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Homework 01

Due Tuesday, September 2, 11:59pm

STAT 400, Fall 2025, D. Unger

Exercise 1

A fair coin is tossed four times, and the sequence of heads and tails is observed.

(a) List each of the 16 sequences in the sample space S.

(a) HHHH HTHH THHH TTHH
HHHT HTTH THHT TTTH
HHTH HTTT THTH TTTT
HHTT HTHT THTT TTHT

(b) What is the probability of observing the specific outcome of {H, T, H, T} after tossing the fair coin four times.

(b) $\frac{1}{16}$

(c) Let events A, B, C, and D be given by A = {at least 3 heads}, B = {at most 2 heads},

C = {heads on the third toss}, and D = {1 head and 3 tails}.

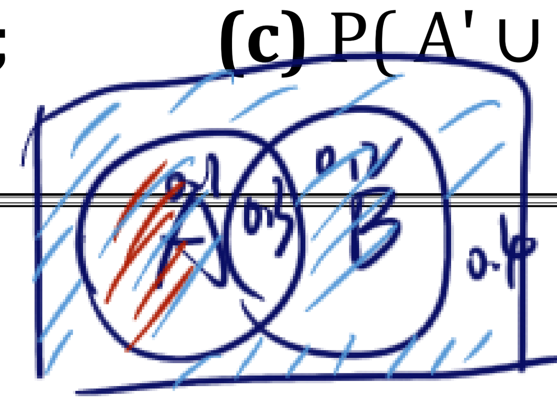
(i) P(A) HHHH HTHH HHHT THHH HHTH $\frac{5}{16}$ 0.0625 (ii) P(B) 1 head 0 head 2 head 4 1 6 $= \frac{11}{16} = 0.6875$ (iii) P(A ∩ B) $= 0$ (iv) P(C) $= \frac{1}{2} = 0.5$

(iv) P(A ∩ C) $= \frac{1}{4} = 0.25$ four in A only not HHTH (v) P(D) $= \frac{4}{16} = \frac{1}{4} = 0.25$ (vi) P(A ∪ C) $= \frac{9}{16} = 0.5625$ all of C and HHTH (vii) P(B ∩ D) $= \frac{1}{4} = 0.25$ all D

Exercise 2

If P(A) = 0.4, P(B) = 0.5, and P(A ∩ B) = 0.3, find ...

(a) P(A ∪ B); **(b)** P(A ∩ B'); **(c)** P(A' ∪ B').



2 (a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 0.4 + 0.5 - 0.3$
$= 0.6$

(b) $P(A \cap B')$ is red one left
so $= 0.1$

(c) $P(A' \cup B') = 0.7$
is the light blue one.

Exercise 3

Suppose

P(A′) = 0.70, P(B) = 0.60, P(C) = 0.40,

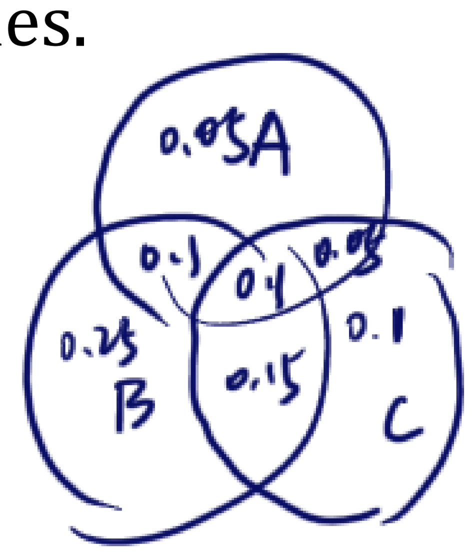
P(A′ ∪ B′) = 0.80, P(A ∩ C) = 0.15, P(B ∩ C) = 0.25, and P(A ∩ B ∩ C) = 0.10.

Find the following probabilities.

(a) P(A ∪ C)

(b) P(A ∩ B)

(c) P(A ∪ B ∪ C)



$\because P(A' \cup B') = 0.8$
$\therefore P(A \cap B) = 1 - P(A' \cup B') = 0.2$
$\because P(A') = 0.7$
$\therefore P(A) = 1 - P(A') = 0.3.$

(a) $P(A \cup C) = P(A) + P(C) - P(A \cap C) = 0.3 + 0.4 - 0.15 = 0.55$

(b) $P(A \cap B) = 1 - P(A' \cup B') = 0.2$

(c) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) = 0.3 + 0.6 + 0.4 - 0.15 - 0.25 - 0.2 + 0.1$
$= 0.8$

4 lab : A referral : B

$P(A'\cap B') = 0.21$

$P(A) = 0.41 \quad P(B) = 0.53$

$P(A\cap B) = P(A'\cap B') + P(A) + P(B) - 1$

$= 0.41 + 0.53 + 0.21 - 1$

$= 0.15$

Exercise 4

During a visit to a primary care physician's office, the probability of having neither lab work nor referral to a specialist is 0.21. Of those coming to that office, the probability of having lab work is 0.41 and the probability of having a referral is 0.53. What is the probability of having both lab work and a referral?

Exercise 5

Find the value of p that would make this a valid probability model.

