Normalization to unit vector. #rvv-eu

$$\hat{a} = \frac{\vec{a}}{a}$$

Derivation +

Any vector can be written as the product of its length and direction:

Vector decomposition into length and direction. #rvv-ei

$$\vec{a} = a\hat{a}$$

Counterclockwise perpendicular vector in 2D. #rvv-en

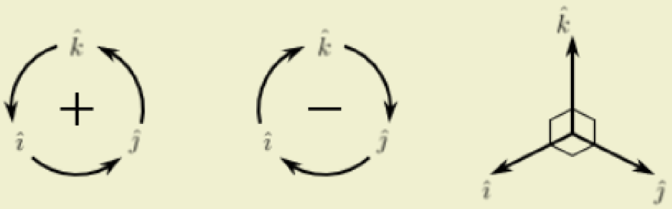
$$\vec{a}^{\perp} = -a_2\,\hat{\imath} + a_1\hat{\jmath}$$

Cross product in components. #rvv-ex

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\,\hat{\imath} + (a_3b_1 - a_1b_3)\,\hat{\jmath} + (a_1b_2 - a_2b_1)\,\hat{k}$$

It is sometimes more convenient to work with cross products of individual basis vectors, which are related as follows.

Cross products of basis vectors. #rvv-eo

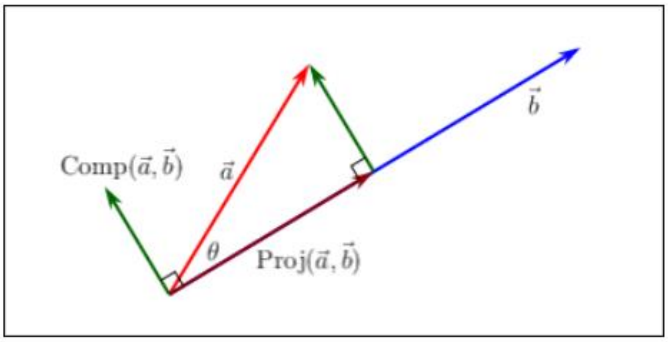


$$\hat{\imath} \times \hat{\jmath} = \hat{k} \qquad \hat{\jmath} \times \hat{k} = \hat{\imath} \qquad \hat{k} \times \hat{\imath} = \hat{\jmath}$$

$$\hat{\jmath} \times \hat{\imath} = -\hat{k} \qquad \hat{k} \times \hat{\jmath} = -\hat{\imath} \qquad \hat{\imath} \times \hat{k} = -\hat{\jmath}$$

Cross product of out-of-plane vector $\hat{k}$ with 2D vector $\vec{a} = a_1\,\hat{\imath} + a_2\,\hat{\jmath}$. #rvv-e9

$$\hat{k} \times \vec{a} = \vec{a}^{\perp}$$



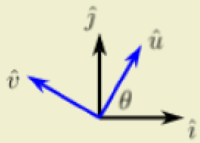
Projection of $\vec{a}$ onto $\vec{b}$ and the complementary projection. #rvv-fm

As we see in the diagram above, the complementary projection is orthogonal to the reference vector:

Complementary projection is orthogonal to the reference. #rvv-er

$$\mathrm{Comp}(\vec{a},\vec{b}) \cdot \vec{b} = 0$$

Change of basis formula in 2D. #rvv-e2



$$\vec{a} = a_i\,\hat{\imath} + a_j\,\hat{\jmath} \qquad \vec{a} = a_u\,\hat{u} + a_v\,\hat{v}$$

$$\begin{aligned} a_i &= \cos\theta\, a_u - \sin\theta\, a_v & \qquad a_u &= \cos\theta\, a_i + \sin\theta\, a_j \\ a_j &= \sin\theta\, a_u + \cos\theta\, a_v & \qquad a_v &= -\sin\theta\, a_i + \cos\theta\, a_j \end{aligned}$$

Scalar triple product formula. #rvi-es

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$$

Vector triple product expansion. #rvi-ev

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

Binet-Cauchy identity. #rvi-eb

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$$

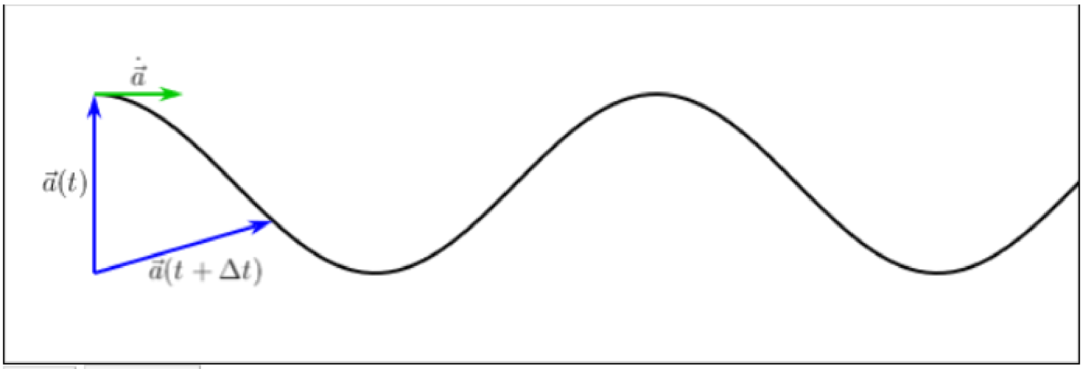
Jacobi's identity. #rvi-ej

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$$

Derivation +

Vector quadruple product expansion. #rvi-eq

$$(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c}) = (\vec{a} \cdot (\vec{b} \times \vec{c}))\vec{a}$$



reset | labels ☑

Show: vector increment $\Delta\vec{a}$ ☐ | approximate derivative $\Delta\vec{a}/\Delta t$ ☐ | exact derivative $\dot{\vec{a}}$ ☑

Leibniz notation is meant to be reminiscent of the definition of a derivative:

$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}.$$

Newton notation is meant to be compact:

$$\dot{y} = \frac{dy}{dt}.$$

$$\begin{aligned} &\text{Newton:} && \dot{y} = y'(x)\dot{x} \\ &\text{Leibniz:} && \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}. \end{aligned}$$

Chain rule for vectors. #rvc-er

$$\frac{d}{dt}\vec{a}(s(t)) = \frac{d\vec{a}}{ds}(s(t))\,\frac{ds}{dt}(t) = \frac{d\vec{a}}{ds}\,\dot{s}$$

- **Example Problem: Chain rule.** #rvc-er

A vector is defined in terms of an angle θ by $\vec{r}(\theta) = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$. If the angle is given by $\theta(t) = t^3$, what is $\dot{\vec{r}}$?

