Module 7

Lecture 21:

Dynamic Force Analysis (DFA) - 2

ME 370 - Mechanical Design 1

"Colibri" by Derek Hugger

** www.youtube.com/watch?v=1scj5sotD-E*

Lecture 21: Dynamic Force Analysis - 2

Today (Reading, Norton Ch 10.1-10.8, 11) **11/5/25**

Activities & Upcoming Deadlines

- **Week 11:**
  - **HW 10 (Motor, Cam, Motion 1):** posted and due Tuesday 11/11
  - **Lab 10 (Motors):** Team post-lab due before lab
  - **Lab 11 (Dynamic Force Analysis with Python)** – Meet in 1001 MEL. Prelab – required
- Project 2:
  - **Project 2 Description** - Follow this document for expected deliverables for P2D2. Submit materials to Gradescope prior to lab 12 (slides + PVA appendix + CAD animation). Demonstration of prototype of entire robot (walker + dispensing mechanism together), grade will depend on level of function
  - **Grading Rubric** and **PPTX template are posted**
  - **Lab 12**: Meet in 1001 MEL for presentation

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DFA 7 Steps

1. Draw complete system
2. Draw free-body diagram of each segment
3. Symbolically write out equations of motion
4. Convert to matrix format
5. Insert known values
6. Invert matric to solve for unknown forces and torques

Force Analysis Procedure: 7 steps to solve for forces and torques

1. **Draw complete system**. Label points, dimensions, external forces & torques, kinematics.

2. **Draw free-body diagram of each segment**. On each link, label local coordinate systems, external joint forces & torques, kinematics, internal forces & torques, and position vectors. Note equal and opposite forces on links at joints. Label forces such that force vector $\vec{F}_{ij}$ represents the force of link *i* **<u>on</u>** link *j* and is applied at the common joint on link *j*.

3. **Symbolically write out equations of motion** for each moving link. (For planar mechanism, 3 EOM)

$$\left.\begin{aligned} &(1)\ \sum F_{i\,x} = m_i A_{CGix} \\ &(2)\ \sum F_{i\,y} = m_i A_{CGiy} \end{aligned}\right\} \sum \vec{F}_i = m_i \vec{A}_{CGi}$$

$$(3)\ \sum T_{i\,z} = I_{CGi}\alpha_i$$

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Force Analysis Procedure (con't)

4. Convert to matrix format [**A**] {**B**} = {**C**}, where

[**A**] contains geometric info (e.g., position vector dimensions),

{**B**} contains all unknowns, and

{**C**} contains dynamic info

(e.g., external forces & torques, PVA analysis results, mass, moments of inertia).

5. Insert known values into [**A**] & {**C**}

Force Analysis Procedure (con't)

6. **Solve for unknown forces and torques** (typically internal joint forces and torques) using

$$\{\mathbf{B}\} = [\mathbf{A}]^{-1}\{\mathbf{C}\}$$

$A\vec{b} = \vec{C} \rightarrow A^{-1}A\vec{b} = A^{-1}C.$

$\vec{b} = A^{-1}C$

Solve for {**B**} using matrix calculator or software programs like MATLAB.

for a particular pose of the mech.

Note: This procedure only provides solution for unknown forces and torques at the specified, instantaneous position. Need to repeat steps 5 & 6 for changing values of parameters in [**A**] *and* {**C**}.

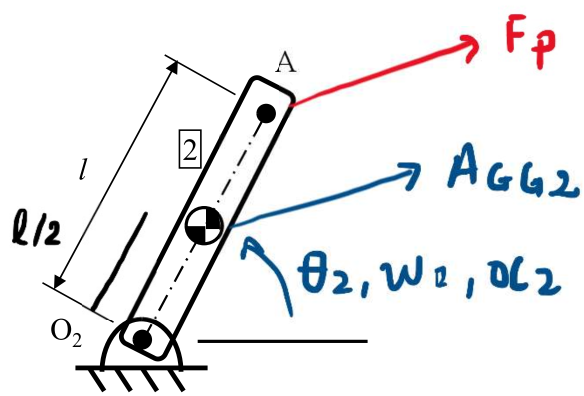
Single Link in Rotation (Ex. 11-1)

- **Given**: A single link of mass m and length l rotates about a fixed point O_2. The link has uniform cross-section. An external force, $\vec{F}_P$, is applied at point A. Assume that, at this instant in time, the following kinematic data are known

  $\theta_2, \omega_2, \alpha_2, \vec{A}_{CG2}$

- **Find**: The reaction force $\vec{F}_{12}$ at O_2 and driving torque T_{12} needed to maintain motion at this instant of time.

