Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) There exists a solution $y(t)$ of $y' = \ln \frac{y^2+1}{2}$ satisfying $y(0) = 0$, $y(3) = 3$.

b) The maximal solution $y(t)$ of the initial value problem $y' = y^2 + t$, $y(0) = 1$ is defined at $t = \frac{1}{2}$.

c) The ODE $(x^4 - 1)y'' + (x^2 - 1)y' + (x - 1)y = 0$ has a nonzero power series solution $y(x) = \sum_{n=0}^{\infty} a_n (x + 2)^n$ which is defined at $x = -4$.

d) Every solution of the system $\mathbf{y}' = \begin{pmatrix} -1 & -3 \\ 3 & 1 \end{pmatrix} \mathbf{y}$ satisfies $\lim_{t\to+\infty} \mathbf{y}(t) = (0, 0)^{\mathsf{T}}$.

e) If $\mathbf{A} \in \mathbb{R}^{3\times3}$ satisfies $\mathbf{A}^3 = \mathbf{A}$ then $\mathrm{e}^{\mathbf{A}t} = \mathbf{I} + \sinh(t)\mathbf{A} + (\cosh t - 1)\mathbf{A}^2$. ($\mathbf{I}$ denotes the 3×3 identity matrix.)

f) Suppose $f, g\colon (0, \infty) \to \mathbb{R}$ are C^1-functions. Then the initial value problem $y' = f(t)g(y)$, $y(1) = 1$ has a solution $y(t)$ that is defined for all $t > 0$.

Question 2 (ca. 9 marks)

Consider the differential equation

$$2x^2y'' + (x^2 - 3x)y' + 2y = 0. \tag{DE}$$

a) Verify that $x_0 = 0$ is a regular singular point of (DE).

b) Determine the general solution of (DE) on $(0, \infty)$.

c) Using the result of b), state the general solution of (DE) on $(-\infty, 0)$ and on $\mathbb{R}$.

Question 3 (ca. 6 marks)

For the initial value problem

$$y' = \frac{y + t}{2y - t}, \qquad y(2) = 2, \tag{H}$$

determine the maximal solution $y(t)$ and its domain.

Hint: The substitution $z(t) = y(t)/t$ transforms (H) into a separable ODE. In order to see this, rewrite y' in terms of z. When solving the separable ODE, the formula $\int \frac{2az + b}{az^2 + bz + c}\,\mathrm{d}z = \ln\left|az^2 + bz + c\right| + C$ may be helpful.

Question 4 (ca. 8 marks)

Consider $\mathbf{A} = \begin{pmatrix} 3 & -2 & -1 \\ -1 & 1 & 0 \\ 8 & -6 & -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

a) Determine a fundamental system of solutions of the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

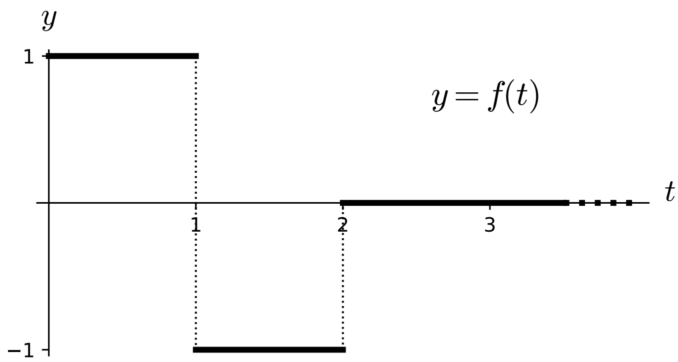
b) Solve the initial value problem $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b},\ \mathbf{y}(0) = (0,0,0)^{\mathsf{T}}$.
Hint: There is a particular solution of the form $\mathbf{y}(t) = \mathbf{w}_0 + t\,\mathbf{w}_1$ ($\mathbf{w}_0, \mathbf{w}_1 \in \mathbb{R}^3$).

Question 5 (ca. 6 marks)

For the function f sketched below, solve the initial value problem

$$y'' + 2\,y' + y = f(t), \quad y(0) = 1,\ y'(0) = 0$$

with the Laplace transform.



Note: For the solution $y(t)$ explicit formulas valid in the intervals $[0,1]$, $[1,2]$, $[2,\infty)$ are required. You *must* use the Laplace transform for the computation.

Question 6 (ca. 6 marks)

a) Determine a real fundamental system of solutions of

$$y''' + y'' - 2\,y = 0.$$

b) Determine the general real solution of

$$y''' + y'' - 2\,y = 1 - 2\,t^3 + \mathrm{e}^{-t}\cos t.$$

Solutions

1 a) False: $y' = \ln\frac{y^2+1}{2}$ has the constant solution $y_1(t) \equiv 1$. Because of continuity, a solution $y_2(t)$ with the indicated property would have to attain the value 1. If $y_2(t_0) = 1$ then, on the domain of $y_2(t)$, we would have two distinct solutions of the IVP $y' = \ln\frac{y^2+1}{2}$, $y(t_0) = 1$, which according to the Existence and Uniqueness Theorem is impossible. $\boxed{2}$