- **Solution**

We can use the chain rule to compute:

$$\begin{aligned}
\frac{d}{dt}\vec{r} &= \frac{d}{d\theta}\vec{r}\,\frac{d}{dt}\theta \\
&= \frac{d}{d\theta}\Big(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}\Big)\,\frac{d}{dt}(t^3) \\
&= \Big(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}\Big)(3t^2) \\
&= -3t^2\sin(t^3)\,\hat{\imath} + 3t^2\cos(t^3)\,\hat{\jmath}.
\end{aligned}$$

Alternatively, we can evaluate $\vec{r}$ as a function of t first and then differentiate it with respect to time, using the scalar chain rule for each component:

$$\begin{aligned}
\frac{d}{dt}\vec{r} &= \frac{d}{dt}\Big(\cos(t^3)\,\hat{\imath} + \sin(t^3)\,\hat{\jmath}\Big) \\
&= -3t^2\sin(t^3)\,\hat{\imath} + 3t^2\cos(t^3)\,\hat{\jmath}.
\end{aligned}$$

Derivative of vector length. #rvc-el

$$\dot{a} = \dot{\vec{a}}\cdot\hat{a}$$

Derivation -

We start with the dot product expression #rvv-ed for length and differentiate it:

$$\begin{aligned}
a &= \sqrt{\vec{a}\cdot\vec{a}} \\
\frac{d}{dt}a &= \frac{d}{dt}\big((\vec{a}\cdot\vec{a})^{1/2}\big) \\
\dot{a} &= \frac{1}{2}(\vec{a}\cdot\vec{a})^{-1/2}(\dot{\vec{a}}\cdot\vec{a} + \vec{a}\cdot\dot{\vec{a}}) \\
&= \frac{1}{2\sqrt{a^2}}(2\dot{\vec{a}}\cdot\vec{a}) \\
&= \dot{\vec{a}}\cdot\hat{a}.
\end{aligned}$$

Derivative of vector direction. #rvc-eu

$$\dot{\hat{a}} = \frac{1}{a}\operatorname{Comp}(\dot{\vec{a}}, \vec{a})$$

Derivation -

We take the definition #rvv-eu for the unit vector and differentiate it:

$$\begin{aligned}\hat{a} &= \frac{\vec{a}}{a} \\ \frac{d}{dt}\hat{a} &= \frac{d}{dt}\left(\frac{\vec{a}}{a}\right) \\ \dot{\hat{a}} &= \frac{\dot{\vec{a}}a - \vec{a}\dot{a}}{a^2} \\ &= \frac{\dot{\vec{a}}}{a} - \frac{\dot{\vec{a}} \cdot \hat{a}}{a^2}\vec{a} \\ &= \frac{1}{a}(\dot{\vec{a}} - (\dot{\vec{a}} \cdot \hat{a})\hat{a}) \\ &= \frac{1}{a}\operatorname{Comp}(\dot{\vec{a}}, \vec{a}).\end{aligned}$$

Derivative of unit vector is orthogonal. #rvc-eu

$$\dot{\hat{a}} \cdot \hat{a} = 0$$

Vector derivative decomposition. #rvc-em

$$\dot{\vec{a}} = \underbrace{\dot{a}\hat{a}}_{\operatorname{Proj}(\dot{\vec{a}},\vec{a})} + \underbrace{a\dot{\hat{a}}}_{\operatorname{Comp}(\dot{\vec{a}},\vec{a})}$$

Derivation -

Differentiating $\vec{a} = a\hat{a}$ and substituting in #rvv-el and #rvv-eu gives

$$\begin{aligned}\dot{\vec{a}} &= \dot{a}\hat{a} + a\dot{\hat{a}} \\ &= (\dot{\vec{a}} \cdot \hat{a})\hat{a} + a\frac{1}{a}\operatorname{Comp}(\dot{\vec{a}}, \hat{a}) \\ &= \operatorname{Proj}(\dot{\vec{a}}, \vec{a}) + \operatorname{Comp}(\dot{\vec{a}}, \vec{a}).\end{aligned}$$

Conversion between cylindrical and Cartesian coordinates #rvy-ec

$$\begin{aligned}x &= r\cos\theta & r &= \sqrt{x^2 + y^2} \\ y &= r\sin\theta & \theta &= \operatorname{atan2}(y, x) \\ z &= z & z &= z\end{aligned}$$

Cylindrical basis vectors #rvy-eb

$$\begin{aligned}\hat{e}_r &= \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \\ \hat{e}_\theta &= -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} \\ \hat{e}_z &= \hat{k}\end{aligned}$$

$$\begin{aligned}\hat{\imath} &= \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta \\ \hat{\jmath} &= \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta \\ \hat{k} &= \hat{e}_z\end{aligned}$$

Derivation #rvy-eb-d -

We write the position vector $\vec{\rho} = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + z\,\hat{k}$ and then use the definition of coordinate basis vectors to find the non-normalized cylindrical basis vectors:

$$\begin{aligned}\vec{e}_r &= \frac{\partial\vec{\rho}}{\partial r} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \\ \vec{e}_\theta &= \frac{\partial\vec{\rho}}{\partial\theta} = -r\sin\theta\,\hat{\imath} + r\cos\theta\,\hat{\jmath} \\ \vec{e}_z &= \frac{\partial\vec{\rho}}{\partial z} = \hat{k}\end{aligned}$$

Both $\vec{e}_r$ and $\vec{e}_z$ are already normalized, and the length of $\vec{e}_\theta$ is r, so we can divide by this to obtain the final normalized basis vector.

To invert the basis change we can solve for $\hat{\imath}$ and $\hat{\jmath}$.

Angular velocity of the cylindrical basis #rvy-ew

$$\vec{\omega} = \dot{\theta}\,\hat{e}_z$$

Time derivatives of cylindrical basis vectors #rvy-et

$$\begin{aligned}\dot{\hat{e}}_r &= \dot{\theta}\,\hat{e}_\theta \\ \dot{\hat{e}}_\theta &= -\dot{\theta}\,\hat{e}_r \\ \dot{\hat{e}}_z &= 0\end{aligned}$$

Derivation #rvy-et-d

We can either directly differentiate the basis vector expressions, or we can recall that $\dot{\hat{e}} = \vec{\omega}\times\hat{e}$ for any basis vector $\hat{e}$. This gives:

$$\begin{aligned}\dot{\hat{e}}_r &= \vec{\omega}\times\hat{e}_r = \dot{\theta}\,\hat{e}_z\times\hat{e}_r = \dot{\theta}\,\hat{e}_\theta \\ \dot{\hat{e}}_\theta &= \vec{\omega}\times\hat{e}_\theta = \dot{\theta}\,\hat{e}_z\times\hat{e}_\theta = -\dot{\theta}\,\hat{e}_r \\ \dot{\hat{e}}_z &= \vec{\omega}\times\hat{e}_\phi = \dot{\theta}\,\hat{e}_z\times\hat{e}_z = 0\end{aligned}$$

where we used the fact that $\hat{e}_r, \hat{e}_\theta, \hat{e}_z$ form a right-handed orthonormal basis to evaluate the cross products.

