Math 286 Differential Equations (Prof. Honold) **Midterm Exam** 2019/04/11

Name: ____________________ Student No.: ____________________ Group A

For each of the following problems, find the correct answer (tick as appropriate!). No justifications are required. Each problem has exactly one correct solution, which is worth 1 mark. Incorrect solutions (including no answer, multiple answers, or unreadable answers) will be assigned 0 marks; there are no penalties.

1. Which of the following ODE's has distinct solutions $y_1, y_2\colon I \to \mathbb{R}$ satisfying $y_1(t_0) = y_2(t_0)$ for some $t_0 \in I$?

   ☐ $y' = |y|$ ☐ $y' = y^2$ ☐ $y' = y\sqrt{t}$ ☐ $ty' = y$ ☐ $y' = ty$

2. $(y\mathrm{e}^{x+y} + 1)\,\mathrm{d}x + (\mathrm{e}^{x+y} - x)\,\mathrm{d}y = 0$ has the integrating factor

   ☐ 0 ☐ 1 ☐ e^{-x} ☐ e^{-y} ☐ e^{-x-y}

3. For the solution $y(t)$ of the IVP $y' = y^3 - y,\ y(0) = \frac{1}{2}$ the limit $\lim_{t\to+\infty} y(t)$ is equal to

   ☐ 0 ☐ 1 ☐ -1 ☐ $+\infty$ ☐ $-\infty$

4. For the solution $y(t)$ of the IVP $y' = t\,y^3,\ y(0) = \frac{1}{2}$ the value $y(1)$ is equal to

   ☐ -1 ☐ $1/\sqrt{2}$ ☐ 1 ☐ $1/\sqrt{3}$ ☐ 2

5. For the solution $y(t)$ of the IVP $y' = 2t(y+1),\ y(0) = 2$ the value $y(1)$ is equal to

   ☐ -1 ☐ $\mathrm{e} - 1$ ☐ $2\mathrm{e} - 1$ ☐ $3\mathrm{e} - 1$ ☐ 1

6. For the solution $y\colon (0, +\infty) \to \mathbb{R}$ of the IVP $t^2y'' + ty' - y = 0,\ y(1) = 0,\ y'(1) = 2$ the value $y(2)$ is equal to

   ☐ $\frac{5}{2}$ ☐ 2 ☐ $\frac{3}{2}$ ☐ 0 ☐ -1

7. For which of the following ODE's does the set of solutions $\phi\colon \mathbb{R} \to \mathbb{R}$ <u>not</u> form a (linear) subspace of $\mathbb{R}^{\mathbb{R}}$?

   ☐ $y' = |t|\,y$ ☐ $y'' = y' + y$ ☐ $y' = t\,|y|$ ☐ $ty' = y$ ☐ $y'' = t(y' - y)$

8. The smallest integer s such that $f_s(x) = \sum_{n=1}^{\infty} \dfrac{x\cos(nx)}{n^s + 1}$ is differentiable on $\mathbb{R}$ is equal to

   ☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4

9. The (real or complex) solution space of $y^{(4)} + (1 + t^2)y' = 0$ has dimension

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10. If $y = y(x)$ solves $y' = \dfrac{xy}{x^2 + y^2}$ then $z = y/x$ solves

    ☐ $z' = \frac{z}{1+z^2}$ ☐ $z' = \frac{z^3}{1+z^2}$ ☐ $z' = -\frac{z^3}{1+z^2}$ ☐ $z' = \frac{z^3}{(1+z^2)x}$

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11. Any solution $y(t)$ of $y'' + 2y' + y = 0$ satisfying $y(0) = 0$ also satisfies

☐ $y'(0) = 0$ ☐ $y'(0) = 1$ ☐ $y'(1) = 0$ ☐ $y'(1) = 1$ ☐ $y'(1) = 2$

12. The sequence $\phi_0, \phi_1, \phi_2, \dots$ of Picard-Lindelöf iterates for the IVP $y' = \sin y,\ y(0) = \pi/2$ has $\phi_2(t)$ equal to

☐ $\pi/2 + t - t^3/6$ ☐ $\pi/2 + \sin t$ ☐ $\pi/2 + \cos t$ ☐ $\pi/2 + t$
☐ $\pi/2 + t + t^2$

13. $y'' - y = \mathrm{e}^t - 2$ has a particular solution $y_p(t)$ of the form

☐ $c_0 + c_1\mathrm{e}^t$ ☐ $c_0 t + c_1\mathrm{e}^t$ ☐ $c_0 e^t + c_1\mathrm{e}^{-t}$ ☐ $(c_0 + c_1 t)\mathrm{e}^t$
☐ $c_0 + c_1 t\mathrm{e}^t$

14. Which of the following points is on exactly one integral curve of the differential equation $(y^2 - y)\,\mathrm{d}x + (x^2 + x)\,\mathrm{d}y = 0$?