(1) *Draw complete system. Label points, dimensions, external forces & torques, kinematics.*

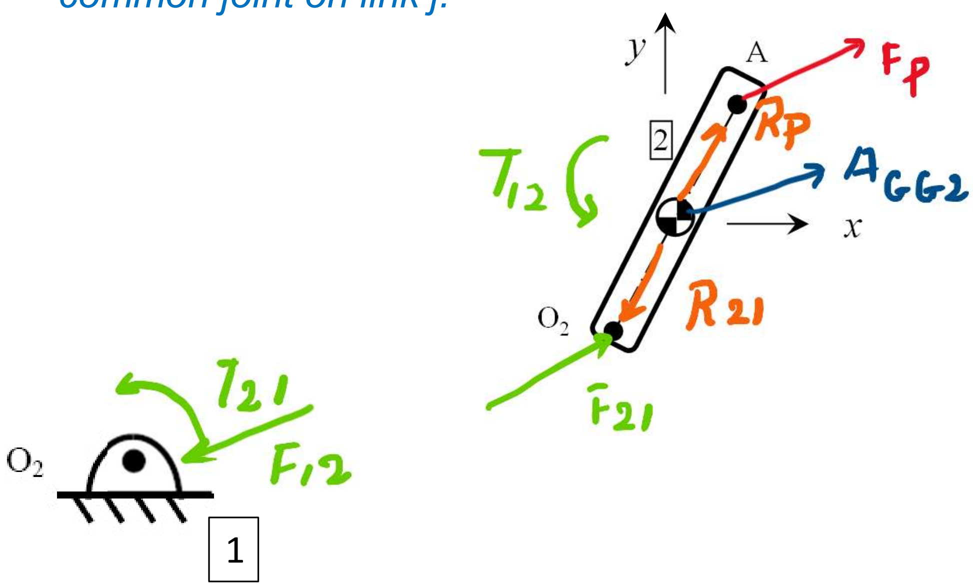


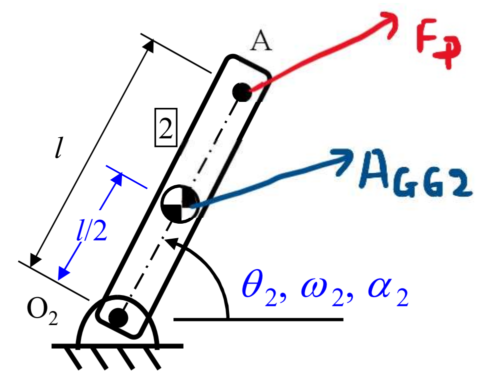
Note: assume that vectors are pointing in direction as shown. True directions will be determined when solved.

Single Link in Rotation (con’t)

(2) Draw free-body diagram of each segment.

- *On each link, label local coordinate systems, external joint forces & torques, kinematics, internal forces & torques, and position vectors.*
- *Note equal and opposite forces on links at joints.*
- *Label forces such that force vector $\vec{F}_{ij}$ represents the force of link i on link j and is applied at the common joint on link j.*





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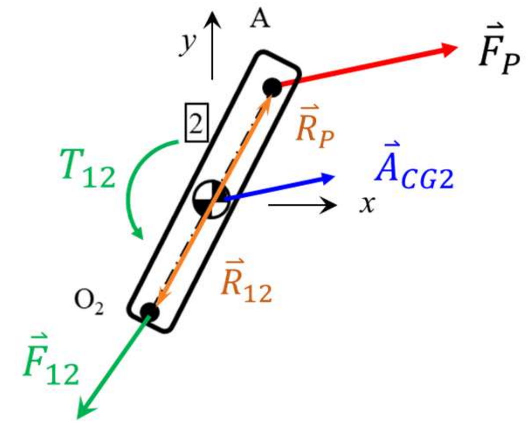
Single Link in Rotation (con't)

(3) Symbolically write out equations of motion for each moving link

- **Assume all unknown forces or torques are positive. True signs will be determined later.**

$$\sum \vec{F}_i = m_i \vec{A}_{CGi} \qquad \sum T_{i\,z} = I_{CGi}\alpha_i$$



Link 2:

$\Sigma \vec{F} =$

$\sum T_z =$

In Cartesian (*x*, *y*) terms these equations become:

$$\sum f_x = F_{12x} + F_{Px} = m_2 A_{CG2x}$$

$$\sum f_y = F_{12y} + F_{Py} = m_2 A_{CG2y}$$

$$\sum T_z = T_{12} + (R_{12x} F_{12y} - R_{12y} F_{12x}) + (R_{Px} F_{Py} - R_{Py} F_{Px}) = I_{CG2}\,\alpha_2$$

Recall: $\vec{R} \times \vec{F} = R_x F_y - R_y F_x$

Single Link in Rotation (con't)

x: $\boxed{F_{12x}} = m_2 A_{CG2x} - F_{Px}$

y: $\boxed{F_{12y}} = m_2 A_{CG2y} - F_{Py}$

T: $-R_{12y}F_{12x} + R_{12x}F_{12y} + \boxed{T_{12}} = I_{CG2}\alpha_2 - R_{Px}F_{Py} + R_{Py}F_{Px}$

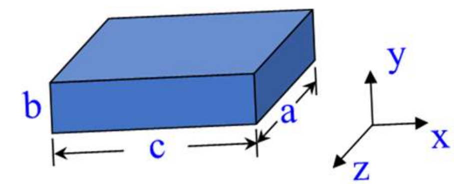
(4) Convert to matrix format [**A**] {**B**} = {**C**}*, where*

[**A**] *contains geometric info (e.g., position vector dimensions),*

{**B**} *contains all unknowns,*

{**C**} *contains dynamic info (e.g., external forces & torques, PVA analysis results, mass, moments of inertia)*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -R_{12y} & R_{12x} & 1 \end{bmatrix} \begin{Bmatrix} F_{12x} \\ F_{12y} \\ T_{12} \end{Bmatrix} = \begin{Bmatrix} m_2 A_{CG2x} - F_{Px} \\ m_2 A_{CG2y} - F_{Py} \\ I_{CG2}\alpha_2 - R_{Px}F_{Py} + .. \end{Bmatrix}$$



$I_x = \frac{m(a^2+b^2)}{12}; I_y = \frac{m(a^2+c^2)}{12}; I_z = \frac{m(b^2+c^2}{12}$

(5) Insert known/given values for variables in [**A**] *&* {**C**}*.*

(6) Solve for unknown forces and torques in {**B**} *(typically internal joint forces and torques) using* {**B**} = [**A**]$^{-1}$ {**C**}*.*

Practice: Fourbar slider-crank (crank = link 2)

Given: Applied external force $\vec{F}_P$ at P. <u>Know kinematics for link 2 ,3 & 4, CGs (e.g., PVA); inertial props</u> (m_i, I_{CGi})

Find: Reaction forces at each pin joint and driving torque T_{12}.

