Module 7

Lecture 21:

Dynamic Force Analysis (DFA) - 2

ME 370 - Mechanical Design 1

“Colibri” by Derek Hugger

** www.youtube.com/watch?v=1scj5sotD-E*

Lecture 21: Dynamic Force Analysis - 2

Today (Reading, Norton Ch 10.1-10.8, 11) **11/5/25**

Activities & Upcoming Deadlines

- **Week 11:**
  - **HW 10 (Motor, Cam, Motion 1):** posted and due Tuesday 11/11
  - **Lab 10 (Motors):** Team post-lab due before lab
  - **Lab 11 (Dynamic Force Analysis with Python)** – Meet in 1001 MEL. Prelab – required
- Project 2:
  - **Project 2 Description** - Follow this document for expected deliverables for P2D2. Submit materials to Gradescope prior to lab 12 (slides + PVA appendix + CAD animation). Demonstration of prototype of entire robot (walker + dispensing mechanism together), grade will depend on level of function
  - **Grading Rubric** and **PPTX template are posted**
  - **Lab 12**: Meet in 1001 MEL for presentation

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DFA topics

- Reading: Chap 10.1-10.8, 11
- Dynamics Fundamentals
  - Newton's laws
  - Mass moment ,Center of gravity
  - Mass moment of inertia, Parallel axis theorem
  - Radius of gyration
- Forward and Inverse Dynamics
- Force Analysis Procedure
  - Free body diagrams and equation development
  - Matrix format and solution
- Examples
  - Single link in rotation
  - Four bar slider crank
  - Gear set

Recall: DFA 6 Steps

1. Draw complete system
2. Draw free-body diagram of each segment
3. Symbolically write out equations of motion
4. Convert to matrix format
5. Insert known values
6. Invert matric to solve for unknown forces and torques

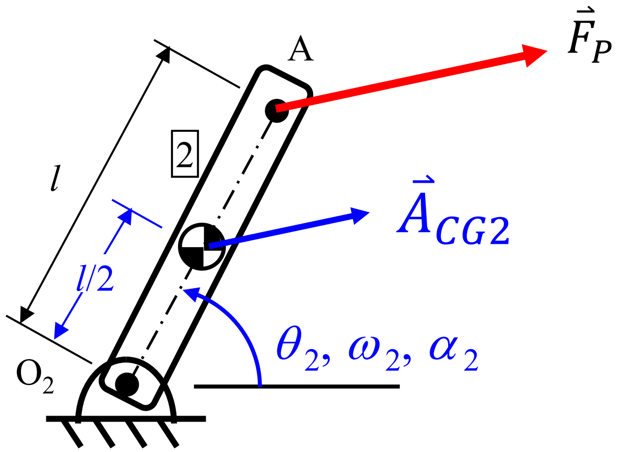
Recall: Single Link in Rotation (Ex. 11-1)

- **Given**: A single link of mass m and length l rotates about a fixed point O_2. The link has uniform cross-section. An external force, $\vec{F}_P$, is applied at point A. Assume that, at this instant in time, the following kinematic data are known

$$\theta_2, \omega_2, \alpha_2, \vec{A}_{CG2}$$

- **Find**: The reaction force $\vec{F}_{12}$ at O_2 and driving torque T_{12} needed to maintain motion at this instant of time.

(1) *Draw complete system. Label points, dimensions, external forces & torques, kinematics.*