b) True. Denoting the maximal domain by (a, b), we have $y'(t) > 0$ for $t \in [0, b)$, i.e., $y(t)$ is increasing on $[0, b)$. Thus, if b is finite, we must have $\lim_{t\uparrow b} y(t) = +\infty$. On the other hand, as long as $0 \leq t \leq 1$ and $y(t)$ exists, it is bounded from above by the solution $z(t)$ of $z' = z^2 + 1$, $z(0) = 1$, which is $z(t) = \tan(t + \pi/4)$ and exists for $t \in [0, \pi/4)$. Hence $b \geq \pi/4 > 1/2$, and $y(1/2)$ is well-defined. $\boxed{2}$

c) False. The point $x_0 = -2$ is an ordinary point, so that nonzero power series solutions $y(x)$ of the indicated form exist, but their guaranteed radius of convergence (and in fact the true radius of convergence) is only the distance from -2 to the nearest singularity of $q(x) = \frac{x-1}{x^4-1} = \frac{1}{(x+1)(x^2+1)}$, which is -1. Thus $R = 1$ and $y(x)$ is not defined at $x = -4$. $\boxed{2}$

d) False. As derived in the lecture, this is true iff the system is asymptotically stable, which in turn is the case iff the eigenvalues of $\mathbf{A} = \left(\begin{smallmatrix}-1 & -3\\ 3 & 1\end{smallmatrix}\right)$ have negative real part. But $\lambda_1 + \lambda_2 = \operatorname{tr}(\mathbf{A}) = -1 + 1 = 0$, contradiction! (In fact $\chi_{\mathbf{A}}(X) = X^2 + 8$, and $\lambda_{1/2} = \pm 2\sqrt{2}\,\mathrm{i}$ are purely imaginary. $\boxed{2}$

e) True. We have $a(\mathbf{A}) = \mathbf{0}$ for $a(X) = X^3 - X = X(X-1)(X+1)$. The ODE $a(\mathrm{D})y = 0$ has the fundamental system 1, e^t, e^{-t}, Hence 1, $\sinh t = \frac{1}{2}\mathrm{e}^t - \frac{1}{2}\mathrm{e}^{-t}$, $\cosh t - 1 = -1 + \frac{1}{2}\mathrm{e}^t + \frac{1}{2}\mathrm{e}^{-t}$ solve the ODE. Since the corresponding Wronski matrix is the 3×3 identity matrix, $\mathrm{e}^{\mathbf{A}t}$ admits the indicated representation; cf. lecture. $\boxed{2}$

f) False. Here is a counterexample: Take $f(t) = 1$, $g(y) = y^2$, so that the ODE is $y' = y^2$. Its solutions are $y(t) = 1/(C - t)$, $C \in \mathbb{R}$. The (maximal) solution satisfying $y(1) = 1$ is the one with $C = 2$, and is defined on $(-\infty, 2)$. Hence no solution of the IVP is defined at $t = 2$ (or at larger t). $\boxed{2}$

Remarks:

a) was solved by most students, using either the indicated argument or the phase line as follows: Since $f(y) = \ln\frac{y^2+1}{2}$ has zeros ± 1, and $f(y) < 0$ for $-1 < y < 1$, the (maximal) solution with $y(0) = 0$ is decreasing and hence cannot satisfy $y(3) = 3$. (More precisely, it has domain $\mathbb{R}$ and satisfies $\lim_{t\to\infty} y(t) = -1$, $\lim_{t\to-\infty} y(t) = 1$.)

b) Only few students were able to solve this question. Many students obtained partial results such as the trivial lower bound 0 or the sharper lower bound $y(t) \geq 1/(1 - t)$, which comes from $y' = y^2 + t \geq y^2$ for $t \geq 0$. But lower bounds cannot prevent the solution from blowing up to $+\infty$ anywhere. The sharper lower bound only gives that $y(t)$ is not defined at $t = 1$.

c) Many students overlooked the singularity at -1 and obtained as guaranteed radius of convergence the distance from -2 to $\pm\mathrm{i}$, which is $\sqrt{5}$. Since the disk $|z+2| < \sqrt{5}$ contains -4, they concluded that the answer is "True". I have assigned 1 mark in this case (but only if the singularity at -1 wasn't found).

   It should be noted that 1 is not a singularity, since the whole ODE can be divided by $x-1$ without changing the solution.

d) was solved by most students, usually by obtaining the general form $\mathbf{y}(t) = c_1\mathrm{e}^{2\sqrt{2}\,\mathrm{i}t}\mathbf{v}_1 + c_2\mathrm{e}^{-2\sqrt{2}\,\mathrm{i}t}\mathbf{v}_2 = c_1'\cos(2\sqrt{2}t)\mathbf{v}_1' + c_2\sin(2\sqrt{2}t)\mathbf{v}_2'$ of the solution first. It was not necessary to compute $\mathbf{v}_1, \mathbf{v}_2$ or $\mathbf{v}_1', \mathbf{v}_2'$, but I have subtracted 0.5 marks if only (scalar?) constants c_1, c_2 were present.

e) was solved by many students, but mostly in other ways. For example, one can use $\mathrm{e}^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{A}^k$ and $\mathbf{A}^3 = \mathbf{A}$, $\mathbf{A}^4 = \mathbf{A}^2$, $\mathbf{A}^5 = \mathbf{A}^3 = \mathbf{A}$, $\mathbf{A}^6 = \mathbf{A}^6 = \mathbf{A}^2$, etc., to express $\mathrm{e}^{\mathbf{A}t}$ as the sum of $\mathbf{I}$, $\sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}\mathbf{A} = \sinh(t)\mathbf{A}$, and $\sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!}\mathbf{A}^2 = (\cosh t - 1)\mathbf{A}^2$; or one can show that $\Phi(t) = \mathbf{I} + \sinh(t)\mathbf{A} + (\cosh t - 1)\mathbf{A}^2$ satisfies $\Phi'(t) = \mathbf{A}\phi(t)$ and $\Phi(0) = \mathbf{I}$, which together characterize $t \mapsto \mathrm{e}^{\mathbf{A}t}$.