☐ $(0,1)$ ☐ $(-1,1)$ ☐ $(0,0)$ ☐ $(-1,0)$ ☐ $(1,1)$

15. Maximal solutions of $y' = y^2 + y + 1$ are defined on an interval of the form

☐ (a,b) ☐ $[a,b]$ ☐ $(a,+\infty)$ ☐ $(-\infty,b)$ ☐ $(-\infty,+\infty)$
with $a, b \in \mathbb{R}$.

Time allowed: 45 min CLOSED BOOK ***Good luck!***

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6. For the solution $y \colon (0, +\infty) \to \mathbb{R}$ of the IVP $t^2y'' + ty' - y = 0,\ y(1) = 0,\ y'(1) = 2$ the value $y(2)$ is equal to

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■ $c_0 + c_1 t\mathrm{e}^t$

14. Which of the following points is on exactly one integral curve of the differential equation $(y^2 - y)\,\mathrm{d}x + (x^2 + x)\,\mathrm{d}y = 0$?

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15. Maximal solutions of $y' = y^2 + y + 1$ are defined on an interval of the form

■ (a,b) ☐ $[a,b]$ ☐ $(a,+\infty)$ ☐ $(-\infty,b)$ ☐ $(-\infty,+\infty)$
with $a, b \in \mathbb{R}$.

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Notes

1. "$ty' = y$" is the only candidate, since it is an implicit ODE to which the Existence and Uniqueness Theorem (EUT) doesn't apply. The other 4 ODE's satisfy the assumptions of the EUT. For $y' = |y|$ compare H25; $y' = y\sqrt{t}$ is 1st-order linear, the square root doesn't matter here, because Lipschitz continutiy with respect to t is not required by the EUT.

2. Although $0\,\mathrm{d}x + 0\,\mathrm{d}y = 0$ is trivially exact, 0 is not an integrating factor, because integrating factors are required to be nonzero everywhere.

3. The phaseline should be used to answer this question.

4. This is a separable equation and can be solved with the standard method: The solution is $y(t) = 1/\sqrt{C - t^2}$, and $y(0) = \frac{1}{2}$ gives $C = 4$.

5. The solution is $y(t) = C\mathrm{e}^{t^2} - 1$, since the associated homogeneous linear ODE is $y' = 2ty$ and $y' = 2t(y+1)$ has the particular solution $y(t) \equiv -1$. The initial condition, which was different for Groups A and B, leads to $C = 3$ ($C = 2$ for Group B) and $y(1) = 3\mathrm{e} - 1$ ($y(1) = 2\mathrm{e} - 1$ for Group B).

6. The Ansatz $y(t) = t^r$ gives $t^2y'' + ty' - y = \big(r(r-1) + r - 1\big)t^r = (r^2 - 1)t^r = 0. \Longrightarrow r_{1/2} = \pm 1$, and the general solution of the Euler equation is $y(t) = c_1 t + c_2 t^{-1}$. The solution of the IVP is then $y(t) = t - t^{-1}$.

7. The only candidate is $y' = t\,|y|$, because the other 4 ODE's are linear and hence its solutions form a subspace of $\mathbb{R}^{\mathbb{R}}$. This is also true for the implicit equation $ty' = y$ (in the lecture we have discussed it in detail for the related case of an Euler equation) and for $y'' = t(y' - y)$ (which is equivalent to $y'' - ty' + ty = 0$). The solutions of $y' = t\,|y|$ are $y(t) \equiv 0$, $y(t) = c\mathrm{e}^{t^2/2}$ for $c > 0$, $y(t) = c\mathrm{e}^{-t^2/2}$ for $c < 0$. They do not form a subspace of $\mathbb{R}^{\mathbb{R}}$, since, e.g., $t \mapsto \mathrm{e}^{t^2/2} - \mathrm{e}^{-t^2/2}$ is not a solution. Thus the problem is indeed well-posed, although you didn't need to know this to find the correct answer.