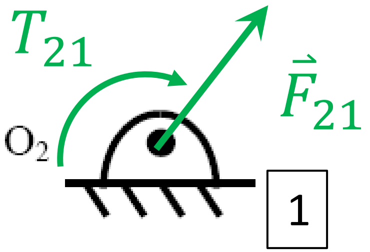


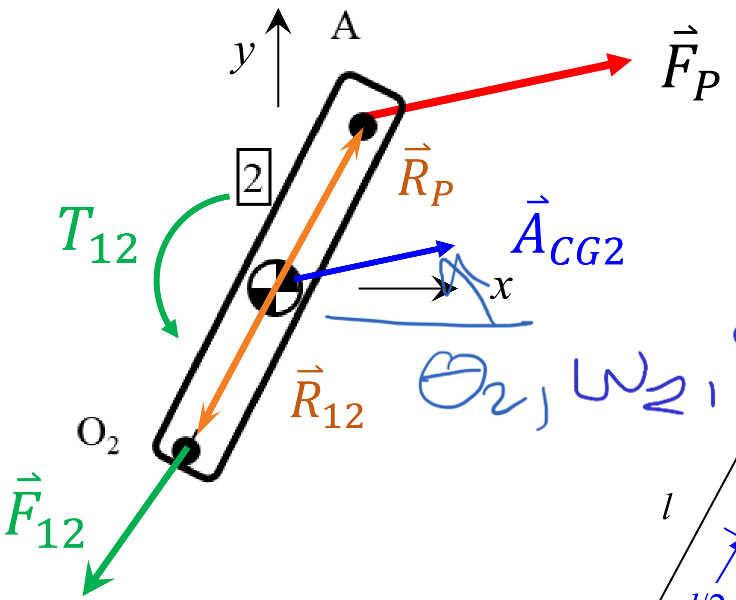
Note: assume that vectors are pointing in direction as shown. True directions will be determined when solved.

Recall: Single Link in Rotation (con't)

(2) Draw free-body diagram of each segment.

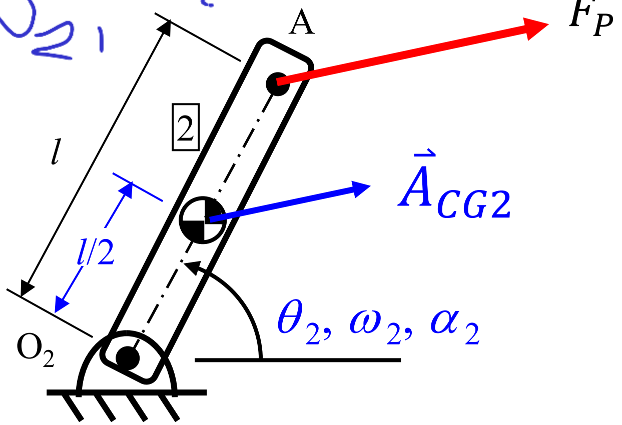
- *On each link, label local coordinate systems, external joint forces & torques, kinematics, internal forces & torques, and position vectors.*
- *Note equal and opposite forces on links at joints.*
- *Label forces such that force vector $\vec{F}_{ij}$ represents the force of link i on link j and is applied at the common joint on link j.*





$\theta_2, \omega_2, \alpha_2$

$$\vec{F}_{12} = -\vec{F}_{21}$$
$$T_{12} = -T_{21}$$



Known	Unknown
Kinematics: $\theta_2, \omega_2, \alpha_2, \vec{A}_{CG_2}$	$\vec{F}_{12} = F_{12x}, F_{12y}$
$\vec{F}_P, \vec{R}_P, \vec{R}_{12}$	T_{12}
m_2, l_2, I_{CG_2}	

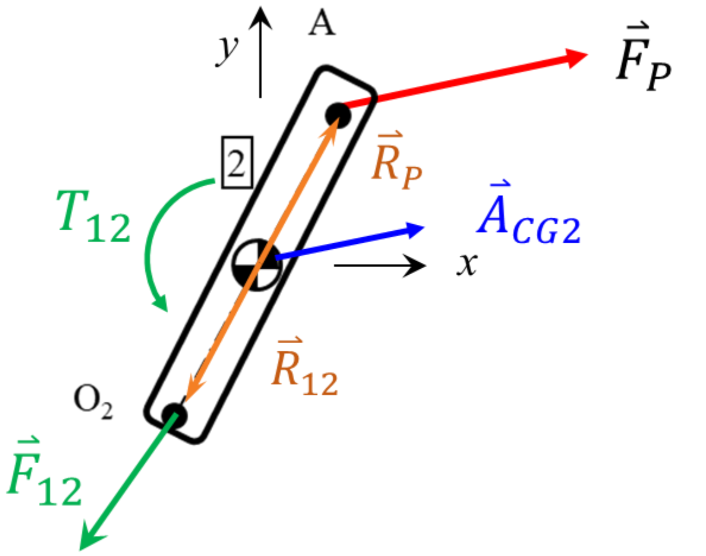
Recall: Single Link in Rotation (con't)

(3) Symbolically write out equations of motion for each moving link

- **Assume all unknown forces or torques are positive. True signs will be determined later.**

$$\sum \vec{F}_i = m_i \vec{A}_{CGi} \qquad \sum T_{i\,z} = I_{CGi}\alpha_i$$

Link 2:



$$\Sigma\vec{F} = \vec{F}_{12} + \vec{F}_P = m_2\vec{A}_{CG2}$$

$$\sum T_z = T_{12} + \left(\vec{R}_{12}\times\vec{F}_{12}\right) + \left(\vec{R}_P\times\vec{F}_P\right) = \boldsymbol{I}_{CG2}\alpha_2$$

