

# Lecture 23

## Module 8:

### Virtual Work, Part 1



ME 370 - Mechanical Design 1

*"Colibri"* by Derek Hugger

\* [www.youtube.com/watch?v=1scj5sotD-E](https://www.youtube.com/watch?v=1scj5sotD-E)

# Virtual Work Topics

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- Energy Method
  - Principle of Virtual Work
  - Total power for system of  $n$  moving links
  - Vector equation for estimating external applied forces and torques
- Examples of solving for input torque
  - Four-bar linkage
  - Simple gear set

(Reading, Norton Ch 10.14-15, 11.10)

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What method could we use if we only wanted to determine external forces and torques (not also internal reaction terms)?

- Use energy (virtual work) methods

## Energy Methods (or Virtual Work Methods)

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We have seen how to use a Newton's second law or 'Force Balance Equations' to analyze the forces or torques on links in a mechanism.

We will now use an Energy/Work balance approach

- Based on principle of virtual work
- Only for determining external forces and torques that produce work (e.g.,  $T_{12}$  or  $F_p$ )
- Not suitable if we also need internal reactions.
- Requires knowledge of accelerations and velocities ← PVA
- Does not require simultaneous solution of large systems of equations

## Recall definition of dot product

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If  $\vec{\mathbf{B}} = b_x \hat{i} + b_y \hat{j} + b_z \hat{k}$

$\vec{\mathbf{C}} = c_x \hat{i} + c_y \hat{j} + c_z \hat{k}$

then  $\vec{\mathbf{B}} \cdot \vec{\mathbf{C}} = b_x c_x + b_y c_y + b_z c_z$

$\Rightarrow$  scalar!

# Definitions

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- Work = dot product of force (or torque) and displacement

$$w = \vec{F} \cdot \vec{R} \quad \text{or} \quad w = \vec{\tau} \cdot \vec{\theta}$$

- Power = dot product of force (or torque) and velocity

$$P = \vec{F} \cdot \vec{v} \quad \text{or} \quad P = \vec{\tau} \cdot \vec{\omega}$$

①

②

Also,

- Power = time rate of change of energy

$$P = \frac{dE}{dt}$$

*the power done by  
external forces  
changes the internal  
Energy of the  
mechanism.*

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# Definitions

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①                                   ②

Also,

- Power = time rate of change of energy

Here we will be interested in external forces, torques (that do work on the system)

$$P = \frac{dE}{dt}$$

# For low-friction pin joints and high-speed mechanisms

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- Assume negligible energy loss due to heat, friction, noise, etc.
- Assume negligible potential energy

$\Rightarrow$  total E = KE only

$$P = \frac{dE}{dt}$$

What are  
expressions for  
 $KE_{trans}$  and  $KE_{rot}$ ?

# For low-friction pin joints and high-speed mechanisms

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- Assume negligible energy loss due to heat, friction, noise, etc.
- Assume negligible potential energy

$\Rightarrow$  total E = KE only

$$P = \frac{dE}{dt}$$

What are  
expressions for  
 $KE_{trans}$  and  $KE_{rot}$ ?

$$P = \frac{d(KE_{trans})}{dt} = \frac{d}{dt} \left( \frac{1}{2} m \vec{v}_{CG}^2 \right) = m \vec{A}_{CG} \cdot \vec{v}_{CG}$$

$$P = \frac{d(KE_{rot})}{dt} = \frac{d}{dt} \left( \frac{1}{2} I_{CG} \boldsymbol{\omega}^2 \right) = I_{CG} \vec{\boldsymbol{\alpha}} \cdot \vec{\boldsymbol{\omega}}$$

③  $P = m \vec{A}_{CG} \cdot \vec{v}_{CG}$

Instantaneous change in energy of mechanism

④  $P = I_{CG} \vec{\boldsymbol{\alpha}} \cdot \vec{\boldsymbol{\omega}}$

# Virtual work

- Called “virtual work” from concept of work done due to forces causing an infinitesimal, or virtual, displacement ( $\delta\vec{R}$ ) such that :

$$\delta w = \vec{F} \cdot \delta\vec{R} + \vec{\tau} \cdot \delta\vec{\theta}$$

- If applied over infinitesimal delta time ( $\delta t$ ) then get instantaneous power of the system:

$$P = \frac{\delta w}{\delta t} = \vec{F} \cdot \frac{\delta\vec{R}}{\delta t} + \vec{\tau} \cdot \frac{\delta\vec{\theta}}{\delta t} = \underbrace{\vec{F} \cdot \vec{v} + \vec{\tau} \cdot \vec{\omega}}_{\text{work on system}}$$

