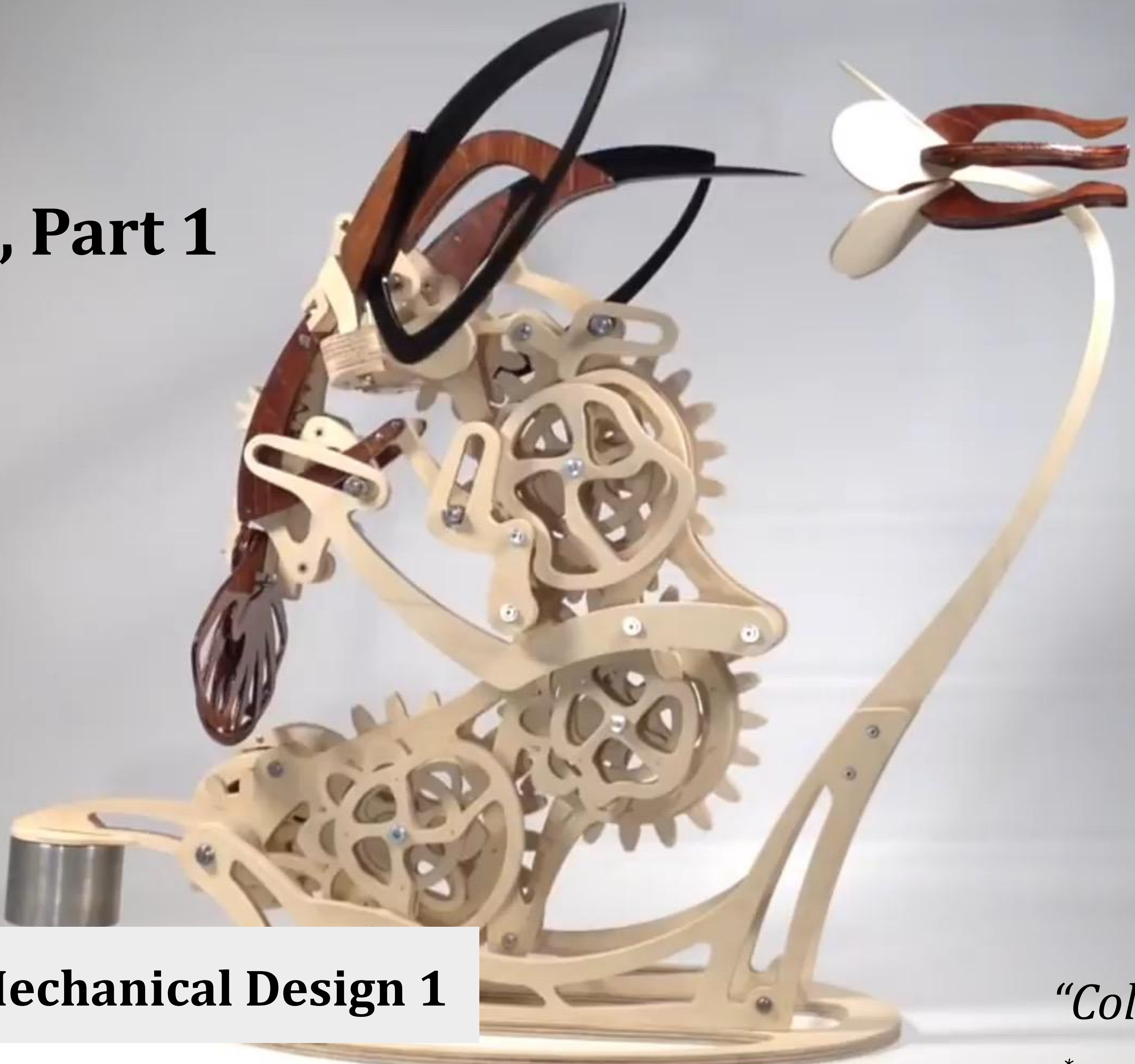
Lecture 23
Module 8:
Virtual Work, Part 1



“Colibri” by Derek Hugger

* www.youtube.com/watch?v=1scj5sotD-E

ME 370 - Mechanical Design 1

Lecture 23: Virtual Work - 1

Today Module 8, Part 1 Virtual Work (Reading, Norton Chapter 10.14-10.15 and 11.10)