In Cartesian (x, y) terms these equations become:

3 EOM for link 2

$$\Sigma\vec{F}_x = F_{12x} + F_{Px} = m_2 A_{CG2x}$$

$$\Sigma\vec{F}_y = F_{12y} + F_{Py} = m_2 A_{CG2y}$$

$$\Sigma T_z = T_{12} + \left(R_{12x}F_{12y} - R_{12y}F_{12x}\right) + \left(R_{Px}F_{Py} - R_{Py}F_{Px}\right) = I_{CG2}\alpha_2$$

Recall: $\vec{R}\times\vec{F} = R_xF_y - R_yF_x$

Single Link in Rotation (con't)

Redistribute terms to put **<u>unknown on left</u>** and **<u>known on right</u>**

$$\sum \vec{F}_x = \overset{?}{F}_{12x} + F_{Px} = m_2 A_{CG2x}$$

$$\sum \vec{F}_y = \overset{?}{F}_{12y} + F_{Py} = m_2 A_{CG2y}$$

$$\sum T_z = \overset{?}{T}_{12} + \left(R_{12x}\overset{?}{F}_{12y} - R_{12y}\overset{?}{F}_{12x}\right) + \left(R_{Px}F_{Py} - R_{Py}F_{Px}\right) = I_{CG2}\alpha_2$$

$$\downarrow$$

x: $\boxed{F_{12x}} = m_2 A_{CG2x} - F_{Px}$

y: $\boxed{F_{12y}} = m_2 A_{CG2y} - F_{Py}$

T: $-R_{12y}F_{12x} + R_{12x}F_{12y} + \boxed{T_{12}} = I_{CG2}\alpha_2 - R_{Px}F_{Py} + R_{Py}F_{Px}$

Single Link in Rotation (con't)

x: $\boxed{F_{12x}} = m_2 A_{CG2x} - F_{Px}$

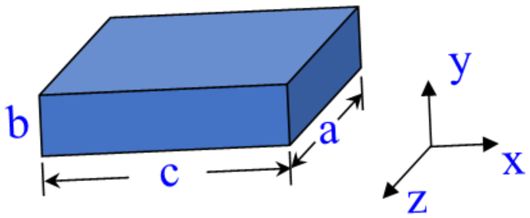
y: $\boxed{F_{12y}} = m_2 A_{CG2y} - F_{Py}$

T: $-R_{12y}F_{12x} + R_{12x}F_{12y} + \boxed{T_{12}} = I_{CG2}\alpha_2 - R_{Px}F_{Py} + R_{Py}F_{Px}$

(4) Convert to matrix format **[A]** **{B}** = **{C}**, *where*

- **[A]** *contains geometric info (e.g., position vector dimensions),*
- **{B}** *contains all unknowns,*
- **{C}** *contains dynamic info (e.g., external forces & torques, PVA analysis results, mass, moments of inertia)*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -R_{12y} & R_{12x} & 1 \end{bmatrix} \begin{Bmatrix} F_{12x} \\ F_{12y} \\ T_{12} \end{Bmatrix} = \begin{Bmatrix} m_2 A_{CG2x} - F_{Px} \\ m_2 A_{CG2y} - F_{Py} \\ I_{CG2}\alpha_2 - R_{Px}F_{Py} + R_{Py}F_{Px} \end{Bmatrix}$$



$I_x = \frac{m(a^2+b^2)}{12}; I_y = \frac{m(a^2+c^2)}{12}; I_z = \frac{m(b^2+c^2)}{12}$

(5) Insert known/given values for variables in **[A]** & **{C}**.

(6) Solve for unknown forces and torques in **{B}** *(typically internal joint forces and torques) using* **{B}** = **[A]**$^{-1}$ **{C}**.

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Practice: Fourbar slider-crank (crank = link 2)

Given: Applied external force $\vec{F}_P$ at P. <u>Know kinematics for link 2 ,3 & 4, CGs (e.g., PVA); inertial props</u> (m_i, I_{CGi})

Find: Reaction forces at each pin joint and driving torque T_{12}.

