

PHYS 214 discussion problem set 1

Introduction

A harmonic wave is written as

$$y(x, t) = A \cos(kx - \omega t + \phi) \quad (1)$$

$y(x, t)$ may represent the relative pressure in a sound wave, the electric field in a light wave, the height of water in a water wave, among many other things. This function tells us the total intensity, averaged over time, is given by

$$I = \frac{A^2}{2} \quad (2)$$

Many types of waves have the property of *superposition*, which means that if source 1 creates a wave $y_1(x, t)$ and source 2 creates a wave $y_2(x, t)$, then the total wave is given by

$$y_{tot}(x, t) = y_1(x, t) + y_2(x, t). \quad (3)$$

In most problems, we know the waves generated by the sources, y_1 and y_2 , and we want to find the intensity of the total wave. In this class, we focus on adding only harmonic waves that have the same k and ω , at some position given x . In class, we found that the sum of two harmonic waves with the same k and ω is

$$y_{tot}(x, t) = A_{tot}(x) \cos(\omega t - \phi_{tot}). \quad (4)$$

Note that the amplitude of this wave depends on x , and we've reversed the signs in the \cos function.

We use **phasors** to determine A_{tot} and ϕ_{tot} from A_1 , A_2 , ϕ_1 , and ϕ_2 . Typically, we will have a collection of wave sources and we would like to compute the average intensity experienced by an observer at some position. This tells us how loud the sound is, how intense the light is, etc. The strategy is:

1. Enumerate the sources and write down their amplitudes and phases (A and ϕ).
2. Map the amplitudes and phases to phasors. The magnitude of the vector is A and the angle from the x axis is ϕ .
3. Use vector addition to add the phasors.
4. Map the phasors back to amplitudes and phases.
5. Compute the intensity using Eqn 2.

1 Phasors practice

Imagine that you are sitting a distance away from a speaker, which is emitting sound waves. The air pressure, evaluated right at your ear is oscillating in time as

$$y(x, t) = A \cos(\omega t - \phi) + P_0, \quad (5)$$

where P_0 is the base pressure (1 atm), which we typically ignore when studying waves. Suppose that $\phi = 0$ and $\omega = \frac{2\pi}{6} \text{ s}^{-1}$

1.1

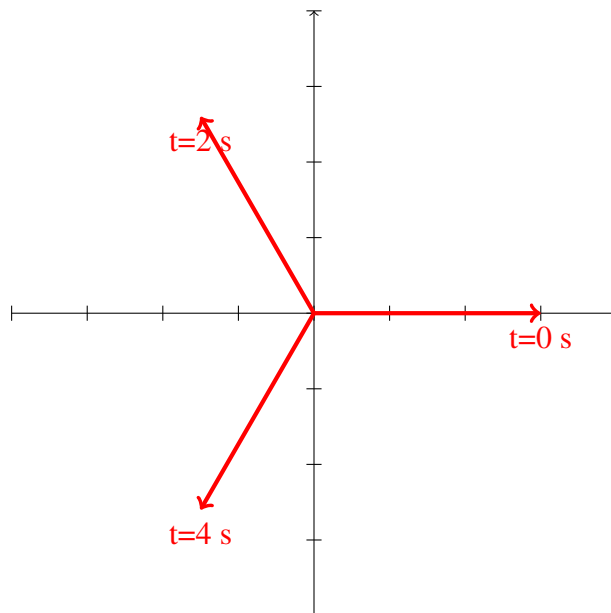
What is the period (T) of the wave (how long does it take for the value of the wave to return to its original value?).

For the period T , $\omega T = 2\pi$. Solving for T , we get

$$T = \frac{2\pi}{\omega} = 6 \text{ s}$$

1.2

Plot the phasor representation of the wave at the three different times $t = 0, 2, 4 \text{ s}$.



1.3

Of the three times that you plotted above, at which one is the deviation of pressure from 1 atm a maximum? Explain your reasoning.

At 0 seconds, the projection onto the x axis is maximal and so the pressure is at a maximum. That means that the maximum *deviation* from ambient pressure is at 0 seconds. It will happen again at 6 seconds.

1.4

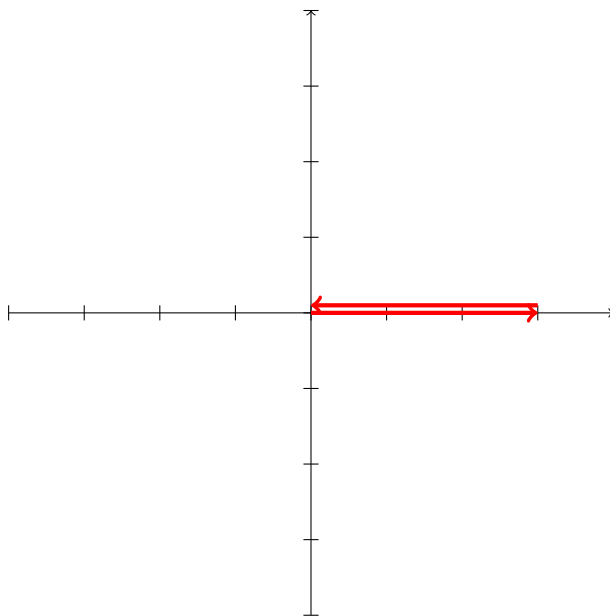
Now we will add a second source.

$$y_1(x, t) = A \cos(\omega t) + P_0 \quad (6)$$

$$y_2(x, t) = A \cos(\omega t - \phi_2) + P_0 \quad (7)$$

The only difference between these two sources is that they are different by a phase, and we are dropping the phase of y_1 because only the *difference* matters for any of these problems. The phase difference is due to the fact that the sources are different distances from you.

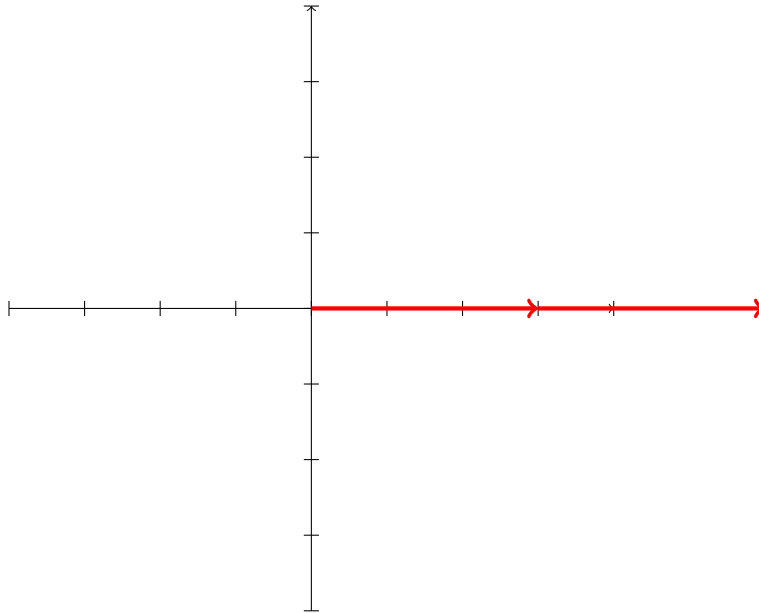
Draw a phasor diagram that shows the two sources interfering *destructively* such that the average intensity at your ear is zero.



The two phasor vectors, shown here at $t = 0$ s, must be anti-parallel. As time increases, the two vectors rotate around the origin, while remaining anti-parallel, which means that the two waves are always interfering destructively.

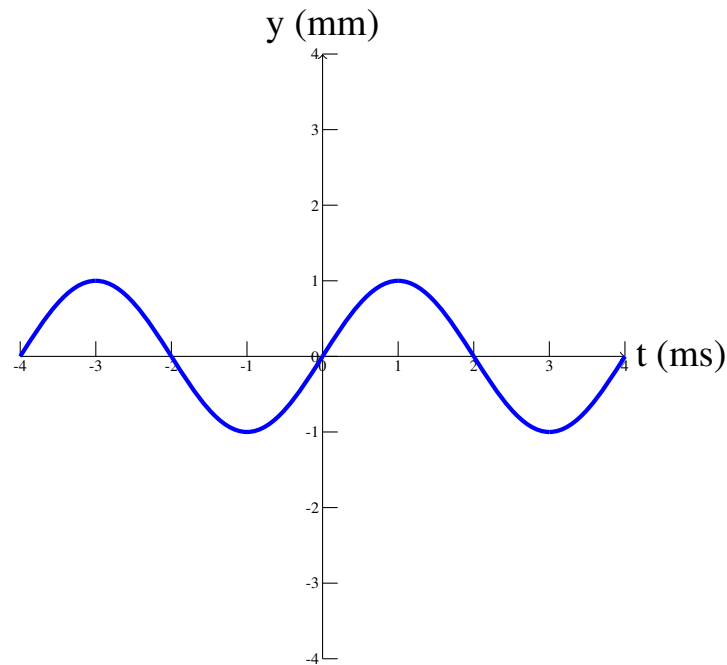
1.5

Draw a phasor diagram that shows the two sources interfering *constructively* such that the average intensity at your ear is maximum.



The two phasor vectors, shown here at $t = 0$ s, must be parallel. As time increases, the two vectors rotate around the origin, while remaining parallel, which means that the two waves are always interfering constructively.

2 Wave properties



An infinitely long string is vibrated, creating a traveling wave. The graph above shows the displacement of the string y as a function of time, at the position $x = 0.3$ m. The y axis is in mm, and the x axis is in ms. The function

$$y(x, t) = A \cos \left(\frac{2\pi x}{\lambda} - \omega t + \phi \right) \quad (8)$$

describes the displacement of the string at any position or time. In this problem, we'll translate between the graph and the equation. From previous measurements, we know the speed of propagation of waves on the string is 300 m/s.

2.1

Find the amplitude A of the wave.

By inspection, the amplitude (max. displacement) is about 1 mm.

2.2

Find the wavelength (λ) of the wave.

The period is $T = 4$ ms. The velocity is $v = \lambda/T$, so $\lambda = 300 \text{ m/s} \times 4 \text{ ms} = 1200 \text{ mm} = 1.2 \text{ m}$

2.3

Find the angular frequency ω of the wave.

The period is $T = 4$ ms. $\omega = \frac{2\pi}{T} = \frac{2\pi}{0.004\text{s}} = 1571 \text{ radians / second}$

2.4

Find the phase ϕ of the wave. By convention, we want the value of ϕ between 0 and 2π . (*Hint: you should write down an equation for the value of the wave at some particular time. It's better to use the maximum of the wave vs the zero, since there are two different zeros.*)

As usual, it's best to solve this algebraically and then plug in numbers only at the end. By inspection, the wave is at its maximum at $t = 1$ ms. So we have $y(x, t = 1\text{ms}) = \cos\left(\frac{2\pi x}{\lambda} - \frac{2\pi t}{T} + \phi\right) = 1$. This means that the argument of the cosine must be multiple of 2π :

$$\begin{aligned}\frac{2\pi x}{\lambda} - \frac{2\pi t}{T} + \phi &= 2\pi n \\ \phi &= 2\pi \left(\frac{t}{T} - \frac{x}{\lambda} + n \right)\end{aligned}$$

where n is any integer. Note that what matters is the ratio between time and the period, and position and the wavelength. Since we want a solution between 0 and 2π , we want to choose n such that the value in the parentheses is between zero and one. Let's evaluate that quantity now.

$$\frac{t}{T} - \frac{x}{\lambda} = \frac{1 \text{ ms}}{4 \text{ ms}} - \frac{0.3 \text{ m}}{1.2 \text{ m}} = 0 \quad (9)$$

In summary, we find that $n = 0$ is fine, and the solution is $\phi = 0$.

2.5

At what position x is the string at a maximum when $t = 0$? Find all solutions.

We can solve this the same way.

$$\begin{aligned}\frac{2\pi x}{\lambda} + \phi &= 2\pi n \\ x &= \frac{\lambda(-\phi + 2\pi n)}{2\pi} = \lambda n - \frac{\lambda\phi}{2\pi} \\ &= \lambda n.\end{aligned}$$

where n is any integer. At $t = 0$, the maximum is at $x = 0.0 \text{ m}$, and all points that are an integer number of wavelengths ($\lambda = 1.2 \text{ m}$) from it.

3 Superposition and the wave equation

The equivalent of $F = ma$ for waves is the *wave equation*:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}, \quad (10)$$

where v is the velocity of the wave. For a function $y(x, t)$, we say that this equation is *satisfied* if the equality holds. The way you check this is to take the derivative on the left and right hand sides, and check to make sure that the two sides are equal. The two sides must be equal for all positions and all times for the function $y(x, t)$ to be a solution.

3.1

Show that $y(x, t) = A \cos(kx - \omega t)$ is a solution to the wave equation. Show the relationship between ω and k that must hold in order for y to be a solution.

Left hand side:

$$\frac{\partial^2 y}{\partial x^2} = -Ak^2 \cos(kx - \omega t)$$

Right hand side:

$$\frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} = -A \frac{\omega^2}{v^2} \cos(kx - \omega t)$$

We can make the right hand side equal the left hand side for **all** x and t if

$$\frac{\omega^2}{v^2} = k^2$$

or

$$\frac{\omega}{|k|} = |v|$$

where $|k|$ is the absolute value of k .

3.2

Use the relationship between ω and k you computed above to derive the relationship between λ and f .

$k = 2\pi/\lambda$ and $\omega = 2\pi f$. Plugging this into the above equation, we get $\lambda f = v$. By convention f and λ are positive.

3.3

Show that if $y_1(x, t)$ and $y_2(x, t)$ are both solutions to the wave equation (the wave equation is true for both of them), then $y_{tot} = y_1(x, t) + y_2(x, t)$ is also a solution to the wave equation.

For this, we start from the left hand side and transform it to the right hand side. We use the fact that $\frac{\partial^2 y_1}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y_1}{\partial t^2}$, and $\frac{\partial^2 y_2}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y_2}{\partial t^2}$, since that was part of the assumptions. We also use the fact that the derivative is *linear*, so that for any functions f and g , $\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$.

$$\begin{aligned}\frac{\partial^2 y_{tot}}{\partial x^2} &= \frac{\partial^2 y_1}{\partial x^2} + \frac{\partial^2 y_2}{\partial x^2} \\ &= \frac{1}{v^2} \frac{\partial^2 y_1}{\partial t^2} + \frac{1}{v^2} \frac{\partial^2 y_2}{\partial t^2} \\ &= \frac{1}{v^2} \left(\frac{\partial^2 y_1}{\partial t^2} + \frac{\partial^2 y_2}{\partial t^2} \right) \\ &= \frac{1}{v^2} \frac{\partial^2 y_{tot}}{\partial t^2}\end{aligned}$$

So the wave equation is satisfied by y_{tot} . We will use this property in the next few weeks to compute the waves generated by multiple sources: we find the waves generated by each source in isolation y_1 and y_2 , and add them together to find the total wave generated.

