

Lecture 23
Module 8:
Virtual Work, Part 1



ME 370 - Mechanical Design 1

"Colibri" by Derek Hugger

** www.youtube.com/watch?v=Iscj5sotD-E*

Virtual Work Topics

- Energy Method
 - Principle of Virtual Work
 - Total power for system of n moving links
 - Vector equation for estimating external applied forces and torques
- Examples of solving for input torque
 - Four-bar linkage
 - Simple gear set

(Reading, Norton Ch 10.14-15, 11.10)

What method could we use if we only wanted to determine external forces and torques (not also internal reaction terms)?

- Use energy (virtual work) methods

Energy Methods (or Virtual Work Methods)

We have seen how to use a Newton's second law or 'Force Balance Equations' to analyze the forces or torques on links in a mechanism.

We will now use an Energy/Work balance approach

- Based on principle of virtual work
- Only for determining external forces and torques that produce work (e.g., T_{12} or F_p)
- Not suitable if we also need internal reactions.
- Requires knowledge of accelerations and velocities ← PVA
- Does not require simultaneous solution of large systems of equations

Recall definition of dot product

$$\text{If } \vec{\mathbf{B}} = b_x \hat{i} + b_y \hat{j} + b_z \hat{k}$$

$$\vec{\mathbf{C}} = c_x \hat{i} + c_y \hat{j} + c_z \hat{k}$$

$$\text{then } \vec{\mathbf{B}} \cdot \vec{\mathbf{C}} = b_x c_x + b_y c_y + b_z c_z$$

$$\Rightarrow \text{scalar!}$$

Definitions

- Work = dot product of force (or torque) and displacement

$$W = \vec{F} \cdot \vec{R} \quad \text{or} \quad W = \vec{\tau} \cdot \vec{\theta}$$

- Power = dot product of force (or torque) and velocity

$$P = \vec{F} \cdot \vec{v} \quad \text{or} \quad P = \vec{\tau} \cdot \vec{\omega}$$

① ②

Also,

- Power = time rate of change of energy

$$P = \frac{dE}{dt}$$

the power done by
external forces
changes the internal
Energy of the
mechanism.

Definitions

- Work = dot product of force (or torque) and displacement

$$W = \vec{\mathbf{F}} \cdot \vec{\mathbf{R}} \quad \text{or} \quad W = \vec{\mathbf{T}} \cdot \vec{\boldsymbol{\theta}}$$

- Power = dot product of force (or torque) and velocity

$$P = \underbrace{\vec{\mathbf{F}} \cdot \vec{\mathbf{v}}}_{(1)} \quad \text{or} \quad P = \underbrace{\vec{\mathbf{T}} \cdot \vec{\boldsymbol{\omega}}}_{(2)}$$

Also,

- Power = time rate of change of energy

$$P = \frac{dE}{dt}$$

Here will be interested in external forces, torques (that do work on the system)

For low-friction pin joints and high-speed mechanisms

- Assume negligible energy loss due to heat, friction, noise, etc.
- Assume negligible potential energy

\Rightarrow total $E = KE$ only

$$P = \frac{dE}{dt}$$

What are
expressions for
 KE_{trans} and KE_{rot} ?

For low-friction pin joints and high-speed mechanisms

- Assume negligible energy loss due to heat, friction, noise, etc.
- Assume negligible potential energy

\Rightarrow total $E = KE$ only

$$P = \frac{dE}{dt}$$

What are
expressions for
 KE_{trans} and KE_{rot} ?

$$P = \frac{d(KE_{trans})}{dt} = \frac{d}{dt} \left(\frac{1}{2} m \vec{v}_{CG}^2 \right) = m \vec{A}_{CG} \cdot \vec{v}_{CG}$$

$$P = \frac{d(KE_{rot})}{dt} = \frac{d}{dt} \left(\frac{1}{2} I_{CG} \omega^2 \right) = I_{CG} \vec{a} \cdot \vec{\omega}$$

③ $P = m \vec{A}_{CG} \cdot \vec{v}_{CG}$

④ $P = I_{CG} \vec{a} \cdot \vec{\omega}$

Instantaneous change in energy of mechanism

Virtual work

- Called “virtual work” from concept of work done due to forces causing an infinitesimal, or virtual, displacement ($\delta \vec{R}$) such that :

$$\delta W = \vec{F} \cdot \delta \vec{R} + \vec{\tau} \cdot \delta \vec{\theta}$$

- If applied over infinitesimal delta time (δt) then get instantaneous power of the system:

$$P = \frac{\delta W}{\delta t} = \vec{F} \cdot \frac{\delta \vec{R}}{\delta t} + \vec{\tau} \cdot \frac{\delta \vec{\theta}}{\delta t} = \underbrace{\vec{F} \cdot \vec{v} + \vec{\tau} \cdot \vec{\omega}}_{\text{work on system}}$$