Activities & Upcoming Deadlines

- **Week 12:**
  - **HW 11 (Motor, Cam, Motion 2):** posted and due Tuesday 11/~~11~~ 18
  - **Lab 11 (Dynamic Force Analysis with Python)** – Post-lab due the night before your lab section during the week of Nov 19 (delayed by 2 weeks due to P2D2 during Lab 12).
  - **Lab 12**: Meet in 1001 MEL for P2D2 presentation

- Project 2:
  - If you need a shaft cut, ask a MakerWorks member for help for free. Do not use a MechSE machinist who charge $67/hour.
  - **Project 2 Description** - Follow this document for expected deliverables for P2D2. Submit materials to Gradescope prior to lab 12 (slides + PVA appendix + CAD animation). Demonstration of prototype of entire robot (walker + dispensing mechanism together), grade will depend on level of function
  - **Complete CATME peer evaluation after finishing P2D2.** – Team member evaluation & self

- **Next lecture:** Module 6, Part 4: Motion Control 2 (Chapter 8).

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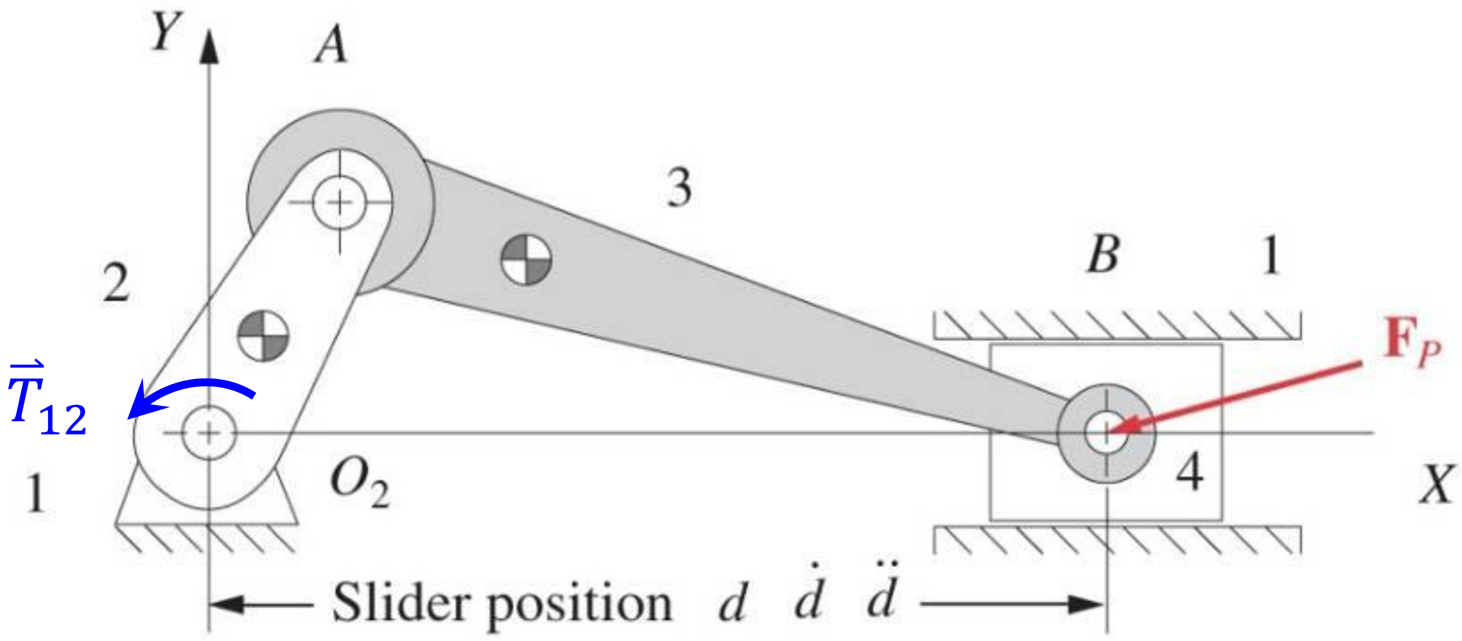
Recall: Class Exercise 1: Piston (slider-crank)

