Lecture 10

PVA, Part 3

“Colibri” by Derek Hugger

* www.youtube.com/watch?v=1scj5sotD-E

ME 370 - Mechanical Design 1

PVA Topics

- Vector notation (Complex and Compact)
- Analytical analysis method
  - **P**osition analysis
  - **V**elocity analysis
  - **A**cceleration analysis
- PVA analysis of a moving point
- Vector loop equation
- Velocity and Acceleration analysis of a four-bar linkage
- PVA analysis of other four-bar mechanisms
  - Offset slider-crank
  - Inverted offset slider-crank
- PVA analysis of mechanisms > four Links

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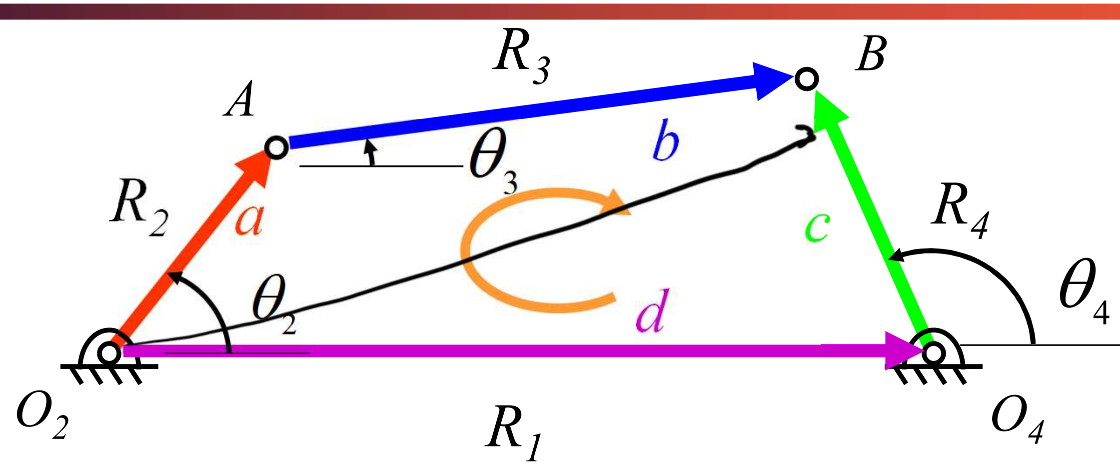
Recall: Example - Position analysis of 4-bar



- Given: a, b, c, d, and θ_2
- Solve for θ_3, θ_4
- Use: Position vector loop equation: (clockwise)

$$\vec{R}_2 + \vec{R}_3 - \vec{R}_4 - \vec{R}_1 = 0$$

$$ae^{j\theta_2} + be^{j\theta_3} - ce^{j\theta_4} - de^{j\theta_1} = 0$$

Recall:

$$\text{e.g., } \vec{R}_2 = \vec{R}_A = \vec{R}_{AO_2} = ae^{j\theta_2}$$

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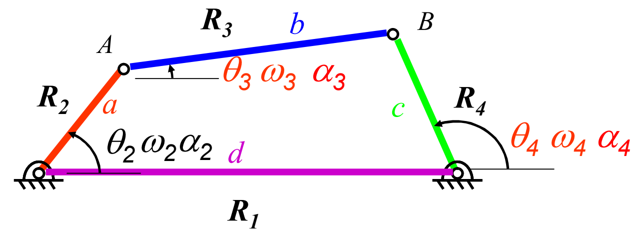
Recall: Solve for θ_3 and θ_4

Use position vector loop equation:

$$\boxed{ae^{j\theta_2} + be^{j\theta_3} - ce^{j\theta_4} - de^{j\theta_1} = 0}$$



Apply Euler's Identity: $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$

$$a(\cos\theta_2 + j\sin\theta_2) + b(\cos\theta_3 + j\sin\theta_3)$$
$$-c(\cos\theta_4 + j\sin\theta_4) - d = 0$$

Recall

$\theta_1 = 0$

Real: $a\cos\theta_2 + b\cos\theta_3 - c\cos\theta_4 - d = 0$ (†)

Im: $a\sin\theta_2 + b\sin\theta_3 - c\sin s\theta_4 = 0$ (‡)

