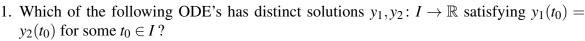
Name:		Student No.:		Group A
cations are required.	Each problem hancluding no answ	ind the correct answer as exactly one correct ser, multiple answers, or	solution, which	is worth 1 mark.
$y_2(t_0)$ for some	$t_0 \in I$?	has distinct solutions y		
y' = y	$y' = y^2$	$y' = y\sqrt{t}$		y' = ty
$2. (ye^{x+y}+1) dx + $	$-(e^{x+y} - x) dy = 0$	has the integrating f e^{-x}	e^{-y}	$\phantom{aaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaa$
3. For the solution	y(t) of the IVP	$y' = y^3 - y$, $y(0) = \frac{1}{2}$	the limit $\lim_{t\to +\infty}$ lim	y(t) is equal to
0	1		+∞	$-\infty$
4. For the solution	y(t) of the IVP	$y' = t y^3, \ y(0) = \frac{1}{2}$ t	the value $y(1)$ is	equal to
	$1/\sqrt{2}$	1	$\boxed{}$ $1/\sqrt{3}$	2
5. For the solution $\boxed{-1}$		$y' = 2t(y+1), \ y(0) =$ $2e - 1$		(1) is equal to
		of the IVP $t^2y'' + ty'$	-y = 0, y(1) = 0	0, y'(1) = 2 the
value $y(2)$ is eq $\frac{5}{2}$	2		0	
7. For which of the subspace of $\mathbb{R}^{\mathbb{R}}$'s does the set of solution	ons $\phi \colon \mathbb{R} \to \mathbb{R} \ \underline{\mathbf{n}}$	ot form a (linear)
1				y'' = t(y' - y)
8. The smallest in	teger s such that	$f_s(x) = \sum_{n=1}^{\infty} \frac{x \cos(nx)}{n^s + 1}$	is differential	ole on $\mathbb R$ is equal
to 0		2	3	4
9. The (real or con	nplex) solution sp	pace of $y^{(4)} + (1+t^2)$	y' = 0 has dim	nension
0	1		3	4
10. If $y = y(x)$ solve	$es y' = \frac{xy}{x^2 + y^2}$	then $z = y/x$ solves		
	z' =	$\frac{z^3}{1+z^2} \qquad \qquad z' =$	$-\frac{z^3}{1+z^2}$	

11. Any solution $y(t)$ of $y'' + 2y' + y = 0$ satisfying $y(0) = 0$ also satisfies
y'(0) = 0 $y'(0) = 1$ $y'(1) = 0$ $y'(1) = 1$ $y'(1) = 1$
12. The sequence $\phi_0, \phi_1, \phi_2, \dots$ of Picard-Lindelöf iterates for the IVP $y' = \sin y$, $y(0) = \pi/2$ has $\phi_2(t)$ equal to
$\boxed{\begin{array}{c} \pi/2 + t - t^3/6 \\ \pi/2 + t + t^2 \end{array}} \boxed{\pi/2 + \sin t} \boxed{\pi/2 + \cos t} \boxed{\pi/2 + \cos t}$
13. $y'' - y = e^t - 2$ has a particular solution $y_p(t)$ of the form
14. Which of the following points is on exactly one integral curve of the differential equation
$(y^2 - y) dx + (x^2 + x) dy = 0$?
15. Maximal solutions of $y' = y^2 + y + 1$ are defined on an interval of the form
Time allowed: 45 min CLOSED BOOK Good luck

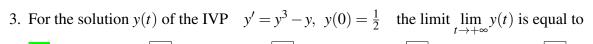
Name: _____ Student No.: ____ Group A

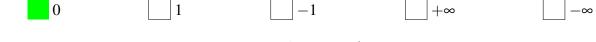
For each of the following problems, find the correct answer (tick as appropriate!). No justifications are required. Each problem has exactly one correct solution, which is worth 1 mark. Incorrect solutions (including no answer, multiple answers, or unreadable answers) will be assigned 0 marks; there are no penalties.