Position, velocity, and acceleration in cylindrical components #rvy-ep

$$\begin{aligned}\vec{\rho} &= r\,\hat{e}_r + z\,\hat{e}_z \\ \vec{v} &= \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta + \dot{z}\,\hat{e}_z \\ \vec{a} &= (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta + \ddot{z}\,\hat{e}_z\end{aligned}$$

Derivation #rvy-ep-d -

From the coordinate expressions we see that the position vector is $\vec{\rho} = r\,\hat{e}_r + z\,\hat{e}_z$. Differentiating this then gives:

$$\vec{v} = \dot{\vec{\rho}} = \frac{d}{dt}\Big(r\,\hat{e}_r + z\,\hat{e}_z\Big) = \dot{r}\,\hat{e}_r + r\,\dot{\hat{e}}_r + \dot{z}\,\hat{e}_z + z\,\dot{\hat{e}}_z$$

and we substitute in the expression for $\dot{\hat{e}}_r$ and $\dot{\hat{e}}_z$ from above. Taking another derivative gives:

$$\begin{aligned}\vec{a} = \dot{\vec{v}} &= \frac{d}{dt}\Big(\dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta + \dot{z}\,\hat{e}_z\Big) \\ &= \ddot{r}\,\hat{e}_r + \dot{r}\,\dot{\hat{e}}_r + (\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{e}_\theta + r\dot{\theta}\,\dot{\hat{e}}_\theta + \ddot{z}\,\hat{e}_z + \dot{z}\,\dot{\hat{e}}_z\end{aligned}$$

and again we can substitute the basis vector derivatives.

Derivative of unit vectors. #rkr-ew

$$\dot{\hat{a}} = \vec{\omega} \times \hat{a}$$

Derivative of general vectors. #rkr-ed

$$\dot{\vec{a}} = \underbrace{\dot{a}\hat{a}}_{\mathrm{Proj}(\dot{\vec{a}},\vec{a})} + \underbrace{\vec{\omega} \times \vec{a}}_{\mathrm{Comp}(\dot{\vec{a}},\vec{a})}$$

Derivation -

Using the same approach as #rvc-em we write $\vec{a} = a\hat{a}$ and differentiate this and use rkr-ew to find:

$$\begin{aligned}\dot{\vec{a}} &= \frac{d}{dt}(a\hat{a}) \\ &= \dot{a}\hat{a} + a\dot{\hat{a}} \\ &= \dot{a}\hat{a} + a(\vec{\omega} \times \hat{a}) \\ &= \dot{a}\hat{a} + \vec{\omega} \times (a\hat{a}) \\ &= \dot{a}\hat{a} + \vec{\omega} \times \vec{a}.\end{aligned}$$

Comparing this to #rvc-em shows that the two components are the projection and the complementary projection, respectively.

Derivative of constant-length vectors. #rkr-el

$$\dot{\vec{a}} = \vec{\omega} \times \vec{a} \qquad \text{if } \vec{a} \text{ is constant length}$$

Derivation -

This can be seen from the fact that $\dot{a} = 0$ if a is constant (a fixed length vector), substituted into #rkr-ed.

Magnitude ω is derivative of angle θ in 2D. #rkr-e2

$$\omega = \dot{\theta}$$

Derivative of rotating vector is orthogonal. #rkr-e2

$$\dot{\vec{a}} \cdot \vec{a} = 0$$

Derivation -

Using #rkr-el and the scalar triple product formula #rvi-es gives:

$$\begin{aligned}\vec{a} \cdot \dot{\vec{a}} &= \vec{a} \cdot (\vec{\omega} \times \vec{a}) \\ &= \vec{\omega} \cdot (\vec{a} \times \vec{a}) \\ &= 0.\end{aligned}$$

Angle θ between rotating vectors is constant. #rkr-e2

$$\theta = \cos^{-1}\left(\frac{\vec{b} \cdot \vec{a}}{ba}\right) = \text{constant}$$

Derivation -

We first consider the dot product $\vec{a} \cdot \vec{b}$ and show that this is not changing with time. We do this by using the scalar triple product formula #rvi-es to find:

$$\begin{aligned}\frac{d}{dt}(\vec{a} \cdot \vec{b}) &= \dot{\vec{a}} \cdot \vec{b} + \vec{a} \cdot \dot{\vec{b}} \\ &= (\vec{\omega} \times \vec{a}) \cdot \vec{b} + \vec{a} \cdot (\vec{\omega} \times \vec{b}) \\ &= \vec{b} \cdot (\vec{\omega} \times \vec{a}) + \vec{b} \cdot (\vec{a} \times \vec{\omega}) \\ &= \vec{b} \cdot (\vec{\omega} \times \vec{a}) - \vec{b} \cdot (\vec{\omega} \times \vec{a}) \\ &= 0.\end{aligned}$$

Now $\vec{a} \cdot \vec{b}$ is constant and the lengths a and b are constant, so the angle θ between the vectors must be constant.

Rotating vectors parallel to $\vec{\omega}$ are constant. #rkr-e2

$$\dot{\vec{a}} = 0 \qquad \text{if } \vec{a} \text{ is rotating and parallel to } \vec{\omega}$$

Derivation -

From #rkr-el we know that the derivative is

$$\dot{\vec{a}} = \vec{\omega} \times \vec{a},$$

but the cross product is zero for parallel vectors, so this the derivative is zero.

Definition of velocity $\vec{v}$ and acceleration $\vec{a}$. #rkv-ev

$$\begin{aligned} \vec{v} &= \dot{\vec{r}} \\ \vec{a} &= \dot{\vec{v}} \end{aligned}$$

The velocity can be decomposed into components parallel and perpendicular to the position vector, reflecting changes in the length and direction of $\vec{r}$.

Decomposition of velocity and acceleration vectors. #rkv-ec

$$\begin{aligned} \vec{v}_{\rm proj} &= \operatorname{Proj}(\vec{v}, \vec{r}) = \dot{r}\hat{r} \\ \vec{v}_{\rm comp} &= \operatorname{Comp}(\vec{v}, \vec{r}) = r\dot{\hat{r}} \\ \vec{a}_{\rm proj} &= \operatorname{Proj}(\vec{a}, \vec{v}) = \dot{v}\hat{v} \\ \vec{a}_{\rm comp} &= \operatorname{Comp}(\vec{a}, \vec{v}) = v\dot{\hat{v}} \end{aligned}$$

Velocity and acceleration in Cartesian basis. #rkv-er

$$\begin{aligned} \vec{r} &= r_1\,\hat{\imath} + r_2\,\hat{\jmath} + r_3\,\hat{k} \\ \vec{v} &= \dot{r}_1\,\hat{\imath} + \dot{r}_2\,\hat{\jmath} + \dot{r}_3\,\hat{k} \\ \vec{a} &= \ddot{r}_1\,\hat{\imath} + \ddot{r}_2\,\hat{\jmath} + \ddot{r}_3\,\hat{k} \end{aligned}$$

Derivation +

Velocity and acceleration in polar basis #rkv-sl

Computing velocity and acceleration in a polar basis must take account of the fact that the basis vectors are not constant.