Simultaneously solve † and ‡ to get:

$$(\theta_4)_{1,2} = 2\tan^{-1}\left[\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\right]$$

$$(\theta_3)_{1,2} = 2\tan^{-1}\left[\frac{-E \pm \sqrt{E^2 - 4DF}}{2D}\right]$$

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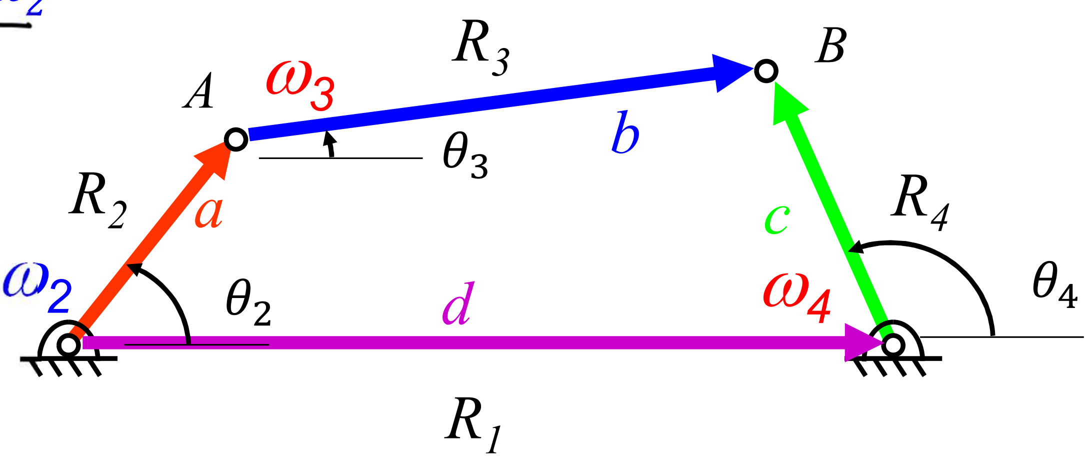
Next step: Velocity analysis of 4-bar

- Given: a, b, c, d, θ_2, and now ω_2
- Solve for θ_3, θ_4, ω_3, and ω_4
- Vector loop equation:



$$\vec{R}_2 + \vec{R}_3 - \vec{R}_4 - \vec{R}_1 = 0$$

$$ae^{j\theta_2} + be^{j\theta_3} - ce^{j\theta_4} - de^{j\theta_1} = 0$$

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Find the velocity vector loop equation

$$\vec{V} = \frac{d\vec{R}}{dt} \quad \Rightarrow \frac{d}{dt}\left(ae^{j\theta_2} + be^{j\theta_3} - ce^{j\theta_4} - de^{j\theta_1}\right) = \frac{d0}{dt} \qquad 0 \qquad \left[\theta_1 = 0 \rightarrow \frac{d\theta_1}{dt} = 0\right]$$

$$ja\frac{d\theta_2}{dt}e^{j\theta_2} + jb\frac{d\theta_3}{dt}e^{j\theta_3} - jc\frac{d\theta_4}{dt}e^{j\theta_4} - jd\cancel{\frac{d\theta_1}{dt}}e^{j\theta_1} = 0$$

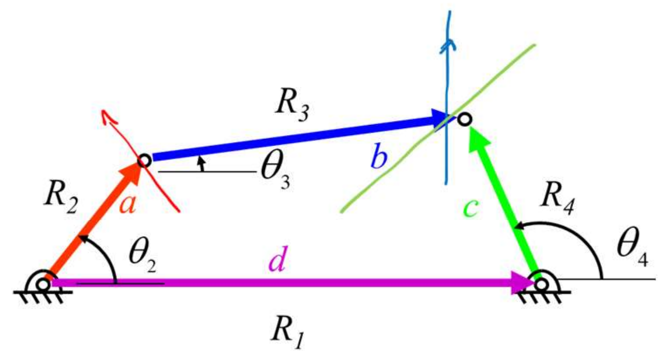
$$\boxed{j\omega_2 ae^{j\theta_2} + j\omega_3 be^{j\theta_3} - j\omega_4 ce^{j\theta_4} = 0}$$ Equation 1

recall, e.g., $\vec{R}_2 = ae^{j\theta_2}$ and $\vec{V}_A = j\omega_2\vec{R}_2$

$$j\omega_2\vec{R}_2 + j\omega_3\vec{R}_3 - j\omega_4\vec{R}_4 = 0$$

Or specifically, $\boxed{\vec{V}_A + \vec{V}_{BA} - \vec{V}_B = 0}$

Velocity Vector Loop Equation $\quad \vec{V}_{BA} = \vec{V}_B - \vec{V}_A$



In-Class Exercise:

Exercise: If $\omega_2 < 0$, $\omega_3 > 0$, $\omega_4 < 0$, which direction do $\vec{V}_A$, $\vec{V}_B$, $\vec{V}_{BA}$ point?