(1) Draw complete system

- *Label points, dimensions, external forces & torques, kinematics.*

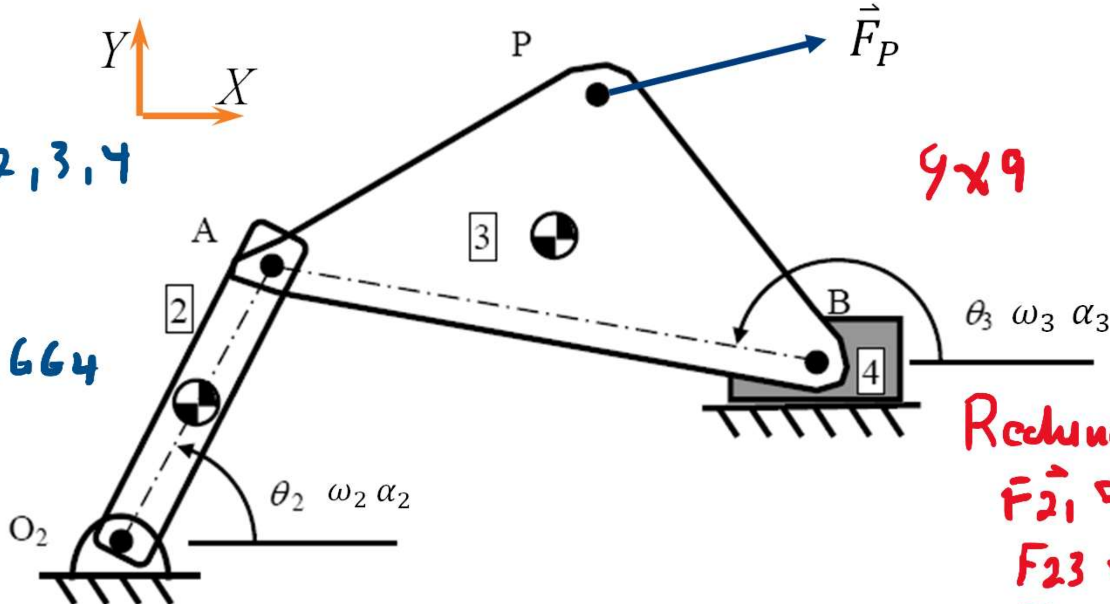
<u>Knowns:</u>

$\vec{F_P}$

m_i, I_{G_i} ; $i = 2, 3, 4$

θ_2, ω_2, α_2

A_{G2}, A_{G3}, A_{G4}



9x9

<u>Unknowns:</u>

T_{12}

$\vec{F_{12}}$: F_{12x}, F_{12y}

$\vec{F_{32}}$: F_{32x}, F_{32y}

$\vec{F_{43}}$: F_{43x}, F_{43y}

F_{14} : F_{14x}, F_{14y}

Redundant

$\vec{F_{21}} = -\vec{F_{12}}$

$F_{23} = -\vec{F_{32}}$

$F_{34} = -\vec{F_{43}}$

$\vec{F_{14}} = -\vec{F_{41}}$

Note: assume that vectors are pointing in direction as shown. True directions will be determined when solved.

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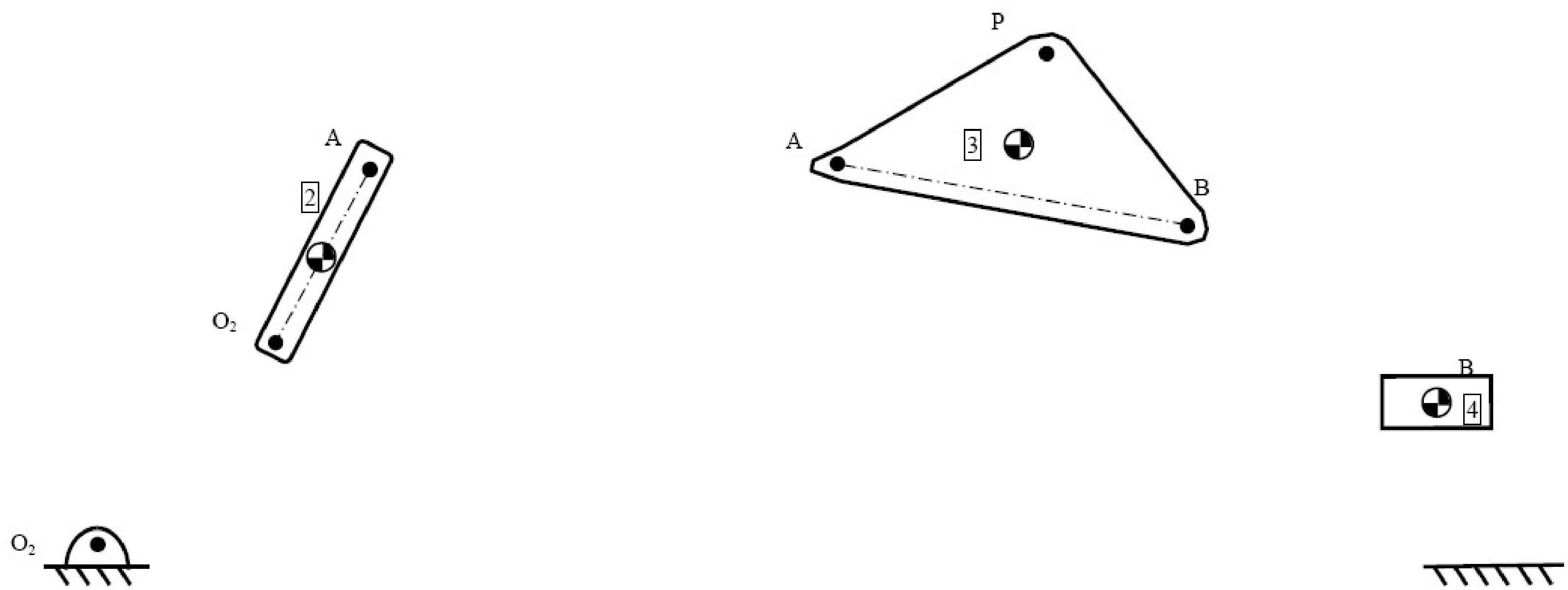
Fourbar slider-crank

(2) Draw free-body diagram of each segment.