(1) Draw complete system

- *Label points, dimensions, external forces & torques, kinematics.*

<u>Knowns:</u>

$\vec{F}_P$

m_i, I_{CGi}

$\theta_2, \omega_2, \cancel{\alpha}_2$

$\theta_3, \omega_3, \alpha_3$

$x_4, \dot{x}_4, \ddot{x}_4$

$\vec{A}_{CG_2}, \vec{A}_{CG_3}, \vec{A}_{CG_4}$

<u>Unknowns:</u>

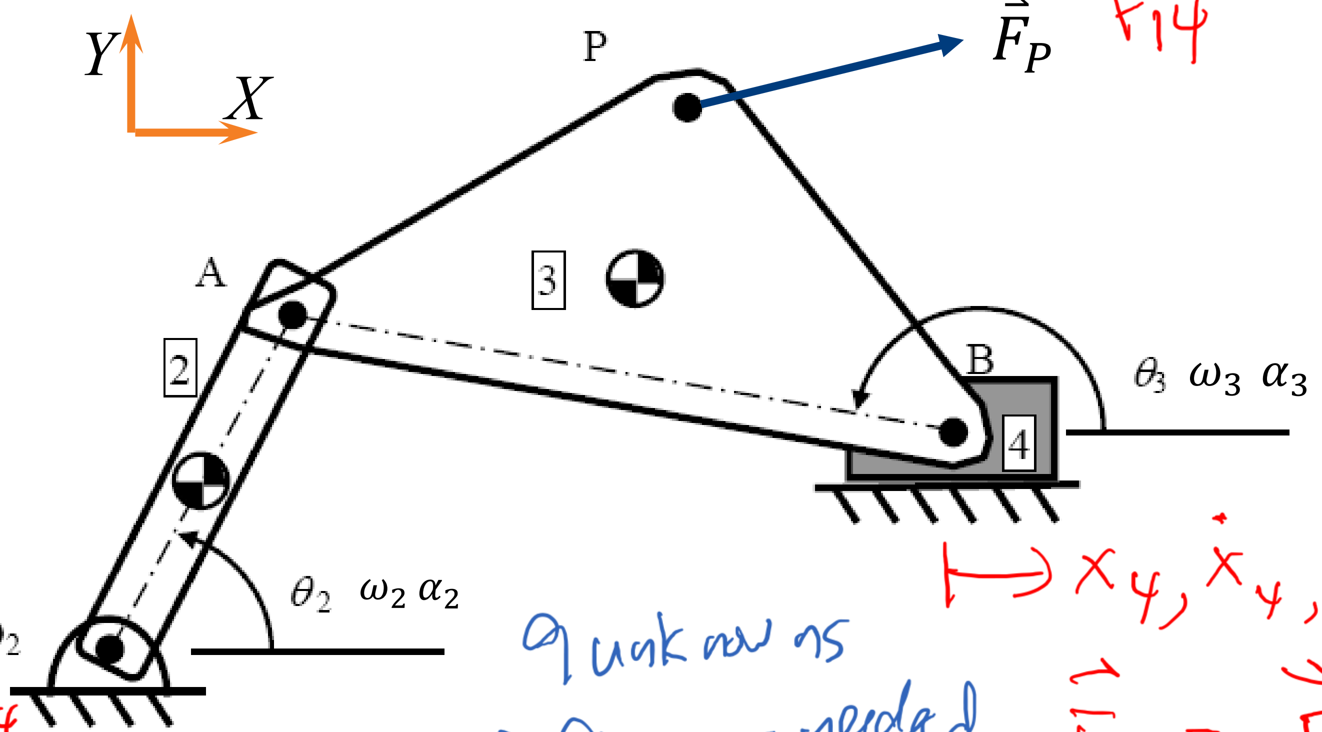
$\vec{F}_{14} = F_{14x}, F_{14y}$

T_{12}

$\vec{F}_{12} = F_{12x}, F_{12y}$

$\vec{F}_{23} = F_{23x}, F_{23y}$

$\vec{F}_{34} = F_{34x}, F_{34y}$



$\mapsto x_4, \dot{x}_4, \ddot{x}_4$

redundant unknown

$\vec{F}_{32} = -\vec{F}_{23}$

$\vec{F}_{43} = -\vec{F}_{34}$

$\vec{F}_{41} = -\vec{F}_{14}$

9 unknowns