PHYS 214 discussion problem set 2: Interference

Introduction

In most problems, we know the waves generated by the sources, y_1 and y_2 , and we want to find the intensity of the total wave. In this class, we focus on adding only waves that have the same k and ω , i.e.

$$y_1(x, t) = A_1 \cos(kx - \omega t + \phi_1) \quad y_2(x, t) = A_2 \cos(kx - \omega t + \phi_2). \quad (1)$$

In class, we found that the sum of two such harmonic waves (with the same k and ω) is also harmonic:

$$y_{\text{tot}}(x, t) = y_1(x, t) + y_2(x, t) = A_{\text{tot}} \cos(kx - \omega t + \phi_{\text{tot}}). \quad (2)$$

Strategy

We use **phasors** to determine A_{tot} and ϕ_{tot} from A_1 , A_2 , ϕ_1 , and ϕ_2 .

Typically, we will have a collection of wave sources and we would like to compute the average intensity experienced by an observer at some position. This tells us how loud the sound is, how intense the light is, etc. The strategy is:

1. Enumerate the sources and write down their amplitudes and phases (A_i and ϕ_i). You will often be given the intensity of each source when the others are turned off.
2. Map the amplitudes and phases to phasors. The magnitude of the vector is A and the angle from the x axis is ϕ .
3. Use vector addition to add the phasors. The most important thing is the *phase difference* between the waves, which determines whether they add constructively or destructively.
4. Map the phasors back to amplitudes and phases to obtain A_{tot} and ϕ_{tot} .
5. Compute the intensity from A_{tot} .

As noted above, the most important question is the phase difference between the waves, which you can obtain by looking at the functional forms for each wave. For example, let us consider two waves that are generated by two sources, separated by a finite distance. Then an observer at a fixed distance r_1 and r_2 from sources 1 and 2, respectively, will experience the superposition of

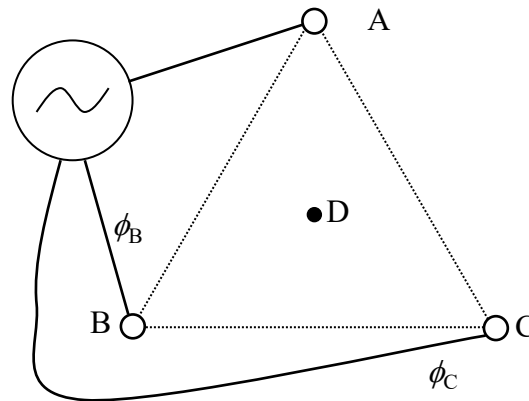
$$y_1(t) = A_1 \cos(kr_1 - \omega t + \phi_1), \quad y_2(t) = A_2 \cos(kr_2 - \omega t + \phi_2). \quad (3)$$

So the phase difference is

$$k(r_2 - r_1) + \phi_2 - \phi_1. \quad (4)$$

1 More phasors practice

An audio oscillator drives three identical speakers, A, B, and C, that are situated at the vertices of an equilateral triangle. The point D is at the center of the triangle. The audio signals from the oscillator pass through two phase shifters, which shift the phases of speakers B and C relative to A. When all phases are set to 0° , the signals at the speakers are in phase. The setup is shown in the figure:



Suppose that the intensity received at point D from each individual speaker is I_0 . In the following questions, you are asked to find the intensity at point D with the speakers operating at various settings of the two phase shifters. In each case, draw the appropriate phasor diagram first.

1. $\phi_B = \phi_C = 0$



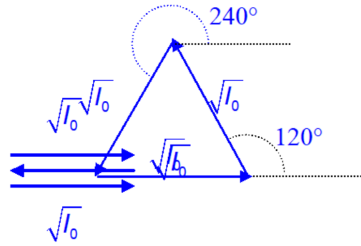
$$\begin{aligned}\sqrt{I} &= 3\sqrt{I_0} \\ I &= 9I_0\end{aligned}$$

2. $\phi_B = 120^\circ, \phi_C = 240^\circ$ The phasor sum of any closed figure is 0.

3. $\phi_B = 180^\circ, \phi_C = 360^\circ$ 360° is equivalent to 0° . Two phasors are in phase, and one is

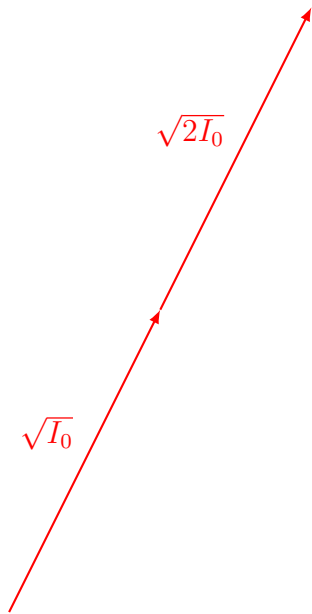
exactly opposite.

$$\begin{aligned}\sqrt{I} &= \sqrt{I_0} - \sqrt{I_0} + \sqrt{I_0} = \sqrt{I_0} \\ I &= I_0\end{aligned}$$



4. We now fix $\phi_B = 90^\circ$ and vary ϕ_C between 0° and 360° . Find the maximum and minimum possible values of the intensity, and determine the corresponding values of the phase ϕ_C .

Largest Sum



To start, add the first two phasors. The phasor diagram is a right

triangle:

$$\sqrt{I_0}^2 + \sqrt{I_0}^2 = \sqrt{2I_0}^2$$

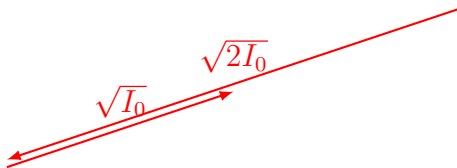
Now consider the third phasor. The largest sum of two phasors occurs when they are parallel, and the smallest when they are antiparallel. So, we want the third phasor to be either parallel to or antiparallel with the sum of the other two. **Smallest Sum** The amplitudes of parallel phasors add:

$$\begin{aligned}\sqrt{I} &= \sqrt{2I_0} + \sqrt{I_0} \\ I &= \left(\sqrt{2I_0} + \sqrt{I_0} \right)^2 = I_0(\sqrt{2} + 1)^2 = 5.83I_0\end{aligned}$$

The amplitudes of antiparallel phasors subtract:

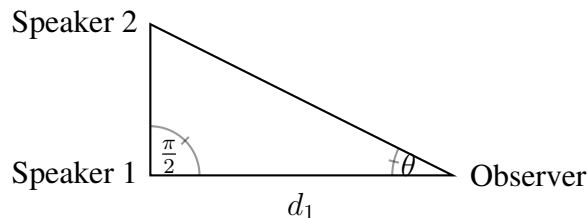
$$\begin{aligned}\sqrt{I} &= \sqrt{2I_0} - \sqrt{I_0} \\ I &= \left(\sqrt{2I_0} - \sqrt{I_0} \right)^2 = I_0(\sqrt{2} - 1)^2 = 0.172I_0\end{aligned}$$

We could have rotated the phasor diagrams to make them horizontal. This is allowed as long as we rotate all phasors by the same amount. To determine what ϕ_C is needed, you must calculate the phase, ϕ_{AB} , of the sum of the first two phasors. Because they have equal amplitudes, the phase is 45° (look at the triangle in the diagram above). Therefore, you will obtain the maximum when ϕ_C is 45° and the minimum when it is 225° (180° larger).



Smallest Sum

2 Two audio sources



As pictured, an observer sits a distance d_1 from one speaker, while there is another speaker at an angle given by θ from the line connecting the observer and the first speaker. This situation is like watching TV with two speakers on a line, and sitting on the left side of the couch. When Speaker 1 is turned off, so they are only hearing the sound from Speaker 2, the observer hears an intensity of 3 W/m^2 . When Speaker 2 is turned off, the observer hears an intensity of 4 W/m^2 .

2.1 Amplitudes

Write down the amplitudes for each speaker A_1 and A_2 in units of $\sqrt{\text{W}}/\text{m}$.

$$I_{rms} = \frac{A^2}{2}$$

so

$$\begin{aligned}A &= \sqrt{2I_{rms}} \\ A_1 &= \sqrt{2 \times 4 \text{ W/m}^2} = \sqrt{8} \sqrt{\text{W}}/\text{m} \\ A_2 &= \sqrt{2 \times 3 \text{ W/m}^2} = \sqrt{6} \sqrt{\text{W}}/\text{m}\end{aligned}$$

2.2 Maximum and minimum intensities

Before computing the intensity that the observer actually experiences, it's useful to compute the bounds. What are the maximum and minimum intensities that the observer could experience? Draw phasor diagrams for each situation.

This is exactly constructive and destructive interference, which we drew last week.

Destructive interference looks like:



Constructive interference looks like:



In the case of destructive interference, the amplitudes subtract, which leads to

$$I_{min} = \frac{(A_1 - A_2)^2}{2} = 0.07 \text{ W/m}^2$$

and

$$I_{max} = \frac{(A_1 + A_2)^2}{2} = 13.9 \text{ W/m}^2$$

So, they can almost cancel out completely, or add up quite a lot.

2.3 Distances

How far is the observer from speaker 2, in terms of d_1 and θ ? Find an algebraic solution.

$$d_1 = d_2 \cos \theta \implies d_2 = \frac{d_1}{\cos \theta}$$

2.4 The relative phase

Assuming that the speakers emit in phase (they both emit sound which is proportional to $\cos(\frac{2\pi x}{\lambda} - \omega t)$), write down a formula for the amplitude of sound wave for each source experienced at the position of the observer, and compute the relative angle between the two signals in terms of d_1 , d_2 , and λ . You will compute $y_1(t)$, $y_2(t)$ and the phase difference $\Delta\phi$.

For the first speaker, the observer is at position d_1 from the speaker. So the equation is

$$y_1(d_1, t) = A_1 \cos(\omega t - \frac{2\pi d_1}{\lambda})$$

$$y_2(d_2, t) = A_2 \cos(\omega t - \frac{2\pi d_2}{\lambda})$$

The phase difference is

$$\Delta\Phi = \frac{2\pi(d_2 - d_1)}{\lambda}$$

2.5 Using electronics to get the loudest sound

If we introduce a delay of δt in the second speaker, then the waves experienced by the observer are

$$y_1(d_1, t) = A_1 \cos\left(\frac{2\pi d_1}{\lambda} - \omega t\right)$$

$$y_2(d_2, t) = A_2 \cos\left(\frac{2\pi d_2}{\lambda} - \omega(t - \delta t)\right).$$

What is the minimum (but still positive) delay δt we should introduce to obtain a maximum intensity? Assume that

- $d_1 = 2$ m
- $d_2 = 3$ m
- $v = 340$ m/s
- $f = 440$ Hz

The phase angle difference is

$$\delta\phi = \frac{2\pi(d_2 - d_1)}{\lambda} + \omega\delta t$$

To have a maximum intensity, we want $\delta\phi = 2\pi n$, where n is an integer. Therefore,

$$\begin{aligned}\delta t &= \frac{2\pi}{\omega} \left(n - \frac{(d_2 - d_1)}{\lambda} \right) \\ &= \frac{1}{f} (n - 1.29)\end{aligned}$$

where we used the fact that $\lambda = v/f = 0.77$ m. The smallest possible delay is thus for $n = 2$, which is 1.6 ms.

3 Light passing through two slits

Recall the two-slit experiment. Light is incident on the plate at the left and passes through the two holes. We view each hole as if it were a single source which is in phase with the other sources. On the screen on the right, fringes of light appear. We want to compute the location of those fringes on the screen.

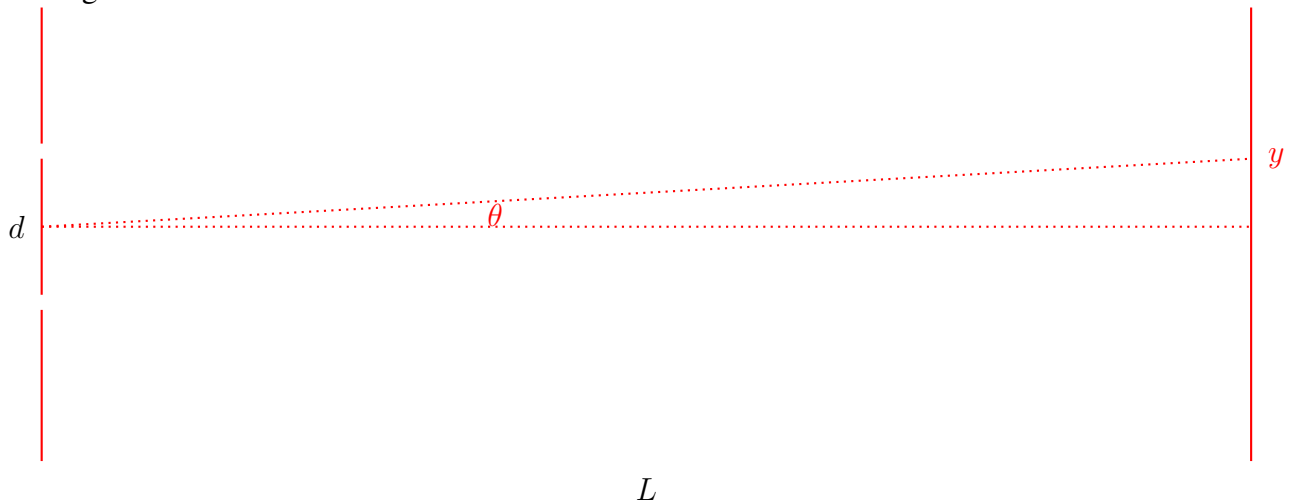
In this case,

- $d = 0.25\text{mm}$
- $L = 2\text{ m}$
- $\lambda = 610\text{ nm}$

Keep in mind that L is much much larger than d . and d is much larger than the size of the slits, which we approximate as being single points. In class, we determined that the difference in distance between the two slits and the observer position at y is $d \sin \theta$.

3.1 Diagram

Draw a diagram of the experimental setup, labeling the distances L , d , position on the screen y , and the angle θ .



3.2 y and L

The position on the screen is measured by the variable y . Write down an algebraic formula for y in terms of θ and L .

Since the middle line is perpendicular to the screen,

$$y = L \tan \theta$$

3.3 Angle of the maxima

What are the angles (values of θ) of maximal intensity? Write an explicit formula for θ_m and compute θ for $m = 1, 2, 3, 4$.

When $d \sin \theta = m\lambda$, there is constructive interference.

$$\theta_m = \sin^{-1} \left(\frac{m\lambda}{d} \right)$$

m	θ
1	0.0024
2	0.0049
3	0.0073
4	0.0097

They are roughly evenly spaced because $m\lambda/d$ is much smaller than 1, and so $\sin^{-1} m\lambda/d \simeq m\lambda/d$.