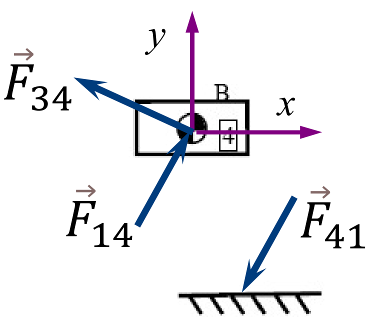
On each link, label local coordinate systems, external joint forces & torques, kinematics, internal forces & torques, and position vectors. Label forces such that force vector $\vec{F}_{ij}$ represents the force of link i ***on*** link j and is applied at the common joint on link j.



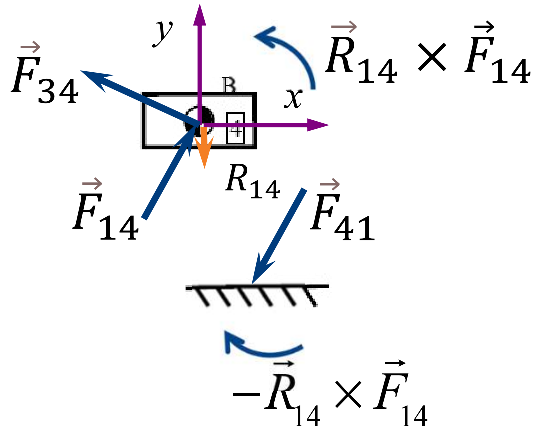
Fourbar slider-crank

How do we best represent forces on sliders, when the pin joint goes through the CG?