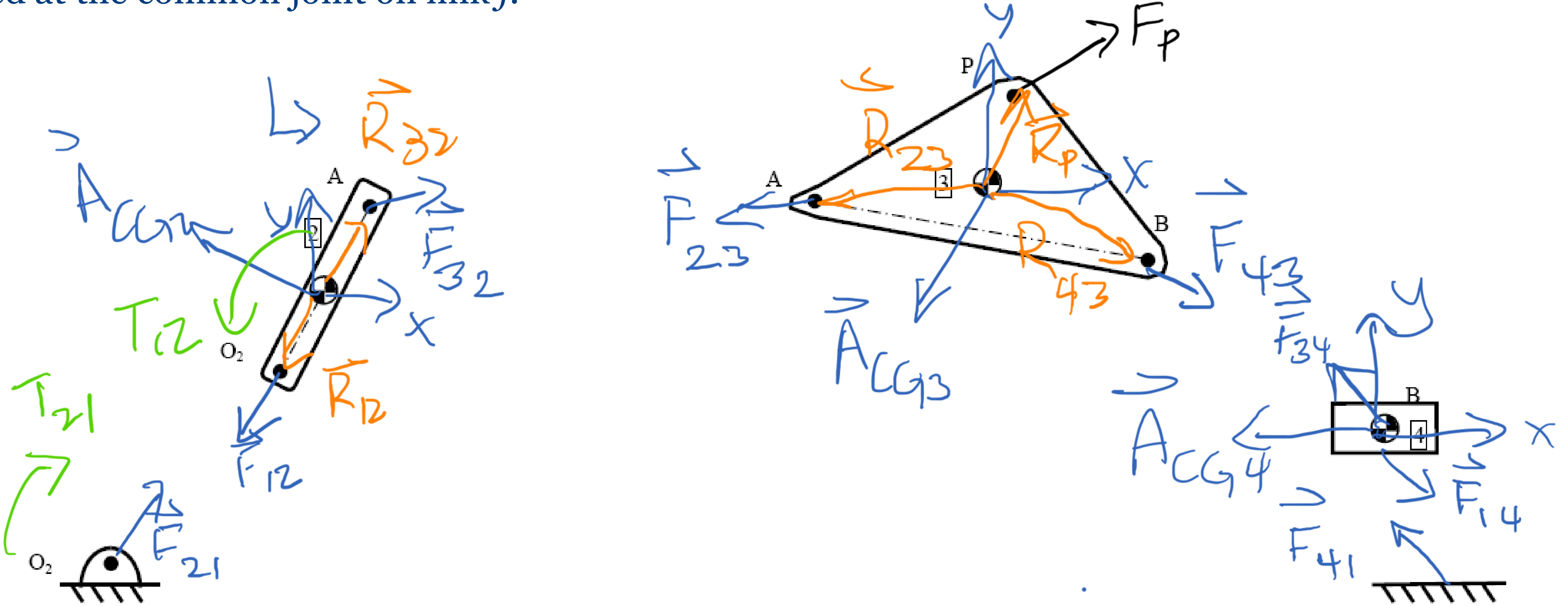
$\Rightarrow$ 9 eqns needed

Note: assume that vectors are pointing in direction as shown. True directions will be determined when solved.

Fourbar slider-crank

(2) Draw free-body diagram of each segment.

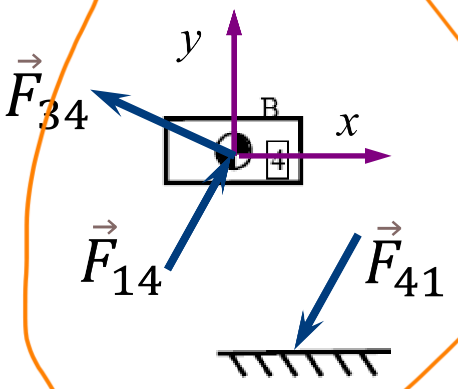
On each link, label local coordinate systems, external joint forces & torques, kinematics, internal forces & torques, and position vectors. Label forces such that force vector $\vec{F}_{ij}$ represents the force of link *i* ***on*** link *j* and is applied at the common joint on link *j*.



Fourbar slider-crank

How do we best represent forces on sliders, when the pin joint goes through the CG?