Hint:

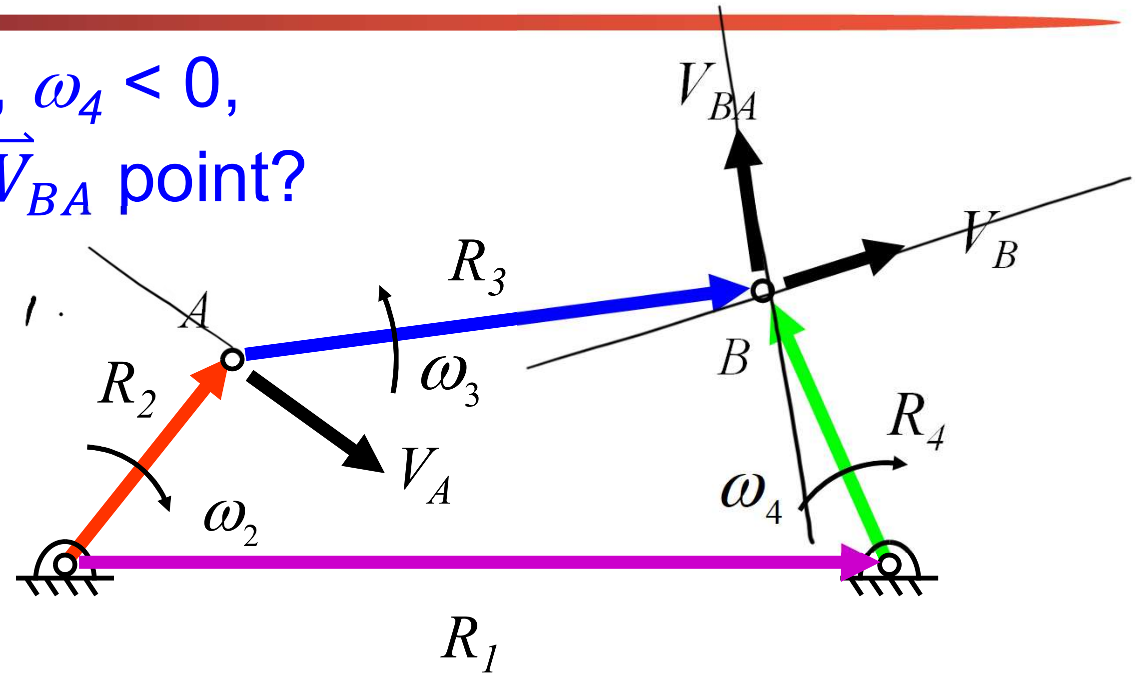
$$\vec{V}_A = j\omega_2\vec{R}_2$$

$$\vec{V}_B = j\omega_4\vec{R}_4$$

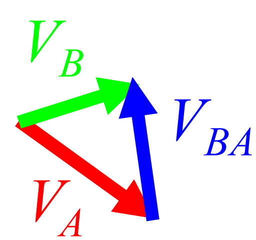
$$\vec{V}_{BA} = j\omega_3\vec{R}_3$$

$$\vec{V}_A + \vec{V}_{BA} - \vec{V}_B = 0$$

1.



3.



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Back to Eqn 1, how to solve for ω_3 and ω_4?

$$\vec{V}_A + \vec{V}_{BA} - \vec{V}_B = 0$$

$$\boxed{j\omega_2 a e^{j\theta_2} + j\omega_3 b e^{j\theta_3} - j\omega_4 c e^{j\theta_4} = 0}$$

Equation 1

Apply Euler's Identity: $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$

$$\boxed{\begin{aligned} &ja\omega_2(\cos\theta_2 + j\sin\theta_2) + jb\omega_3(\cos\theta_3 + j\sin\theta_3) \\ &- jc\omega_4(\cos\theta_4 + j\sin\theta_4) = 0 \end{aligned}}$$

Equation 2

Recall: $j^2 = -1$

Real(Eqn 2): $-a\omega_2\sin\theta_2 - b\omega_3\sin\theta_3 + c\omega_4\sin\theta_4 = 0$

Im(Eqn 2): $a\omega_2\cos\theta_2 + b\omega_3\cos\theta_3 - c\omega_4\cos\theta_4 = 0$

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How to solve for ω_3 and ω_4?