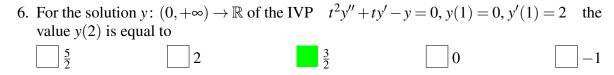


$$y' = |y| \qquad y' = y^2 \qquad y' = y\sqrt{t} \qquad y' = y$$

2.
$$(ye^{x+y}+1)dx+(e^{x+y}-x)dy=0$$
 has the integrating factor
$$0 1 e^{-x} e^{-y}$$







7. For which of the following ODE's does the set of solutions $\phi \colon \mathbb{R} \to \mathbb{R}$ <u>not</u> form a (linear) subspace of $\mathbb{R}^{\mathbb{R}}$? $y' = |t|y \qquad y'' = y' + y \qquad y' = t|y| \qquad ty' = y \qquad y'' = t(y' - y)$

8. The smallest integer
$$s$$
 such that $f_s(x) = \sum_{n=1}^{\infty} \frac{x \cos(nx)}{n^s + 1}$ is differentiable on \mathbb{R} is equal



9. The (real or complex) solution space of
$$y^{(4)} + (1+t^2)y' = 0$$
 has dimension 0 1 2 3 4

11. Any solution $y(t)$ of $y'' + 2y' + y'(0) = 0$ $y'(0) = 1$	y = 0 satisfying $y(0) = 0$ also s y'(1) = 0 $y'(1) = 0$	
12. The sequence $\phi_0, \phi_1, \phi_2, \dots$ of F $\pi/2$ has $\phi_2(t)$ equal to		P $y' = \sin y$, $y(0) = t$
13. $y'' - y = e^t - 2$ has a particular $c_0 + c_1 e^t$ $c_0 + c_1 t e^t$	· · ·	
14. Which of the following points is $(y^2 - y) dx + (x^2 + x) dy = 0$? $(0,1) \qquad (-1,1)$	on exactly one integral curve of the $(0,0)$ (-1)	
15. Maximal solutions of $y' = y^2 + \cdots + y^2 = y^2 + y^$	$+y+1$ are defined on an interval $(a,+\infty)$ $(-\infty,b)$	
Time allowed: 45 min	CLOSED BOOK	Good luck!

Notes

- 1. "ty' = y" is the only candidate, since it is an implicit ODE to which the Existence and Uniqueness Theorem (EUT) doesn't apply. The other 4 ODE's satisfy the assumptions of the EUT. For y' = |y| compare H25; $y' = y\sqrt{t}$ is 1st-order linear, the square root doesn't matter here, because Lipschitz continutiy with respect to t is not required by the EUT.
- 2. Although 0 dx + 0 dy = 0 is trivially exact, 0 is not an integrating factor, because integrating factors are required to be nonzero everywhere.
- 3. The phaseline should be used to answer this question.
- 4. This is a separable equation and can be solved with the standard method: The solution is $y(t) = 1/\sqrt{C-t^2}$, and $y(0) = \frac{1}{2}$ gives C = 4.
- 5. The solution is $y(t) = Ce^{t^2} 1$, since the associated homogeneous linear ODE is y' = 2ty and y' = 2t(y+1) has the particular solution $y(t) \equiv -1$. The initial condition, which was different for Groups A and B, leads to C = 3 (C = 2 for Group B) and y(1) = 3e 1 (y(1) = 2e 1 for Group B).
- 6. The Ansatz $y(t) = t^r$ gives $t^2y'' + ty' y = (r(r-1) + r 1)t^r = (r^2 1)t^r = 0. \Longrightarrow r_{1/2} = \pm 1$, and the general solution of the Euler equation is $y(t) = c_1t + c_2t^{-1}$. The solution of the IVP is then $y(t) = t t^{-1}$.
- 7. The only candidate is y' = t |y|, because the other 4 ODE's are linear and hence its solutions form a subspace of $\mathbb{R}^{\mathbb{R}}$. This is also true for the implicit equation ty' = y (in the lecture we have discussed it in detail for the related case of an Euler equation) and for y'' = t(y' y) (which is equivalent to y'' ty' + ty = 0). The solutions of y' = t |y| are $y(t) \equiv 0$, $y(t) = ce^{t^2/2}$ for c > 0, $y(t) = ce^{-t^2/2}$ for c < 0. They do not form a subspace of $\mathbb{R}^{\mathbb{R}}$, since, e.g., $t \mapsto e^{t^2/2} e^{-t^2/2}$ is not a solution. Thus the problem is indeed well-posed, although you didn't need to know this to find the correct answer.
- 8. For checking the differentiability of $f_s(x)$ one has to look at the series of derivatives, which is