   Some students concluded from $\mathbf{A}^3 = \mathbf{A}$ that $\mathbf{A}^2 = \mathbf{I}$, which is wrong for non-invertible matrices.

f) was solved completely only by a few students. At least you should have been able to recall from the lecture that separable ODE's $y' = f(t)g(y)$ may have maximal solutions whose domain is strictly smaller than the domain of f, which when quoted correctly was worth 1 mark.

$$\sum_1 = 12$$

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2 a) The explicit form of (DE) is

$$y'' + \left(\frac{1}{2} - \frac{3}{2x}\right)y' + \frac{1}{x^2}\,y = 0$$

   $p(x) := \frac{1}{2} - \frac{3}{2x}$ has a pole of order 1 at 0, and $q(x) := \frac{1}{x^2}$ has a pole of order 2 at 0. This shows that 0 is a regular singular point of (DE). $\boxed{1}$

   Alternatively, use that the limits defining p_0, q_0 below are finite.

b) From a) we have $p_0 = \lim_{x\to 0} x\,p(x) = -3/2$, $q_0 = \lim_{x\to 0} x^2\,q(x) = 1$. (These coefficients can just be read off from the explicit form.)
   $\Longrightarrow$ The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = r^2 - \frac{5}{2}r + 1 = (r-2)(r-1/2) = 0.$$

$\Longrightarrow$ The exponents at the singularity $x_0 = 0$ are $r_1 = 2$, $r_2 = 1/2$. Since $r_1 - r_2 \notin \mathbb{Z}$, there exist two fundamental solutions y_1, y_2 of the form

$$y_1(x) = x^2 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+2},$$
$$y_2(x) = x^{1/2} \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} b_n x^{n+1/2} \qquad \boxed{1}$$

with normalization $a_0 = b_0 = 1$.

First we determine $y_1(x)$. We have

$$\begin{aligned}
0 &= 2x^2 y_1'' + (x^2 - 3x) y_1' + 2\,y_1 \\
&= 2x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_n x^n + (x^2 - 3x) \sum_{n=0}^{\infty} (n+2) a_n x^{n+1} + 2 \sum_{n=0}^{\infty} a_n x^{n+2} \\
&= \sum_{n=0}^{\infty} \left[2(n+2)(n+1) - 3(n+2) + 2\right] a_n x^{n+2} + \sum_{n=0}^{\infty} (n+2) a_n x^{n+3} \\
&= \sum_{n=0}^{\infty} (2n^2 + 3n) a_n x^{n+2} + \sum_{n=1}^{\infty} (n+1) a_{n-1} x^{n+2} \\
&= \sum_{n=1}^{\infty} \left[n(2n+3) a_n + (n+1) a_{n-1}\right] x^{n+2}.
\end{aligned}$$

Equating coefficients gives the recurrence relation

$$a_n = -\frac{n+1}{n(2n+3)} a_{n-1} \quad \text{for } n = 1, 2, 3, \dots, \qquad \boxed{1}$$

and with $a_0 = 1$ further $a_n = (-1)^n \frac{n+1}{5 \cdot 7 \cdot 9 \cdots (2n+3)}$ for $n \geq 1$.

$$\begin{aligned}
\Longrightarrow y_1(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{n+1}{5 \cdot 7 \cdot 9 \cdots (2n+3)}\, x^{n+2} \qquad \boxed{1\tfrac{1}{2}} \\
&= x^2 - \frac{2}{5}\, x^3 + \frac{3}{5 \cdot 7}\, x^4 - \frac{4}{5 \cdot 7 \cdot 9}\, x^5 + \frac{5}{5 \cdot 7 \cdot 9 \cdot 11}\, x^6 \mp \cdots
\end{aligned}$$

(For $n = 1$ the product in the denominator is understood as the the empty product 1.)

For the determination of $y_2(x)$ we repeat the process with exponents decrreased by

1.5 :

$$\begin{aligned}
0 &= 2x^2\,y_2'' + (x^2-3x)y_2' + 2\,y_2\\
&= 2x^2\sum_{n=0}^{\infty}\left(n+\tfrac{1}{2}\right)\left(n-\tfrac{1}{2}\right)b_n x^{n-3/2} + (x^2-3x)\sum_{n=0}^{\infty}\left(n+\tfrac{1}{2}\right)b_n x^{n-1/2} + 2\sum_{n=0}^{\infty} b_n x^{n+1/2}\\
&= \sum_{n=0}^{\infty}\left[2\left(n+\tfrac{1}{2}\right)\left(n-\tfrac{1}{2}\right) - 3\left(n+\tfrac{1}{2}\right)+2\right]b_n x^{n+1/2} + \sum_{n=0}^{\infty}\left(n+\tfrac{1}{2}\right)b_n x^{n+3/2}\\
&= \sum_{n=0}^{\infty}(2n^2-3n)b_n x^{n+1/2} + \sum_{n=1}^{\infty}\left(n-\tfrac{1}{2}\right)b_{n-1}x^{n+1/2}\\
&= \sum_{n=1}^{\infty}\left[n(2n-3)b_n + \left(n-\tfrac{1}{2}\right)b_{n-1}\right]x^{n+1/2}.
\end{aligned}$$