3.4 Positions of the maxima

What are the corresponding values of y_m ?

m	y_m (meters)
1	0.0049
2	0.0098
3	0.0146
4	0.0195

So the maxima are spaced around half a cm apart from one another.

3.5 Changing the pattern

You may have noticed that the values of θ are very small. In this case, they are so small that you can approximate that $\sin \theta \simeq \theta$ and $\tan \theta \simeq \theta$. Derive a simple expression using this approximation for the maxima y_m in terms of m , λ , L , and d . Use this expression to suggest a way to change the apparatus to increase the separation between the maxima $y_{m+1} - y_m$.

$$y_m \simeq L \frac{m\lambda}{d}$$

The spacing between two maxima is given by

$$y_{m+1} - y_m \simeq L \frac{\lambda}{d}$$

You could increase the spacing by either increasing L or decreasing d (either are acceptable answers). You could also increase λ , but that's not a property of the apparatus.

4 Interferometer

An interferometer experiment is set up, with a laser incident on a half-silvered mirror that splits the laser into two arms of length L_1 and L_2 . Each arm has a mirror at the end of it that reflects the light back to the half-silvered mirror, which then splits the light again. We observe the power coming out of the apparatus at a detector.

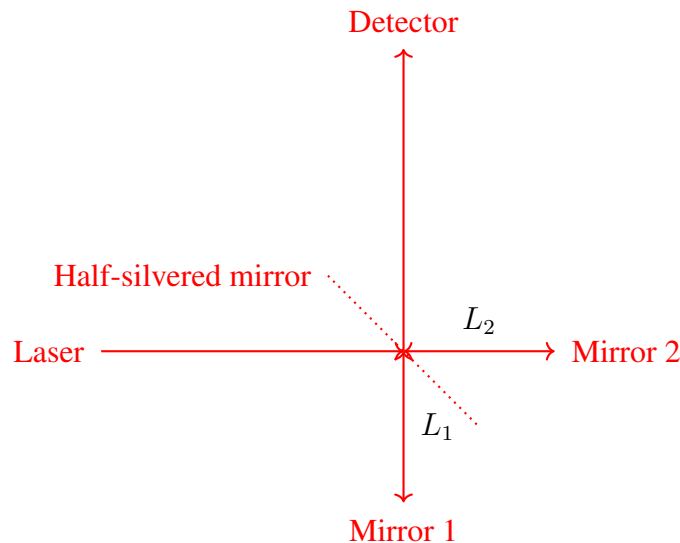
Suppose that we slowly move one of the mirrors. The power observed at the detector increases, decreases, and then increases again. We are going to use this fact to measure how much we moved the mirror.

For this, assume that the setup is as follows

- $L_1 = 3$ m
- $L_2 = 3$ m
- $\lambda = 400$ nm

4.1 Diagram

Draw a diagram of the experimental setup, labeling the laser, the half-silvered mirror, the lengths of each arm L_1 and L_2 , the mirrors at each end of the arms, and the detector. You can reference the notes from the class to do this.



4.2 Phase difference

Compute the algebraic formula for the phase difference between the two paths in terms of L_1 , L_2 , and λ .

The phase difference is given by

$$\phi = \frac{2\pi(2L_1 - 2L_2)}{\lambda}$$

The factors of 2 are there because the light travels up and back down the arms.

4.3 Measuring displacements of the mirror

Now we move mirror 2 by Δx . Let's write the length of the second arm as $L_2 + \Delta x$. Since at the beginning, $L_1 = L_2$, the phase difference is zero and there is constructive interference. We increase Δx until another maximum is found. How much did we move the mirror?

$$2\Delta x = \lambda$$

$$\Delta x = \lambda/2 = 200nm$$

4.4 Intensity as a function of displacement

If the intensity of light at the detector is given by I_0 when the signal is at a maximum, write a formula for the intensity as a function of Δx .

Since the two sources (paths) are the same amplitude, we can use the formula given in the equation sheet:

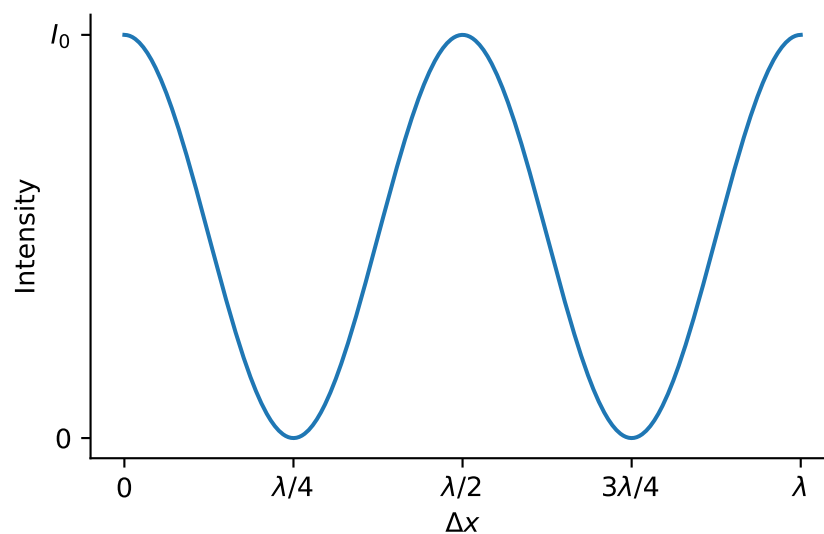
$$I = I_0 \cos^2 \left(\frac{\phi}{2} \right)$$

So, using the result from 3.2:

$$I(\Delta x) = I_0 \cos^2 \left(\frac{2\pi 2\Delta x}{2\lambda} \right) = I_0 \cos^2 \left(\frac{2\pi \Delta x}{\lambda} \right)$$

4.5 Graph the intensity

Draw a graph of the intensity as a function of Δx from 0 to λ , marking where each minimum and maximum is.



PHYS 214 discussion problem set 3: Single slit diffraction and the properties of photons

Introduction

This week we have two independent topics:

- Diffraction: Classical interference of waves.
- Photons: this is our first "quantum" observation. Energy can only be added or removed from the electromagnetic field in discrete amounts

Single slit diffraction

In class we computed the zeros of the intensity from a single linear slit as begin $a \sin \theta_0 = \lambda$. This is where the phasors, integrated over the whole slit, sum up to zero. For a circular aperture, the zero in the 2D spot is given by $D \sin \theta_0 = 1.22\lambda$, where D is the diameter of the aperture.

Photons

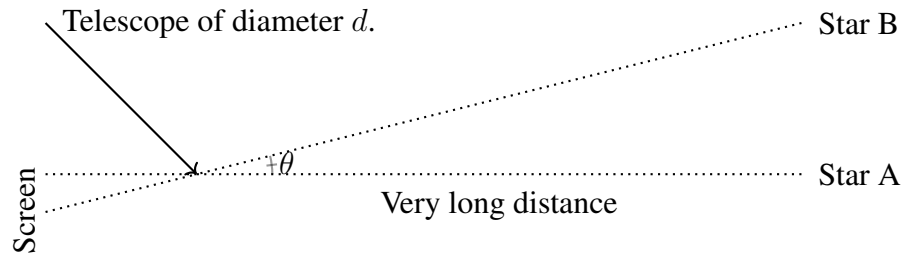
Light can be added or removed from the electric field in discrete amounts that depend on the frequency, $E = hf$. h is a new quantity that is fundamental in physics, called Planck's constant. On your equation sheet, we give you Planck's constant in J·s, and also in eV·s. An eV is an "Electron-Volt", which is the amount an electron's energy changes when it is moved across a potential difference of one Volt. In Quantum Mechanics, it is often useful to use eV as the unit of energy instead of J "Joules". 1 eV is about 1.6×10^{-19} J, because the charge of the electron is 1.6×10^{-19} C. The momentum of a photon is given by $p_{\text{photon}} = E/c = h/\lambda$. Remember that $c = f \cdot \lambda$, where c is the speed of light.

In these problems, we count the number of photons emitted per second by a light source with a given power P , using the simple equation $N_{\text{photons}} = \frac{P}{hf}$, since the units of power are units of energy (J or eV) per second, and hf is the photon's energy.

The photoelectric effect can be understood using photons. A piece of material absorbs a photon, gaining energy hf . That energy is transferred to an electron, which is emitted by the material. We use energy conservation to write down that $hf = \Phi + KE_{\text{electron}}$. The quantity Φ is called the work function and is the potential energy of the electron inside the material; it's the barrier that must be overcome to remove the electron from the material.

In atomic recoil experiments the reverse process is studied. Atoms that are in an excited state can transition to a lower-energy state by emitting a photon. By momentum conservation, the emitted photon's momentum must be balanced by the atom's recoil momentum.

1 Diffraction limited telescope:



Let's idealize a telescope as a spherical opening. We are using the telescope to produce an image of two stars on a screen. We approximate the stars as point sources and the optical lenses as perfect. Without diffraction, the stars would produce pointlike spots on the screen.

For this case, assume that

- The angle the stars are separated by, θ , is 0.005 degrees
- The diameter d of the telescope is 2 cm.
- The screen is $L = 1.2$ m away from the telescope.
- Approximate the wavelength as $\lambda = 650$ nm

1.1 Distance between the spots

Ignoring diffraction for the moment, what is the separation Δy between the two spots on the screen? Show the algebraic solution and then plug in the numbers.

$$\Delta y = L \tan \theta = 0.1\text{mm}$$

1.2 Size of the spots

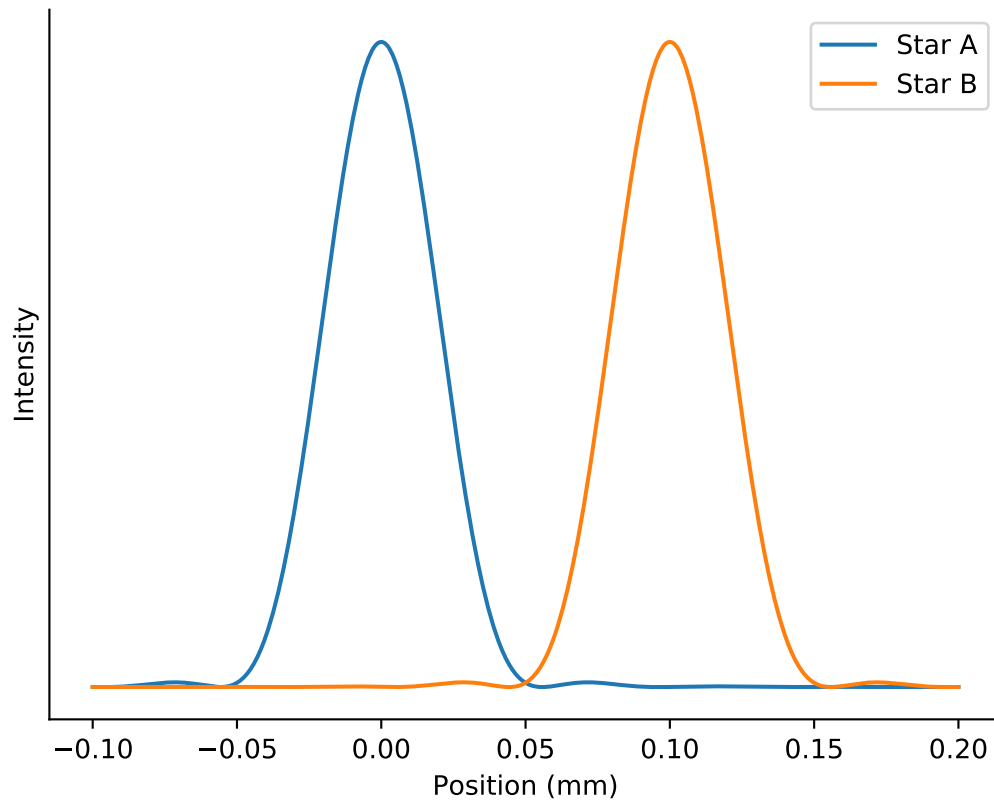
Diffraction will cause each spot to be a bit wider. Compute the radius of each spot, defined by the distance from the maximum intensity to where interference causes the intensity to drop to zero.

$$\theta_{min} = \sin^{-1} 1.22\lambda/d$$

$$\delta y_{min} = L \tan \theta_{min} = L \tan \sin^{-1} 1.22\lambda/d = 0.05\text{mm}$$

1.3 Draw the spots

As a function of y , sketch the intensity on the screen. Label the position of the centers of each spot and their width.



1.4 Resolution

Suppose the telescope opening were only 1 mm in diameter, roughly the size of your pupil. Would the spots now be distinct? Why?

If $d = 1$ mm, then $\delta y_{min} = 0.95$ mm. So, reducing the opening of the telescope by a factor of 20 increases the widths of the spots by about a factor of 20. Hence, they would overlap so much that the stars would not appear distinct.

2 Energy and momentum carried by photons

2.1 Energy of photons

A red laser pointer emits light of wavelength about 635 nm. What is the energy of a photon at that wavelength in Joules?

$$E_{\text{photon}} = \frac{hc}{\lambda} = \frac{6.626 \times 10^{-34} \text{ J s} \times 2.998 \times 10^8 \text{ m/s}}{635 \times 10^{-9} \text{ m}} = 3.13 \times 10^{-19} \text{ J}$$

2.2 Rate of photons emitted by a laser pointer

Now consider a typical laser pointer that emits light at about 1 mW (power). Find the rate (dN/dt) at which photons are emitted by the laser.

A Watt is 1 J/s. So if we divide the power by the energy of a photon, we will get the number of photons emitted per second.

$$\frac{dN}{dt}_{\text{photon}} = \frac{1 \times 10^{-3} \text{ W}}{3.13 \times 10^{-19} \text{ J/photon}} = 3.19 \times 10^{15} \text{ photons/s}$$

2.3 Momentum of photons

Now let's consider the momentum of a photon. What is the momentum of a photon of wavelength 635 nm, in SI units kg m/s?

$$p = h/\lambda = \frac{6.626 \times 10^{-34} \text{ J s}}{636 \times 10^{-9} \text{ m}} = 1.04 \times 10^{-27} \text{ kg m/s}$$

2.4 Force exerted by a laser pointer

Now let's find the force that a laser pointer exerts on an object when the photons are absorbed. In introductory physics you probably learned that $F = ma$, but the more general equation is actually $F = \frac{dp}{dt}$. So the force on an object is given by the rate of change of the momentum. Use the above information to find the force exerted by a laser pointer, in newtons (N). To an order of magnitude, how much power would the laser pointer need to emit in order to exert a force of about 1 N, which is large enough that you could perceive it?