$$\sum_{n=1}^{\infty} \frac{\cos(nx) - nx\sin(nx)}{n^s + 1},$$

and prove that this series converges uniformly on \mathbb{R} (it doesn't) or on all intervals of the form [-R,R], R>0 (it does). On [-R,R] we can estimate as follows:

$$\left|\frac{\cos(nx) - nx\sin(nx)}{n^s + 1}\right| \le \frac{1 + nR}{n^s + 1} \le \frac{2nR}{n^s} = \frac{2R}{n^{s-1}}$$

for n sufficiently large (such that $nR \ge 1$). Applying Weierstrass's Criterion gives uniform convergence on [-R,R] for $s \ge 3$. For s = 2 the function $f_s : \mathbb{R} \to \mathbb{R}$ is still well-defined but not differentiable at $x = 2k\pi$ for integers $k \ne 0$. This should be clear from the example $x \mapsto \sum_{n=1}^{\infty} \cos(nx)/n^2$ considered in the lecture. (Replacing n^2 by $n^2 + 1$ is inessential and the additional factor x certainly doesn't help with differentiability when $x \ne 0$.)

9. The solution space of an explicit (scalar) *n*-th order linear ODE has dimension *n*, regardless of whether the coefficients depend on *t* or are zero (as the coefficient of *y* in the case under consideration).

10.
$$y' = \frac{y/x}{1 + (y/x)^2} = \frac{z}{1 + z^2} \Longrightarrow z' = (y/x)' = (y'x - y)/x^2 = \frac{1}{x} \left(\frac{z}{1 + z^2} - z\right) = \frac{1}{x} \frac{-z^3}{1 + z^2}$$

11. The general solution is $y(t) = c_1 e^{-t} + c_2 t e^{-t}$. The value y(0) = 0 gives $c_1 = 0$, and hence $y'(t) = c_2(1-t)e^{-t}$, y'(1) = 0. We can also argue without any computation as follows: "y'(0) = 0" is ruled out, because the EUT says, e.g., there is a solution with y(0) = 0, y'(0) = 1. The other wrong answers are ruled out, because the all-zero solution doesn't satisfy them.

- 12. $\phi_0(t) = \pi/2$; $\phi_1(t) = \pi/2 + \int_0^t \sin(\pi/2) ds = \pi/2 + t$; $\phi_2(t) = \pi/2 + \int_0^t \sin(\pi/2 + s) ds = \pi/2 + [-\cos(\pi/2 + s)]_0^t = \pi/2 \cos(\pi/2 + t) = \pi/2 + \sin t$
- 13. A particular solution of y'' y = -2 is y = 2, and the correct "Ansatz" for obtaining a solution of $y'' y = e^t$ is $y(t) = c_1 t e^t$ (because $\lambda = 1$ is a root of multiplicity 1 of the characteristic polynomial $X^2 1$). Superposition then gives a solution of $y'' y = e^t 2$ of the form $c_0 + c_1 t e^t$, viz. $y(t) = 2 + \frac{1}{2} t e^t$.
- 14. The singular points are the solutions (x,y) of $y^2 y = x^2 + x = 0$, viz. the 4 combinations of $x \in \{-1,0\}$, $y \in \{0,1\}$. This leaves (1,1) as the only non-singular point, which according to the general theory must be on a unique integral curve.
- 15. Since $y^2 + y + 1 = 0$ has no real roots, the corresponding canonical form is $z' = z^2 + 1$, which is solved by $z(t) = \tan(t + C)$. Hence the domain of any maximal solution is a bounded open interval. We see this also when trying to compute the solution directly: $\frac{\mathrm{d}y}{y^2 + y + 1} = \frac{\mathrm{d}y}{(y + 1/2)^2 + 3/4} = \mathrm{d}t$. The integral is of the form $a \arctan(by + c) = t + C$, which yields $y(t) = a' \tan(b't + C') + c'$ (with fixed a', b', c' and variable C') and hence a bounded domain. The case a' = 0 cannot occur.