Laplace Transform

Convolution

Theorem: Let $F(s) = \mathcal{L}\{f(t)\}$, $G(s) = \mathcal{L}\{g(t)\}$ exist then

$$\mathcal{L}\{f \star g\} := \mathcal{L}\{\int_0^t f(\tau)g(t-\tau)d\tau\} = F(s)G(s)$$

$h = f * g$ $\qquad H(s) = F(s)\,G(s)$

- Thus solution to an ODE with any function $u(t)$ can be determined by finding the solution when the input is impulse (i.e. $\delta(t)$).

Initial / Final Value Theorem

Initial Value Theorem: Let $\mathcal{L}\{f(t)\}$, $\mathcal{L}\{\dot{f}(t)\}$ and $\lim_{s\to\infty} sF(s)$ exist then

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

Final Value Theorem: Let $\mathcal{L}\{f(t)\}$, $\mathcal{L}\{\dot{f}(t)\}$ exist and real parts of poles of $sF(s)$ are negative then

$$f(\infty) := \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

In the left half plane / at most one zero

2nd-Order System

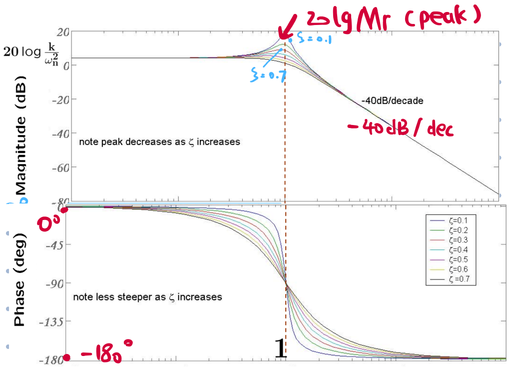
m [kg] k [kg/s²] b(c) [kg/s]

Bode-plots of 2nd order systems ($\zeta < 1$)

$G(s) = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2}$.

This implies that at *resonance* (where $M(\omega)$ is maximum):

- ⋆ resonance frequency ω_r is given by $\omega_r = \omega_n\sqrt{1-2\zeta^2}$
- ⋆ peak value is given by $M_r = M(\omega_r) = \frac{\frac{k}{\omega_n^2}}{2\zeta\sqrt{1-\zeta^2}}$
  - when $\zeta \le \frac{1}{\sqrt{2}} \approx 0.707$
  - no peak when $\zeta > \frac{1}{\sqrt{2}}$
- ⋆ note M_r goes to infinity as ζ goes to zero



Stability of System

The system is stable iff $\mathrm{Re}(\lambda_i) < 0$

1. Roots of characteristic Equation of ODE
2. Poles of Transfer function $G(s)$
3. Eigenvalues of Matrix A in state-space

Bode Plots

- Free response: when there is no input term, i.e. no forcing term
- Forced response: when initial conditions are zero $IC = 0$
- Therefore transfer function tells us about the forced response

Frequency Response

Transient x_{tr}: $\lim_{t\to\infty} x_{tr} = 0$,

Steady State x_{ss}: $\lim_{t\to\infty} x(t) = x_{ss}$

Transfer Function $\boxed{G(s) = X(s)/U(s)}$

$$G(s) = \frac{A}{B}\frac{(s-a_1)(s-a_2)\cdots}{(s-b_1)(s-b_2)\cdots}$$

Poles: $b_1, b_2, \ldots$ Zeros: $a_1, a_2 \ldots$

Freq Resp Func: $G(j\omega)$

Stable Linear System, Sinusoidal

Condition ① $G(s)$ stable ② $U(t) = A\sin(\omega t)$

$$\boxed{\begin{aligned} X_{ss}(t) &= AM\sin(\omega t + \phi) \\ \phi &= \angle G(j\omega),\ M = |G(j\omega)| \end{aligned}}$$

* Transient 一律忽略. 如 e^{-at}, $\delta(t)$ …
常数一律为 $A\sin(0t + \frac{\pi}{2})$

M is amp magnitude . ϕ is phase shift

Draw Bode Plots

dB/dec (slope): 每乘10倍, $20\lg M(\omega)$ 加多少

$\boxed{20\lg M(\omega)\ [\mathrm{dB}]\ \text{vs}\ \omega\ [\mathrm{rad/s}]}$

Initial Value $= 20\lg(G(0))$ (m 为重根数)