Here we obtain the recurrence relation

$$b_n = -\frac{n-\frac{1}{2}}{n(2n-3)}\,b_{n-1} = -\frac{2n-1}{2n(2n-3)}\,b_{n-1}\quad\text{for } n=1,2,3,\dots, \qquad \boxed{1}$$

and with $b_0=1$ further $b_n = (-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)(-1)1\cdot 3\cdots(2n-3)} = (-1)^{n-1}\frac{2n-1}{2\cdot 4\cdot 6\cdots(2n)}$ for $n\geq 1$.

$$\begin{aligned}
\implies\ y_2(x) &= x^{1/2} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n-1}{2\cdot 4\cdot 6\cdots(2n)}\,x^{n+1/2} \qquad \boxed{1\tfrac{1}{2}}\\
&= x^{1/2} + \frac{1}{2}\,x^{3/2} - \frac{3}{2\cdot 4}\,x^{5/2} + \frac{5}{2\cdot 4\cdot 6}\,x^{7/2} - \frac{7}{2\cdot 4\cdot 6\cdot 8}\,x^{9/2} \mp\cdots
\end{aligned}$$

Alternative solution: We use the general recurrence relation for the rational functions $a_n(r)$, viz. $a_0(r)=1$ and

$$a_n(r) = -\frac{1}{F(r+n)}\sum_{k=0}^{n-1}\left[(r+k)p_{n-k}+q_{n-k}\right]a_k(r)\quad\text{for } n\geq 1.$$

Since $F(r)=(r-2)(r-1/2)$ and all coefficients p_i, q_i except for p_0, q_0 and $p_1=1/2$ are zero, we obtain

$$\begin{aligned}
a_n(r) &= -\frac{(r+n-1)p_1}{(r+n-2)(r+n-1/2)}\,a_{n-1}(r)\\
&= -\frac{r+n-1}{(r+n-2)(2r+2n-1)}\quad\text{for } n\geq 1.
\end{aligned}$$

Thus the coefficients $a_n(2)$ of $y_1(x)$ satisfy the recurrence relation $a_n(2) = -\frac{n+1}{n(2n+3)}\,a_{n-1}(2)$ (the same as for a_n above) and the coefficients $a_n(1/2)$ of $y_2(x)$ satisfy the recurrence relation $a_n(1/2) = -\frac{n-1/2}{(n-3/2)2n}\,a_{n-1}(1/2) = -\frac{2n-1}{(2n-3)2n}\,a_{n-1}(1/2)$ (the same as for b_n above). The rest of the computation remains the same.

The general (real) solution on $(0,\infty)$ is then $y(x) = c_1\,y_1(x) + c_2\,y_2(x)$, $c_1, c_2\in\mathbb{R}$. $\boxed{\tfrac{1}{2}}$

That solutions are defined on the whole of $(0,\infty)$, is guaranteed by the analyticity of $p(x)$, $q(x)$ in $\mathbb{C}\setminus\{0\}$, but follows also readily from the easily established fact that the radius of convergence of both power series is ∞. $\boxed{\tfrac{1}{2}}$

c) The solution on $(-\infty,0)$ is $y(x)=c_1\,y_1(x)+c_2\,y_2^-(x)$ with the same power series $y_1(x)$ as in b) and

$$y_2^-(x)=(-x)^{1/2}\sum_{n=0}^{\infty}\frac{(-1)^{n-1}(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\,x^n. \qquad \boxed{1}$$

(This is <u>not</u> the same as $y_2(-x)$, which has negative coefficients when written in terms of powers of $-x$.)

Since $y_1(x)$ is analytic at zero but $y_2(x)$ is not, the general solution on $\mathbb{R}$ is $y(x)=c\,y_1(x)$, $c\in\mathbb{R}$. $\boxed{1}$

Remarks: a) was solved by virtually all students.

Most students also essentially solved b), sometimes with a little penality for missing the general solution $y=c_1y_1+c_2y_2$, the radius of convergence of y_1 and y_2, or not simplyfing the formulas obtained for the coefficients a_n, b_n. Some students made more serious errors like starting the recurrence relation at $n=0$ (if $a_n=u_na_{n-1}$ and $a_0=1$ then $a_n=u_na_{n-1}=u_nu_{n-1}a_{n-2}=u_nu_{n-1}\cdots u_1a_0=u_nu_{n-1}\cdots u_1\,!$), or using "half-factorials" like $\left(n-\frac{1}{2}\right)!$ which haven't been defined.