The rate of change of the photon momentum is given by the rate at which photons are emitted and the momentum of each photon, assuming that the photons are absorbed by the material:

$$\frac{dp}{dt} = \frac{dN}{dt}_{\text{photon}} * p_{\text{photon}} = 3.32 \times 10^{-12} \text{ N}$$

Just working with orders of magnitude, since 1 mW results in a force of 10^{-12} N, it would take 3×10^{11} mW or 0.3×10^9 W or 3×10^8 W to create a force of 1 N.

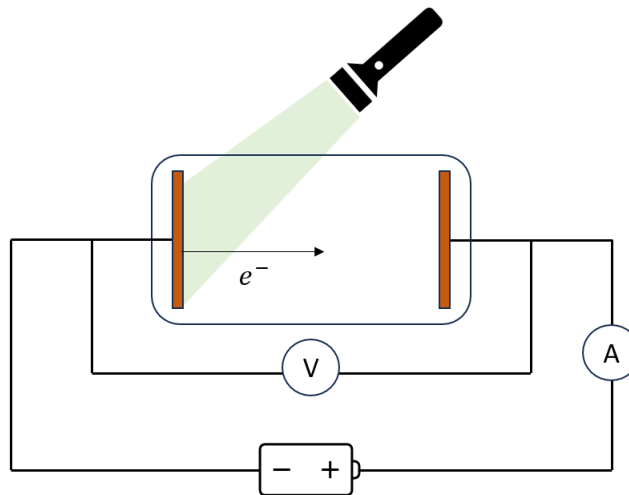
3 Photoelectric effect

A metallic surface is illuminated by light. There is another metal plate a few cm away, which is held at an electric potential of V . The plate is connected to a circuit which measures the current of electrons.

When light of wavelength 520 nm is incident on the metal surface, electrons are emitted from the metal surface and absorbed by the second plate, which creates a current. When the potential of the second plate V is increased to $V_{\text{stop}} = 0.3 \text{ V}$, the current stops.

3.1 Circuit diagram

Sketch a diagram of the circuit described above.



3.2 Photon energy

What is the energy of each photon in electron volts?

The easiest way is to use

$$E_{\text{photon}} = \frac{hc}{\lambda} = \frac{4.135 \times 10^{-15} \text{ eV s} \times 2.998 \times 10^8 \text{ m/s}}{520 \times 10^{-9} \text{ m}} = 2.38 \text{ eV}$$

3.3 Potential energy of an electron

In electron volts, what is the potential energy of an electron on the second plate; that is, an electron in a potential of V_{stop} ? Keep in mind that the charge of an electron is e .

The potential energy is the charge times the electric potential, so it's $eV_{\text{stop}} = 0.3 \text{ eV}$.

3.4 Kinetic energy of the electron

Describe what happens to the kinetic energy of an electron, after it is emitted from the metal surface. Determine the initial kinetic energy of the electron (as it is emitted from the metallic surface), when the potential of the second plate is adjusted so that it equals V_{stop} .

The electron's total energy is conserved. It is emitted with an initial kinetic energy, which, as it moves towards the second plate, turns into potential energy. Hence, the kinetic energy "at the bottom of the hill" is equal to the potential energy the electron acquires when it reaches the second plate and stops.

In short, the kinetic energy of the electron is $E_{\text{kin}} = eV_{\text{stop}} = 0.3 \text{ eV}$.

3.5 Work function

Notice that there is some missing energy; the photon which is absorbed has more energy than the electron that was emitted. That missing energy is the energy it takes to remove the electron from the metallic surface, called the work function Φ . Use energy conservation to set up an equation that includes the energy of the photon, the work function, and the kinetic energy of the emitted electron. Use this to solve for the work function of the metal.

$$hf = \Phi + KE$$

$$\phi = hf - KE = (2.38 - 0.3)\text{eV} = 2.08\text{eV}$$

3.6 Comparison to real materials

Here are some approximate work functions of elemental solids.

Element	Φ (eV)
Au	5.3
Al	4.1
Cr	4.5
Ce	2.9
Cs	2.0
Fe	4.7

Which metal is most likely your sample and why?

The computed work function is closest to Cesium (Cs), so that is the most likely metal.

3.7 Photon frequency

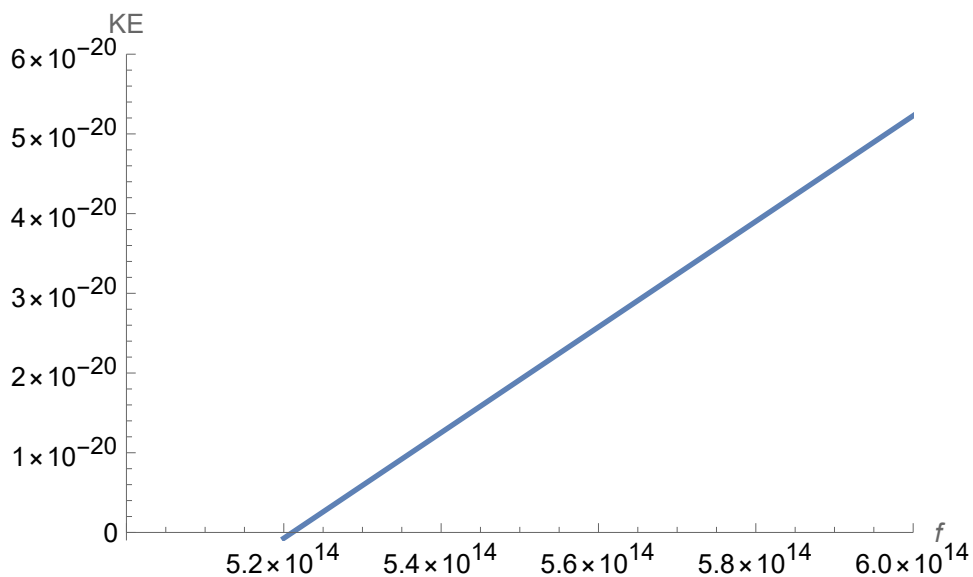
What happens to the initial kinetic energy of the electrons if we change the wavelength of the light? Determine the longest possible wavelength that still results in the emission of electrons from the metal surface. Sketch the dependence of the electron kinetic energy on the photon frequency.

If we change the wavelength of light, the initial kinetic energy will change linearly. To determine the longest possible wavelength, use

$$E_{\text{photon}} = hf = \Phi$$

$$f = \frac{\phi}{h} = \frac{2.08 \times 1.66 \times 10^{-19}}{6.626 \times 10^{-34}} = 5.2 \times 10^{14} \text{ Hz}$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{5.02 \times 10^{14}} = 5.97 \times 10^{-7} = 597 \text{ nm}$$



4 Atomic recoil

An atom can make a transition from a higher energy state to a lower energy state by emitting a photon. By momentum conservation, the atom must recoil against the emitted photon. Assume that a Rubidium atom at rest (mass $m_{\text{Rb}} = 87 \text{ amu}$; $1 \text{ amu} = 1.66 \times 10^{-27} \text{ kg}$) emits a photon of wavelength $\lambda = 795 \text{ nm}$.

4.1 photon momentum and energy

Determine the momentum and energy of the photon in SI units.

We just use the formula:

$$p = \frac{h}{\lambda} = \frac{6.626 \times 10^{-34} \text{ kg } \frac{\text{m}^2}{\text{s}}}{795 \times 10^{-9} \text{ m}} = 8.33 \times 10^{-28} \text{ kg } \frac{\text{m}}{\text{s}}$$

Use the formula:

$$E = \frac{hc}{\lambda} = pc = (8.33 \times 10^{-28})(2.998 \times 10^8) \text{ J} = 2.5 \times 10^{-19} \text{ J}$$

Converting to eV units yields:

$$E = \frac{2.5 \times 10^{-19} \text{ J}}{1.609 \times 10^{-19} \text{ J/eV}} = 1.55 \text{ eV}$$

4.2 Recoil momentum and velocity

Determine the recoil momentum and recoil velocity of the Rubidium atom in SI units.

Momentum is conserved, so the momentum of the atom is equal-and-opposite to the photon momentum:

$$\|p_{\text{atom}}\| = 8.33 \times 10^{-28} \text{ kg } \frac{\text{m}}{\text{s}}$$

Divide the atomic momentum by the atomic mass:

$$v_{\text{atom}} = \frac{p_{\text{atom}}}{m_{\text{atom}}} = \frac{8.33 \times 10^{-28} \text{ kg } \frac{\text{m}}{\text{s}}}{87 \times 1.6 \times 10^{-27} \text{ kg}} = 0.0057 \text{ m/s}$$

4.3 Kinetic energy ratio

Determine the ratio of the kinetic energy of the Rubidium atom to the energy of the photon, $KE_{\text{Rb}}/E_{\text{photon}}$, and interpret the result.

$$KE_{\text{Rb}} = \frac{p_{\text{Rb}}^2}{2m_{\text{Rb}}} = 2.40 \times 10^{-30} \text{ J}$$

So now we just take the ratio:

$$\frac{KE_{\text{Rb}}}{E_{\text{photon}}} = \frac{2.40 \times 10^{-30} \text{ J}}{2.5 \times 10^{-19} \times J} \approx 10^{-11}$$

That is to say, when the atom transitions into a lower-energy state, almost all of the resulting energy gets carried-off by the emitted photon. However small, the atomic recoil energy can be measured very precisely in modern experiments using atom interferometers.

PHYS 214 discussion problem set 4: The wave function and probability density

Introduction

In order to describe interference in quantum mechanics, we had to introduce a way to compute the *probability* that a single particle is observed. This method uses the wave function as a mathematical object that exhibits interference and can be used to compute probability. The wave function is a function of position and time that returns a complex number. The absolute value squared of the wave function is proportional to probability.

Complex numbers

Complex numbers are 2-dimensional vectors that can be multiplied. The x direction is called the real part, and the y direction is called the imaginary part. We write

$$\begin{aligned}i &= \sqrt{-1} \\z &= x + iy \\z^* &= x - iy \\|z|^2 &= zz^* \\e^{i\theta} &= \cos \theta + i \sin \theta\end{aligned}$$

Probability density

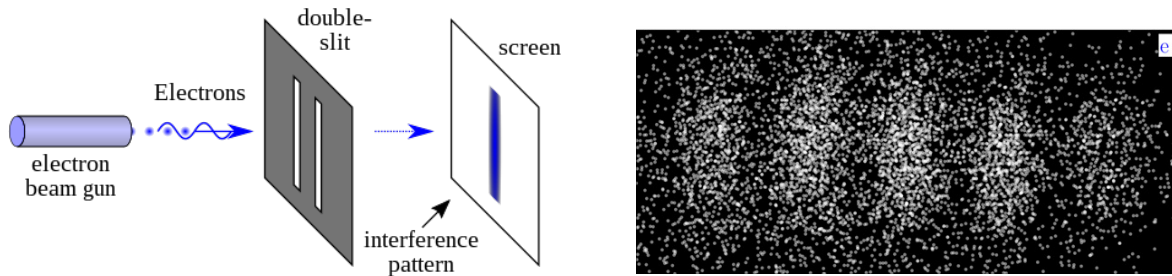
The wave function allows us to compute the probability density $|\Psi(x)|^2 = \rho(x)$. The probability density is the probability per unit length, which must be integrated to find the probability. This is like the relationship between total power and intensity.

$$P(x_1 < x < x_2) = \int_{x_1}^{x_2} \rho(x) dx$$

(continued on next page)

1 Electron interference

The wave nature of matter particles can be illuminated in double slit experiments. The schematic of a double slit experiment using a beam of electrons is shown in the left panel of the figure below. The right panel shows the emerging interference pattern when single electrons are sent through the double slit (see also the article on Controlled double-slit electron diffraction).



The electrons have a kinetic energy of $E_e = 600$ eV, and the two slits in this experiment are separated by $d = 272$ nm.

1.1 Wavelength

Find the de Broglie wavelength of the electrons.

Use

$$E_e = \frac{h^2}{2m\lambda^2} \implies \lambda = \frac{h}{\sqrt{2m_e E_e}}$$

The de Broglie wavelength of the electrons is $\lambda = 5 \times 10^{-11}$ m or 0.05 nm.

1.2 Momentum

Determine the momentum p_e of the electrons.

Use

$$p_e = \frac{h}{\lambda} = \frac{\sqrt{2m_e E_e}}{\hbar} \hbar = \sqrt{2m_e E_e}$$

The momentum of the electrons is $p_e = 1.323 \times 10^{-23}$ kg m/s.

1.3 First interference peak

Find the angle of the first peak (maximum intensity).

Use

$$d \sin \theta = m \lambda \implies \theta = \arcsin \left(\frac{h}{d \sqrt{2m_e E_e}} \right)$$

The angle of the first peak is $\theta = 0.01055$ degrees or 0.00018 rad.

1.4 Photon energy

If the electrons are replaced by photons of the same wavelength, what is the photon energy?

Use

$$E_{\text{photon}} = pc = \frac{hc}{\lambda} = \frac{\sqrt{2m_e E_e}}{\hbar} \hbar c = \sqrt{2m_e c^2 E_e}$$

The energy of the photons is $E_{\text{photon}} = 3.97 \times 10^{-15} \text{ J}$ or 24.7 keV.

2 Probability and Probability density

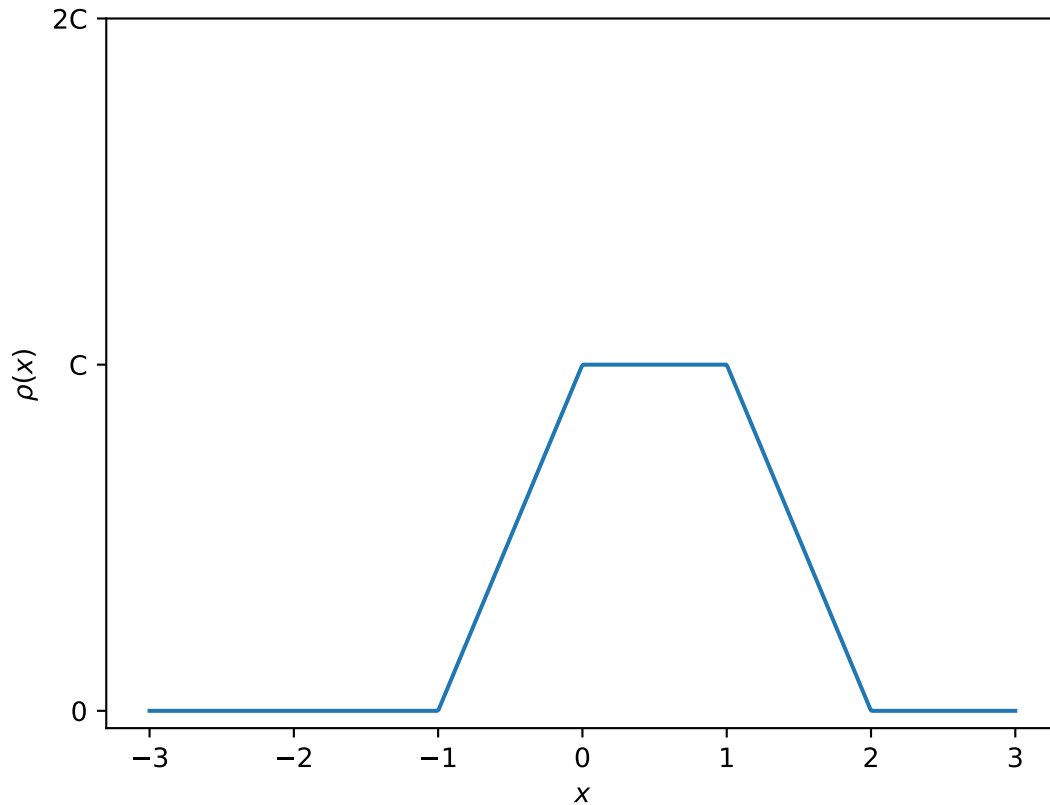
Suppose the probability density for finding an electron along a line is given by

$$\rho(x) = \begin{cases} C(1+x) & -1 < x < 0 \\ C & 0 < x < 1 \\ C(2-x) & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}, \quad (1)$$

where C is an unknown constant.