(a) Suppose S = {0, 2, 4, 6, 8, ... } (i.e, even non-negative integers) and

$$P(0) = p, \qquad P(k) = \frac{1}{3^k}, \qquad k = 2, 4, 6, 8, \ldots .$$

(b) Suppose S = {1, 2, 3, 4, ...} (i.e., positive integers) and

$$P(1) = p, \qquad P(k) = \frac{(\ln 3)^k}{k!}, \qquad k = 2, 3, 4, \ldots .$$

(a) $\Sigma P(k) = \frac{1}{3^2} + \frac{1}{3^4} + \cdots$

$= \frac{1}{3^2} \times \left(\frac{1-0}{1-\frac{1}{3^2}}\right)$

$= \frac{1}{8}$

$\therefore p = 1 - \Sigma P(k) = \frac{7}{8} = 0.875$

(b) $\because$ Maclawin Function $\therefore \sum_{k=0}^{\infty} \frac{(\ln 3)^k}{k!} = e^{\ln 3} = 3$

$\therefore p = 1 - \sum_{k=2}^{\infty} \frac{(\ln 3)^k}{k!}$

$= 1 + \frac{(\ln 3)^0}{0!} + \frac{(\ln 3)^1}{1!} - \sum_{k=0}^{\infty} \frac{(\ln 3)^k}{k!}$

$= \ln 3 - 1 = 0.0986$

Exercise 6

Suppose S = {0, 1, 2, 3, ...} and P(k) = $\frac{1/3}{(3/2)^k}$, for $k \in$ S.

(a) Find P[{2}]

(b) Find P[outcome is less than 2]

(c) Find P[outcome is greater than 2]

(d) Prove that P[S] = 1. In other words, show that this probability function obeys the second property of the definition of probability.

(a) $P[\{2\}] = \frac{\frac{1}{3}}{\left(\frac{3}{2}\right)^2} = \frac{4}{9\times 3} = \frac{4}{27} = 0.148$

(b) P [out less than 2] $= \frac{\frac{1}{3}}{\left(\frac{3}{2}\right)^0} + \frac{\frac{1}{3}}{\left(\frac{3}{2}\right)^1} = \frac{2}{9} + \frac{1}{3} = \frac{5}{9} = 0.556$

(c) P[out larger than 2] $= \frac{\frac{1}{3}}{\left(\frac{3}{2}\right)^3} + \frac{\frac{1}{3}}{\left(\frac{3}{2}\right)^4} + \cdots$

$= \left(\frac{2}{3}\right)^3 \frac{1}{3} \left(\frac{1-0}{1-\frac{2}{3}}\right)$

$= \frac{8}{81} \times 3$

$= \frac{8}{27} = 0.296$

(d) we can add up previous three

$P_{total} = \frac{4}{27} + \frac{5}{9} + \frac{8}{27} = \frac{12}{27} + \frac{5}{9} = 1$

so it obeys the second property.

Exercise 7

Suppose S = {1, 2, 3,...} and P(k) = $c \cdot \frac{1}{\pi^{2k}}$ for k = 1, 2, 3, ...

(a) Find the value of c that makes this a valid probability distribution.

(b) Find P(outcome is even).

(a) $\Sigma P(k) = c\cdot\left(\frac{1}{\pi^2} + \frac{1}{\pi^4} + \frac{1}{\pi^6} + \cdots\right)$

$= \frac{c}{\pi^2}\left(\frac{1-0}{1-\frac{1}{\pi^2}}\right) = \frac{c}{\pi^2 - 1}$

$\therefore c = \pi^2 - 1$

$= 8.870$

(b) P(out even)

$= P(2) + P(4) + \cdots$

$= c\cdot\left(\frac{1}{\pi^4} + \frac{1}{\pi^8} + \frac{1}{\pi^{12}} + \cdots\right)$

$= \frac{c}{\pi^4 - 1}$

when c is $\pi^2 - 1$

then out is $\frac{\pi^2-1}{\pi^4-1} = \frac{1}{\pi^2+1} = 0.0920$

Exercise 8

Suppose S = {1, 2, 3, …} and P(k) = $c \cdot \frac{3^k}{k!}$, for k = 1, 2, 3, … .

(a) Find the value of c that makes this is a valid probability distribution.

(b) Find P(outcome is greater than 3).

(a) $\Sigma P(k) = c \cdot \sum_{1}^{\infty} \frac{3^k}{k!}$

$= c \cdot \left(e^3 - \frac{3^0}{0!} \right)$

$= c \cdot (e^3 - 1)$

$\therefore c = \frac{1}{e^3 - 1}$

$= 0.0524$

(b) $\Sigma P(\text{out} > 3)$

$= c \cdot \left(\sum_{4}^{\infty} \frac{3^k}{k!} \right)$

$= c \left(e^3 - \frac{3^3}{3!} - \frac{3^2}{2!} - \frac{3^1}{1!} - \frac{3^0}{0!} \right)$

$= c \cdot \left(e^3 - \frac{27}{6} - \frac{9}{2} - 3 - 1 \right)$

$= c \cdot (e^3 - 13)$

as $c = \frac{1}{e^3 - 1}$

so $\Sigma P = \frac{e^3 - 13}{e^3 - 1} = 0.371$