Real(Eqn 2): $-a\omega_2 \sin\theta_2 - b\omega_3 \sin\theta_3 + c\omega_4 \sin\theta_4 = 0$

Im(Eqn 2): $a\omega_2 \cos\theta_2 + b\omega_3 \cos\theta_3 - c\omega_4 \cos\theta_4 = 0$

➔ 2 equations, 2 unknowns: ω_3 and ω_4

Given or known: $a, b, c, d, \theta_2, \omega_2, \theta_3, \theta_4$

⇨ Solve simultaneously using trig identities

Get

$$\boxed{\begin{aligned}\omega_3 &= \frac{a\omega_2 \sin(\theta_4 - \theta_2)}{b\sin(\theta_3 - \theta_4)} \\ \omega_4 &= \frac{a\omega_2 \sin(\theta_2 - \theta_3)}{c\sin(\theta_4 - \theta_3)}\end{aligned}}$$

See Norton 6.7

ω_3 and ω_4 allow us to solve for

$$\vec{V}_B = j\omega_4 \vec{R}_4$$

$$\vec{V}_{BA} = j\omega_3 \vec{R}_3$$

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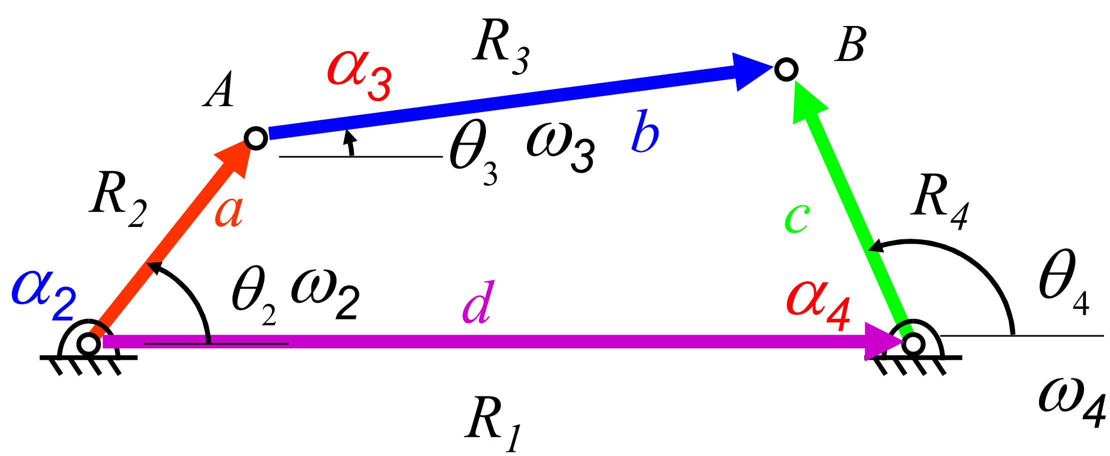
Acceleration analysis

- Given: a, b, c, d, θ_2, ω_2, and now α_2
- Solve for θ_3, θ_4, ω_3, ω_4, α_3, and α_4



Find the acceleration equation

$$\vec{A} = \frac{d\vec{V}}{dt}$$

Use Equation 1 $\quad j\omega_2 a e^{j\theta_2} + j\omega_3 b e^{j\theta_3} - j\omega_4 c e^{j\theta_4} = 0$

$$\frac{d}{dt}\left(j\omega_2 a e^{j\theta_2} + j\omega_3 b e^{j\theta_3} - j\omega_4 c e^{j\theta_4}\right) = 0$$

Product Rule

$$\boxed{\begin{aligned}&\left(j\alpha_2 a e^{j\theta_2} - \omega_2^2 a e^{j\theta_2}\right) + \left(j\alpha_3 b e^{j\theta_3} - \omega_3^2 b e^{j\theta_3}\right)\\ &-\left(j\alpha_4 c e^{j\theta_4} - \omega_4^2 c e^{j\theta_4}\right) = 0\end{aligned}}$$