Case A



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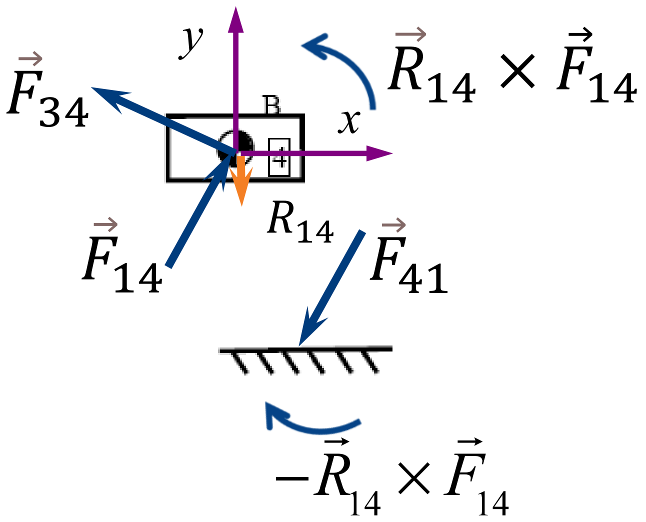
Case B



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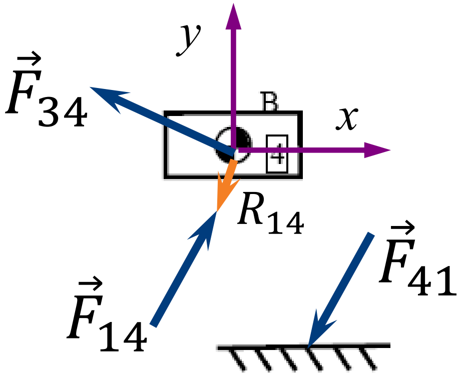
Case C



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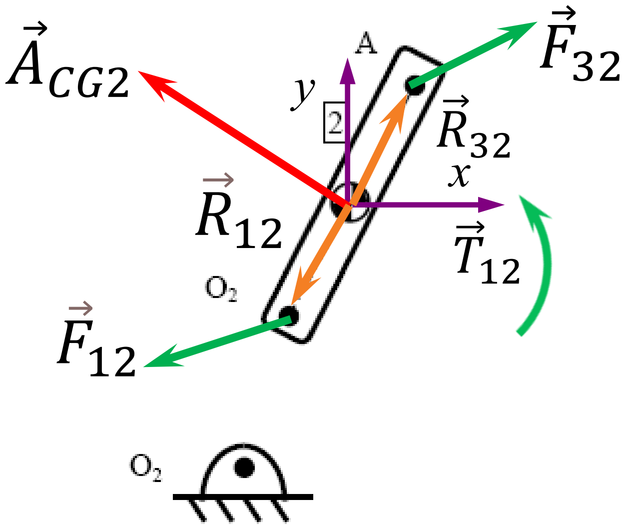
Join Code: **370**

Fourbar slider-crank

(3) Symbolically write out equations of motion for each moving link

For Link 2: $\sum \vec{F}_i = m_i \vec{A}_{CGi}$ $\sum T_{i\,z} = I_{CGi}\alpha_i$



$$\sum \vec{\mathbf{F}} = \vec{\mathbf{F}}_{12} + \vec{\mathbf{F}}_{32} = m_2 \vec{\mathbf{A}}_{CG2}$$

$$\sum T_z = T_{12} + \left(\vec{\mathbf{R}}_{12} \times \vec{\mathbf{F}}_{12}\right) + \left(\vec{\mathbf{R}}_{32} \times \vec{\mathbf{F}}_{32}\right) = I_{CG2}\alpha_2$$

In Cartesian (x, y) terms these equations become:

$$\boxed{F_{12x}} + \boxed{F_{32x}} = m_2 A_{CG2x}$$

$$\boxed{F_{12y}} + \boxed{F_{32y}} = m_2 A_{CG2y}$$

$$\boxed{T_{12}} + \left(R_{12x}F_{12y} - R_{12y}F_{12x}\right) + \left(R_{32x}F_{32y} - R_{32y}F_{32x}\right) = I_{CG2}\alpha_2$$

5 unknowns but only 3 equations

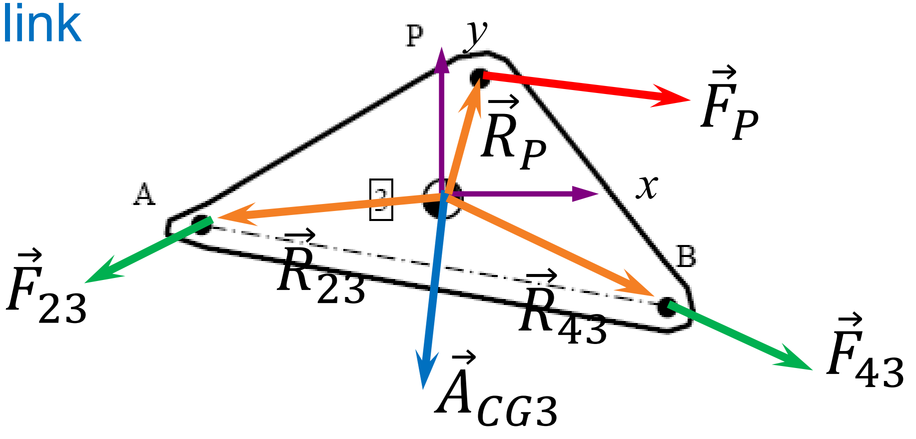
Recall: $\vec{R} \times \vec{F} = R_x F_y - R_y F_x$