For a given time-varying force applied to link 4 ($\vec{F}_P$(t)), we want to know the output torque with respect to time ($\vec{T}_{12}$(t)). Assume (a) each link has mass m_i and length l_i, (b) the PVA analysis for this mechanism is already solved, including piston kinematics ($d, \dot{d}, \ddot{d}$), and (c) there is friction at link 4, so: $F_{14x} = \pm\mu F_{14y}$ $= -\underline{\mathrm{Sign}(\dot{d})}\,\mu F_{14y}$

Solve for output torque $\vec{T}_{12}$ via Dynamic Force Analysis (perform steps 1-3)



Recall: Convert to matrix format [A] {B} = {C}

Link 2:

1 $F_{12x} + F_{32x} = m_2 a_{CG2x}$

2 $F_{12y} + F_{32y} = m_2 a_{CG2y}$

3 $T_{21} + (R_{12x}F_{12y} - R_{12y}F_{12x}) + (R_{32x}F_{32y} - R_{32y}F_{32x}) = I_{CG2}\alpha_2$

Link 3:

4 $F_{43x} - F_{32x} = m_3 a_{CG3x}$

5 $F_{43y} - F_{32y} = m_3 a_{CG3y}$

6 $(R_{43x}F_{43y} - R_{43y}F_{43x}) - (R_{32x}F_{32y} - R_{32y}F_{32x}) = I_{CG3}\alpha_3$

Link 4: No ΣT

$F_{14x} - F_{43x} + F_{px} = m_4 a_{CG4x}$

7 $F_{14y} - F_{43y} + F_{px} = m_4 a_{CG4y} = 0$

8 $F_{14x} = \pm\mu F_{14y} = -\text{sign}(\dot{d})_{\mu F_{14y}}$

9 unknowns, 8 equations (matrix)

+ 1 equation ($F_{14x} = \pm\mu F_{14y}$)

[A] {B} unknowns {C}

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
-R_{12_y} & R_{12_x} & -R_{32_y} & R_{32_x} & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & R_{23_y} & -R_{23_x} & -R_{43_y} & R_{43_x} & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -\mu SGN\left(\dot{d}\right) & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0
\end{bmatrix}
\times
\begin{bmatrix}
F_{12_x} \\ F_{12_y} \\ F_{32_x} \\ F_{32_y} \\ F_{43_x} \\ F_{43_y} \\ F_{14_y} \\ T_{12}
\end{bmatrix}
=
\begin{bmatrix}
m_2 a_{G_{2x}} \\ m_2 a_{G_{2y}} \\ I_{G_2}\alpha_2 \\ m_3 a_{G_{3x}} \\ m_3 a_{G_{3y}} \\ I_{G_3}\alpha_3 \\ m_4 a_{G_{4x}} - F_{P_x} \\ -F_{P_y}
\end{bmatrix}
$$

Virtual Work Topics

- Energy Method
  - Principle of Virtual Work
  - Total power for system of *n* moving links
  - Vector equation for estimating external applied forces and torques
- Examples of solving for input torque
  - Four-bar linkage
  - Simple gear set

(Reading, Norton Ch 10.14-15, 11.10)

What method could we use if we only wanted to determine external forces and torques (not also internal reaction terms)?

- Use energy (virtual work) methods

Energy Methods (or Virtual Work Methods)

We have seen how to use a Newton's second law or 'Force Balance Equations' to analyze the forces or torques on links in a mechanism.

We will now use an Energy/Work balance approach

- Based on principle of virtual work
- Only for determining <u>external</u> forces and torques that produce work (e.g., T_{12} or F_p)
- <u>Not suitable if we also need internal reactions</u>.
- Requires knowledge of <u>accelerations and velocities</u> ← PVA
- Does not require simultaneous solution of large systems of equations

Recall definition of dot product

$$\text{If } \vec{\mathbf{B}} = b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}$$

$$\vec{\mathbf{C}} = c_x\hat{\imath} + c_y\hat{\jmath} + c_z\hat{k}$$

$$\text{then } \vec{\mathbf{B}} \cdot \vec{\mathbf{C}} = b_x c_x + b_y c_y + b_z c_z$$

$$\Longrightarrow \text{scalar!}$$

Definitions

- Work = dot product of force (or torque) and displacement

$W = \vec{F} \cdot \vec{R}$ or $W = \vec{T} \cdot \vec{\theta}$

- Power = dot product of force (or torque) and velocity

$P = \vec{F} \cdot \vec{V}$ (1) or $P = \vec{T} \cdot \vec{\omega}$ (2) ← $\vec{F}$ and $\vec{T}$ are only external values that do work

Also,

- Power = time rate of change of energy

$P = \frac{dE}{dt}$

F

Assumption

For low-friction pin joints and high-speed mechanisms

- Assume negligible energy loss due to heat, friction, noise, etc.
- Assume negligible potential energy

$$\Longrightarrow \text{total E} = \text{KE only} \qquad P = \frac{dE}{dt}$$

What are expressions for KE_{trans} and KE_{rot}?