Many students had problems with c). The solution on $\mathbb{R}$ is not just a combination of the solutions on $(0,\infty)$ and $(-\infty,0)$, because the ODE needs to be satisfied also at $x=0$. In particular, a solution on $\mathbb{R}$ needs to be defined and differentiable at $x=0$. Also, the term "analytic function" still seems to be a foreign word for quite a few students, who claimed "none of y_1, y_2 are analytic at $x=0$ and hence the only solution on $\mathbb{R}$ is $y=0$".

$$\sum_2 = 10$$

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3 Suppressing the argument t as usual, we have $y'=\frac{y/t+1}{2y/t-1}=\frac{z+1}{2z-1}$ and hence

$$z'=\left(\frac{y}{t}\right)'=\frac{y't-y}{t^2}=\frac{y'-z}{t}=\frac{1}{t}\left(\frac{z+1}{2z-1}-z\right)=\frac{-2z^2+2z+1}{t(2z-1)}. \qquad \boxed{2}$$

This is a separable equation and can be solved by the usual method, noting that $y(2)=2$

corresponds to $z(2)=1$:

$$\begin{aligned}
\frac{2z-1}{-2z^2+2z+1}\,\mathrm{d}z &= \frac{\mathrm{d}t}{t}\\
\int_1^z \frac{2\zeta-1}{-2\zeta^2+2\zeta+1}\,\mathrm{d}\zeta &= \int_2^t \frac{\mathrm{d}\tau}{\tau}\\
\left[-\frac{1}{2}\ln\left|-2\zeta^2+2\zeta+1\right|\right]_1^z &= \left[\ln|\tau|\right]_2^t\\
-\frac{1}{2}\ln\left(-2z^2+2z+1\right) &= \ln t-\ln 2=\ln\frac{t}{2}\\
\ln\left(-2z^2+2z+1\right) &= -2\ln\frac{t}{2}\\
-2z^2+2z+1 &= \mathrm{e}^{-2\ln\frac{t}{2}}=\frac{4}{t^2}\\
2z^2-2z-1+\frac{4}{t^2} &= 0\\
z &= \frac{1}{4}\left(2\pm\sqrt{4-8\left(\frac{4}{t^2}-1\right)}\right)=\frac{1}{2}\left(1\pm\sqrt{3-\frac{8}{t^2}}\right) \qquad \boxed{3}
\end{aligned}$$

Since $z(2)=1$, the correct sign is '+'. The solution of (H) is then

$$y(t)=t\,z(t)=\frac{t}{2}\left(1+\sqrt{3-\frac{8}{t^2}}\right) \qquad \boxed{1}$$

with maximal domain determined by $3-8/t^2>0$, i.e., $t>\sqrt{8/3}=\frac{2\sqrt{2}}{\sqrt{3}}$ (since it must be an interval containing $t=2$). $\boxed{1}$.

Remarks: This question was answered in full only by 2-3 students. Most students obtained the solution only in implicit form, for which we have assigned at most 4.5 marks (solving the quadratic was worth one mark, resolving $\pm$ in the solution a further 0.5 marks, and the maximal domain 1 mark). Even those who obtained the correct expression for $y(t)$ often missed the maximal domain, which must be an interval, and must not contain the end point $\sqrt{8/3}$.

$$\sum_3 = 7$$

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4 a) The characteristic polynomial of $\mathbf{A}$ is

$$\chi_{\mathbf{A}}(X)=\begin{vmatrix} X-3 & 2 & 1\\ 1 & X-1 & 0\\ -8 & 6 & X+2\end{vmatrix}=\begin{vmatrix} X-3 & 2 & 1\\ 1 & X-1 & 0\\ -2+X-X^2 & 2-2X & 0\end{vmatrix}$$

$$=\begin{vmatrix} X-3 & 2 & 1\\ 1 & X-1 & 0\\ X-X^2 & 0 & 0\end{vmatrix}=(-1)(X-X^2)(X-1)=X(X-1)^2.$$

$\Longrightarrow$ The eigenvalues of $\mathbf{A}$ are $\lambda_1=0$ with algebraic multiplicity 1 and $\lambda_2=1$ with algebraic multiplicity 2. $\boxed{2}$

$$\mathbf{A}-0\,\mathbf{I}=\begin{pmatrix}3&-2&-1\\-1&1&0\\8&-6&-2\end{pmatrix}\to\begin{pmatrix}-3&2&1\\1&-1&0\\0&0&0\end{pmatrix}$$

(Substitute $X=0$ in the computation above.)
$\Longrightarrow$ The eigenspace corresponding to $\lambda_2=1$ is one-dimensional and generated by $\mathbf{v}_2=(1,1,1)^\mathsf{T}$. (This is also clear from the fact that $\mathbf{A}$ has constant row sums zero.)

$$\mathbf{A}-\mathbf{I}=\begin{pmatrix}2&-2&-1\\-1&0&0\\8&-6&-3\end{pmatrix}\to\begin{pmatrix}-2&2&1\\1&0&0\\0&0&0\end{pmatrix}$$

(Substitute $X=1$ in the computation above.)
$\Longrightarrow$ The eigenspace corresponding to $\lambda_1=1$ is one-dimensional and generated by $\mathbf{v}_2=(0,1,-2)^\mathsf{T}$.

A further generalized eigenvector $\mathbf{v}_3$ can be found by solving $(\mathbf{A}-\mathbf{I})\mathbf{v}_3=\mathbf{v}_2$:

$$\left(\begin{array}{ccc|c}2&-2&-1&0\\-1&0&0&1\\8&-6&-3&-2\end{array}\right)\to\left(\begin{array}{ccc|c}-1&0&0&1\\0&-2&-1&2\\0&-6&-3&6\end{array}\right)\to\left(\begin{array}{ccc|c}-1&0&0&1\\0&-2&-1&2\\0&0&0&0\end{array}\right),$$

e.g., $\mathbf{v}_3=(-1,-1,0)^\mathsf{T}$.