- At any instant, the rate of change of energy in the system must balance the rate of work done on the system

$$\vec{F} \cdot \vec{v} + \vec{\tau} \cdot \vec{\omega} = m \vec{F}_{GG} \cdot \vec{v}_{GG} + I_{GG} \vec{\alpha} \cdot \vec{\omega}$$

(1)      (2)      (3)      (4)  
power of External forces = change in KE

# Virtual work

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- Called “virtual work” from concept of work done due to forces causing an infinitesimal, or virtual, displacement ( $\delta\vec{R}$ ) such that :

$$\delta W = \vec{F} \cdot \delta\vec{R}$$

- If applied over infinitesimal delta time ( $\delta t$ ) then get instantaneous power of the system:

$$P = \frac{\delta W}{\delta t} = \vec{F} \cdot \frac{\delta\vec{R}}{\delta t} = \vec{F} \cdot \vec{v}$$

- At any instant, the rate of change of energy in the system must balance the rate of work done on the system

$$\vec{F} \cdot \vec{v} + \vec{T} \cdot \vec{\omega} = m\vec{A}_{CG} \cdot \vec{v}_{CG} + I_{CG}\vec{a} \cdot \vec{\omega}$$

(1)                  (2)                  (3)                  (4)

Rate of work done by  
external forces and torques

Rate of change of (kinetic)  
energy in the system

# Virtual work

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- Total power for system of  $n$  moving links:
- Assuming fixed link is link 1

$$\sum_{i=2}^n (\vec{\mathbf{F}}_i \cdot \vec{\mathbf{v}}_i) + \sum_{i=2}^n (\vec{\mathbf{T}}_i \cdot \vec{\boldsymbol{\omega}}_i) = \sum_{i=2}^n (\bar{m}_i \vec{\mathbf{A}}_{CGi} \cdot \vec{\mathbf{v}}_{CGi}) + \sum_{i=2}^n (I_{CGi} \vec{\mathbf{a}}_i \cdot \vec{\boldsymbol{\omega}}_i)$$

# Virtual work

- Total power for system of  $n$  moving links:
- Assuming fixed link is link 1

**Due to external  
forces and torques  
on links**

$$\sum_{i=2}^n (\vec{F}_i \cdot \vec{v}_i) + \sum_{i=2}^n (\vec{T}_i \cdot \vec{\omega}_i)$$

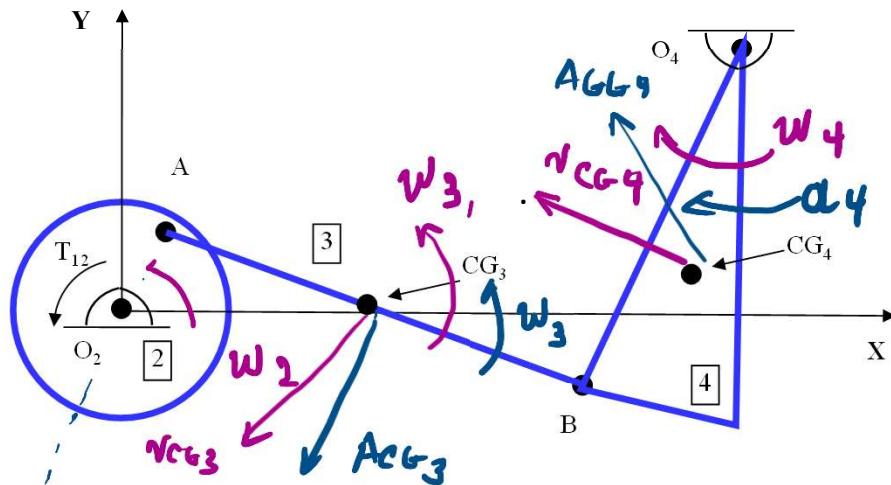
Velocity at point of application of external force, not CG!

**Due to inertial  
properties of links**

$$\sum_{i=2}^n (\bar{m}_i \vec{A}_{CGi} \cdot \vec{v}_{CGi}) + \sum_{i=2}^n (I_{CGi} \vec{a}_i \cdot \vec{\omega}_i)$$

# Exercise 1

4-bar linkage where link 2 is a disk that is driven by a **constant-velocity motor**, solve for motor torque  $T_{12}$



Given:

Link	$m_i$ (kg)	$I_{CGi}$ (kg m <sup>2</sup> )	$\omega_i$ (rad/s)	$\alpha_i$ (rad/s <sup>2</sup> )	$\vec{v}_{CGi}$ (m/s)	$\vec{A}_{CGi}$ (m/s <sup>2</sup> )
2	4.53	0.023	-24 k	?	?	?
3	1.81	0.008	4.9 k	241 k	1.585 i - 0.418 j	-24.40 i - 13.58 j
4	3.63	0.035	7.8 k	-129 k	0.997 i + 0.323 j	-18.95 i + 2.46 j