Velocity and acceleration in polar basis. #rkv-ep

$$\begin{aligned} \vec{r} &= r\,\hat{e}_r \\ \vec{v} &= \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta \\ \vec{a} &= (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta \end{aligned}$$

Tangential/normal basis vectors. #rkt-eb

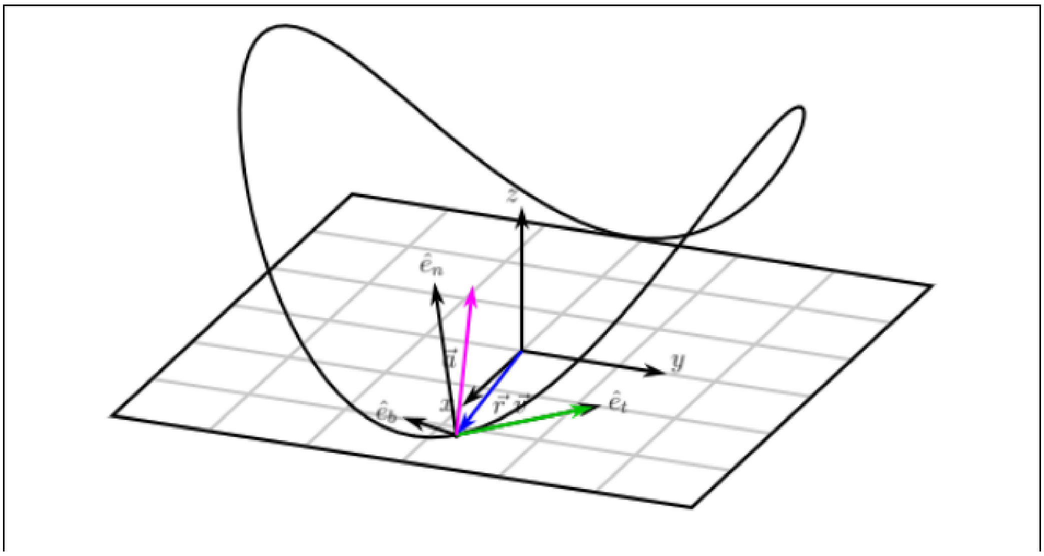
$$\hat{e}_t = \hat{v} \qquad \text{tangential basis vector}$$

$$\hat{e}_n = \frac{\dot{\hat{e}}_t}{\|\dot{\hat{e}}_t\|} = \frac{\operatorname{Comp}(\vec{a}, \vec{v})}{\|\operatorname{Comp}(\vec{a}, \vec{v})\|} \qquad \text{normal basis vector}$$

$$\hat{e}_b = \hat{e}_t \times \hat{e}_n \qquad \text{binormal basis vector}$$

Derivation

The tangential basis vector $\hat{e}_t$ points tangential to the path, the normal basis vector $\hat{e}_n$ points perpendicular (normal) to the path towards the instantaneous center of curvature, and the binormal basis vector $\hat{e}_b$ completes the right-handed basis.



Curvature and torsion. #rkt-ek

To better understand the geometry of the tangential/normal basis, we can use the *curvature* κ to describe the curving of the path, and the *torsion* τ to describe the rotation of the basis about the path. These quantities are defined by:

$$\kappa = \frac{d\hat{e}_t}{ds} \cdot \hat{e}_n = \frac{1}{v} \dot{\hat{e}}_t \cdot \hat{e}_n \qquad \text{curvature}$$

$$\rho = \frac{1}{\kappa} \qquad \text{radius of curvature}$$

$$\tau = -\frac{d\hat{e}_b}{ds} \cdot \hat{e}_n = -\frac{1}{v} \dot{\hat{e}}_b \cdot \hat{e}_n \qquad \text{torsion}$$

$$\sigma = \frac{1}{\tau} \qquad \text{radius of torsion}$$

Velocity and acceleration in tangential/normal basis. #rkt-ev

$$\begin{aligned}\vec{v} &= \dot{s}\,\hat{e}_t \\ \vec{a} &= \ddot{s}\,\hat{e}_t + \frac{\dot{s}^2}{\rho}\hat{e}_n\end{aligned}$$

Derivation -

From the definition #rkt-eb of $\hat{e}_t$ we see that $v\,\hat{e}_t = v\hat{v} = \vec{v}$, which is the first equation above. Differentiating this and using $v = \dot{s}$ from #rkt-es gives

$$\begin{aligned}\vec{v} &= \dot{s}\,\hat{e}_t \\ \vec{a} = \dot{\vec{v}} &= \ddot{s}\,\hat{e}_t + \dot{s}\,\dot{\hat{e}}_t \\ &= \ddot{s}\,\hat{e}_t + \dot{s}\big(v\kappa\,\hat{e}_n\big) \\ &= \ddot{s}\,\hat{e}_t + \frac{\dot{s}^2}{\rho}\,\hat{e}_n,\end{aligned}$$

where we used the derivative #rkt-ed of $\hat{e}_t$ in terms of the curvature κ, and the definition #rkt-ek of the radius of curvature to give $\kappa = 1/\rho$.

Radius of curvature ρ for velocity $\vec{v}$ and acceleration $\vec{a}$ with angle θ between them. #rkt-er

$$\rho = \frac{v^2}{a_n} = \frac{v^2}{|a\sin\theta|}$$

Curvature of parametric curve $\vec{r}(u)$ in 3D. #rkt-ec

$$\kappa = \frac{\|\vec{r}\,' \times \vec{r}\,''\|}{\|\vec{r}\,'\|^3}$$

Derivation +

While the above formula can be used in 2D by taking the third component to be zero, it can also be written in an explicitly 2D form:

Curvature of parametric curve $x = x(u)$, $y = y(u)$ in 2D. #rkt-e2

$$\kappa = \frac{|x''y' - y''x'|}{(x'^2 + y'^2)^{3/2}}$$

Circular motion (constant r). #rke-ec

$$\begin{aligned}r &= \text{constant} \\ \vec{v} &= r\omega\,\hat{e}_\theta \\ \vec{a} &= -r\omega^2\,\hat{e}_r + r\alpha\,\hat{e}_\theta\end{aligned}$$

Constant linear acceleration. #rke-er

$$\begin{aligned} a &= a_0 = \text{constant} \\ v &= v_0 + a_0 t \\ x &= x_0 + v_0 t + \frac{1}{2} a_0 t^2 \end{aligned}$$

Circular motion (constant r). #rke-ec

$$\begin{aligned} r &= \text{constant} \\ \vec{v} &= r\omega\,\hat{e}_\theta \\ \vec{a} &= -r\omega^2\,\hat{e}_r + r\alpha\,\hat{e}_\theta \end{aligned}$$

Note that for circular motion the velocity v is linear in both the radius r and angular velocity ω. The tangential acceleration is linear in both the radius r and angular acceleration α, while the radial acceleration is linear in r but quadratic in ω.