The corresponding fundamental system of solutions is:

$$\mathbf{y}_1(t)=\begin{pmatrix}1\\1\\1\end{pmatrix},$$
$$\mathbf{y}_2(t)=\mathrm{e}^t\begin{pmatrix}0\\1\\-2\end{pmatrix},$$
$$\mathbf{y}_3(t)=\mathrm{e}^t\begin{pmatrix}-1\\-1\\0\end{pmatrix}+t\,\mathrm{e}^t\begin{pmatrix}0\\1\\-2\end{pmatrix}. \qquad \boxed{3}$$

Changing signs in $\mathbf{y}_3(t)$, i.e., choosing $(0,-1,2)$ as generator of the eigenspace for $\lambda_2=1$ makes the figures slightly simpler.

b) $\mathbf{y}(t)=\mathbf{w}_0+t\,\mathbf{w}_1$ is a solution iff $\mathbf{w}_1=\mathbf{y}'(t)=\mathbf{A}\mathbf{y}(t)+\mathbf{b}=\mathbf{A}\mathbf{w}_0+t\,\mathbf{A}\mathbf{w}_1+\mathbf{b}$, which is equivalent to $\mathbf{w}_1=\mathbf{A}\mathbf{w}_0+\mathbf{b}\wedge\mathbf{A}\mathbf{w}_1=\mathbf{0}$. Thus we need to solve $\mathbf{A}^2\mathbf{w}_0+\mathbf{A}\mathbf{b}=\mathbf{0}$. $\boxed{1}$

$$\mathbf{A}^2=\begin{pmatrix}3&-2&-1\\-4&3&1\\14&-10&-4\end{pmatrix},\quad \mathbf{A}\mathbf{b}=\begin{pmatrix}-2\\1\\-6\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}3&-2&-1&2\\-4&3&1&-1\\14&-10&-4&6\end{array}\right)\to\left(\begin{array}{ccc|c}-1&1&0&1\\-4&3&1&-1\\-2&2&0&2\end{array}\right)\to\left(\begin{array}{ccc|c}-1&1&0&1\\-4&3&1&-1\\0&0&0&0\end{array}\right)$$

The solution with $x_1 = 0$ is $x_2 = 1$, $x_3 = -4$, i.e., $\mathbf{w}_0 = (0, 1, -4)^\mathsf{T}$, giving

$$\mathbf{w}_1 = \begin{pmatrix} 3 & -2 & -1 \\ -1 & 1 & 0 \\ 8 & -6 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix},$$

$$\mathbf{y}_p(t) = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + t \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}. \qquad \boxed{2}$$

The general solution of $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$ is $\mathbf{y}(t) = \mathbf{y}_p(t) + c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + c_3\mathbf{y}_3(t)$. In order to satisfy the required initial condition, (c_1, c_2, c_3) needs to solve

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & -2 & 0 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -2 & 1 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$\Longrightarrow c_3 = 2$, $c_2 = -1$, $c_1 = 2$, and the final answer is

$$\begin{aligned} y(t) &= 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \mathrm{e}^t \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + 2\,\mathrm{e}^t \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + 2t\,\mathrm{e}^t \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + t \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 + 2t - 2\,\mathrm{e}^t \\ 3 + 2t - 3\,\mathrm{e}^t + 2t\,\mathrm{e}^t \\ -2 + 2t + 2\,\mathrm{e}^t - 4t\,\mathrm{e}^t \end{pmatrix}. \qquad \boxed{2} \end{aligned}$$

Remarks: Most students obtained $\chi_{\mathbf{A}}(X)$ correctly and earned the first 2 marks—often without justification, which we haven't penalized this time. A few students multiplied the first row of the polynomial matrix with $X+2$ in order to eliminate the $(i, j) = (1, 3)$ entry, which is possible but multiplies the determinant by $X + 2$ and needs to be accounted for at the end. Otherwise the result, viz. $(X + 2)X(X - 1)^2$ is wrong (-1 mark).

The 2 marks for the 2 fundamental solutions corresponding to the 2 eigenvectors were also earned by most students. But many students had problems with the 3rd fundamental solution. Often the chain of generalized eigenvectors defining the solution didn't start with a generalized eigenvector but with an arbitrarily chosen vector in $\mathbb{R}^3$, such as $(1, 0, 0)^\mathsf{T}$ (probably mislead by the 5×5 example in a homework exercise, where such a choice was possible). As the generalized eigenspace for $\lambda = -1$ is 2-dimensional and generated by $(0, 1, -2)$ and $(1, 1, 0)$, i.e., parametrized as $(t, s + t, -2s)^\mathsf{T}$ with $s, t \in \mathbb{R}$, this was easy for us to detect. (Even simpler, a correct chain should have length 2 and stop with a nonzero mutiple of $(0, 1, -2)^\mathsf{T}$.)

b) was admittedly not straightforward, but with the given hint could at least be solved without any magic. Those who ignored the hint, and there were many, could at most earn 0.5 marks for b) if the form $\mathbf{y} = \mathbf{y}_p + \mathbf{y}_h$ of the general solution was mentioned. A few students solved b) completely (5 marks) or obtained at least $\mathbf{y}_p(t)$ (3 marks), while another few earned the 1st mark but then got stuck at the condition $\mathbf{w}_1 = \mathbf{A}\mathbf{w}_0 + \mathbf{b} \wedge \mathbf{A}\mathbf{w}_1 = \mathbf{0}$. The difficulty here is to realize that $\mathbf{w}_1$ must be a multiple of $(1, 1, 1)^\mathsf{T}$, cf. a), but need not be equal to $(1, 1, 1)^\mathsf{T}$. (In fact $\mathbf{w}_1 = 2(1, 1, 1)^\mathsf{T}$ is uniquely determined, and $\mathbf{w}_0$ may differ from $(0, 1, -4)^\mathsf{T}$ by any multiple of $(1, 1, 1)^\mathsf{T}$.