经过 m 个 |zero| $k + 20m$ dB/dec

m 个 |pole| $k - 20m$ dB/dec

$\boxed{\phi(\omega)\ [\mathrm{rad}]\ \text{vs}\ \omega\ [\mathrm{rad/s}]}$

Initial Value $= 0$ (m 为重根数)

经过 m 个 |zero| $k + 90^\circ m$ (degree)

m 个 |pole| $k - 90^\circ m$ (degree)

过渡大致由前一个10倍到后一个10倍

Superposition Method

$$G(s) = -\frac{10s^2}{(s+100)(s+1000)}$$

$$G(j\omega) = \frac{10\omega^2}{(j\omega+100)(j\omega+1000)}$$

$$20\lg M(\omega) = 20\lg|G(j\omega)|$$

$$= 20\lg\frac{|10\omega^2|}{|j\omega+100|\,|j\omega+1000|}$$

$$= 20\lg 10 + 40\lg\omega - 20\lg|j\omega+100|$$

$$- 20\lg|j\omega+1000|$$

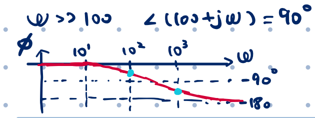
$$\phi(\omega) = \angle G(j\omega) = \angle\cancel{10\omega^2}$$

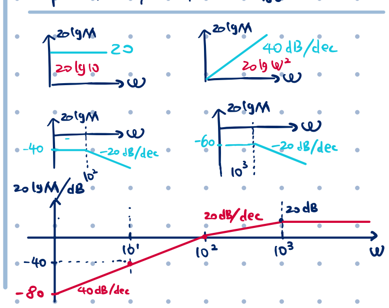
$$- \angle(100+j\omega) - \angle(1000+j\omega)$$

Analysis: Same for $1000+j\omega$

$\omega \ll 100$. $\angle(100+j\omega) = 0$

$\omega \gg 100$ $\angle(100+j\omega) = 90^\circ$





$F(s) = L\{f(t)\} = \int_0^\infty f(t) e^{-st} dt$

Existence: $\exists M, a, \forall t \geq 0$. that $|x(t)| \leq M e^{at}$

FUNCTIONS $u_s(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$

Pulse func $u_p(t) = \begin{cases} 1/t_1 & 0 \leq t \leq t_1 \\ 0 & \text{else} \end{cases}$

Unit pulse $\delta(t) = \lim_{t_1 \to 0} u_p(t)$

f(t)	F(s)	f(t)	F(s)
$u_s(t)$	$\frac{1}{s}$	$\sin(at)\, u_s(t)$	$\frac{a}{s^2+a^2}$
$u_s(t-c)$	$e^{-cs}\frac{1}{s}$	$\cos(at)\, u_s(t)$	$\frac{s}{s^2+a^2}$
$t\, u_s(t)$	$1/s^2$	$e^{at}\sin(bt)\, u_s(t)$	$\frac{b}{(s-a)^2+b^2}$
$t^2 u_s(t)$	$2/s^3$	$e^{at}\cos(bt)\, u_s(t)$	$\frac{s-a}{(s-a)^2+b^2}$
$t^n u_s(t)$	$n!/s^{n+1}$	$e^{-at} u_s(t)$	$\frac{1}{s+a}$
$\delta(t)$	1 (one)	$u_p(t)$	$\frac{1}{st_1}(1-e^{-st_1})$

f(t)	F(s)	f(t)	F(s)
$u_s(t-c) f(t-c)$	$e^{-cs}F(s)$	$f'(t)$	$sF(s) - f(0)$
$e^{ct} f(t)$	$F(s-c)$	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$t f(t)$	$(-1) F'(s)$	$\frac{1}{t} f(t)$	$\int_s^\infty F(s)\, ds$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$f(ct)$	$\frac{1}{c} F(\frac{s}{c})$

$m\ddot{x} + c\dot{x} + kx = f(t)$

$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = u(t),$

Natual Freq $\omega_n = \sqrt{k/m}$ $\zeta = \sqrt{\frac{c^2}{4mk}}$

Oscillating

Transient to SS

Inhomo. step response. underdamped

Input: $B u_s(t)$, $x_{ss} = B/\omega_n^2$. $T = \frac{2\pi}{\omega_d}$

rise time t_r

peak time t_p

x_{ss} 2% setting t_s

Max overshoot M_p

ζ dampiy ratio