- At any instant, the rate of change of energy in the system must balance the rate of work done on the system

$$\underbrace{\vec{F} \cdot \vec{v}}_{(1)} + \underbrace{\vec{\tau} \cdot \vec{\omega}}_{(2)} = m \underbrace{\vec{A}_{CG}}_{(3)} \cdot \underbrace{\vec{v}_{CG}}_{(4)} + I_{CG} \vec{\alpha} \cdot \vec{\omega}$$

power of External forces = change in KE

Virtual work

- Called “virtual work” from concept of work done due to forces causing an infinitesimal, or virtual, displacement ($\delta \vec{R}$) such that :

$$\delta W = \vec{F} \cdot \delta \vec{R}$$

- If applied over infinitesimal delta time (δt) then get instantaneous power of the system:

$$P = \frac{\delta W}{\delta t} = \vec{F} \cdot \frac{\delta \vec{R}}{\delta t} = \vec{F} \cdot \vec{v}$$

- At any instant, the rate of change of energy in the system must balance the rate of work done on the system

$$\underbrace{\vec{F} \cdot \vec{v}}_{(1)} + \underbrace{\vec{T} \cdot \vec{\omega}}_{(2)} = m \underbrace{\vec{A}_{CG}}_{(3)} \cdot \underbrace{\vec{v}_{CG}}_{(4)} + I_{CG} \underbrace{\vec{a}}_{(4)} \cdot \underbrace{\vec{\omega}}_{(4)}$$

Rate of work done by
external forces and torques

Rate of change of (kinetic)
energy in the system

Virtual work

- Total power for system of n moving links:
- Assuming fixed link is link 1

$$\sum_{i=2}^n (\vec{\mathbf{F}}_i \cdot \vec{\mathbf{v}}_i) + \sum_{i=2}^n (\vec{\mathbf{T}}_i \cdot \vec{\boldsymbol{\omega}}_i) = \sum_{i=2}^n (\bar{m}_i \vec{\mathbf{A}}_{CGi} \cdot \vec{\mathbf{v}}_{CGi}) + \sum_{i=2}^n (I_{CGi} \vec{\mathbf{a}}_i \cdot \vec{\boldsymbol{\omega}}_i)$$

Virtual work

- Total power for system of n moving links:
- Assuming fixed link is link 1

Due to external
forces and torques
on links

Due to inertial
properties of links

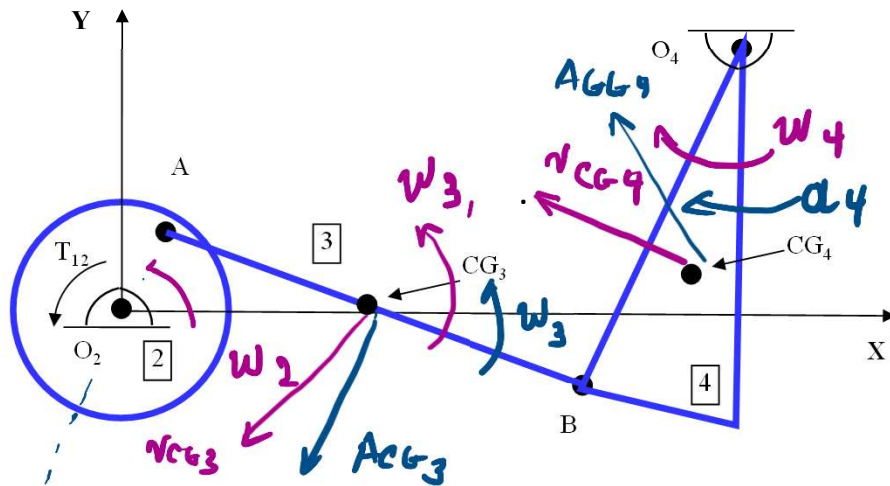
$$\sum_{i=2}^n (\vec{\mathbf{F}}_i \cdot \vec{\mathbf{v}}_i) + \sum_{i=2}^n (\vec{\mathbf{T}}_i \cdot \vec{\boldsymbol{\omega}}_i) = \sum_{i=2}^n (\bar{m}_i \vec{\mathbf{A}}_{CGi} \cdot \vec{\mathbf{v}}_{CGi}) + \sum_{i=2}^n (I_{CGi} \vec{\mathbf{a}}_i \cdot \vec{\boldsymbol{\omega}}_i)$$

Velocity at point of application of external force, not CG!

Final solution is summation of scalar entities.