Fourbar slider-crank

(3) Symbolically write out equations of motion for each moving link

For Link 3:

$$\sum \vec{F}_i = m_i \vec{A}_{CGi} \qquad \sum T_{i\,z} = I_{CGi}\alpha_i$$



$$\sum \vec{\mathbf{F}} = \vec{\mathbf{F}}_P + \vec{\mathbf{F}}_{23} + \vec{\mathbf{F}}_{43} = m_3 \vec{\mathbf{A}}_{CG3}$$

$$\sum T_z = (\vec{\mathbf{R}}_P \times \vec{\mathbf{F}}_P) + (\vec{\mathbf{R}}_{23} \times \vec{\mathbf{F}}_{23}) + (\vec{\mathbf{R}}_{43} \times \vec{\mathbf{F}}_{43}) = I_{CG3}\alpha_3$$

Recall: $\vec{\mathbf{F}}_{23} = -\vec{\mathbf{F}}_{32}$

$$-F_{32x} \boxed{+ F_{43x}} + F_{Px} = m_3 A_{CG3x}$$

$$-F_{32y} \boxed{+ F_{43y}} + F_{Py} = m_3 A_{CG3y}$$

$$(-R_{23x}F_{32y} + R_{23y}F_{32x}) + (R_{43x}F_{43y} - R_{43y}F_{43x}) + (R_{Px}F_{Py} - R_{Py}F_{Px}) = I_{CG3}\alpha_3$$

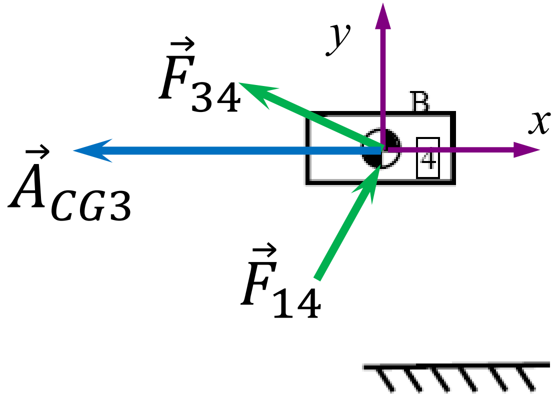
Add 2 more unknowns, and 3 more equations

Fourbar slider-crank

(3) Symbolically write out equations of motion for each moving link

For Link 4:

$$\sum \vec{F}_i = m_i \vec{A}_{CGi} \qquad \sum T_{i\,z} = I_{CGi}\alpha_i$$



$$\left.\begin{aligned} \alpha_4 &= 0 \\ A_{CG4y} &= 0 \\ R_{ij} &= 0 \end{aligned}\right\} \text{No torque balance necessary!}$$

Friction on slider:

$$F_{14x} = \pm\mu F_{14y} = -\operatorname{sign}(V_{CG4x})\,\mu F_{14y}$$

$$F_{14x} - F_{43x} = m_4 A_{CG4x} \rightarrow \pm\mu \boxed{F_{14y}} - F_{43x} = m_4 A_{CG4x}$$

$$\boxed{F_{14y}} - F_{43y} = 0$$

Add 1 unknown, and 2 more equations

→ 8 unknowns, 8 equations

(note: originally 9 unknowns, but can find F_{14x} through friction equation)