In the case of constant angular acceleration, the angular components function like rectilinear motion but in a circle, giving the explicit formulas:

Constant angular acceleration. #rke-ea

$$\begin{aligned} r &= r_0 = \text{constant} \\ \alpha &= \alpha_0 = \text{constant} \\ \omega &= \omega_0 + \alpha_0 t \\ \theta &= \theta_0 + \omega_0 t + \frac{1}{2} \alpha_0 t^2 \end{aligned}$$

Rigid body point relations. #rkg-er

$$\begin{aligned}
\vec{r}_Q &= \vec{r}_P + \vec{r}_{PQ} \\
\vec{v}_Q &= \vec{v}_P + \vec{\omega} \times \vec{r}_{PQ} \\
\vec{a}_Q &= \vec{a}_P + \vec{\alpha} \times \vec{r}_{PQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{PQ})
\end{aligned}$$

Points P and Q are two locations on a rigid body. Vectors $\vec{\omega}$ and $\vec{\alpha}$ are the angular velocity and angular acceleration of the rigid body.

Derivation -

The first equation is simply the definition of the offset vector $\vec{r}_{PQ}$. Differentiating the first equation gives

$$\begin{aligned}
\vec{r}_Q &= \vec{r}_P + \vec{r}_{PQ} \\
\dot{\vec{r}}_Q &= \dot{\vec{r}}_P + \dot{\vec{r}}_{PQ} \\
\vec{v}_Q &= \vec{v}_P + \vec{\omega} \times \vec{r}_{PQ},
\end{aligned}$$

where the derivative of $\vec{r}_{PQ}$ comes from the rotation formula, given that this offset vector is simply rotating with the rigid body.

If we differentiate again then we obtain

$$\begin{aligned}
\vec{v}_Q &= \vec{v}_P + \vec{\omega} \times \vec{r}_{PQ} \\
\dot{\vec{v}}_Q &= \dot{\vec{v}}_P + \dot{\vec{\omega}} \times \vec{r}_{PQ} + \vec{\omega} \times \dot{\vec{r}}_{PQ} \\
\vec{a}_Q &= \vec{a}_P + \vec{\alpha} \times \vec{r}_{PQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{PQ}),
\end{aligned}$$

where we use the fact that $\vec{\alpha} = \dot{\vec{\omega}}$.

If a point M on a rigid body has zero velocity then it is called the *instantaneous center of rotation*, because the velocity of all points on the body will be given by simple rotation about M with the angular velocity $\vec{\omega}$ of the body. In 2D we can always find the instantaneous center with the following equation, although it might be outside of the physical body.

Instantaneous center of rotation M in 2D. #rkg-ei

$$\vec{r}_{PM} = \frac{1}{\omega^2}\vec{\omega} \times \vec{v}_P = \frac{1}{\omega}\vec{v}_P^{\perp}$$

Point P has velocity $\vec{v}_P$ and is attached to a rigid body rotating with angular velocity $\vec{\omega}$.

Derivation -

The instantaneous center M has the property that the point on the rigid body at M has zero velocity. We thus want:

$$0 = \vec{v}_M = \vec{v}_P + \vec{\omega} \times \vec{r}_{PM}.$$

Taking another cross product by $\vec{\omega}$ gives:

$$\begin{aligned}
0 &= \vec{\omega} \times \left(\vec{v}_P + \vec{\omega} \times \vec{r}_{PM}\right) \\
&= \vec{\omega} \times \vec{v}_P + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{PM}) \\
&= \vec{\omega} \times \vec{v}_P - \omega^2 \vec{r}_{PM} \text{ (because } \vec{\omega} \perp \vec{r}_{PM}) \\
\vec{r}_{PM} &= \frac{1}{\omega^2}\vec{\omega} \times \vec{v}_P.
\end{aligned}$$

Here we used the fact that if $\vec{r}_{PM}$ lies in the $\hat{\imath}, \hat{\jmath}$ plane and $\vec{\omega}$ is in the $\hat{k}$ direction, then from #rvv-e9 we have $\vec{\omega} \times (\vec{\omega} \times \vec{r}_{PM}) = \omega^2 \vec{r}_{PM}^{\perp\perp} = -\omega^2 \vec{r}_{PM}$, as in the derivation of #rkg-e2.

If M is an instantaneous center, so it has zero velocity, then the velocity of any other point on the rigid body is given by the following equation.

Velocity from the instantaneous center M. #rkg-ev

$$\vec{v}_Q = \vec{\omega} \times \vec{r}_{MQ}$$

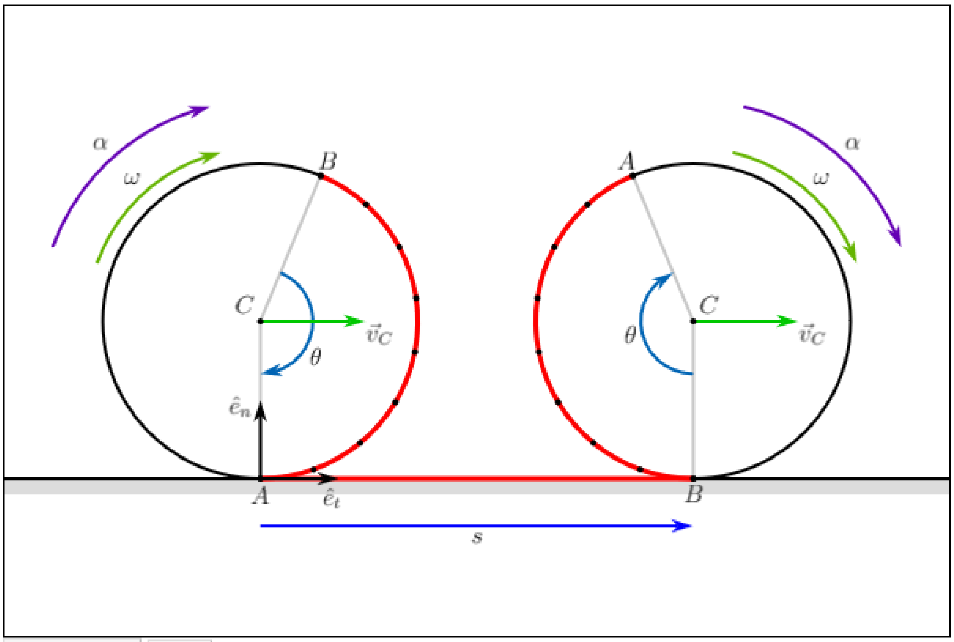
Point M is the instantaneous center of rotation for a rigid body rotating with angular velocity $\vec{\omega}$, and Q is any point on the body.

Rolling without slipping on stationary ground surfaces. #rko-er

$$\text{Contact point } P \text{ has zero velocity:} \quad \vec{v}_P = 0$$

Rolling on a 2D flat surface #rko-sf

The most common and also the simplest form of rolling occurs on a flat surface.



animation ☐ reset

show paths ☐ reverse direction ☐

Geometry and variables for rolling without slipping on a flat surface. #rko-ff

While rolling, the velocity and acceleration are directly connected to the angular velocity and angular acceleration, as shown by the next equations.