$= \frac{1}{2} m v_{CG}^2 \qquad = \frac{1}{2} I_{CG} \omega^2$

$P = \frac{d}{dt}(KE_{trans}) = \frac{d}{dt}\left(\frac{1}{2} m \vec{V}_{CG}^{\,2}\right) = m\vec{A}_{CG} \cdot \vec{V}_{CG}$

$P = \frac{d}{dt}(KE_{rot}) = \frac{d}{dt}\left(\frac{1}{2} I_{CG} \vec{\omega}^{\,2}\right) = I_{CG}\vec{\alpha} \cdot \vec{\omega}$

3 $P = m\vec{A}_{CG} \cdot \vec{V}_{CG}$

4 $P = I_{CG}\vec{\alpha} \cdot \vec{\omega}$

Instantaneous change in energy

F

Virtual work

- Called "virtual work" from concept of work done due to forces causing an infinitesimal, or virtual, displacement $(\delta\vec{\mathbf{R}})$ such that :

$$\delta W = \vec{F} \cdot \delta\vec{R}$$

- If applied over infinitesimal delta time $(\delta \boldsymbol{t})$ then get instantaneous power of the system:

$$P = \frac{\delta W}{\delta t} = \vec{F} \cdot \frac{\delta \vec{R}}{\delta t} = \vec{F} \cdot \vec{V}$$

- At any instant, the rate of change of energy in the system must balance the rate of work done on the system

$$\underbrace{\underset{①}{\vec{F} \cdot \vec{V}} + \underset{②}{\vec{T} \cdot \vec{\omega}}}_{\text{Rate of work by ext F \& T}} = \underbrace{\underset{③}{m \vec{A}_{CG} \cdot \vec{V}_{CG}} + \underset{④}{I_{CG} \vec{\alpha} \cdot \vec{\omega}}}_{\text{Rate of change of (kinetic) energy}}$$

F

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Virtual work

- Total power for system of *n* moving links:
- Assuming fixed link is link 1

Due to ext forces · or torques

due to inertial ~~inter~~ props of links

$$\sum_{i=2}^{n}(\vec{\mathbf{F}}_i \cdot \vec{\mathbf{v}}_i) + \sum_{i=2}^{n}(\vec{\mathbf{T}}_i \cdot \vec{\boldsymbol{\omega}}_i) = \sum_{i=2}^{n}(\vec{m}_i \vec{\mathbf{A}}_{CGi} \cdot \vec{\mathbf{v}}_{CGi}) + \sum_{i=2}^{n}(I_{CGi}\vec{\boldsymbol{a}}_{\boldsymbol{i}} \cdot \vec{\boldsymbol{\omega}}_{\boldsymbol{i}})$$