(4) Convert to matrix format [A] {B} = {C},

$$Link2: \quad \boxed{F_{12x}} + \boxed{F_{32x}} = m_2 A_{CG2x}$$

$$\boxed{F_{12y}} + \boxed{F_{32y}} = m_2 A_{CG2y}$$

$$\boxed{T_{12}} + \left(R_{12x}F_{12y} - R_{12y}F_{12x}\right) + \left(R_{32x}F_{32y} - R_{32y}F_{32x}\right) = I_{CG2}\alpha_2$$

$$Link3: \quad -F_{32x} + \boxed{F_{43x}} + F_{Px} = m_3 A_{CG3x}$$

$$-F_{32y} + \boxed{F_{43y}} + F_{Py} = m_3 A_{CG3y}$$

$$\left(-R_{23x}F_{32y} + R_{23y}F_{32x}\right) + \left(R_{43x}F_{43y} - R_{43y}F_{43x}\right)$$

$$+ \left(R_{Px}F_{Py} - R_{Py}F_{Px}\right) = I_{CG3}\alpha_3$$

$$Link4:$$

$$F_{14x} - F_{43x} = m_4 A_{CG4x} \rightarrow \pm\mu F_{14y} - F_{43x} = m_4 A_{CG4x}$$

$$\boxed{F_{14y}} - F_{43y} = 0$$

Recall: $F_{14x} = \pm\mu F_{14y} = -\operatorname{sign}(V_{4x})\,\mu F_{14y}$

Write out unknowns above [A] for bookkeeping

$$\begin{array}{cccccccc} F_{12x} & F_{12y} & F_{32x} & F_{32y} & F_{43x} & F_{43y} & F_{14x} & T_{12} \\ \end{array}$$

$$\begin{bmatrix} \; & \; & \; & \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; & \; & \; & \; \end{bmatrix} \begin{Bmatrix} F_{12x} \\ F_{12y} \\ F_{32x} \\ F_{32y} \\ F_{43x} \\ F_{43y} \\ F_{14y} \\ T_{12} \end{Bmatrix} = \begin{Bmatrix} \; \\ \; \\ \; \\ \; \\ \; \\ \; \\ \; \\ \; \end{Bmatrix}$$

(4) Convert to matrix format [A] {B} = {C},

$$Link2: \quad \boxed{F_{12x}} + \boxed{F_{32x}} = m_2 A_{CG2x}$$

$$\boxed{F_{12y}} + \boxed{F_{32y}} = m_2 A_{CG2y}$$

$$\boxed{T_{12}} + \left(R_{12x}F_{12y} - R_{12y}F_{12x}\right) + \left(R_{32x}F_{32y} - R_{32y}F_{32x}\right) = I_{CG2}\alpha_2$$

$$Link3: \quad -F_{32x} + \boxed{F_{43x}} + F_{Px} = m_3 A_{CG3x}$$

$$-F_{32y} + \boxed{F_{43y}} + F_{Py} = m_3 A_{CG3y}$$

$$\left(-R_{23x}F_{32y} + R_{23y}F_{32x}\right) + \left(R_{43x}F_{43y} - R_{43y}F_{43x}\right) + \left(R_{Px}F_{Py} - R_{Py}F_{Px}\right) = I_{CG3}\alpha_3$$

$$Link4:$$

$$F_{14x} - F_{43x} = m_4 A_{CG4x} \rightarrow \pm\mu F_{14y} - F_{43x} = m_4 A_{CG4x}$$

$$\boxed{F_{14y}} + F_{43y} = 0$$

Recall: $F_{14x} = \pm\mu F_{14y} = -\,\mathrm{sign}(V_{4x})\,\mu F_{14y}$

Write out unknowns above [A] for bookkeeping

$$\begin{array}{c}
\begin{array}{cccccccc} F_{12x} & F_{12y} & F_{32x} & F_{32y} & F_{43x} & F_{43y} & F_{14x} & T_{12} \end{array} \\
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
-R_{12y} & R_{12x} & -R_{32y} & R_{32x} & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & R_{23y} & -R_{23x} & -R_{43y} & R_{43x} & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & \pm\mu & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0
\end{bmatrix}
\end{array}
\begin{Bmatrix} F_{12x} \\ F_{12y} \\ F_{32x} \\ F_{32y} \\ F_{43x} \\ F_{43y} \\ F_{14y} \\ T_{12} \end{Bmatrix}
=
\begin{Bmatrix} m_2 A_{CG2x} \\ m_2 A_{CG2y} \\ I_{CG2}\alpha_2 \\ m_3 A_{CG3x} - F_{Px} \\ m_3 A_{CG3y} - F_{Py} \\ I_{CG3}\alpha_3 - R_{Px}F_{Py} + R_{Py}F_{Px} \\ m_4 A_{CG4x} \\ 0 \end{Bmatrix}$$

Fourbar slider-crank

(5) Insert known/given values for variables in **[A]** *&* **{C}**.

(6) Solve for unknown forces and torques in **{B}** *(typically internal joint forces and torques) using* **{B}** = **[A]**$^{-1}$ **{C}**.