Center velocity and acceleration while rolling on a flat surface. #rko-ef

$$\begin{aligned}\vec{v}_C &= r\omega\,\hat{e}_t \\ \vec{a}_C &= r\alpha\,\hat{e}_t\end{aligned}$$

We begin by observing that the sign conventions in Figure #rko-ff mean that $\vec{\omega} = -\omega\,\hat{e}_b$. Now rolling without slipping means the contact point A must instantaneously have zero velocity, so using #rkg-er gives:

$$\begin{aligned}\vec{v}_C &= \vec{v}_A + \vec{\omega} \times \vec{r}_{AC} \\ &= (-\omega\,\hat{e}_b) \times r\,\hat{e}_n \\ &= r\omega\,\hat{e}_t.\end{aligned}$$

Because the surface is flat, the tangential basis vector $\hat{e}_t$ is constant, and the radius r is also constant. Differentiating the velocity expression thus results in:

$$\begin{aligned}\vec{v}_C &= r\omega\,\hat{e}_t \\ \dot{\vec{v}}_C &= r\dot{\omega}\,\hat{e}_t \\ \vec{a}_C &= r\alpha\,\hat{e}_t.\end{aligned}$$

Distance-angular relationships for rolling on a flat surface. #rko-eh

$$\begin{aligned} s &= r\theta \\ \dot{s} &= r\omega \\ \ddot{s} &= r\alpha \end{aligned}$$

Derivation -

The contact point P and center C are offset by the constant vector $r\,\hat{e}_n$, so $\dot{s} = v_P = v_C = r\omega$, from #rko-ef. Because r is constant, differentiating the velocity expression gives $\ddot{s} = r\dot{\omega} = r\alpha$, while integrating with zero initial displacement gives $s = r\theta$.

While rolling, the contact point will have zero velocity but will have a centripetal acceleration towards the rolling center:

Contact point P velocity and acceleration while rolling on a flat surface. #rko-eo

$$\begin{aligned} \vec{v}_P &= 0 \\ \vec{a}_P &= \omega^2\,\vec{r}_{PC} \end{aligned}$$

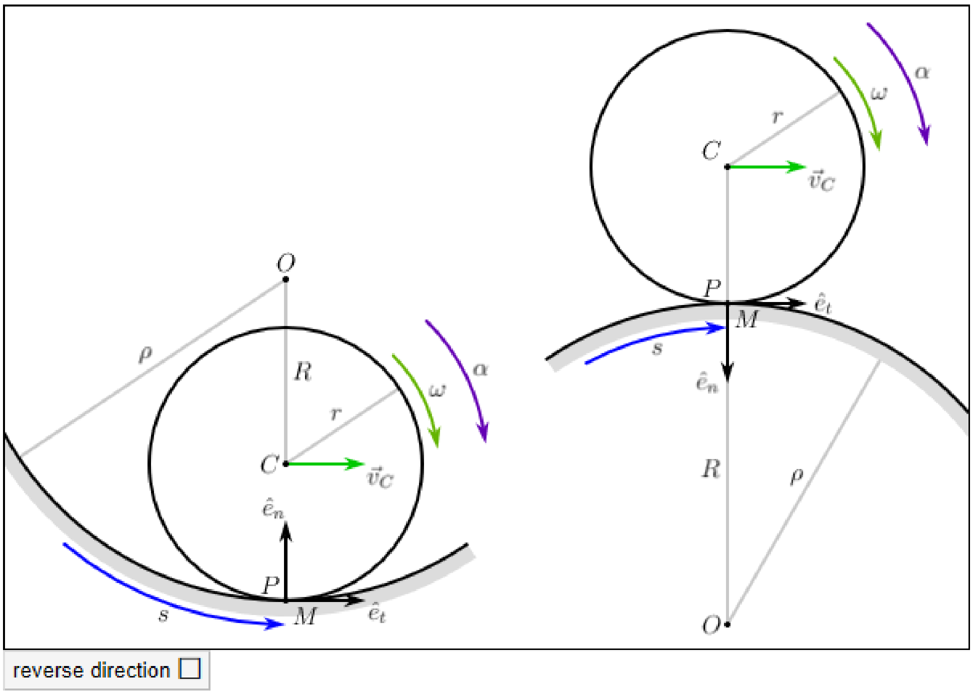
Derivation -

By definition of non-slip rolling contact, the point of contact P has zero velocity. The acceleration can be computed from the center C with #rkg-e2:

$$\begin{aligned} \vec{a}_P &= \vec{a}_C + \vec{\alpha} \times \vec{r}_{CP} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{CP}) \\ &= \alpha r\,\hat{e}_t + (-\alpha\,\hat{e}_b) \times (-r\,\hat{e}_n) + (-\omega\,\hat{e}_b) \times \Big((-\omega\,\hat{e}_b) \times (-r\,\hat{e}_n)\Big) \\ &= \alpha r\,\hat{e}_t - \alpha r\,\hat{e}_t + \omega^2 r\,\hat{e}_n \\ &= \omega^2\,\vec{r}_{PC}. \end{aligned}$$

Rolling on a 2D curved surface #rko-sc

When a circular rigid body rolls without slipping on a surface which is itself curved, the radius of curvature of the surface affects the acceleration (but not velocity) of points on the rolling body.



Geometry and variables for rolling without slipping on a curved surface. The left diagram shows rolling on the inside of the curve, while the right diagram is rolling on the outside of the curve. #rko-fc

Because there are two different geometries for rolling on a curved surface (inside and outside), there are different sign conventions and variable definitions in the two cases, as listed below.

Geometric quantities for rolling on a curved surface. #rko-eg

$$\left.\begin{aligned} R &= \rho - r \\ \vec{\omega} &= -\omega\,\hat{e}_b \\ \vec{\alpha} &= -\alpha\,\hat{e}_b \end{aligned}\right\} \text{when rolling on the inside of a curved surface}$$

$$\left.\begin{aligned} R &= \rho + r \\ \vec{\omega} &= \omega\,\hat{e}_b \\ \vec{\alpha} &= \alpha\,\hat{e}_b \end{aligned}\right\} \text{when rolling on the outside of a curved surface}$$

Center velocity and acceleration while rolling on a curved surface. #rko-ec

$$\begin{aligned} \vec{v}_C &= r\omega\,\hat{e}_t \\ \vec{a}_C &= r\alpha\,\hat{e}_t + \frac{(r\omega)^2}{R}\,\hat{e}_n \end{aligned}$$

Distance-angular relationships for rolling on a curved surface. #rko-ew

$$\begin{aligned}\omega &= \frac{R}{\rho r}\dot{s} \quad \text{on any curved surface} \\ \alpha &= \frac{R}{\rho r}\ddot{s} \quad \text{on a circular surface}\end{aligned}$$

Contact point P velocity and acceleration while rolling on a curved surface. #rko-ep

$$\begin{aligned}\vec{v}_P &= 0 \\ \vec{a}_P &= \frac{\rho}{R}\omega^2\vec{r}_{PC}\end{aligned}$$

Derivation -

The derivations for rolling on the inside or outside of a surface are different. Here we show the rolling inside case, as the outside case is very similar.