1. Draw in angular and translational velocities and accelerations for each link
2. Use virtual work derived power equation to solve for  $T_{12}$

$$\sum_{i=2}^4 (\vec{F}_i \cdot \vec{v}_i) + \sum_{i=2}^4 (\vec{T}_i \cdot \vec{\omega}_i) = \sum_{i=2}^4 (\vec{m}_i \vec{A}_{CGi} \cdot \vec{v}_{CGi}) + \sum_{i=2}^4 (I_{CGi} \vec{a}_i \cdot \vec{\omega}_i)$$

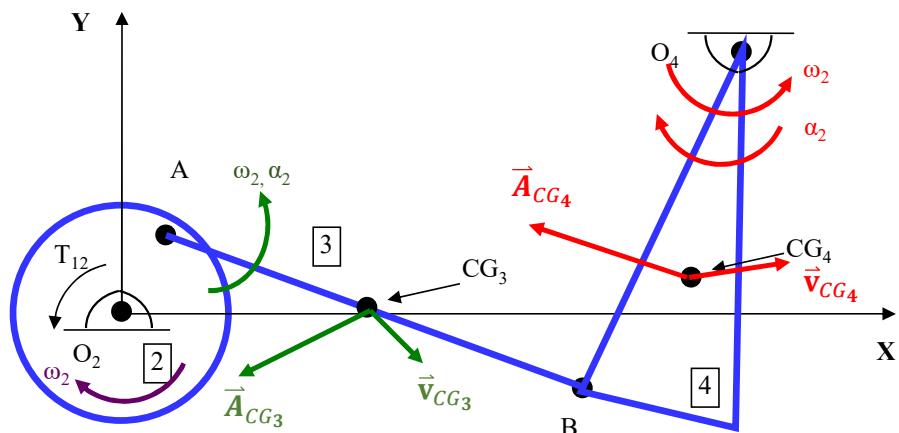
*Power of External Forces*

*rate of change of KE*

$$\vec{T}_{12} \cdot \vec{\omega}_2 = m_2 \vec{A}_{CG2} \cdot \vec{v}_{CG2} + T_{CG2} \vec{a}_2 \cdot \vec{\omega}_2 + m_3 \vec{A}_{CG3} \cdot \vec{v}_{CG3} + T_{CG3} \vec{a}_3 \cdot \vec{\omega}_3 \\ m_4 \vec{A}_{CG4} \cdot \vec{v}_{CG4} + T_{CG4} \vec{a}_4 \cdot \vec{\omega}_4$$

# Exercise 1

4-bar linkage where link 2 is a disk that is driven by a **constant-velocity motor**, solve for motor torque  $T_{12}$



Given:

Link	$m_i$ (kg)	$I_{cg_i}$ (kg m <sup>2</sup> )	$\omega_i$ (rad/s)	$\alpha_i$ (rad/s <sup>2</sup> )	$\vec{v}_{CG_i}$ (m/s)	$\vec{A}_{CG_i}$ (m/s <sup>2</sup> )
2	4.53	0.023	-24 k	? 0	? 0	? 0
3	1.81	0.008	4.9 k	241 k	1.585 i - 0.418 j	-24.40 i - 13.58 j
4	3.63	0.035	7.8 k	-129 k	0.997 i + 0.323 j	-18.95 i + 2.46 j

1. Draw in known angular and translational velocities and accelerations for each link
2. Use virtual work derived power equation to solve for  $T_{12}$

$$\sum_{i=2}^4 (\vec{F}_i \cdot \vec{v}_i) + \sum_{i=2}^4 (\vec{T}_i \cdot \vec{\omega}_i) = \sum_{i=2}^4 (\vec{m}_i \vec{A}_{CGi} \cdot \vec{v}_{CGi}) + \sum_{i=2}^4 (I_{CGi} \vec{a}_i \cdot \vec{\omega}_i)$$

$$T_{12}\omega_2 = (m_2 \vec{A}_{CG2} \cdot \vec{v}_{CG2} + m_3 \vec{A}_{CG3} \cdot \vec{v}_{CG3} + m_4 \vec{A}_{CG4} \cdot \vec{v}_{CG4}) + (I_{CG2} \alpha_2 \omega_2 + I_{CG3} \alpha_3 \omega_3 + I_{CG4} \alpha_4 \omega_4)$$

F Plug in values from table:  $T_{12} = 6.3\hat{k} = 6.3 \text{ Nm (ccw)}$

What are external  $\vec{F}_i$ ?

No applied external forces  $\vec{F}_i = 0$

What are unknown in table?

(From inspection should get the following answers:

$V_{cg2} = A_{cg2} = 0$  since  $O_2$  doesn't move.

$\alpha_2 = 0$  since link 2 is driven by

"constant velocity motor" as noted in description.