2.1 Diagram

Sketch the probability density as a function of x .



2.2 Normalization

Find the value of the normalization C .

We know that

$$1 = \int_{-\infty}^{\infty} \rho(x) dx, \quad (2)$$

which means that the area under the curve we sketched above must be equal to one. Since the curve is a trapezoid, we have that its area is given by $1/2(a+b)h$, where $h = C$ is the height,

$a = 1$ is the length of the short base, and $b = 3$ is the length of the long base. We thus have

$$1 = \int_{-\infty}^{\infty} \rho(x) dx = \frac{1}{2} C (1 + 3) = 2C \quad (3)$$

And therefore

$$C = \frac{1}{2} \quad (4)$$

2.3 Average position

Suppose that we prepare many electrons with this same probability density $\rho(x)$, and we measure the position of each of them. We then average our results together to define the average position \bar{x} of the electron. What do we find for \bar{x} ?

We could approach this problem either graphically or analytically. Graphically, we note that the probability density $\rho(x)$ is symmetric about $1/2$, so that when we average all our observations we will find roughly the same number of electrons at positions x and $1/2 - x$. Upon adding together the positions, we will find

$$\bar{x} = \frac{1}{2} \quad (5)$$

Analytically, we find \bar{x} by summing up the position of the electron times the probability of finding the electron at that position. This means integrating

$$\bar{x} = \int_{-\infty}^{\infty} x \rho(x) dx \quad (6)$$

$$= \frac{1}{2} \left(\int_{-1}^0 x(1+x) dx + \int_0^1 x dx + \int_1^2 x(2-x) dx \right) \quad (7)$$

$$= \frac{1}{2} \left(-\frac{1}{6} + \frac{1}{2} + \frac{2}{3} \right) \quad (8)$$

$$= \frac{1}{2} \quad (9)$$

3 Probability density from the wave function

Suppose that an electron has a wave function given by

$$\Psi_1(x) = \begin{cases} C_1(1 + ix^2) & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (10)$$

where C_1 is a normalization constant.

3.1 Normalization

Find the value of C_1 so that the wave function is normalized. Choose the positive real solution.

The equation is

$$|C_1|^2 \int_{-1}^1 (1 + ix^2)(1 - ix^2) dx = 1$$

Which then leads to

$$|C_1|^2 = \frac{5}{12}$$

So C_1 should be $\sqrt{5/12}$.

3.2 Probability

What is the probability that the electron will be found at a position $x > 0$?

You can do this by inspection; since it is symmetric, the probability is $\frac{1}{2}$. You can also do the integral

$$|C_1|^2 \int_0^1 (1 + ix^2)(1 - ix^2) dx$$

3.3 The role of the i

Compare the probability density of

$$\Psi_2(x) = \begin{cases} \sqrt{\frac{15}{56}}(1+x^2) & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}. \quad (11)$$

to Ψ_1 . Why is the normalization different?

Between $-1 < x < 1$,

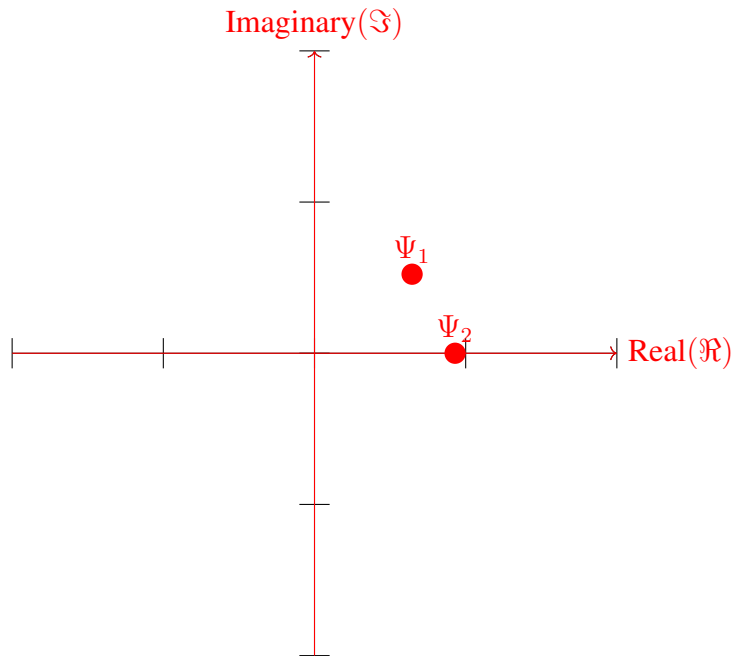
$$\rho_1(x) = \frac{5}{12}(1+x^4)$$

While

$$\rho_2(x) = \sqrt{15/56}(1+x^2)^2 = \frac{15}{56}(1+2x^2+x^4)$$

3.4 Plotting wave function value

Compute the values $\Psi_1(0.9)$ and $\Psi_2(0.9)$ and plot them on the complex plane below. Make sure to label the real and imaginary axes.



4 Superposition of momentum states

Suppose that an electron confined to a large box of length L has a wave function given by

$$\Psi(x) = C \sin\left(\frac{500\pi x}{L}\right) \quad (12)$$

4.1 Euler's formula

The wave function of an electron with momentum $\hbar k$ is e^{ikx} . Using Euler's formula, write the wave function $\Psi(x)$ as the sum of two momentum states (e^{ikx}). You will want to use these identities:

$$\begin{aligned} e^{i\theta} &= \cos(\theta) + i \sin(\theta) \\ e^{-i\theta} &= \cos(\theta) - i \sin(\theta) \end{aligned}$$

Subtract the equations to get

$$\sin(kx) = \frac{e^{ikx} - e^{-ikx}}{2i}$$

So

$$\Psi(x) = \left(\frac{C}{2i}\right) (e^{i500\pi x/L} - e^{-i500\pi x/L})$$

4.2 Measuring the momentum

This wave function is a *superposition* of two states with different momenta. Suppose that we measure the momentum of the electron by placing a very sensitive piece of foil in the box. What are the possible values that we could measure? Write it in terms of L , \hbar , and π .

For an electron, $p = \hbar k$, where the wave function is e^{ikx} . Reading the values of k from above, we get

$$p = \pm \hbar \frac{500\pi}{L}$$

4.3 Probabilities of the momentum

What are the probabilities of getting each of those values?

Both momentum states have equal amplitudes, so they are equal probability, 1/2 each.

4.4 Energy

What are the possible values of the electron energy? Express your answer in terms of \hbar , L and the electron mass m_e

$$E = \frac{p^2}{2m_e} = \frac{\hbar^2 k^2}{2m_e} = \frac{\hbar^2}{2m_e} \left(\frac{500\pi}{L} \right)^2$$

So we can measure only one possible energy, even though the momentum can have two possible values.

PHYS 214 discussion problem set 5: Wave functions with definite properties

Last week we learned how to compute the probability of observing a particle at a given position and time given the wave function. The wave function can tell you the probability of measuring *any* physical quantity, not just the position. In this section, we'll learn how to do that computation. For now, we will not consider the time dependence of wave functions for simplicity, just assume that the wave function is at some particular time.

The general strategy is to write a *general* wave function as a sum of wave functions with definite values of each quantity. A wave function has a definite value if we can only obtain one value with probability one when we measure the quantity. These wave functions are called *eigenstates*. Just because a wave function is an eigenstate of one quantity does not mean that it is an eigenstate of another. Examples of eigenstates that we've seen in this course:

Momentum A wave function that is written as Ae^{ikx} has definite momentum $\hbar k$.

Energy A wave function that satisfies the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2\Psi(x)}{dx^2} + U(x)\Psi(x) = E\Psi(x) \quad (1)$$

has definite energy E . Note that this equation must be true for *all* values of x with just one value of E ; i.e., E is not a function of x .

(examples follow on the next page)

Strategy example

Suppose that Ψ_1 and Ψ_2 are energy eigenstates with energy E_1 and E_2 . That means that they satisfy Equation 1, which you can verify by plugging them into the left-hand and right-hand sides and comparing. Suppose further that the particle is described using the wave function

$$\Psi_{tot} = a\Psi_1 + b\Psi_2, \quad (2)$$

where $|a|^2 + |b|^2 = 1$. There is a probability $|a|^2$ to observe energy E_1 and probability $|b|^2$ to observe energy E_2 . You can generalize this to superpositions of more eigenstates.

Heisenberg Uncertainty Principle

The Heisenberg uncertainty principle comes directly from the fact that states with definite momentum are given by Ae^{ikx} . This wave function is very spread out in space, and so any wave function that is *not* spread out must be a superposition of many momentum eigenstates. This is summarized in the famous equation

$$\Delta x \Delta p \geq \frac{\hbar}{2}, \quad (3)$$

which is to say that the range of values one can measure for position times the range of values one can measure for momentum must be greater than $\frac{\hbar}{2}$ for any wave function.

Infinite square well

The infinite square well is an idealized system in which we can solve for all energy eigenfunctions. There are discrete solutions

$$\Psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad (4)$$

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2mL^2} \quad (5)$$

Quantized energy levels

A wave function can only be a superposition of energy eigenstates that exist. So if it happens that the solutions of the Schrödinger equation are discrete, then that means that only the E_n 's above can be ever observed. This gives rise to the different emission colors of atoms you may have seen in lecture demos.

1 Heisenberg Uncertainty Principle

In this class, we will use the Heisenberg uncertainty principle fairly approximately, which we will follow here. The strict definition is that Δx and Δp represent the *standard deviation* of the position and momentum, as you may have seen in statistics courses. In this problem, we will be more loose with the definitions.

A common wave function is the Gaussian:

$$\Psi(x) = Ce^{-\frac{x^2}{4a^2}}, \quad (6)$$

where a is a parameter that controls the width of the wave function and $C = \frac{1}{\sqrt{a(2\pi)^{1/4}}}$ is a normalization constant, which we will not need to solve for here. This is the ground state of a particle in a harmonic oscillator (quadratic) potential well. We will assume that the particle is an electron for this problem.

1.1 Probability density

Write down a formula for the probability density in terms of C and a .

$$\rho(x) = |\Psi(x)|^2 = |C|^2 e^{\frac{-x^2}{2a^2}}$$

1.2 Spread

Find the values of x such that the probability density is $|C|^2 e^{-2} \simeq 0.135|C|^2$. We will estimate the spread Δx of the wave function as the distance between those two points. It turns out that for this wave function, there is about a 95% probability that the electron will be found between those two points.¹

Solving

$$\begin{aligned} |C|^2 e^{\frac{-x^2}{2a^2}} &= |C|^2 e^{-2}, \\ \frac{-x^2}{2a^2} &= -2 \\ x &= \pm 2a \end{aligned}$$

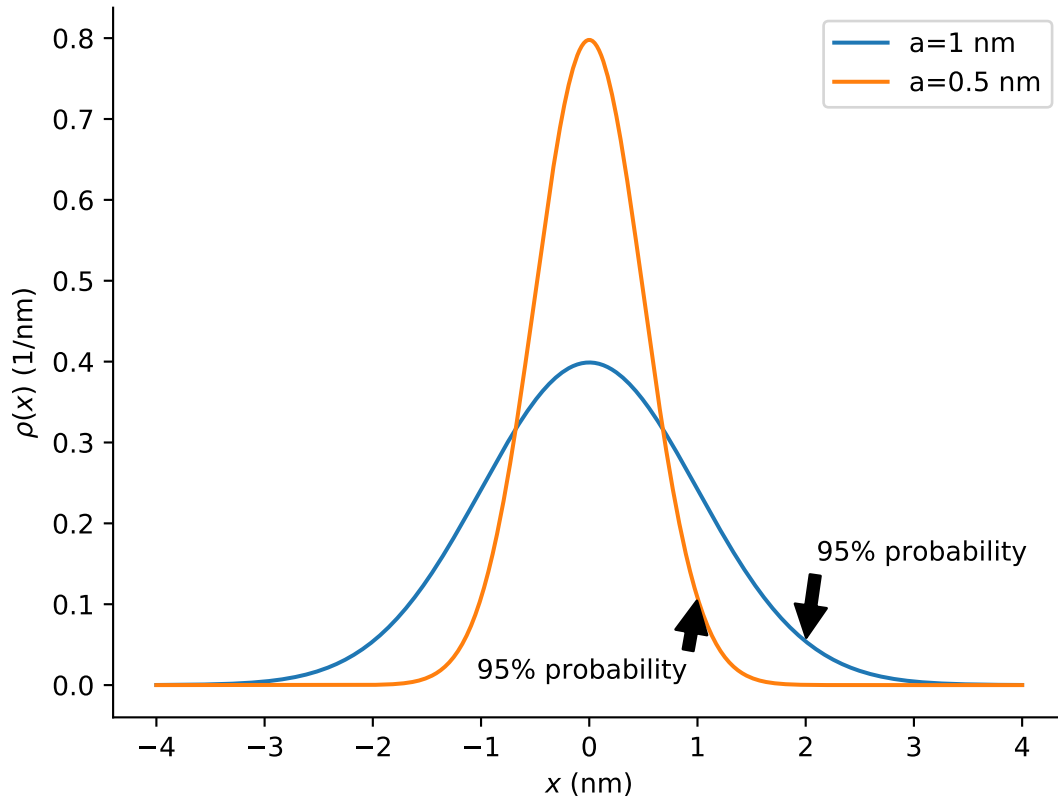
So the spread is

$$\Delta x = 4a$$

¹This is the normal distribution you may have seen in statistics.

1.3 Diagram

Sketch the probability density as a function of x for $a = 1$ nm and $a = 0.5$ nm. Indicate where the probability density reaches a factor of $1/e^2$ of its maximal value. For which value is the wave function more localized to the region around $x = 0$?