By definition the contact point P has zero velocity. The acceleration can be computed from the center C:

$$\begin{aligned}\vec{a}_P &= \vec{a}_C + \vec{\alpha} \times \vec{r}_{CP} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{CP}) \\ &= r\alpha\,\hat{e}_t + \frac{(r\omega)^2}{R}\,\hat{e}_n + (-\alpha\,\hat{e}_b) \times (-r\,\hat{e}_n) + (-\omega\,\hat{e}_b) \times \Big((-\omega\,\hat{e}_b) \times (-r\,\hat{e}_n)\Big) \\ &= r\alpha\,\hat{e}_t + \frac{(r\omega)^2}{R}\,\hat{e}_n - r\alpha\,\hat{e}_t + r\omega^2\,\hat{e}_n \\ &= \frac{(r+R)}{R}r\omega^2\,\hat{e}_n \\ &= \frac{\rho}{R}\omega^2\,\vec{r}_{PC}.\end{aligned}$$

Angular momentum #rep-sg

Angular momentum is defined with respect to a given base point O. For a point mass, the angular momentum is the cross product between the position vector and the linear momentum:

Angular momentum about fixed base point O of a point mass at P. #rep-eg

$$\vec{H}_O = \vec{r}_{OP} \times \vec{p}_P = \vec{r}_{OP} \times m\vec{v}_P$$

When a force is applied to a body it produces a *moment* about any given fixed base point O:

Moment about fixed point O of a force acting at P. #rep-eo

$$\vec{M}_O = \vec{r}_{OP} \times \vec{F}_P$$

The angular momentum and applied moment are related by:

Moment equation about fixed base point O. #rep-et

$$\vec{M}_O = \dot{\vec{H}}_O$$

Center of mass #reg-sc

Total mass of a body. #reg-em

$$m = \int_V \rho \, dV$$

Center of mass C. #reg-ec

$$\vec{r}_C = \frac{1}{m} \int_V \rho \vec{r} \, dV$$

Moment of inertia #reg-si

Moment of inertia about axis $\hat{a}$ through point P. #reg-ei

$$I_{P,\hat{a}} = \int_V \rho r^2 \, dV$$

Here r is the distance from the axis through P in direction $\hat{a}$.

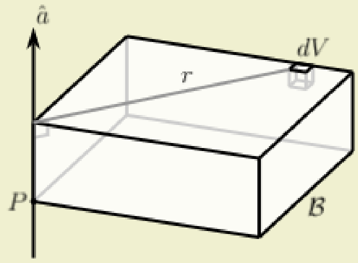
Rigid body equations for rotation about axis $\hat{a}$. #reg-ee

$$\begin{aligned} \sum_i \vec{F}_i &= m\vec{a}_C \\ \sum_i \vec{M}_{C,i} &= I_{C,\hat{a}}\vec{\alpha} \\ \text{or} \sum_i \vec{M}_{O,i} &= I_{O,\hat{a}}\vec{\alpha} \qquad \text{if } O \text{ is a fixed point} \end{aligned}$$

Moment of inertia about axis $\hat{a}$ through point P. #rem-ei

$$I_{P,\hat{a}} = \iiint_{\mathcal{B}} \rho r^2 \, dV$$

(units: kg m^2)



The distance r is the perpendicular distance to dV from the axis through P in direction $\hat{a}$.

+ **Warning: Mass moments of inertia are different to area moments of inertia.** #rem-wm

Observe that the moment of inertia is proportional to the mass, so that doubling the mass of an object will also double its moment of inertia. In addition, the moment of inertia is proportional to the square of the size of the object, so that doubling every dimension of an object (height, width, etc) will cause it to have four times the moment of inertia.

Moments of inertia about coordinate axes through point P. #rem-ec

$$\begin{aligned} I_{P,x} &= I_{P,xx} = I_{P,\hat{\imath}} = \iiint_{\mathcal{B}} \rho (y^2 + z^2) \, dx \, dy \, dz \\ I_{P,y} &= I_{P,yy} = I_{P,\hat{\jmath}} = \iiint_{\mathcal{B}} \rho (z^2 + x^2) \, dx \, dy \, dz \\ I_{P,z} &= I_{P,zz} = I_{P,\hat{k}} = \iiint_{\mathcal{B}} \rho (x^2 + y^2) \, dx \, dy \, dz \end{aligned}$$

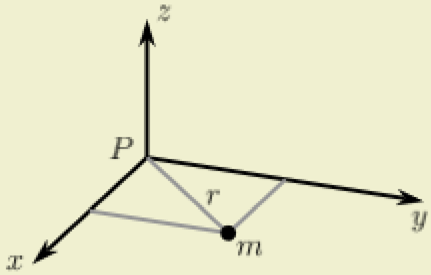
The coordinates (x, y, z) in the body are measured from point P.

Parallel axis theorem. #rem-el

$$I_{P,\hat{a}} = I_{C,\hat{a}} + m \, d^2$$

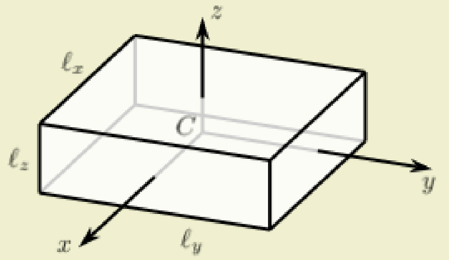
Here d is the perpendicular distance between the axes through P and C in direction $\hat{a}$, so $d = \| \operatorname{Comp}(\vec{r}_{CP}, \hat{a}) \|$.

Point mass: moments of inertia #rem-ep



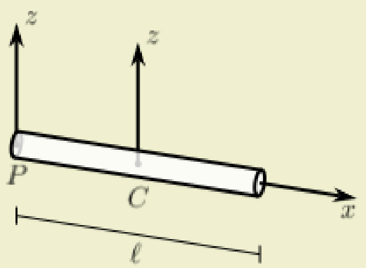
$$I_{P,z} = mr^2$$

Rectangular prism: moments of inertia #rem-er



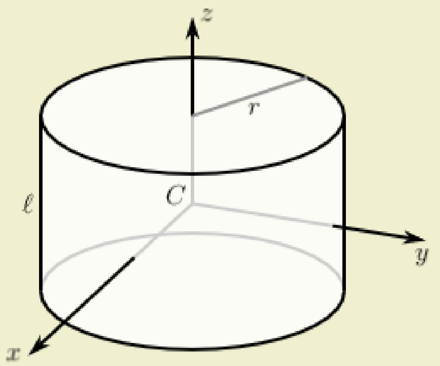
$$I_{C,x} = \frac{1}{12}m({\ell_y}^2 + {\ell_z}^2)$$
$$I_{C,y} = \frac{1}{12}m({\ell_z}^2 + {\ell_x}^2)$$
$$I_{C,z} = \frac{1}{12}m({\ell_x}^2 + {\ell_y}^2)$$