8. For checking the differentiability of $f_s(x)$ one has to look at the series of derivatives, which is
$$\sum_{n=1}^{\infty} \frac{\cos(nx) - nx\sin(nx)}{n^s + 1},$$
and prove that this series converges uniformly on $\mathbb{R}$ (it doesn't) or on all intervals of the form $[-R, R]$, $R > 0$ (it does). On $[-R, R]$ we can estimate as follows:
$$\left|\frac{\cos(nx) - nx\sin(nx)}{n^s + 1}\right| \leq \frac{1 + nR}{n^s + 1} \leq \frac{2nR}{n^s} = \frac{2R}{n^{s-1}}$$
for n sufficiently large (such that $nR \geq 1$). Applying Weierstrass's Criterion gives uniform convergence on $[-R, R]$ for $s \geq 3$. For $s = 2$ the function $f_s \colon \mathbb{R} \to \mathbb{R}$ is still well-defined but not differentiable at $x = 2k\pi$ for integers $k \neq 0$. This should be clear from the example $x \mapsto \sum_{n=1}^{\infty} \cos(nx)/n^2$ considered in the lecture. (Replacing n^2 by $n^2 + 1$ is inessential and the additional factor x certainly doesn't help with differentiability when $x \neq 0$.)

9. The solution space of an explicit (scalar) n-th order linear ODE has dimension n, regardless of whether the coefficients depend on t or are zero (as the coefficient of y in the case under consideration).

10. $y' = \frac{y/x}{1 + (y/x)^2} = \frac{z}{1 + z^2} \Longrightarrow z' = (y/x)' = (y'x - y)/x^2 = \frac{1}{x}\left(\frac{z}{1+z^2} - z\right) = \frac{1}{x}\frac{-z^3}{1+z^2}$

11. The general solution is $y(t) = c_1\mathrm{e}^{-t} + c_2 t\mathrm{e}^{-t}$. The value $y(0) = 0$ gives $c_1 = 0$, and hence $y'(t) = c_2(1 - t)\mathrm{e}^{-t}$, $y'(1) = 0$. We can also argue without any computation as follows: "$y'(0) = 0$" is ruled out, because the EUT says, e.g., there is a solution with $y(0) = 0$, $y'(0) = 1$. The other wrong answers are ruled out, because the all-zero solution doesn't satisfy them.

12. $\phi_0(t) = \pi/2$; $\phi_1(t) = \pi/2 + \int_0^t \sin(\pi/2)\,\mathrm{d}s = \pi/2 + t$; $\phi_2(t) = \pi/2 + \int_0^t \sin(\pi/2+s)\,\mathrm{d}s = \pi/2 + [-\cos(\pi/2+s)]_0^t = \pi/2 - \cos(\pi/2+t) = \pi/2 + \sin t$

13. A particular solution of $y'' - y = -2$ is $y = 2$, and the correct „Ansatz" for obtaining a solution of $y'' - y = \mathrm{e}^t$ is $y(t) = c_1 t\mathrm{e}^t$ (because $\lambda = 1$ is a root of multiplicity 1 of the characteristic polynomial $X^2 - 1$). Superposition then gives a solution of $y'' - y = \mathrm{e}^t - 2$ of the form $c_0 + c_1 t\mathrm{e}^t$, viz. $y(t) = 2 + \frac{1}{2}t\mathrm{e}^t$.

14. The singular points are the solutions (x, y) of $y^2 - y = x^2 + x = 0$, viz. the 4 combinations of $x \in \{-1, 0\}$, $y \in \{0, 1\}$. This leaves $(1, 1)$ as the only non-singular point, which according to the general theory must be on a unique integral curve.

15. Since $y^2 + y + 1 = 0$ has no real roots, the corresponding canonical form is $z' = z^2 + 1$, which is solved by $z(t) = \tan(t + C)$. Hence the domain of any maximal solution is a bounded open interval. We see this also when trying to compute the solution directly: $\frac{\mathrm{d}y}{y^2+y+1} = \frac{\mathrm{d}y}{(y+1/2)^2+3/4} = \mathrm{d}t$. The integral is of the form $a \arctan(by + c) = t + C$, which yields $y(t) = a' \tan(b't + C') + c'$ (with fixed a', b', c' and variable C') and hence a bounded domain. The case $a' = 0$ cannot occur.