1.4 Uncertainty principle

Use Heisenberg's Uncertainty Principle to estimate the spread in velocities of the electron for the two values of a above in m/s. On average, would you measure the electron as moving quickly or slowly, compared to everyday velocities? Keep in mind that typical highway speeds are around 30 m/s.

$$\begin{aligned}\Delta x &= 4a \\ \Delta x \Delta p &= \hbar/2 \\ \Delta v = \Delta p/m &= \frac{\hbar}{2\Delta x m} = \frac{\hbar}{8am}\end{aligned}$$

So for 1 nm spread, $\Delta v = 14,471$ m/s, and for 0.5 nm spread, $\Delta v = 28,942$ m/s. A typical measurement would thus result in a very high velocity for the electron, compared to normal everyday velocities.

2 Superposition and Measurement

Consider an electron in a potential well that at $t = 0$ is in a particular superposition of the ground state, first ($n = 2$), and second ($n = 3$) excited states:

$$\Psi(x) = N (2\psi_1(x) + 5\psi_2(x) + 3i\psi_3(x))$$

Here N is the normalization factor and $\psi_n(x)$ ($n = 1, 2, 3$) are eigenstate wave functions of the potential well with energies E_n ($n = 1, 2, 3$), respectively. The eigenfunctions $\psi_n(x)$ are normalized and orthogonal.

2.1 Compute N to normalize the wave function Ψ

First write down the condition that the wave function Ψ must satisfy, then use the fact that the eigenfunctions are normalized and orthogonal.

The wave function Ψ must be normalized so that

$$\int dx |\Psi(x)|^2 = 1$$

The orthonormality of the eigenfunctions ψ_n can be expressed as:

$$\int dx \psi_n(x) \psi_m^*(x) = \delta_{nm}$$

where $\delta_{nm} = 1$ if $n = m$ and $\delta_{nm} = 0$ if $n \neq m$.

Writing $\Psi(x)$ in terms of the $\psi_n(x)$ as specified in the problem statement yields

$$\begin{aligned} \int dx |\Psi(x)|^2 &= |N|^2 \int dx |2\psi_1(x) + 5\psi_2(x) + 3i\psi_3(x)|^2 \\ &= |2N|^2 \int dx |\psi_1(x)|^2 + |5N|^2 \int dx |\psi_2(x)|^2 + |3iN|^2 \int dx |\psi_3(x)|^2 \\ &\quad + \sum_{n \neq m=1}^3 C_{nm} \int dx \psi_n(x) \psi_m^*(x) \end{aligned}$$

where the sum in the last term is for the cross terms, so $n \neq m$ and C_{nm} are complex-valued constants. Using the orthonormality properties of the ψ_n , we see that the cross terms vanish, and the individual integrals over the $|\psi_n(x)|^2$ are all equal to unity, yielding

$$\begin{aligned} \int dx |\Psi(x)|^2 &= 4|N|^2 + 25|N|^2 + (3iN)(-3iN^*) = 4|N|^2 + 25|N|^2 + 9|N|^2 = 1 \\ N &= \frac{1}{\sqrt{38}} \end{aligned}$$

2.2 Energy measurement

Now suppose that the energy of the electron is measured.

1. Find the probability for the energy measurement to return the values E_1 , E_2 , and E_3 , respectively.

The probability of an energy measurement to return the value E_n is the same as the probability for the electron to be in the state n . Both are given $P(E_n) = |\int dx \psi_n^*(x) \Psi(x)|^2$.

The probability of finding the electron in state $n = 1$ is then

$$\begin{aligned} P(E_1) &= \left| \int dx \psi_1^*(x) \Psi(x) \right|^2 \\ &= \left| 2N \int dx \psi_1^*(x) \psi_1(x) + 5N \int dx \psi_1^*(x) \psi_2(x) + 3iN \int dx \psi_1^*(x) \psi_3(x) \right|^2 \\ &= |2N + 0 + 0|^2 = 4|N|^2 = \frac{4}{38} \end{aligned}$$

Similarly, $P(E_2) = \frac{25}{38}$ and $P(E_3) = \frac{9}{38}$

The easy trick is that the probability of finding the electron in state n is given by the modulus of the normalized coefficient ($|c_n|^2$) for that state in the superposition sum, $\Psi(x)$.

2. If the energy measurement returns the value E_2 , find the wave function of the electron right after the measurement.

$$\Psi(x) = \psi_2(x)$$

3 Energy eigenstates

The time-independent Schrödinger equation for a free particle is

$$-\frac{\hbar^2}{2m} \frac{d^2\Psi(x)}{dx^2} = E\Psi(x), \quad (7)$$

since the potential energy $U = 0$.

3.1 Showing that a wave function satisfies the Schrödinger equation

Show that for a free particle, a particle with momentum $\hbar k$ (in a momentum eigenstate) is also an energy eigenstate, and compute its energy.

If $p = \hbar k$, then $\Psi(x) = Ae^{ikx}$. Left hand side:

$$\begin{aligned} -\frac{\hbar^2}{2m} \frac{d^2\Psi(x)}{dx^2} &= -\frac{\hbar^2}{2m} A(-k^2)e^{ikx} \\ &= \frac{\hbar^2 k^2}{2m} \Psi(x). \end{aligned}$$

So this satisfies the Schrödinger equation if we set $E = \frac{\hbar^2 k^2}{2m}$.

3.2 Relation to classical mechanics

Is the relationship between energy and momentum you derived consistent with the classical mechanics relationship for a free particle, that is, $E = \frac{p^2}{2m}$? Explain how.

Yes.

$$E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m},$$

which is the relationship that we derived earlier. This works because for a free particle, all of its energy is due to momentum.

3.3 Adding a potential U

Suppose that the particle is now in a potential well, so that $U = \frac{1}{2}\kappa x^2$. Show that a momentum eigenstate does **NOT** satisfy Schrödinger's equation and is thus not an energy eigenstate.

The left hand side is now

$$-\frac{\hbar^2 k^2}{2m}\Psi(x) + \frac{1}{2}\kappa x^2\Psi(x)$$

We would like to set that equal to $E\Psi(x)$, but that is impossible for all values of x , since there is still the $\frac{1}{2}\kappa x^2$ term. So this is not an energy eigenstate.

3.4 Measurement

Suppose a particle is prepared so that it has the wave function $A e^{ikx}$ when it is placed in the harmonic potential well of part 3, and its total energy is subsequently measured.

1. First, describe the wave function of the particle in terms of the eigenstate wave functions of the harmonic potential well.

As we showed in part 3, the momentum eigenstate wave function $A e^{ikx}$ is not an eigenstate of the harmonic potential. This means that the particle with this wave function when placed into the well is in a superposition of the well's eigenstates, ψ_n :

$$A e^{ikx} = \sum_n c_n \psi_n(x)$$

where the coefficients can be calculated (in principle), if the eigenfunctions $\psi_n(x)$ are known.

2. Then consider what happens to the particle when its energy is measured. Suppose you did this entire process (put a particle in a momentum eigenstate and then put it in the well) many times. Would you expect to get the same energy every time? Why or why not?

You would not get the same energy every time. Since the particle is in a superposition of (many) energy eigenstates, there are many possibilities for the measurement of energy, each with probability $P(E_n) = |c_n|^2$.

4 Transitions for a particle in a box

An electron is confined to an infinite potential well of length 0.5 nm.

4.1 Energy level diagram

Draw an energy level diagram for the first four energies of this system, labelling each in eV. An energy level diagram is a tower of lines, with the energies marked on them.

$$E_n = \frac{\pi^2 \hbar^2 n^2}{2mL^2} = n^2 E_1$$

$$E_1 = \frac{(1.054 \times 10^{-34} \text{ Js})^2 \pi^2}{2(9.109 \times 10^{-31} \text{ kg})(0.5 \times 10^{-9} \text{ m})^2} = 2.407 \times 10^{-19} \text{ J} = 1.503 \text{ eV}$$

$$\text{————— } E_4 = 16E_1 = 24 \text{ eV}$$

$$\text{————— } E_3 = 9E_1 = 13.5 \text{ eV}$$

$$\text{————— } E_2 = 4E_1 = 6.0 \text{ eV}$$

$$\text{————— } E_1 = 1.5 \text{ eV}$$

4.2 Energy conservation

Suppose that the electron is kept very cold, so that it starts in the ground state (the lowest energy state). We shine light on the system. Write down an energy balance equation for an initial state of a photon plus the electron in the ground state, and a final condition of the electron in an excited state.

$$hf + E_1 = E_n$$

4.3 Absorption of light

Use the energy balance equation above to compute what photon energies can be absorbed by the system, using just the states you enumerated in part 1.

$$hf = E_n - E_1$$

This means that the system could absorb photons of energies 4.5, 12.0, or 22.5 eV.

4.4 Emission of light

Now suppose we put energy into the system. We might do this by running an electric current through it, which can give the electron energy. For simplicity, suppose that the electron is now in the second excited state ($n = 3$). Use energy conservation to determine what energy photons could be emitted by the system. This is how quantum mechanics explains that atoms only emit in very specific colors of light.

Now the energy balance equation is

$$E_3 = E_n + hf$$

This is only possible for $n < 3$, so there are two possibilities: 7.5 eV or 12 eV.

4.5 Changing L

Now suppose the width L is changed and you perform an experiment to find the new value of L . If you shine light on the system, only certain frequencies of the light will be absorbed. If the longest wavelength of light absorbed is $\lambda = 710$ nm (red color), find the width L of the well.

The smallest energy difference between two states of a particle in a box is the energy difference between $n = 1$ and $n = 2$. We get

$$\begin{aligned} hf &= E_2 - E_1 \\ &= 2^2 E_1 - E_1 \\ &= (4 - 1)E_1 = 3 \frac{\pi^2 \hbar^2}{2mL^2} \end{aligned}$$

Since $\lambda = \frac{c}{f}$ and $\hbar = \frac{h}{2\pi}$. The equation can be simplified to

$$\frac{c}{\lambda} = \frac{3\hbar}{4mL^2}$$

With some algebra, we get

$$L = \sqrt{\frac{3\pi\hbar\lambda}{4mc}} = 8.02 \times 10^{-10} \text{ m} = 0.8 \text{ nm}$$

PHYS 214 discussion problem set 6: Energy Levels and the Harmonic Oscillator

Introduction

Energy levels are determined by the time-independent Schrödinger equation

The time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m} \frac{d^2\Psi(x)}{dx^2} + U(x)\Psi(x) = E\Psi(x). \quad (1)$$

This equation tells us **whether a wave function is an energy eigenstate**. If it is true for all x , then the wave function is an energy eigenstate. This means that if we measure its energy, we will obtain E with probability 1.

For every potential energy function, there is a set of allowed energies and energy eigenstates. In this class, you have seen the free particle ($U=0$), infinite square well, and harmonic oscillator potentials. Each of these has a distinct set of energy levels, which are the values of E for which there exists a solution to the time independent Schrödinger equation. This allows us to probe quantum systems, and determine something about U just by looking at what energies we can measure. We usually do that by sending in photons and seeing what energies the system can absorb, or seeing what energy the system can emit. This tells us the difference between energy levels, since, for example, $hf_{\text{absorbed}} = E_f - E_i$.

Multiple particles

In this class, we consider multiple electron systems, and will be ignoring any interactions between them. Under these conditions, we construct multiple particle energy eigenstates by filling up one-particle eigenstates. Each one-particle energy eigenstate can hold two electrons—one with spin up and one with spin down. Every distinct filling leads to a different many-particle energy eigenstate. Just like with one particle, the lowest energy eigenstate with N particles is the ground states. That consists of filling up all the lowest energy single-particle eigenstates.

1 Simple harmonic oscillator energy levels

We'll consider the motion of a simple harmonic oscillator in classical mechanics versus quantum mechanics.

In both cases the potential energy is $U(x) = \frac{1}{2}kx^2$.

1.1 Classical frequency

Show that $x(t) = A \sin(\omega t)$ is a solution to Newton's equation $F = ma$, and find ω in terms of k and m .

$$F = -\frac{dU}{dx} = -kx = m\frac{d^2x}{dt^2} = -Am\omega^2 \sin(\omega t) = -m\omega^2 x$$

This equation can be satisfied if $\omega = \sqrt{k/m}$.

1.2 Classical energy

Now let us look at the energy of the classical oscillator, whose motion is described by $x(t) = A \sin(\omega t)$.

- What is the total energy, expressed in terms of A , m and ω ?
- What are the position and momentum of the particle in the lowest allowed classical energy state?

The potential energy of the particle is given by

$$U = \frac{1}{2}kx(t)^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t). \quad (2)$$

Similarly, the kinetic energy is

$$KE = \frac{1}{2}m \left(\frac{dx}{dt} \right)^2 = \frac{1}{2}mA^2\omega^2 \cos^2(\omega t). \quad (3)$$

Adding these together and making use of the identity $\cos^2(\omega t) + \sin^2(\omega t) = 1$, we find

$$E = \frac{1}{2}m\omega^2 A^2. \quad (4)$$

We see that, classically, the total energy depends continuously on the amplitude A of oscillation. The lowest energy state then has $A = 0$, $E = 0$. This corresponds to a particle with trajectory $x(t) = 0$ — a particle at rest at the origin. In the classical ground state, we can say that the particle has definite position $x = 0$ and definite momentum $p = 0$.

1.3 Energy level diagram

Now we turn to the quantum description. In the energy level diagram below for a simple harmonic oscillator, evaluate the energies of the first 4 states in terms of \hbar and $\omega = \sqrt{k/m}$.

$$\begin{array}{l}
 \text{————— } E_3 = \frac{7}{2}\hbar\omega \\
 \text{————— } E_2 = \frac{5}{2}\hbar\omega \\
 \text{————— } E_1 = \frac{3}{2}\hbar\omega \\
 \text{————— } E_0 = \frac{1}{2}\hbar\omega
 \end{array}$$

1.4 Quantum states

Let's compare the quantum states to the classical ground state above.

- (a) Compare the ground state energy of the quantum harmonic oscillator with that of the classical harmonic oscillator. Does the ground state energy depend on the spring constant?

The quantum ground state has energy $E_0 = \frac{1}{2}\hbar\omega > 0$, whereas the classical ground state energy is $E = 0$. Yes, the quantum ground state energy depends on k and on the mass of the system, because $\omega = \sqrt{k/m}$.

- (b) If a quantum harmonic oscillator is in its ground state, what can you say about the result of an energy measurement?

In this case, the wavefunction is given by

$$\psi_0(x) = Ce^{-\frac{\sqrt{mk}}{2\hbar}x^2} = Ce^{-\frac{m\omega}{2\hbar}x^2}. \quad (5)$$

An energy measurement will return the ground state energy, E_0 . After the measurement, the system will still be in the ground state with the wave function given above.