$$\sum_4 = 10$$

5 Writing $Y(s)=\mathcal{L}\{y(t)\}$, $F(s)=\mathcal{L}\{f(t)\}$, and applying the Laplace transform to both sides of the ODE gives

$$\begin{aligned}\mathcal{L}\{y''+2\,y'+y\}&=s^2\,Y(s)-s\,y(0)+2\bigl(s\,Y(s)-y(0)\bigr)+Y(s)\\&=(s^2+2s+1)Y(s)-s-2=\mathcal{L}\bigl\{f(t)\bigr\}=F(s).\end{aligned}$$

Further we have

$$\begin{aligned}f(t)&=\mathrm{u}(t)-\mathrm{u}(t-1)-\bigl(\mathrm{u}(t-1)-\mathrm{u}(t-2)\bigr)\\&=\mathrm{u}(t)-2\mathrm{u}(t-1)+\mathrm{u}(t-2), && \boxed{1}\\ \implies F(s)&=\frac{1-2\,\mathrm{e}^{-s}+\mathrm{e}^{-2s}}{s}. && \boxed{1}\\ \implies Y(s)&=\frac{s+2}{(s+1)^2}+\frac{1-2\,\mathrm{e}^{-s}+\mathrm{e}^{-2s}}{s(s+1)^2} && \boxed{1}\\&=\frac{1}{s}+\frac{-2\,\mathrm{e}^{-s}+\mathrm{e}^{-2s}}{s(s+1)^2}\\&=\frac{1}{s}+\left(\frac{1}{s}-\frac{1}{s+1}-\frac{1}{(s+1)^2}\right)\left(-2\,\mathrm{e}^{-s}+\mathrm{e}^{-2s}\right) && \boxed{1}\end{aligned}$$

The inverse Laplace transform of $\frac{1}{s}-\frac{1}{s+1}-\frac{1}{(s+1)^2}$ is $1-\mathrm{e}^{-t}-t\,\mathrm{e}^{-t}$.

$$\begin{aligned}\implies y(t)&=1-2\,\mathrm{u}_1(t)\left(1-\mathrm{e}^{1-t}+(1-t)\mathrm{e}^{1-t}\right)+\mathrm{u}_2(t)\left(1-\mathrm{e}^{2-t}+(2-t)\mathrm{e}^{2-t}\right)\\&=1-2\,\mathrm{u}_1(t)\left(1-t\,\mathrm{e}^{1-t}\right)+\mathrm{u}_2(t)\left(1+\mathrm{e}^{2-t}-t\,\mathrm{e}^{2-t}\right) && \boxed{1}\\&=\begin{cases}1 & \text{for } 0\leq t\leq 1,\\ -1+2t\,\mathrm{e}^{1-t} & \text{for } 1\leq t\leq 2,\\ 2t\,\mathrm{e}^{1-t}+\mathrm{e}^{2-t}-t\,\mathrm{e}^{2-t} & \text{for } t\geq 2.\end{cases} && \boxed{1}\end{aligned}$$

The 3rd expression can also be written as $(2t+\mathrm{e}-t\,\mathrm{e})\mathrm{e}^{1-t}$.
Remarks: Many students overlooked that the initial conditions are not $y(0)=y'(0)=0$ this time and obtained $Y(s)=\frac{1-2\,\mathrm{e}^{-s}+\mathrm{e}^{-2s}}{s(s+1)^2}$. For a correct solution of this modified IVP we have assigned 4 marks. Also, the stated simplification of $Y(s)$ (with first summand $1/s$) was often overlooked (no penalty), and as a result the expression for $y(t)$ became very long and prone to errors. Of those who obtained the correct solution $y(t)$, many didn't understand the request for a solution free of heaviside functions (or hadn't enough time to compute it), losing the last mark. Other errors were omitting the factors $\mathrm{u}_1(t)$, $\mathrm{u}_2(t)$, or translating only part of the arguments of $1-\mathrm{e}^{-t}-t\,\mathrm{e}^{-t}$ by one or two, such as inverting $\frac{\mathrm{e}^{-s}}{(s+1)^2}$ into $t\,\mathrm{e}^{-(t-1)}$ (it should be $(t-1)\mathrm{e}^{-(t-1)}$ for $t\geq 1$, and $\mathrm{u}(t-1)(t-1)\mathrm{e}^{-(t-1)}$ in general).

$$\sum_5 = 6$$

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6 a) The characteristic polynomial is

$$\begin{aligned}a(X)&=X^3+X^2-2\\&=(X-1)(X^2+2\,X+2)\\&=(X-1)(X+1-\mathrm{i})(X+1+\mathrm{i}).\end{aligned}$$

with zeros $\lambda_1 = 1$, $\lambda_2 = -1 + \mathrm{i}$, $\lambda_3 = -1 - \mathrm{i}$, all of multiplicity 1. $\boxed{1}$