Rod: moments of inertia #rem-eo



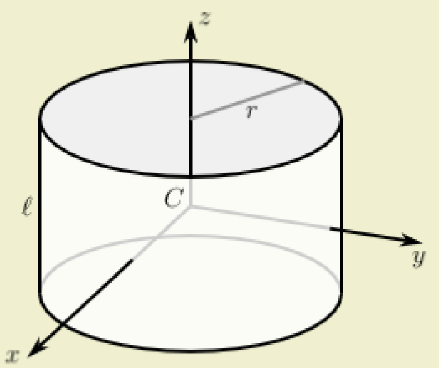
$$I_{C,z} = \frac{1}{12}m\ell^2$$
$$I_{P,z} = \frac{1}{3}m\ell^2$$
$$I_{C,x} = I_{P,x} = 0$$

Solid cylinder or disk: moments of inertia #rem-ek



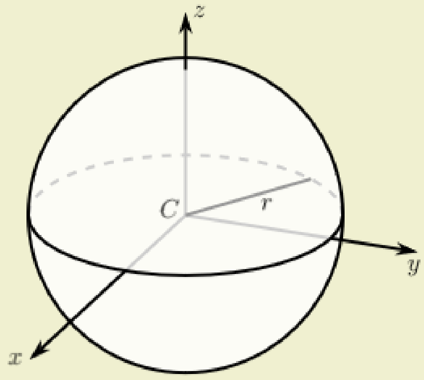
$$I_{C,x} = I_{C,y} = \frac{1}{12}m(3r^2 + \ell^2)$$
$$I_{C,z} = \frac{1}{2}mr^2$$

Hollow cylinder or hoop: moments of inertia #rem-eh



$$I_{C,x} = I_{C,y} = \frac{1}{12}m(6r^2 + \ell^2)$$
$$I_{C,z} = mr^2$$

Solid ball: moments of inertia #rem-eb

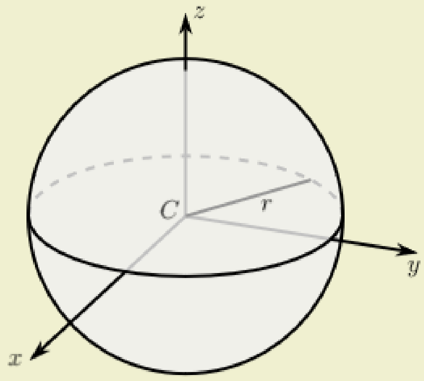


$$I_C = \frac{2}{5} m r^2$$

All axes through C have the same moment of inertia.

Derivation +

Hollow sphere: moments of inertia #rem-ew



$$I_C = \frac{2}{3} m r^2$$

Kinetic and potential energy. #ren-ee

$$E = T + V$$
$$\text{total energy} = \text{kinetic energy} + \text{potential energy}$$

Kinetic energy of a point mass. #ren-ep

$$T = \frac{1}{2} m v^2$$

Kinetic energy of an arbitrary body. #ren-eb

$$T = \iiint_B \frac{1}{2} \rho v^2 \, dV$$

Kinetic energy of a rigid body about the center of mass C. #ren-er

$$T = \frac{1}{2} m v_C^2 + \frac{1}{2} I_{C,\hat{\omega}} \omega^2$$

Derivation +

Kinetic energy of a rigid body about the instantaneous center M. #ren-em

$$T = \frac{1}{2} I_{M,\hat{\omega}} \omega^2$$

Derivation +

Kinetic energy of a rigid body about an arbitrary body point Q. #ren-ea

$$T = \frac{1}{2} m v_Q^2 + m \vec{v}_Q \cdot (\vec{\omega} \times \vec{r}_{QC}) + \frac{1}{2} I_{Q,\hat{\omega}} \omega^2$$

Potential energy #ren-sg

Gravitational potential energy with uniform gravitational acceleration g. #ren-eg

$$V = m g h_C$$

The distance h_C is the height of the center of mass of the body above the reference height.

Work and power #ren-sw

Work done by a force $\vec{F}$. #ren-ef

$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$$

Work done by a constant force $\vec{F}$. #ren-ec

$$W = \vec{F} \cdot \Delta\vec{r}$$

The change in position is $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$.

Derivation +

Forces of constraint do zero work. #ren-et

$$W = 0$$

The work W is by a force of constraint F.

Derivation +

Power transferred by a force $\vec{F}$. #ren-eo

$$P = \vec{F} \cdot \vec{v}$$

The velocity $\vec{v}$ is the velocity of the point where $\vec{F}$ is applied.

Work in terms of power. #ren-et

$$W = \int_{t_1}^{t_2} P\,dt = \int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\,dt$$

The velocity $\vec{v}$ is the velocity of the point where $\vec{F}$ is applied.

Work done by a moment M. #ren-em

$$\begin{aligned} W &= \int_{\theta_1}^{\theta_2} M\,d\theta \\ &= \int_{t_1}^{t_2} M\dot{\theta}\,dt \\ &= M\,\Delta\theta \text{ (for constant } M\text{)} \end{aligned}$$

The rotation angle θ is measured around the same axis $\vec{a}$ about which the moment M is applied.

Work-energy principle. #ren-ew

$$W = \Delta E = E_2 - E_1$$

The increase in energy ΔE is the change from the initial energy E_1 to final energy E_2, and W is the total external work done on the system.

Limit on friction force magnitude. #rfr-el

$$F \leq \mu N$$

Variables F and N are the magnitude of the friction force $\vec{F}$ and normal force $\vec{N}$, and μ is the coefficient of friction.

Possible states of motion with friction. #ren-ep

	relative motion	$\vec{F}$ magnitude	$\vec{F}$ direction
Sticking	$v_{Px} = 0$ and $a_{Px} = 0$	$F \leq \mu N$	any direction
Transition	$v_{Px} = 0$ and $a_{Px} = 0$	$F = \mu N$	any direction
Slipping	$v_{Px} \neq 0$	$F = \mu N$	opposes $\vec{v}_P$
About to slip	$v_{Px} = 0$ and $a_{Px} \neq 0$	$F = \mu N$	opposes $\vec{a}_P$

Point P is the contact point, x is the tangential contact direction, $\vec{F}$ is the friction force, $\vec{N}$ is the normal force, and μ is the coefficient of friction.

Analysis method for systems that might start to slip #rfr-sa

For a system that currently has zero relative contact velocity, we might not know whether it will stick or start to slip. To determine this, we can check the following cases in any order. Exactly one of the checks will pass, and that is the motion state that will physically occur.

	Assume	Check
Sticking	$v_{Px} = 0$ and $a_{Px} = 0$	$F \leq \mu N$
About to slip	$F = \mu N$	$\vec{F}$ opposes $\vec{a}_P$