(\text{since } 3\,y(1)-1=-\tfrac{2}{3}\right)\\
y=y(x) &= \sqrt{\frac{1-\frac{2}{3}\,\mathrm{e}^{3(1-x^2)}}{3}} && \boxed{2}
\end{aligned}$$

The domain I of $y(x)$ is determined by

$$\begin{aligned} &1-\frac{2}{3}\,\mathrm{e}^{3(1-x^2)}\geq 0\\ \iff\quad &3(1-x^2)\leq \ln\frac{3}{2}\\ \iff\quad &x^2\geq 1-\frac{1}{3}\ln\frac{3}{2}\approx 0.87 \end{aligned}$$

Since I must be an interval containing 1, we obtain $I=\left[\sqrt{0.87},\infty\right)$. $\boxed{1}$

More precisely $I=(a,\infty)$ with $a=\sqrt{1-\frac{1}{3}\ln\frac{3}{2}}\approx 0.929970410262577$. Since $y(x)$ is not differentiable at a, we exclude a from the domain.

c) No. The integral curves of (DF) are the contours of $F(x,y)=\frac{1}{2}\,x^2+\frac{1}{6}\,\ln|3y^2-1|$. From $\mathrm{d}F(x,y)=x\,\mathrm{d}x+\frac{y}{3y^2-1}\,\mathrm{d}y$ we get $F_x=x$, $F_y=\frac{y}{3y^2-1}$, $F_{xx}=1$, $F_{xy}=F_{yx}=0$, and $F_{yy}=-\frac{3y^2+1}{(3y^2-1)^2}$. Since $F_x(0,0)=F_y(0,0)=0$, $F_{xx}(0,0)=1$, $F_{yy}(0,0)=-1$, the origin $(0,0)$ is a saddle point of F and hence contained in two distinct integral curves. $\boxed{+2}$

Remarks: Many students didn't notice that the ODE is separable and determined an integrating factor using the criteria from the lecture: $M(x,y)=x(3\,y^2-1)$, $N(x,y)=y$ satisfy $M_y-N_x=6xy-0=6xy$, $\frac{M_y-N_x}{N}=6x=g(x)$, $\frac{M_y-N_x}{M}=\frac{6y}{3\,y^2-1}=g(y)$, and hence there exist integrating factors of both forms $\mu(x)$, $\mu(y)$.

In the 1st case, $\mu(x)$ satisfies $\mu'=6x\,\mu$, so that we can take $\mu(x)=\mathrm{e}^{3x^2}$. The corresponding exact equation $x\,\mathrm{e}^{3x^2}(3y^2-1)\,\mathrm{d}x+y\,\mathrm{e}^{3x^2}\,\mathrm{d}y=0$ has implicit solution $F(x,y):=\frac{1}{2}\,y^2\,\mathrm{e}^{3x^2}-\frac{1}{6}\,\mathrm{e}^{3x^2}=C$, $C\in\mathbb{R}$, which can also be written as $\mathrm{e}^{3x^2}(3y^2-1)=C'$ with $C'=6C$ and, taking logs, as $3x^2+\log|3y^2-1|=C''$ with $C''=\ln|C'|$ (whereby the curves with parameters $\pm C'$ are joined and the one with $C'=0$ goes missing). Up to a constant factor this is the same as my solution.

In the 2nd case, $\mu(y)$ satisfies $\mu'=-\frac{6y}{3\,y^2-1}\,\mu$, so that we can take $\mu(y)=\frac{1}{3\,y^2-1}$. The corresponding exact equation is the same as in my solution.

The first integrating factor $\mu(x)=\mathrm{e}^{3x^2}$ has the advantage that it is $\neq 0$ everywhere, and hence no solutions are lost by proceeding to the corresponding exact equation.

In b) several students didn't provide the explicit solution $y(x)$, which also requires to decide on the sign of the square root. The corresponding two marks were only assigned for the final expiclit expression. Also, several students miscalculated the domain of $y(x)$, which must be (i) interval, (ii) contain the initial argument $x=1$, and (iii) be maximal with respect to these properties. Often the approximation $\sqrt{\frac{13}{15}}\approx 0.93$ for the left end point of the domain was used, which is better than 0.87 but wasn't honored with additional marks.