Exercise 1

4-bar linkage where link 2 is a disk that is driven by a **constant-velocity motor**, solve for motor torque T_{12}



Given:

Link	m_i (kg)	I_{CGi} (kg m ²)	ω_i (rad/s)	α_i (rad/s ²)	\vec{v}_{CGi} (m/s)	\vec{a}_{CGi} (m/s ²)
2	4.53	0.023	-24 k	?	?	?
3	1.81	0.008	4.9 k	241 k	1.585 i - 0.418 j	-24.40 i - 13.58 j
4	3.63	0.035	7.8 k	-129 k	0.997 i + 0.323 j	-18.95 i + 2.46 j

1. Draw in angular and translational velocities and accelerations for each link
2. Use virtual work derived power equation to solve for T_{12}

$$\sum_{i=2}^4 (\vec{F}_i \cdot \vec{v}_i) + \sum_{i=2}^4 (\vec{T}_i \cdot \vec{\omega}_i) = \sum_{i=2}^4 (\vec{m}_i \vec{A}_{CGi} \cdot \vec{v}_{CGi}) + \sum_{i=2}^4 (I_{CGi} \vec{a}_i \cdot \vec{\omega}_i)$$

Power of External forces

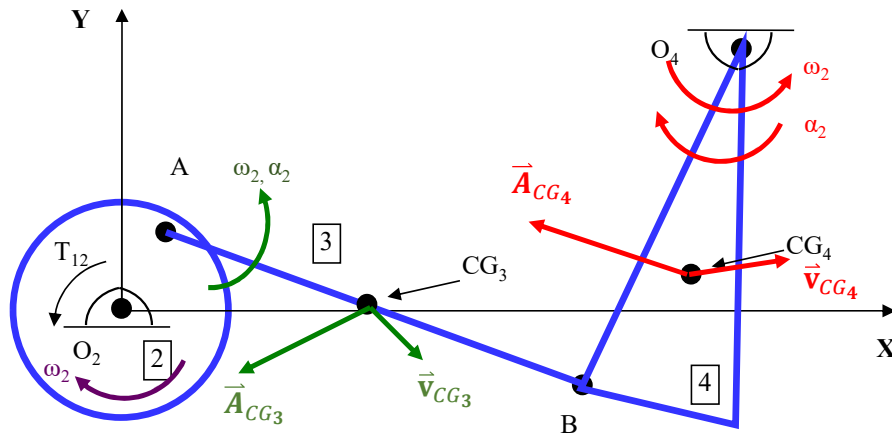
rate of change of KE

$$\cancel{T_{12} \cdot \vec{\omega}_2} = \cancel{m_2 A_{CG2} \cdot \vec{v}_{CG2}} + \cancel{T_{CG2} \vec{a}_2 \cdot \vec{\omega}_2} + m_3 A_{CG3} \cdot \vec{v}_{CG3} + I_{CG3} \vec{a}_3 \cdot \vec{\omega}_3 + m_4 A_{CG4} \cdot \vec{v}_{CG4} + I_{CG4} \vec{a}_4 \cdot \vec{\omega}_4$$

Exercise 1

4-bar linkage where link 2 is a disk that is driven by a **constant-velocity motor**, solve for motor torque T_{12}

Given:



Link	m_i (kg)	I_{CGi} (kg m ²)	ω_i (rad/s)	α_i (rad/s ²)	\vec{v}_{CGi} (m/s)	\vec{A}_{CGi} (m/s ²)
2	4.53	0.023	-24 k	? 0	? 0	? 0
3	1.81	0.008	4.9 k	241 k	1.585 i - 0.418 j	-24.40 i - 13.58 j
4	3.63	0.035	7.8 k	-129 k	0.997 i + 0.323 j	-18.95 i + 2.46 j

1. Draw in known angular and translational velocities and accelerations for each link
2. Use virtual work derived power equation to solve for T_{12}

$$\sum_{i=2}^4 (\vec{F}_i \cdot \vec{v}_i) + \sum_{i=2}^4 (\vec{T}_i \cdot \vec{\omega}_i) = \sum_{i=2}^4 (\vec{m}_i \vec{A}_{CGi} \cdot \vec{v}_{CGi}) + \sum_{i=2}^4 (I_{CGi} \vec{a}_i \cdot \vec{\omega}_i)$$

$$T_{12} \omega_2 = (m_2 \vec{A}_{CG2} \cdot \vec{v}_{CG2} + m_3 \vec{A}_{CG3} \cdot \vec{v}_{CG3} + m_4 \vec{A}_{CG4} \cdot \vec{v}_{CG4}) + (I_{CG2} a_2 \omega_2 + I_{CG3} a_3 \omega_3 + I_{CG4} a_4 \omega_4)$$

Plug in values from table: $T_{12} = 6.3 \hat{k} = 6.3 \text{ Nm (ccw)}$

What are external \vec{F}_i ?

No applied external forces $\vec{F}_i = 0$

What are unknown in table?

(From inspection should get the following answers:

$V_{CG2} = A_{CG2} = 0$ since O_2 doesn't move.

$\alpha_2 = 0$ since link 2 is driven by "constant velocity motor" as noted in description.