- (c) What can we say about the result of a measurement of position of the particle in the lowest allowed quantum energy state?

To consider position measurements, we need the probability density:

$$\rho(x) = |\psi_0(x)|^2 = |C|^2 e^{-\frac{\sqrt{mk}}{\hbar}x^2} = |C|^2 e^{-\frac{m\omega}{\hbar}x^2}. \quad (6)$$

. We see that the probability density is a Gaussian function centered at $x = 0$, which is greater than zero for all values of x . This means that there is a nonzero probability of finding the

particle in any interval on the x -axis, however, in regions far from $x = 0$, the probability is very small. The probability of finding the particle in the interval $[x_1, x_2]$ is given by:

$$P(x_1, x_2) = \int_{x_1}^{x_2} dx |\psi_0(x)|^2 = |C|^2 \int_{x_1}^{x_2} dx e^{-\frac{m\omega}{\hbar}x^2}, \quad (7)$$

and the probability of finding the particle between x and $x + \delta x$ is given by $\rho(x) \delta x$. In last week's discussion problem 5.1, we defined the spread Δx as the interval around $x = 0$ where the probability of finding the particle is 95%. Using the result from problem 5.1, we have $\Delta x = 4a$ where in this case, $a = \sqrt{\hbar/(m\omega)}$.

In contrast, the only allowed position of the classical HO when it has its lowest allowed energy is $x = 0$.

- (d) What can we say about the result of a measurement of momentum of the particle in the lowest allowed quantum energy state?

The momentum-space probability density is also a Gaussian function, and hence, a momentum measurement can, in principle yield any value of p , however, however the probability density for large momenta is very small. Hence, similar to the position case, there is a range of probable momenta, Δp , for which the probability is 95%.

- (e) If we increase the mass of the particle but keep ω fixed, how do the results of position measurements in the ground state change?

If we increase m but keep ω fixed, $\sqrt{m\hbar} = m\omega$ increases. From the form of the wavefunction, we see that this causes the position probability density to become more sharply peaked, decreasing the width of $\rho(x)$. Using the result from problem 5.1, we have $\Delta x = 4a$ where in this case, $a = \sqrt{\hbar/(m\omega)}$. Hence, Δx , the spread of probable outcomes of a position measurement, decreases.

- (f) If we increase the mass of the particle but keep ω fixed, how do the results of momentum measurements in the ground state change?

By Heisenberg's Uncertainty Principle, the spread of probable outcomes of a momentum measurement Δp must then increase, $\Delta p \geq \hbar/(2\Delta x)$.

2 Infrared absorption

The atoms within a molecule can vibrate in many different “modes”, as illustrated by this Wikipedia animation: Molecular Vibrations of CH₂. Each vibrational mode is well described as a harmonic oscillator with its own spring constant and, thus, its own frequency $f_{\text{mode}} = \sqrt{k/m}/2\pi$. Here are the oscillator frequencies for the vibrational modes of some common molecules:

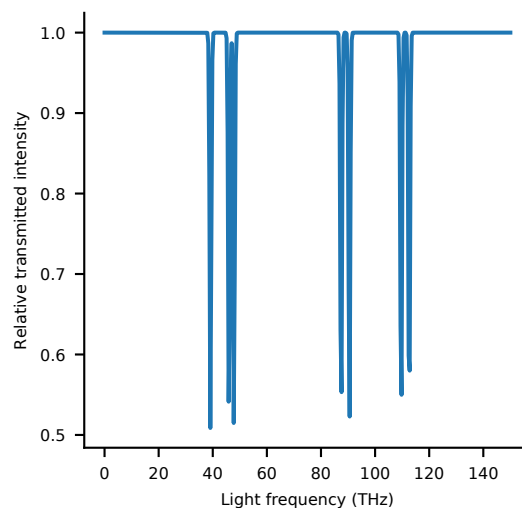
Molecule	Frequency (THz)
Methane (CH ₄)	39.2, 46.0, 87.5, 90.5
Ozone (O ₃)	21.1, 31.2, 33.3
Oxygen (O ₂)	47.4
Water (H ₂ O)	47.8, 109.6, 112.6
Carbon dioxide (CO ₂)	20.0, 40.0, 70.4

2.1

Write down the relationship between the oscillator frequency f_{mode} of a molecule in a particular mode, the spacing between the energy levels $E_1 - E_0$ associated with that mode, and the frequency of light corresponding to that energy difference f_{light} .

$$hf_{\text{mode}} = E_1 - E_0 = \hbar\omega = hf_{\text{light}}$$

2.2



Now suppose that we pass infrared light of varying frequency through a gas made up of a mixture of two types of molecules. The y axis here is the proportion of light that passes through the gas. When it is less than one, it means that the gas has absorbed some of the infrared light. What two molecules are in the mixture?

Here you can read off the frequencies that appear in the table above. They occur around 39, 46, 47, 88, 90, 110, and 112 THz. This is consistent with water and methane. NOTE: O_2 could also be a possibility here. However, since we are restricted to just two molecules and there are other frequencies that appear, we know it has to be water and methane.

2.3

Write down a formula for the spring constant k in terms of the frequency of light absorbed by the molecule.

$$\begin{aligned} hf_{\text{light}} &= \hbar\omega = \hbar\sqrt{\frac{k}{m}} \\ 2\pi f_{\text{light}} &= \sqrt{\frac{k}{m}} \\ k &= m(2\pi f_{\text{light}})^2 \end{aligned}$$

2.4

The diatomic molecule O_2 has only one vibrational mode. Compute its spring constant of O_2 in units of N/m. Shock absorbers in a car have a spring constant of about 5000 N/m. The O_2 molecule has an effective mass of 1.33×10^{-26} kg.

- Is an O_2 molecule stiffer or softer than shock absorbers?
- The frequency is much larger than the fundamental frequency of a shock absorber. Why is that?

Plugging in values from above,

$$k = 1.33 \times 10^{-26} \text{ kg} (2\pi \times 47.4 \times 10^{12} \text{ s}^{-1})^2 = 1180 \text{ N/m}$$

It is thus somewhat softer than shock absorbers, which are pretty stiff! The frequency is much higher because the mass is much lower.

3 Many electrons in many levels

3.1 Energy level diagram

The lowest 8 energy eigenstates of a particular system are found to be 0.1, 0.2, 0.3, 0.4, 4.4, 4.5, 4.6, 4.7 eV. Draw an energy level diagram for this system, labelling the energies.

$$\begin{array}{c} \text{=====} \\ \text{=====} \\ \text{=====} \\ \text{=====} \end{array} E = 4.4, 4.5, 4.6, 4.7 \text{ eV}$$

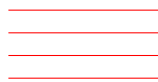
$$\begin{array}{c} \text{=====} \\ \text{=====} \\ \text{=====} \\ \text{=====} \end{array} E = 0.1, 0.2, 0.3, 0.4 \text{ eV}$$

3.2 $n = 6$

Suppose that we place 6 electrons in the system.

- Draw an energy level diagram showing the ground state of the entire system. (Remember, electrons are spin-1/2 particles and can have two possible orientations.)
- Use the energy level diagram to determine the lowest energy photon that the system could absorb to get to an excited state.

For clarity, this diagram is not quite to scale.



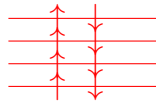
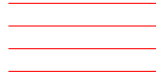
The lowest energy photon that could be absorbed is equal to the difference between the highest occupied and lowest unoccupied state. In this case, that's $E_{\text{photon}} = 0.1 \text{ eV}$.

3.3 $n = 8$

Now suppose that we place 8 electrons in the system instead of 6.

- Draw an energy level diagram showing the ground state of the entire system.
- Use the energy level diagram to determine the lowest energy photon that the system could absorb to get to an excited state.

For clarity, this diagram is not quite to scale.



The lowest energy photon that could be absorbed is equal to the difference between the highest occupied and lowest unoccupied state. In this case, that's $E_{\text{photon}} = 4 \text{ eV}$.

3.4 Adding electrons

The energy needed to add an electron to a system is $E(n + 1) - E(n)$. Compare this energy between $n = 6$ and $n = 8$. In which situation would it take less work to add electrons (e.g. to conduct electricity)?

When $n = 6$,

$$E(n = 6) = (2 \times 0.1 + 2 \times 0.2 + 2 \times 0.3) \text{ eV} = 1.2 \text{ eV} .$$

When $n = 7$, we have:

$$E(n = 7) = (2 \times 0.1 + 2 \times 0.2 + 2 \times 0.3 + 0.4) \text{ eV} = 1.6 \text{ eV} .$$

The energy needed (work) to add a 7th electron to a 6-electron system is given by the difference, $E(7) - E(6) = 0.4 \text{ eV}$, which is the same as the lowest available energy level for the 7th electron.

For $n = 8$, $E(n + 1) - E(n) = 4.4 \text{ eV}$. In this case, it takes a lot more work to add electrons to the system, which means a higher barrier to conducting electricity.

4 Superposition and energy transitions (optional)

Consider an electron in a potential well that is in a particular superposition of the first ($n = 2$) and second ($n = 3$) excited states:

$$\Psi(x) = N [2\psi_2(x) - 3i\psi_3(x)]$$

4.1 Normalization

Determine N , assuming the eigenstate wave functions $\psi_2(x)$ and $\psi_3(x)$ are individually normalized and orthogonal.

The wave function Ψ must satisfy the normalization condition

$$\int dx |\Psi(x)|^2 = 1 = |N|^2 \int dx |2\psi_2(x) - 3i\psi_3(x)|^2$$

Since $\psi_2(x)$ and $\psi_3(x)$ are individually normalized and orthogonal, the normalization condition above yields:

$$|N|^2 [4 + 9] = 1.$$

Hence we find that $N = 1/\sqrt{13}$.

For a general superposition,

$$\Psi(x) = c_1\psi_1(x) + c_2\psi_2(x) + c_3\psi_3(x) + c_4\psi_4(x) + \dots = \sum_n c_n\psi_n(x),$$

the normalization condition becomes

$$|c_1|^2 + |c_2|^2 + |c_3|^2 + |c_4|^2 + \dots = \sum_n |c_n|^2 = 1,$$

where the sum goes over as many (or as few) terms as needed to describe the wave function in question.

In the case at hand, we have $c_1 = 0$, $c_2 = 2N$, $c_3 = 3iN$, and $c_n = 0$ for all $n > 3$. We can determine N from the normalization condition above, yielding the same result as before, so $c_2 = 2/\sqrt{13}$ and $c_3 = 3i/\sqrt{13}$.

4.2 Energy transitions

Suppose, the energies of the ground, first, and second excited states are $E_1 = 2$ eV, $E_2 = 4$ eV, and $E_3 = 6$ eV. In this system, the electron can transition from the excited states, $n = 2$ or $n = 3$ to the ground state ($n = 1$) by emitting a photon. If for some reason the electron **cannot** transition from the $n = 3$ to $n = 2$ state, what are the possible energies of photons that might be emitted by this system?

The allowed photon energies are $E_{\text{photon}} = E_2 - E_1 = 2$ eV, and $E_{\text{photon}} = E_3 - E_1 = 4$ eV. No other photon energies are possible.

4.3 Transition probabilities

Let's assume that the probabilities for the $n = 2 \rightarrow n = 1$ and $n = 3 \rightarrow n = 1$ transitions are (roughly) equal. If the electron starts in the state Ψ , it is in a superposition of the $n = 2$ and $n = 3$ eigenstates. What is the relative probability for the electron to transition to ground state from $n = 2$ versus $n = 3$, $P(2 \rightarrow 1)/P(3 \rightarrow 1)$?

The transition probabilities are now simply proportional to the probabilities of the electron to be in either the $n = 2$ state (ψ_2) or the $n = 3$ state (ψ_3). Each probability is equal to the modulus of the corresponding coefficient. Hence,

$$\frac{P(2 \rightarrow 1)}{P(3 \rightarrow 1)} = \frac{|2|^2}{|3i|^2} = \frac{4}{9}.$$

Note that the relative probability doesn't require knowledge of the normalization N .

4.4 Average energy of emitted photons

Assuming we have a large ensemble of systems in the state Ψ , what is the average energy of the emitted light?

The average energy \overline{E} is given by the weighted sum of the allowed photon energies, ($E_3 - E_1$ and $E_2 - E_1$) weighted by the probabilities of finding the electron in the corresponding initial energy state:

$$\overline{E} = (E_2 - E_1) P(E_2) + (E_3 - E_1) P(E_3) = 2 \text{ eV} \left| \frac{2}{\sqrt{13}} \right|^2 + 4 \text{ eV} \left| \frac{3i}{\sqrt{13}} \right|^2 = 3.38 \text{ eV}.$$

Note that the normalization $N = 1/\sqrt{13}$ is needed here. Also note that **none** of the emitted photons actually have an energy of \overline{E} . Roughly 31% of the photons have energy $E_{\text{photon}} = 2 \text{ eV}$, and the other 69% have $E_{\text{photon}} = 4 \text{ eV}$.

PHYS 214 discussion problem set 7: Polarization, spin systems, and band structure

1 Polarization problem

We consider shining light on a system of three filters:

Left Horizontal filter \rightarrow right-handed circular filter \rightarrow vertical filter

Right Horizontal filter \rightarrow right-handed circular filter \rightarrow horizontal filter

1.1 Wave functions

Write out the polarization wave functions after each step.

Horizontal filter: Ψ_h

Right-handed circular filter: $\frac{1}{\sqrt{2}}(\Psi_h + i\Psi_v)$

Vertical filter: Ψ_v

1.2 Probabilities

Compute the following probabilities. Make sure to *show* how you got the probabilities by taking the dot product between the different states.