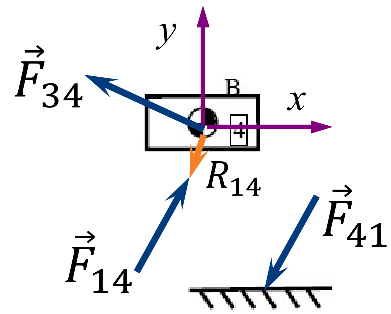
Case A



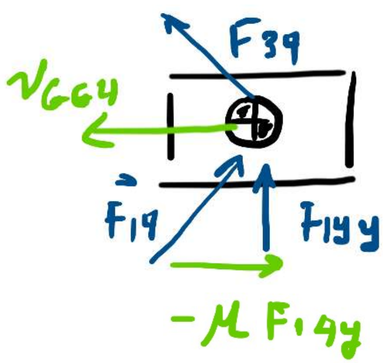
Case B



Case C



Join Code: **370**



(normal to the contact) answer is A

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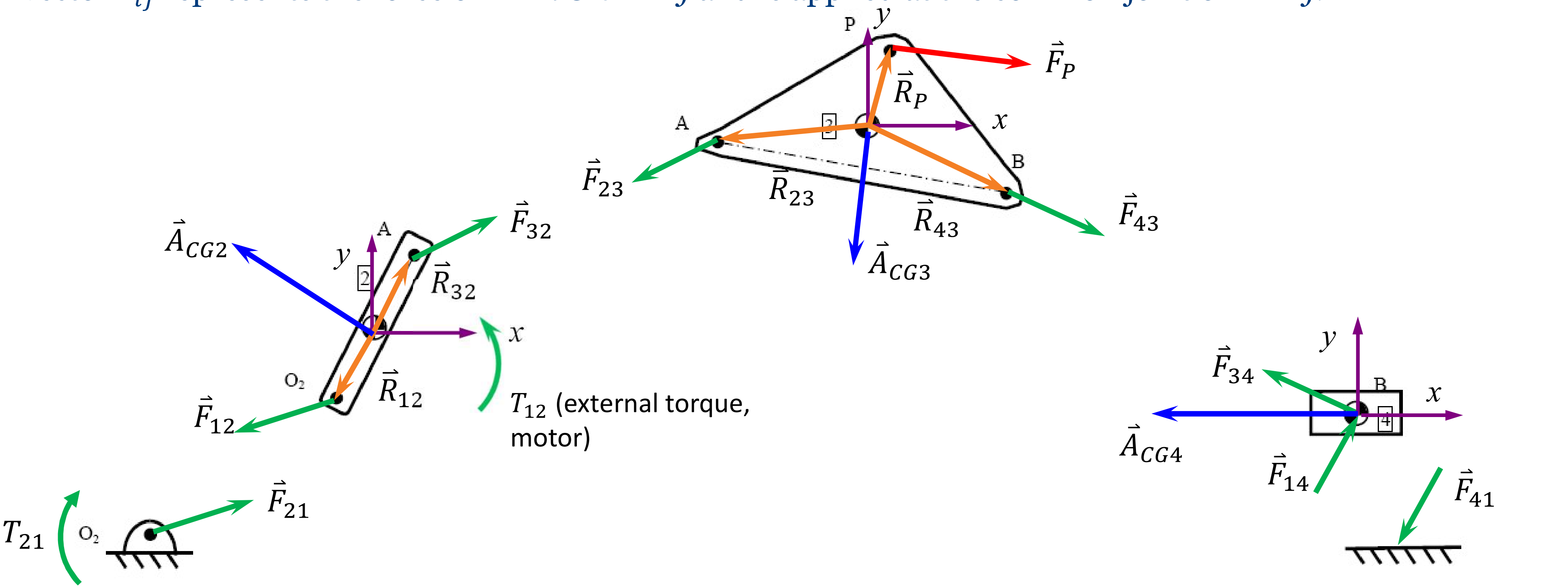
Fourbar slider-crank

(2) Draw free-body diagram of each segment.

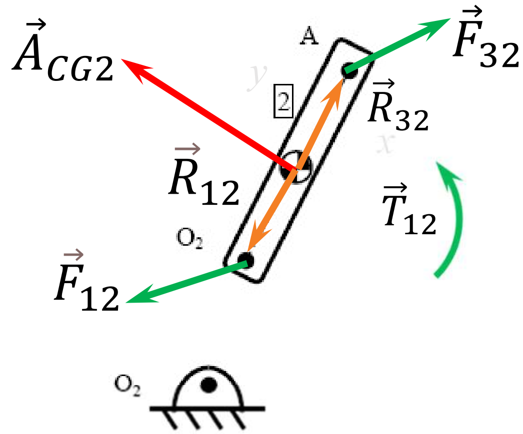
On each link, label local coordinate systems, external joint forces & torques, kinematics, internal forces & torques, and position vectors. Note equal and opposite forces on links at joints. Label forces such that force vector $\vec{F}_{ij}$ represents the force of link i ***on*** link j and is applied at the common joint on link j.



Fourbar slider-crank

(3) Symbolically write out equations of motion for each moving link

For Link 2: $\sum \vec{F}_i = m_i \vec{A}_{CGi}$ $\sum T_{i\,z} = I_{CGi}\alpha_i$



$$\sum \vec{\mathbf{F}} = \vec{\mathbf{F}}_{12} + \vec{\mathbf{F}}_{32} = m_2 \vec{\mathbf{A}}_{CG2}$$

$$\sum T_z = T_{12} + \left(\vec{\mathbf{R}}_{12} \times \vec{\mathbf{F}}_{12}\right) + \left(\vec{\mathbf{R}}_{32} \times \vec{\mathbf{F}}_{32}\right) = I_{CG2}\alpha_2$$

in cartesian

5 unknown, 3 Eqs

recall $\vec{R} \times \vec{F} = R_x F_y - R_y F_x$

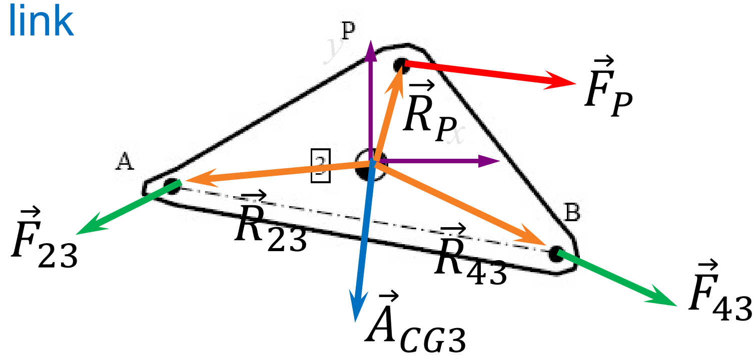
1) $\underline{F_{12x}} + \underline{F_{32x}} = m A_{CG2x}$