Equation 3

$$\left(j\alpha_2 \vec{R}_2 - \omega_2^2 \vec{R}_2\right) + \left(j\alpha_3 \vec{R}_3 - \omega_3^2 \vec{R}_3\right) - \left(j\alpha_4 \vec{R}_4 - \omega_4^2 \vec{R}_4\right) = 0$$

Or specifically,

$$\left(\vec{A}_A^t + \vec{A}_A^n\right) + \left(\vec{A}_{BA}^t + \vec{A}_{BA}^n\right) - \left(\vec{A}_B^t + \vec{A}_B^n\right) = 0$$

$$\boxed{\vec{A}_A + \vec{A}_{BA} - \vec{A}_B = 0}$$

Acceleration Vector Loop Equation

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How to solve for α_3 and α_4?

- Use Equation 3
- Apply Euler's Identity
- Separate into Real and Imaginary parts
- Solve simultaneously

$$\alpha_3 = \frac{CD - AF}{AE - BD} \qquad \alpha_4 = \frac{CE - BF}{AE - BD}$$

where

$$A = c\sin\theta_4$$

$$B = b\sin\theta_3$$

$$C = a\alpha_2 \sin\theta_2 + a\omega_2^2 \cos\theta_2 + b\omega_3^2 \cos\theta_3 - c\omega_4^2 \cos\theta_4$$

$$D = c\cos\theta_4$$

$$E = b\cos\theta_3$$

$$F = a\alpha_2 \cos\theta_2 - a\omega_2^2 \sin\theta_2 - b\omega_3^2 \sin\theta_3 + c\omega_4^2 \sin\theta_4$$

See Norton 7.3

In-Class Exercise:

Exercise: If $\omega_2 < 0$, $\omega_3 > 0$, $\omega_4 < 0$, which direction do $\vec{V}_A$, $\vec{V}_B$, $\vec{V}_{BA}$ point?

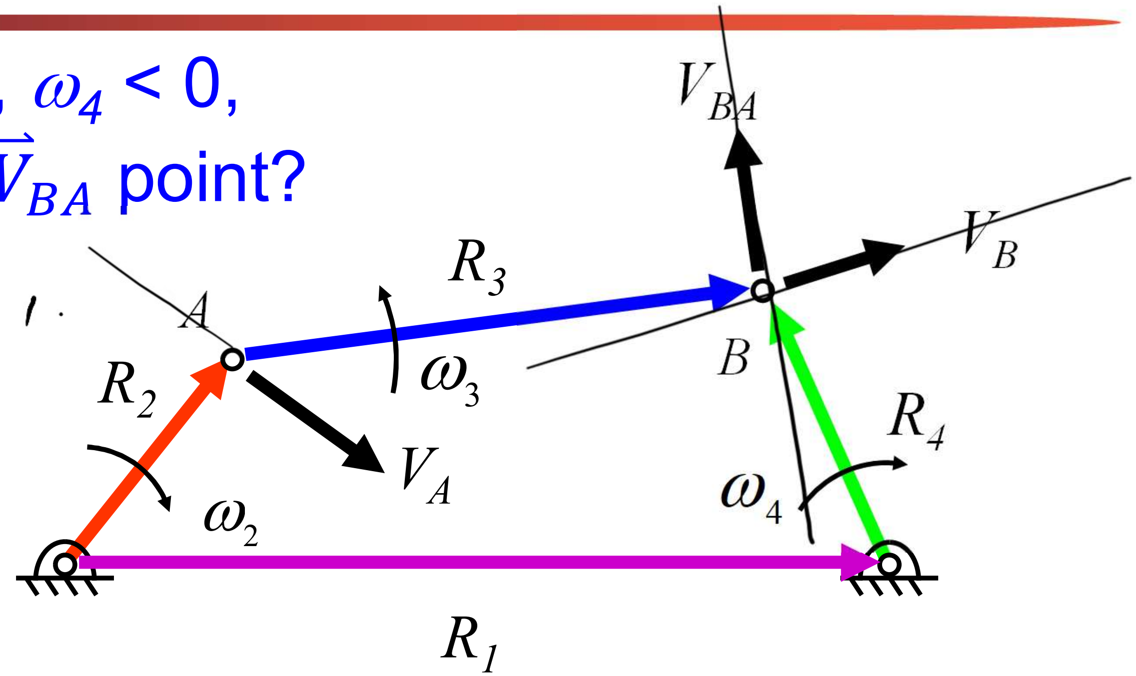
Hint:

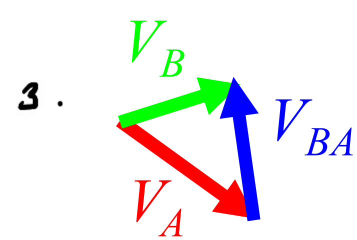
$$\vec{V}_A = j\omega_2\vec{R}_2$$

$$\vec{V}_B = j\omega_4\vec{R}_4$$

$$\vec{V}_{BA} = j\omega_3\vec{R}_3$$

$$\vec{V}_A + \vec{V}_{BA} - \vec{V}_B = 0$$





Review: The PVA Analysis Steps

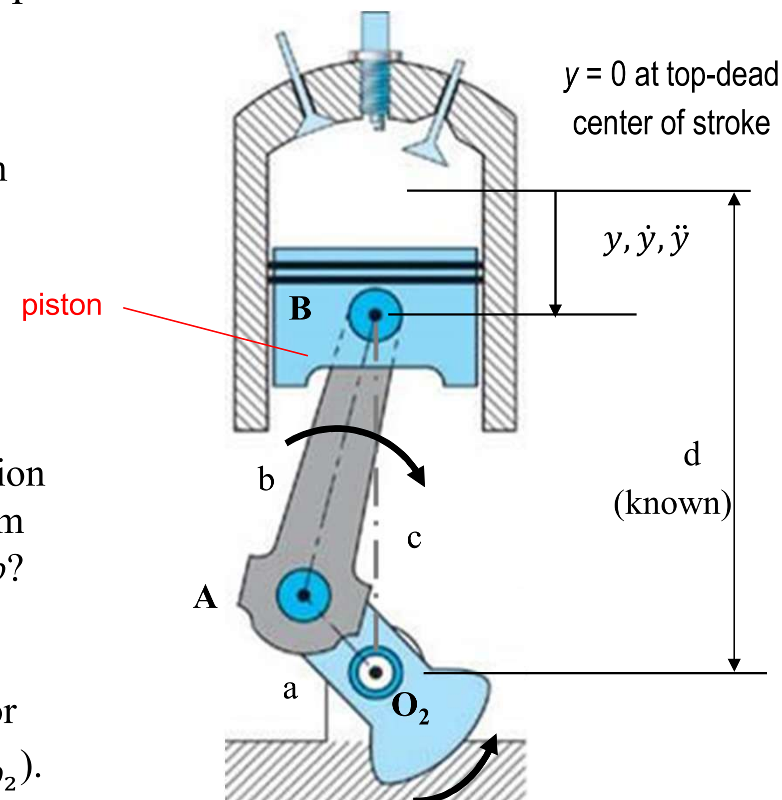
- For position, velocity, or acceleration, recall

1. Derive vector loop equation in compact vector form (for P, V, or A)
2. Use Euler's Identity to find the real and imaginary components
3. Solve equations simultaneously to get unknown values

Breakout Room Exercise 1

Consider the cylinder for a four-stroke internal combustion engine. Assume that the translational position, velocity and acceleration of the piston $(y(t), \dot{y}(t), \ddot{y}(t))$ are known.

a. Sketch the vector representation for the mechanism. Carefully label on the sketch, all points, vectors, length variables, and angles. Label each position vector using notation such as $\vec{R}_{AO_2}$ which points from O_2 to A.
b. List all unknown output variables for position analysis.
c. Derive a vector loop equation for <u>position</u> in terms of the compact vector form, e.g., $ae^{j\theta_i(t)}$. Define the loop equation using the normal loop direction defined in class.
d. Develop two scalar equations that can be solved to find the unknown position variables, but that do not contain *e* or *j*. Do you have enough equations from step *c* to solve for your unknown position variables that you listed in step *b*? If not, write any additional equations that you will need to solve for all position variables.
e. Using the equation that you found in the step d, derive a vector equation for <u>velocity</u> in *compact vector form*, and in terms of position vectors (e.g., $\vec{R}_{AO_2}$).