1. Horizontally polarized light passing through a right-handed circular filter.

$$|\frac{1}{\sqrt{2}}(\Psi_h + i\Psi_v)^* \cdot \Psi_h| = |\frac{1}{\sqrt{2}}(\Psi_h^* - i\Psi_v^*) \cdot \Psi_h|^2 = |\frac{1}{\sqrt{2}}\Psi_h^* \Psi_h + 0|^2 = 1/2$$

2. Right-handed circularly polarized light passing through a horizontal filter.

$$|(\Psi_h)^* \cdot \frac{1}{\sqrt{2}}(\Psi_h + i\Psi_v)|^2 = |\frac{1}{\sqrt{2}}\Psi_h^* \Psi_h + 0|^2 = 1/2$$

3. Right-handed circularly polarized light passing through a vertical filter.

$$|(\Psi_v)^* \cdot \frac{1}{\sqrt{2}}(\Psi_h + i\Psi_v)|^2 = |\frac{1}{\sqrt{2}}\Psi_v^* i\Psi_v + 0|^2 = 1/2$$

4. Horizontally polarized light passing through a vertical filter.

$$|\Psi_v^* \cdot \Psi_h|^2 = 0$$

1.3 Total probabilities

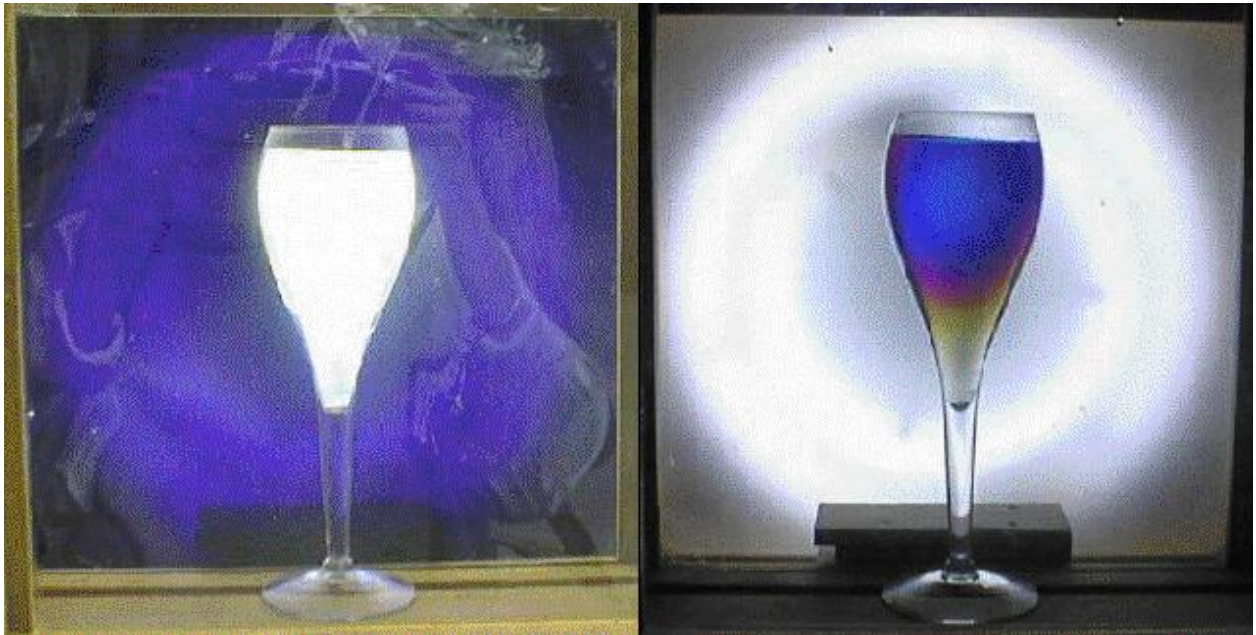
Keeping in mind that 50% of the light is always blocked by the initial horizontal filter, fill in the table. We have used the notation h : horizontal filter, rc right-hand circular filter, v vertical filter.

Sequence of filters	Total probability of light passing through
$h \rightarrow rc \rightarrow v$	
$h \rightarrow rc \rightarrow h$	
$h \rightarrow v$	
$h \rightarrow h$	

Sequence of filters	Total probability of light passing through
$h \rightarrow rc \rightarrow v$	1/8
$h \rightarrow rc \rightarrow h$	1/8
$h \rightarrow v$	0
$h \rightarrow h$	1/2

1.4 Corn syrup

We now consider an experiment that you may have seen in the lecture as a demo.



Here, between two linear polarizing filters, is placed a glass containing corn syrup. The corn syrup acts as a quarter wave plate, converting linearly polarized light into circularly polarized, without reducing the intensity. In the set-up on the left side, the two linear filters are oriented so that the polarization directions are orthogonal. On the right, the polarization directions of the two linear filters are aligned.

Discuss with your table and instructor why the wine glass appears to glow in the left image, but in the right image, it appears dark.

On the left, the background is a linear filter followed by a filter with its polarization direction orthogonal to the initial filter, which blocks almost all light. So by contrast the wine glass looks bright. On the other hand, on the right, the background is a linear filter followed by another linear filter oriented the same, so it is brighter than the wine glass. In both cases, the amount of light passing through the wine glass is the same; it's really about the contrast.

2 Stern Gerlach experiment

Sketch the Stern-Gerlach experiment, and briefly describe how it works. Discuss with your table and instructor.

In its simplest form, the Stern-Gerlach experiment consists of three main regions: the oven and a collimator that produces a collimated beam of neutral atoms; the magnet, which generates the inhomogeneous magnetic field; and a detector for the atoms, as depicted in the figure below. The

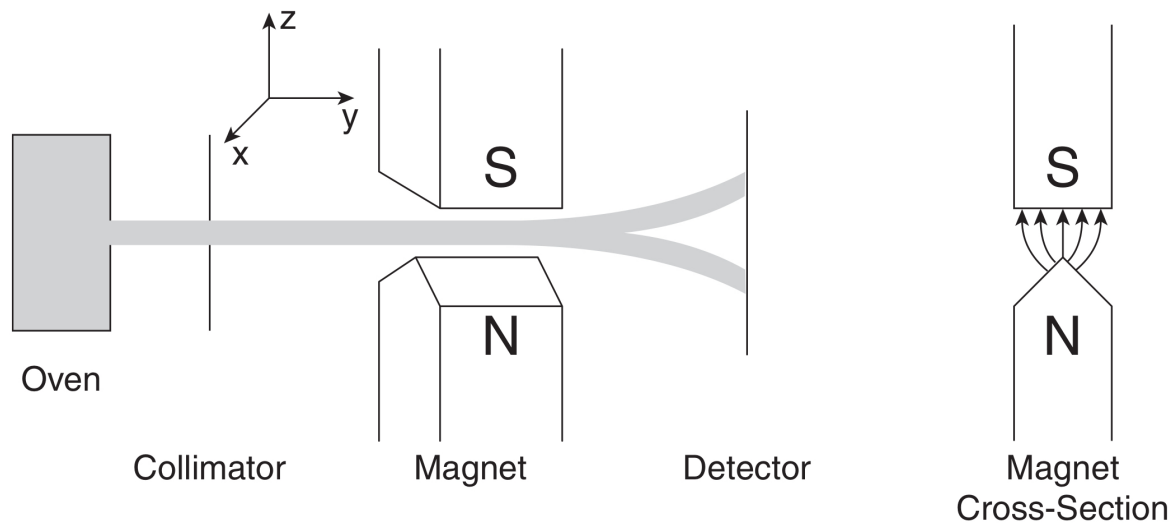


Figure 1: McIntyre, D. H. (2012). Quantum Mechanics: A Paradigms Approach. Cambridge: Cambridge University Press, p.2

neutral atoms interact with the magnetic field in the z -direction, leading to a magnetic force in the z -direction that depends on the spin property of the atoms. This interaction separates the atoms with different spins to produce a split pattern at the detector.

2.1 Probabilities

We now consider combinations of Stern-Gerlach filters through which a beam of Silver atoms is guided. We denote, for example, a filter that selects $+x$ direction as $F_x(+)$. Assume that the atom starts out with a random spin.

1. $F_x(+)$

probability = $\frac{1}{2}$ and the state collapses to $\uparrow_x = \frac{1}{\sqrt{2}}(\uparrow + \downarrow)$

2. $F_x(+)$ \rightarrow $F_x(-)$

From part 1, the atoms pass through $F_x(+)$ with probability = $\frac{1}{2}$ and the state collapses to \uparrow_x . Then probability = $|\downarrow_x \cdot \uparrow_x|^2 = 0$

3. $F_x(+) \rightarrow F_x(+)$

From part 1, the atoms pass through $F_x(+)$ with probability $= \frac{1}{2}$ and the state collapses to \uparrow_x . The probability of the second filter $= |\uparrow_x \cdot \uparrow_x|^2 = 1$. Then the total probability $= \frac{1}{2} \times 1 = \frac{1}{2}$

4. $F_y(+) \rightarrow F_x(+)$

The atoms pass through $F_y(+)$ with probability $= \frac{1}{2}$ and the state collapses to $\uparrow_y = \frac{1}{\sqrt{2}}(\uparrow + i\downarrow)$. The probability of the second filter $= |\uparrow_x \cdot \uparrow_y|^2 = |\frac{1}{\sqrt{2}}(\uparrow + \downarrow) \cdot \frac{1}{\sqrt{2}}(\uparrow + i\downarrow)|^2 = \frac{1}{2}$. Then the total probability $= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

5. $F_y(+) \rightarrow F_x(+) \rightarrow F_z(-)$

From part 4, the atoms pass through the two filters with probability $= \frac{1}{4}$ and the state collapses to $\uparrow_x = \frac{1}{\sqrt{2}}(\uparrow + \downarrow)$.

The probability of the third filter $F_z(-)$, $= |\downarrow \cdot \uparrow_x|^2 = |\downarrow \cdot \frac{1}{\sqrt{2}}(\uparrow + \downarrow)|^2 = \frac{1}{2}$

Then the total probability $= \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$.

6. $F_z(+) \rightarrow F_x(+) \rightarrow F_z(-)$

The atoms pass through $F_z(+)$ with probability $= \frac{1}{2}$ and the state collapses to \uparrow .

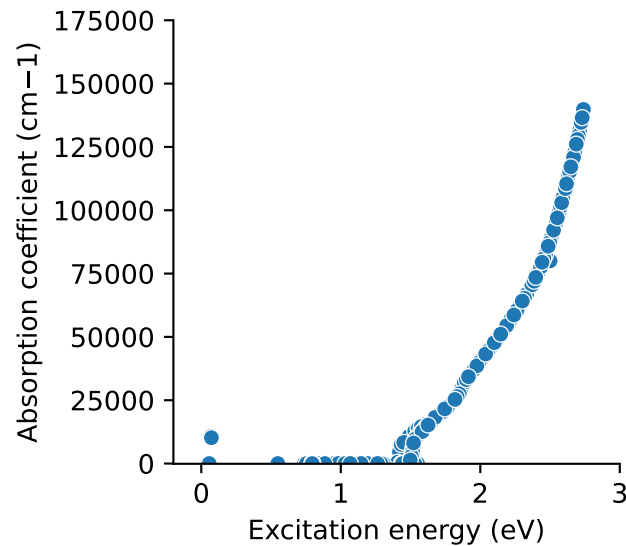
The probability of the second filter $= |\uparrow_x \cdot \uparrow|^2 = |\frac{1}{\sqrt{2}}(\uparrow + \downarrow) \cdot \uparrow|^2 = \frac{1}{2}$ and the state collapses to $\uparrow_x = \frac{1}{\sqrt{2}}(\uparrow + \downarrow)$

The probability of the third filter $F_z(-)$ is $= |\downarrow \cdot \uparrow_x|^2 = |\downarrow \cdot \frac{1}{\sqrt{2}}(\uparrow + \downarrow)|^2 = \frac{1}{2}$

Then the total probability $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

3 Gaps and absorption

Here is data from the Springer Materials database, showing the absorption coefficient of GaAs, a material that is often use in optoelectronics applications such as solar cells, light emitting diodes, and light sensors. The absorption coefficient measures how well the material absorbs light of a given frequency. If it is small, then the material is more transparent, if it is large, then the material is more opaque.



From the absorption plot, estimate the gap of the material. Explain why you made the estimate you did.

The gap is around 1.5 eV. This is where the absorption really starts to take off, so it corresponds to the band gap.

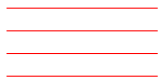
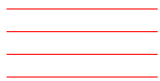
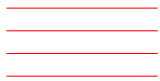
4 Band filling

It is possible to control the number of electrons in a material by a technique called “gating.” In this process, a piece of metal is placed in contact with the material, and by controlling the electric field, electrons can be pulled into the material or pushed out of it. As we do this, we can monitor the number of electrons per atom that are inserted in the system.

4.1 Band structure diagram

Draw a band structure diagram such that there is an “isolated band:” a single band, with a gap to bands below it and bands above it.

In the band diagram below, each set of four energy levels forms a band. The isolated band in the middle is the conduction band.

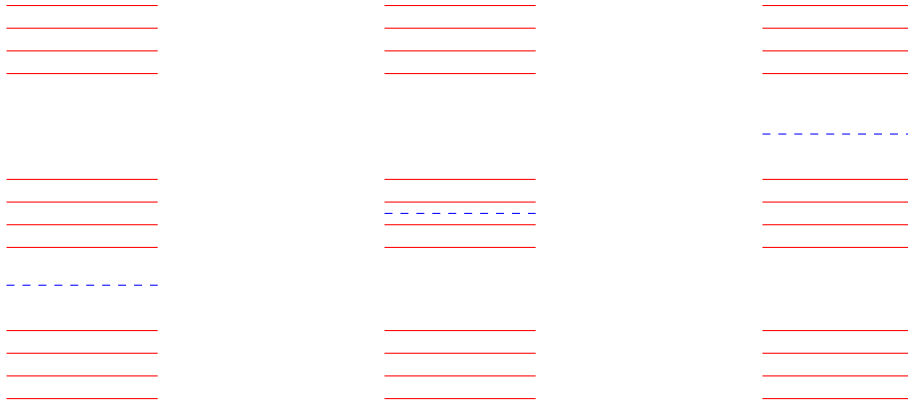


A real-world example of such a system is twisted bilayer graphene. However, twisted bilayer graphene has four isolated bands, while the picture here shows only one.

4.2 The measurement

Now consider the following experiment: we start our system such that the Fermi level is below the isolated band. The system is insulating. We then flow electrons into the system. It becomes conducting as the band is partially filled. Eventually it becomes insulating again. Indicate the initial and final Fermi levels on the graph above.

The initial Fermi level is below the isolated band and the final Fermi level is above the isolated band. In the insulating systems, the Fermi level is in the gap. Below, the dashed blue lines represent the Fermi level. The diagrams on the left and the right are insulating, while the center one is conducting (the isolated/conduction band is partially filled).



4.3 Computing the number of electrons in a band

Let's suppose that we have 1 g of the material in our system, and that there are 9.5×10^{21} atoms per gram of this material. If it takes one Coulomb of charge to perform the above experiment, compute

1. How many energy levels total are in the band.

A total charge of 1 Coulomb corresponds to 6.2422×10^{18} electrons, which are added to the material. Since each energy level can accommodate two electrons, the number of energy levels filled is

$$N_{\text{levels}} = 3.12 \times 10^{18}$$

2. How many atoms are in the material.

Since there is one gram of material, there are simply 9.5×10^{21} atoms.

3. How many energy levels per atom are in the band.

The number of energy levels per atom in the band is

$$\frac{N_{\text{levels}}}{N_{\text{atoms}}} = 3.2 \times 10^{-4}.$$

This surprisingly small number means that there is one energy level for every $\sim 3,000$ atoms. This means that this material must have a large unit cell. Examples, which are being studied in current experiments are twisted bilayer systems in which the effective unit cell can be thousands of atoms