2) $\underline{F_{12y}} + \underline{F_{32y}} = m A_{CG2y}$

3) $\underline{T_{12}} + (R_{12x} F_{12y} - R_{12y} F_{12x}) + (R_{32x} F_{32y} - R_{32y} F_{32x}) = I_{CG2}\alpha_2$

Fourbar slider-crank

(3) Symbolically write out equations of motion for each moving link



For Link 3: $\sum \vec{F}_i = m_i \vec{A}_{CGi}$ $\qquad \sum T_{i\,z} = I_{CGi}\alpha_i$

$$\sum \vec{\mathbf{F}} = \vec{\mathbf{F}}_P + \vec{\mathbf{F}}_{23} + \vec{\mathbf{F}}_{43} = m_3 \vec{\mathbf{A}}_{CG3}$$

$$\sum T_z = (\vec{\mathbf{R}}_P \times \vec{\mathbf{F}}_P) + (\vec{\mathbf{R}}_{23} \times \vec{\mathbf{F}}_{23}) + (\vec{\mathbf{R}}_{43} \times \vec{\mathbf{F}}_{43}) = I_{CG}\ \alpha_3$$

Recall: $\vec{\mathbf{F}}_{23} = -\vec{\mathbf{F}}_{32}$

$F_{23x} + F_{43x} + F_{px} = m_3 A_{CG3x}$

(new variables are introduced with positive sign)

$$-F_{32x} \boxed{+ F_{43x}} + F_{Px} = m_3 A_{CG3x}$$

Add 2 more unknowns, and 3 more equations

$$-F_{32y} \boxed{+ F_{43y}} + F_{Py} = m_3 A_{CG3y}$$

$$(-R_{23x}F_{32y} + R_{23y}F_{32x}) + (R_{43x}F_{43y} - R_{43y}F_{43x}) + (R_{Px}F_{Py} - R_{Py}F_{Px}) = I_{CG3}\alpha_3$$

Torque of F_{32} $\qquad$ torque of F_{43} $\qquad$ Torque of Ext force

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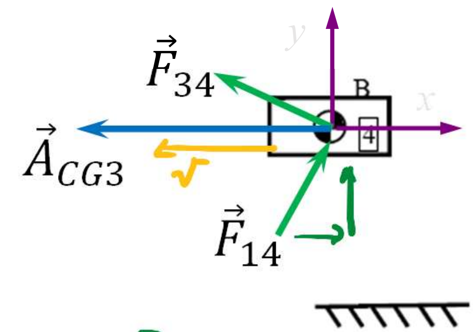
Fourbar slider-crank

(3) Symbolically write out equations of motion for each moving link

For Link 4: $\sum \vec{F}_i = m_i \vec{A}_{CGi}$ $\sum T_{i\,z} = I_{CGi}\alpha_i$ → does not rotate so

all force pass through CG

$0 = 0$



$$\alpha_4 = 0$$

$$\underline{A_{CG4y} = 0}$$

$$R_{ij} = 0$$

No torque balance necessary!

Friction on slider:

$F_{14x} = -\mathrm{sign}(v)\,\mu F_{14y}$

not an unknown ← $\underline{F_{14x}} = \pm\mu F_{14y} = -\,\mathrm{sign}(V_{CG4x})\,\mu F_{14y}$

$$F_{14x} - F_{43x} = m_4 A_{CG4x} \rightarrow \pm\mu \boxed{F_{14y}} - F_{43x} = m_4 A_{CG4x}$$ → $\Sigma f_x = m A_{CG4x}$

$$\boxed{F_{14y}} - F_{43y} = \underline{0}$$

Add 1 unknown, and 2 more equations

→ 8 unknowns, 8 equations

(note: originally 9 unknowns, but can find F_{14x} through friction equation)