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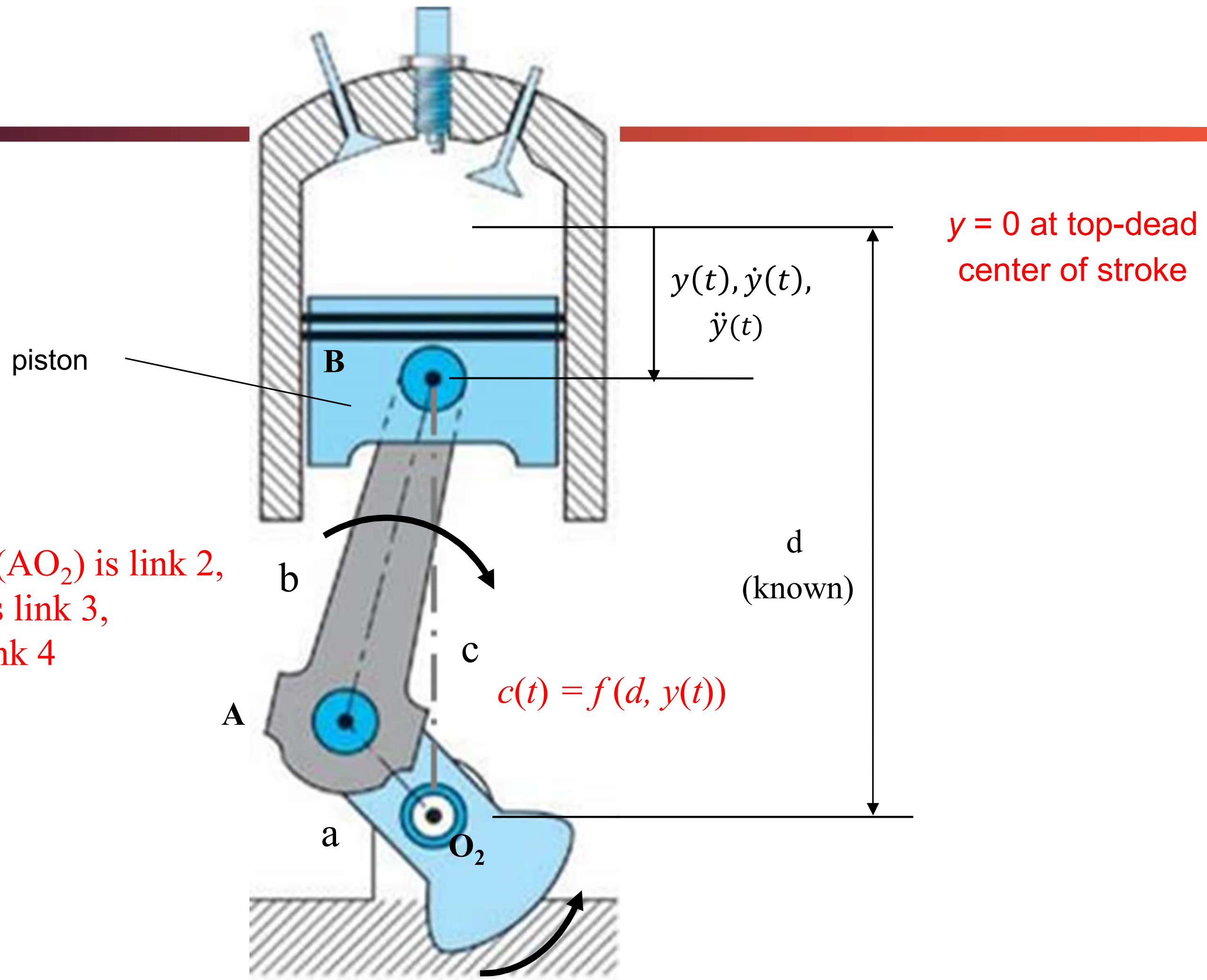
Hints



Assume crank (AO_2) is link 2,
coupler (AB) is link 3,
and piston is link 4

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Breakout Room Exercise 1

a. Sketch the vector representation for the mechanism. Carefully label on the sketch, all points, vectors, length variables, and angles. Label each position vector using notation such as $\vec{R}_{AO_2}$ which points from O_2 to A.