$\Longrightarrow$ A complex fundamental system of solutions is e^t, $\mathrm{e}^{(-1+\mathrm{i})t}$, $\mathrm{e}^{(-1-\mathrm{i})t}$, and the corresponding real fundamental system is

$$\mathrm{e}^t, \quad \mathrm{e}^{-t}\cos t, \quad \mathrm{e}^{-t}\sin t. \qquad \boxed{1\tfrac{1}{2}}$$

b) In order to obtain a particular solution $y_p(t)$ of the inhomogeneous equation, we solve the two equations $a(\mathrm{D})y_i = b_i(t)$ for $b_1(t) = 1 - 2\,t^3$, $b_2(t) = \mathrm{e}^{-t}\mathrm{e}^{\mathrm{i}t} = \mathrm{e}^{(-1+\mathrm{i})t}$. Superposition then yields the particular solution $y_p(t) = y_1(t) + \operatorname{Re} y_2(t)$.

(1) Since $\mu = 0$ is not a root of $a(X)$, the correct Ansatz is $y_1(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$.

$$\begin{aligned} y_1''' + y_1'' - 2\,y_1 &= 6c_3 + 2c_2 + 6c_3 t - 2(c_0 + c_1 t + c_2 t^2 + c_3 t^3) \\ &= 6c_3 + 2c_2 - 2c_0 + (6c_3 - 2c_1)t - 2c_2 t^2 - 2c_3 t^3 \overset{!}{=} 1 - 2\,t^3 \end{aligned}$$

$\Longrightarrow$ $c_3 = 1$, $c_2 = 0$, $c_1 = 3c_3 = 3$, $c_0 = (6c_3 + 2c_2 - 1)/2 = 5/2$, so that $y_1(t) = \frac{5}{2} + 3\,t + t^3$. $\boxed{1}$

(2) Since $\mu = -1 + \mathrm{i}$ is a zero of $a(X)$ of multiplicity 1, the correct Ansatz is $y_2(t) = ct\,\mathrm{e}^{(-1+\mathrm{i})t}$.

$$\begin{aligned} y_2''' + y_2'' - 2\,y_2 &= (\mathrm{D} - 1)(\mathrm{D} + 1 + \mathrm{i})(\mathrm{D} + 1 - \mathrm{i})\left[ct\,\mathrm{e}^{(-1+\mathrm{i})t}\right] \\ &= c(\mathrm{D} - 1)(\mathrm{D} + 1 + \mathrm{i})\mathrm{e}^{(-1+\mathrm{i})t} \\ &= c(\mathrm{D} - 1)\left[2\mathrm{i}\,\mathrm{e}^{(-1+\mathrm{i})t}\right] \\ &= c\,2\mathrm{i}(-2 + \mathrm{i})\mathrm{e}^{(-1+\mathrm{i})t} = c(-2 - 4\mathrm{i})\mathrm{e}^{(-1+\mathrm{i})t} \end{aligned}$$

$\Longrightarrow c = \frac{1}{-2-4\mathrm{i}} = \frac{-2+4\mathrm{i}}{2^2+4^2} = \frac{-1+2\mathrm{i}}{10} \Longrightarrow y_2(t) = \frac{-1+2\mathrm{i}}{10}\,t\,\mathrm{e}^{(-1+\mathrm{i})t}$. $\boxed{1\tfrac{1}{2}}$

Putting things together gives

$$\begin{aligned} y_p(t) &= \frac{5}{2} + 3\,t + t^3 + \operatorname{Re}\left(\frac{-1 + 2\mathrm{i}}{10}\,t\,\mathrm{e}^{(-1+\mathrm{i})t}\right) \\ &= \frac{5}{2} + 3\,t + t^3 - \frac{1}{10}\,t\,\mathrm{e}^{-t}\cos t - \frac{1}{5}\,t\,\mathrm{e}^{-t}\sin t. \end{aligned} \qquad \boxed{1}$$

The general real solution is then

$$y(t) = y_p(t) + c_1\mathrm{e}^t + c_2\mathrm{e}^{-t}\cos t + c_3\mathrm{e}^{-t}\sin t, \quad c_1, c_2, c_3 \in \mathbb{R}. \qquad \boxed{1}$$

Remarks: Most students solved a) successfully. Some students misread the characteristic polynomial as $X^3 + X^2 - 2\,X$ (confusing y and y'), which in fact is a very common error, and could then earn only 1 mark if the associated fundamental system, which is real, was determined correctly.

A lot more students had problems with b). Either they did not know the correct Ansatz to compute a particular solution, or used a real „Ansatz“, viz. $t \mapsto At\,\mathrm{e}^{-t}\cos t + Bt\,\mathrm{e}^{-t}\sin t$, which complicates the matter and ultimately lead to computational errors in most cases. Also we noticed many correct solutions containing $-\frac{1}{2}$ and $3 + 3\,t + t^3$ as summands, which reveals that students had applied the superposition principle with the three functions 1, $-2t^3$, $\mathrm{e}^{-t}\cos t$. As noted in the lecture, this is unnecessary and causes only additional work.

The constant c can also be computed at $\frac{1}{a'(-1+\mathrm{i})}$: We have $a'(X) = 3\,X^2 + 2\,X$, $a'(-1+\mathrm{i}) = 3(-2\,\mathrm{i}) + 2(-1+\mathrm{i}) = -2 - 4\,\mathrm{i}$, giving the same as above.

$$\sum_{6} = 7$$

$$\sum_{\text{Final Exam}} = 52$$