(4) Convert to matrix format [**A**] {**B**} = {**C**},

$$Link2: \quad \boxed{F_{12x}} + \boxed{F_{32x}} = m_2 A_{CG2x}$$

$$\boxed{F_{12y}} + \boxed{F_{32y}} = m_2 A_{CG2y}$$

$$\boxed{T_{12}} + \left(R_{12x}F_{12y} - R_{12y}F_{12x}\right) + \left(R_{32x}F_{32y} - R_{32y}F_{32x}\right) = I_{CG2}\alpha_2$$

$$Link3: \quad -F_{32x} + \boxed{F_{43x}} + F_{Px} = m_3 A_{CG3x}$$

$$-F_{32y} + \boxed{F_{43y}} + F_{Py} = m_3 A_{CG3y}$$

$$\left(-R_{23x}F_{32y} + R_{23y}F_{32x}\right) + \left(R_{43x}F_{43y} - R_{43y}F_{43x}\right)$$

$$+ \left(R_{Px}F_{Py} - R_{Py}F_{Px}\right) = I_{CG3}\alpha_3$$

$$Link4:$$

$$F_{14x} - F_{43x} = m_4 A_{CG4x} \rightarrow \pm\mu F_{14y} - F_{43x} = m_4 A_{CG4x}$$

$$\boxed{F_{14y}} + F_{43y} = 0$$

Recall: $F_{14x} = \pm\mu F_{14y} = -\,\mathrm{sign}(V_{4x})\,\mu F_{14y}$

Write out unknowns above [A] for bookkeeping

$$\begin{array}{c}
\begin{array}{cccccccc} F_{12x} & F_{12y} & F_{32x} & F_{32y} & F_{43x} & F_{43y} & F_{14x} & T_{12} \end{array} \\
\left[\begin{array}{cccccccc}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & &
\end{array}\right]
\end{array}
\begin{Bmatrix} F_{12x} \\ F_{12y} \\ F_{32x} \\ F_{32y} \\ F_{43x} \\ F_{43y} \\ F_{14y} \\ T_{12} \end{Bmatrix}
=
\begin{Bmatrix} m_2 A_{CG2x} \\ m_2 A_{CG2y} \\ \\ \\ \\ \\ \\ \\ \end{Bmatrix}$$

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(4) Convert to matrix format [**A**] {**B**} = {**C**},

$$Link2: \quad \boxed{F_{12x}} + \boxed{F_{32x}} = m_2 A_{CG2x}$$

$$\boxed{F_{12y}} + \boxed{F_{32y}} = m_2 A_{CG2y}$$

$$\boxed{T_{12}} + \left(R_{12x}F_{12y} - R_{12y}F_{12x}\right) + \left(R_{32x}F_{32y} - R_{32y}F_{32x}\right) = I_{CG2}\alpha_2$$

$$Link3: \quad -F_{32x} + \boxed{F_{43x}} + F_{Px} = m_3 A_{CG3x}$$

$$-F_{32y} + \boxed{F_{43y}} + F_{Py} = m_3 A_{CG3y}$$

$$\left(-R_{23x}F_{32y} + R_{23y}F_{32x}\right) + \left(R_{43x}F_{43y} - R_{43y}F_{43x}\right) + \left(R_{Px}F_{Py} - R_{Py}F_{Px}\right) = I_{CG3}\alpha_3$$

$$Link4:$$

$$F_{14x} - F_{43x} = m_4 A_{CG4x} \rightarrow \pm\mu F_{14y} - F_{43x} = m_4 A_{CG4x}$$

$$\boxed{F_{14y}} - F_{43y} = 0$$

Recall: $F_{14x} = \pm\mu F_{14y} = -\,\mathrm{sign}(V_{4x})\,\mu F_{14y}$

Write out unknowns above [A] for bookkeeping

$$\begin{array}{cccccccc}
F_{12x} & F_{12y} & F_{32x} & F_{32y} & F_{43x} & F_{43y} & F_{14x} & T_{12} \\
\end{array}$$

$$\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
-R_{12y} & R_{12x} & -R_{32y} & R_{32x} & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & R_{23y} & -R_{23x} & -R_{43y} & R_{43x} & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & \pm\mu & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0
\end{bmatrix}
\begin{Bmatrix}
F_{12x} \\ F_{12y} \\ F_{32x} \\ F_{32y} \\ F_{43x} \\ F_{43y} \\ F_{14y} \\ T_{12}
\end{Bmatrix}
=
\begin{Bmatrix}
m_2 A_{CG2x} \\ m_2 A_{CG2y} \\ I_{CG2}\alpha_2 \\ m_3 A_{CG3x} - F_{Px} \\ m_3 A_{CG3y} - F_{Py} \\ I_{CG3}\alpha_3 - R_{Px}F_{Py} + R_{Py}F_{Px} \\ m_4 A_{CG4x} \\ 0
\end{Bmatrix}$$