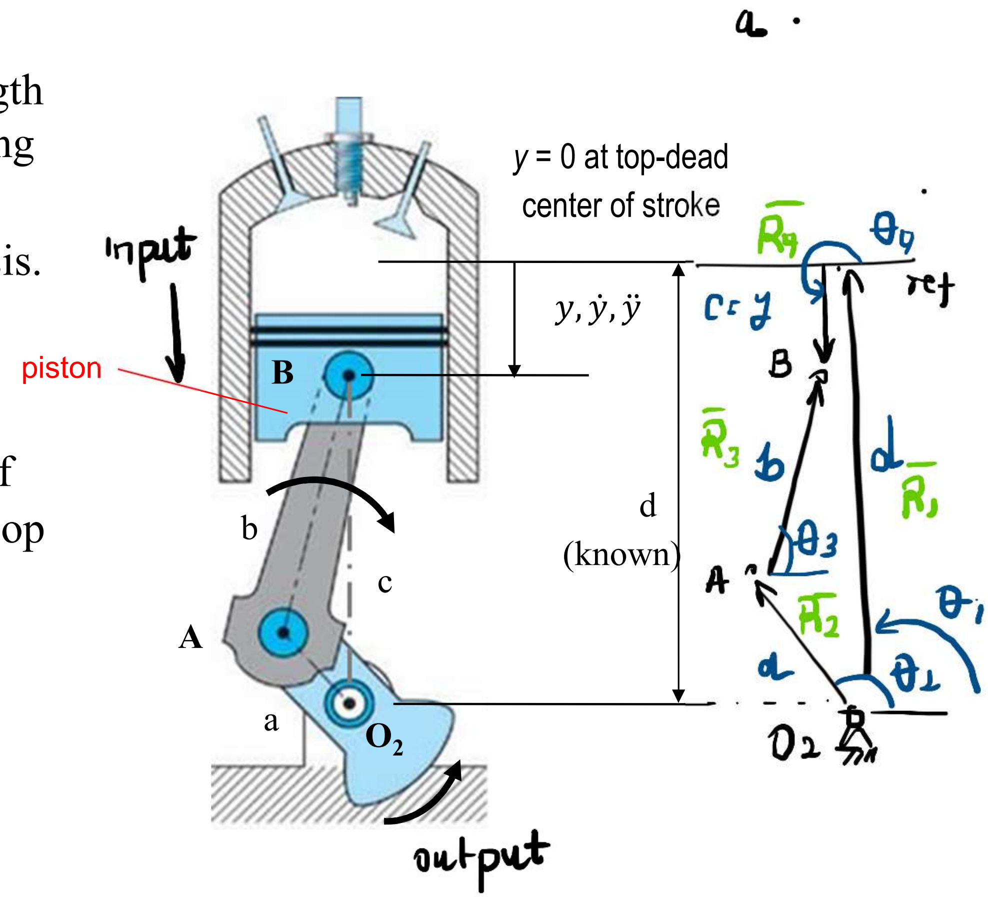
b. List all unknown output variables for position analysis.

θ_2 , θ_3

c. Derive a vector loop equation for <u>position</u> in terms of the compact vector form, e.g., $ae^{j\theta_i(t)}$. Define the loop equation using the normal loop direction defined in class.

$$\vec{R}_2 + \vec{R}_3 = \vec{R}_1 + \vec{R}_4$$

$$a e^{\theta_2 j} + b e^{\theta_3 j} = d e^{\theta_1 j} + y(t) e^{j\theta_4}.$$



Breakout Room Exercise 1

d. Develop two scalar equations that can be solved to find the unknown position variables, but that do not contain *e* or *j*. Do you have enough equations from step *c* to solve for your unknown position variables that you listed in step *b*? If not, write any additional equations that you will need to solve for all position variables.

$$ae^{\theta_2 j} + be^{\theta_3 j} = de^{\theta_1 j} + y(t)e^{j\theta_4}.$$

$$(a\cos\theta_2 + a\sin\theta_2 j) + (b\cos\theta_3 + b\sin\theta_3 j) =$$

$$(d\cos\theta_1 + d\sin\theta_1 j) + (y\cos\theta_5 + y\sin\theta_5 j)$$

solve θ_2, θ_3

e. Derive a vector equation for <u>velocity</u> in *compact vector form*, and in terms of position vectors (e.g., $\vec{R}_{AO_2}$).

$$\dot{\theta}_2 j \vec{R}_2 + \dot{\theta}_3 j \vec{R}_3 = \cancel{\dot{\theta}_1 j \vec{R}_1} \rightarrow 0 + \dot{y}(t) \underbrace{e^{j\theta_4}}_{\text{constant}}$$