In c) many students claimed the answer is "Yes" – perhaps because it was so in Q5 of the first sample final exam. But the answer here is different, as can also be seen in the following more direct way: The 0-contour of F is $e^{3x^2}(3y^2-1)=-1$, or $3y^2=1-\mathrm{e}^{-3x^2}$, which by symmetry cannot have a smooth parametrization near the origin.

$$\sum_5 = 6+2$$

6 a) The characteristic polynomial is

$$\begin{aligned} a(X) &= X^5 + 4\,X^4 + 24\,X^3 + 40\,X^2 + 100\,X \\ &= X(X^4 + 4\,X^3 + 24\,X^2 + 40\,X + 100) \\ &= X(X^2 + 2\,X + 10)^2 \\ &= X(X + 1 - 3\mathrm{i})^2(X + 1 + 3\mathrm{i})^2 \end{aligned}$$

with zeros $\lambda_1 = 0$ of multiplicity 1 and $\lambda_2 = -1 + 3\mathrm{i}$, $\lambda_3 = -1 - 3\mathrm{i}$ of multiplicity 2.
$\Longrightarrow$ A complex fundamental system of solutions is 1, $\mathrm{e}^{(-1+3\mathrm{i})t}$, $t\,\mathrm{e}^{(-1+3\mathrm{i})t}$, $\mathrm{e}^{(-1-3\mathrm{i})t}$, $t\,\mathrm{e}^{(-1-3\mathrm{i})t}$ and the corresponding real fundamental system is

$$1, \quad \mathrm{e}^{-t}\cos(3t), \quad \mathrm{e}^{-t}\sin(3t), \quad t\,\mathrm{e}^{-t}\cos(3t), \quad t\,\mathrm{e}^{-t}\sin(3t). \qquad \boxed{2}$$

b) In order to obtain a particular solution $y_p(t)$ of the inhomogeneous equation, we solve the two systems $a(\mathrm{D})y_i = b_i(t)$ for $b_1(t) = 200\,t$, $b_2(t) = \mathrm{e}^{-t}$. Superposition then yields the particular solution $y_p(t) = y_1(t) - y_2(t)$.

(1) Since $\mu = 0$ is a zero of multiplicity 1 of $a(X)$, the proper Ansatz in this case is $y_1(t) = c_0 t + c_1 t^2$. Substituting it into the ODE we get

$$40(2c_1) + 100(c_0 + 2c_1 t) = 200\,t.$$

$\Longrightarrow c_1 = 1$, $c_0 = -\frac{4}{5}$, and $y_1(t) = t^2 - \frac{4}{5}\,t$. $\boxed{1}$

(2) Since $\mu = -1$ is not a root of $a(X)$, we can take $y_2(t) = \frac{1}{a(-1)}\,\mathrm{e}^{-t} = -\frac{1}{81}\,\mathrm{e}^{-t}$. $\boxed{1}$

$\Longrightarrow y_p(t) = t^2 - \frac{4}{5}\,t + \frac{1}{81}\,\mathrm{e}^{-t}$ is a particular solution. $\boxed{1}$

The general real solution is then

$$\begin{aligned} y(t) = c_1 + c_2\,\mathrm{e}^{-t}\cos(3t) + c_3\,\mathrm{e}^{-t}\sin(3t) + c_4\,t\,\mathrm{e}^{-t}\cos(3t) + c_5\,t\,\mathrm{e}^{-t}\sin(3t) \\ + t^2 - \tfrac{4}{5}\,t + \tfrac{1}{81}\,\mathrm{e}^{-t} \qquad \text{with } c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}. \quad \boxed{1} \end{aligned}$$

c) The Laplace transform of the right-hand side of the ODE in b) is

$$200\,\mathcal{L}\{t\} - \mathcal{L}\left\{e^{-t}\right\} = \frac{200}{s^2} - \frac{1}{s+1} = \frac{-s^2 + 200\,s + 200}{s^2(s+1)}. \qquad \boxed{1}$$

Using the formulas for the Laplace transform of the derivatives of $y(t)$ and the given initial conditions, this implies $s^5\,Y(s) + 4s^4\,Y(s) + 24s^3\,Y(s) + 40s^2\,Y(s) + 100s\,Y(s) = \frac{-s^2+200\,s+200}{s^2(s+1)}$, i.e.,

$$Y(s) = \frac{-s^2 + 200\,s + 200}{s^3(s+1)(s^2 + 2s + 10)^2}. \qquad \boxed{1}$$

Remarks: This question was generally solved well. Some of the more frequent errors were the following: Sometimes the multiplicity of the roots of $a(X)$ wasn't taken into account when stating the fundamental system (and hence only three fundamental solutions were obtained); sometimes an incorrect „Ansatz“ was used for computing a particular solution in b), which lead to a wrong result; and some students tried to solve c) by first determining the coeffients c_1, c_2, c_3, c_4, c_5 in b) from the initial conditions and then computing $\mathcal{L}\{y(t)\}$ directly, which isn't wrong but by way too costly.

$$\sum_{6} = 8$$

$$\sum_{\text{Final Exam}} = 45 + 7$$