velocity at point of application of force, not of CG

ext

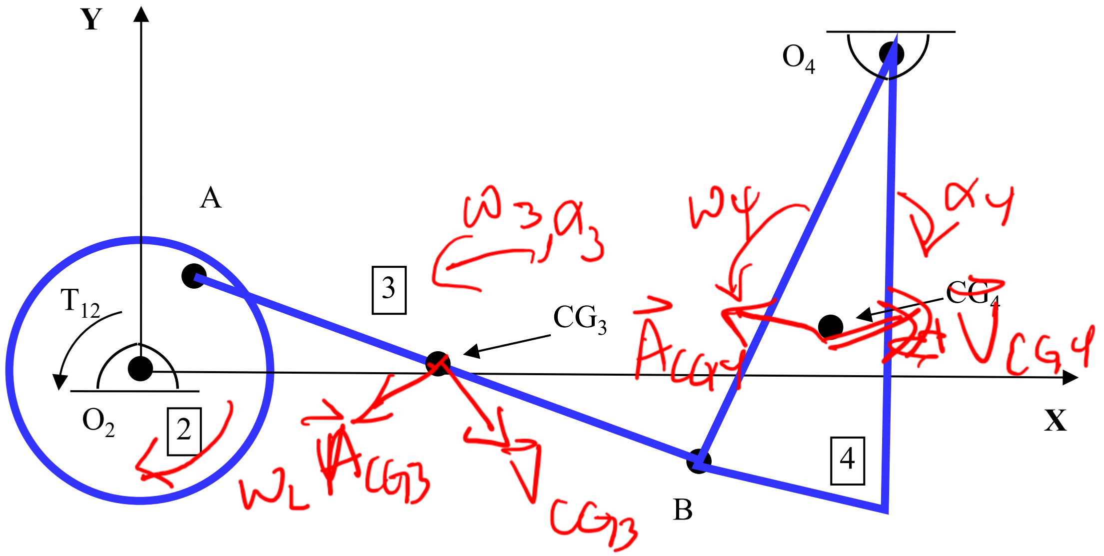
Dot products → sum of scalar values

F

Exercise 1

$\omega_2 = -24k$ rad/s

4-bar linkage where link 2 is a disk that is driven by a constant-velocity motor, solve for motor torque T_{12}



Given:

Link	m_i (kg)	I_{cgi} (kg m^2)	ω_i (rad/s)	α_i (rad/s^2)	$\vec{v}_{CG_i}$ (m/s)	$\vec{A}_{CG_i}$ (m/s^2)
2	4.53	0.023	-24 **k**	? 0	? 0	? 0
3	1.81	0.008	4.9 **k**	241 **k**	1.585 **i** - 0.418 **j**	-24.40 **i** - 13.58 **j**
4	3.63	0.035	7.8 **k**	-129 **k**	0.997 **i** + 0.323 **j**	-18.95 **i** + 2.46 **j**

1. Draw in angular and translational velocities and accelerations for each link
2. Use virtual work derived power equation to solve for T_{12}

$$\sum_{i=2}^{4}\cancel{(\vec{\mathbf{F}}_i\cdot\vec{\mathbf{v}}_i)}^{0} + \sum_{i=2}^{4}(\vec{\mathbf{T}}_i\cdot\vec{\boldsymbol{\omega}}_i) = \sum_{i=2}^{4}(\vec{m}_i\vec{\mathbf{A}}_{CGi}\cdot\vec{\mathbf{v}}_{CGi}) + \sum_{i=2}^{4}(I_{CGi}\vec{\boldsymbol{a}}_i\cdot\vec{\boldsymbol{\omega}}_i)$$

$$T_{12}\,\omega_2 = m_3\vec{A}_{CG_3}\cdot\vec{V}_{CG_3} + m_4\vec{A}_{CG_4}\cdot\vec{V}_{CG_4} + I_{CG_3}\vec{\alpha}_3\cdot\vec{\omega}_3 + I_{CG_4}\vec{\alpha}_4\cdot\vec{\omega}_4$$

solve for $T_{12} = 6.3\,\hat{k} = 6.3$ Nm (ccw)

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Class Exercise

Create a kinematic diagram of your Project 2 robot leg (if working with a team, select one member's design for this exercise).

1. Number all links—**excluding any redundant or passive links**—and label their centers of gravity.
2. **Assume that the position, velocity, and acceleration (PVA) of the links at the drawing configuration are known.**
3. Draw the angular and translational velocities and accelerations for each link.
4. **Assume that the foot contact forces are known** (sufficient to support the walker's weight and sustain forward acceleration while overcoming friction).
5. Using the **virtual work–derived power equation**, relate the **power done by the external forces and torques**—the input torque at the effective crank and the contact forces at the foot—to the **rate of change of the mechanical system's kinetic energy**.

$$\sum_{i=2}^{4}(\vec{\mathbf{F}}_i \cdot \vec{\mathbf{v}}_i) + \sum_{i=2}^{4}(\vec{\mathbf{T}}_i \cdot \vec{\boldsymbol{\omega}}_i) = \sum_{i=2}^{4}(\vec{m}_i \vec{\mathbf{A}}_{CGi} \cdot \vec{\mathbf{v}}_{CGi}) + \sum_{i=2}^{4}(I_{CGi}\vec{\boldsymbol{a}}_i \cdot \vec{\boldsymbol{\omega}}_i)$$