Degradation modeling under multiple failure modes

This is a statistical method.

1. Model Framework

We assume there are K failure modes. Under failure mode k, with $k=1,\dots,K$, we define the failure time T_l of unit l as the time when the underlying degradation status $g_l(t)$ exceeds a predefined failure threshold D_k.

$$T_l = \arg\min_t \; g_l(t) > D_k \qquad \text{Eq. (1)}$$

① Modeling health index (HI): $y_{lk}(t) = \psi(x_l(t))\, w_k$. — a fusion model

$y_{lk}(t)$: HI of unit l at time t under failure mode k

$x_l(t)$: the vector of sensor signals $x_l(t) = [x_{l1}(t), \dots, x_{lm}(t)] \in R^{1\times m}$

$\psi(\cdot)$: a basis function, e.g. linear, kernel, etc.

w_k : fusion coefficient under failure mode k, $w_k \in R^{m\times 1}$

② Modeling degradation status: $g_{lk}(t) = \phi(t)\Gamma_{lk}$

$g_{lk}(t)$: degradation status of unit l at time t under failure mode k

$\phi(t)$: basis function in terms of time t, e.g, quadratic $\phi(t) = [1, t, t^2]$, $p=3$
$\phi(t) \in R^{1\times p}$

Γ_{lk}: degradation parameter of unit l under failure mode k, $\Gamma_{lk} \in R^{p\times 1}$

Failure mode: $Z_l \sim \text{Multinomial}(\pi_1, \dots, \pi_K)$

Degradation parameter: $\forall l$, $\Gamma_{lk} \equiv \Gamma_l \mid Z_l = k \sim N(\mu_k, \Sigma_k)$, $k=1,\dots,K$

Noise term: $\varepsilon_{lk}(t) \equiv \varepsilon_l(t) \mid Z_l = k \sim N(0, \sigma_k^2)$

③ The relationship between HI and degradation status

$$y_{lk}(t) = \psi(x_l(t))\, w_k = g_{lk}(t) + \varepsilon_{lk}(t)$$
$$= \phi(t)\Gamma_{lk} + \varepsilon_{lk}(t)$$

2. Parameter Estimation

Matrix form: $y_{\ell k} = \psi(x_\ell) w_k = \Phi_\ell \Gamma_{\ell k} + \varepsilon_{\ell k}$

Historical units: sensor signals until failure. (totally n_ℓ time points)

(known data) failure time t_{n_ℓ}

Note: ① the failure modes of historical units are unknown

② The failure threshold D_k is unknown, but we set arbitrary value of D_k, e.g. $D_k = 1$

EM Algorithm:

Complete data likelihood:

$$L(y, z, \Gamma) = p(y \mid \Gamma, z)\, p(\Gamma \mid z)\, p(z)$$

$$= \prod_{\ell=1}^{L} p(y_\ell \mid \Gamma_\ell, z_\ell)\, p(\Gamma_\ell \mid z_\ell)\, p(z_\ell)$$

Define: $\rho_{\ell k} = \begin{cases} 1, & \text{if } z_\ell = k \\ 0 & \text{if } z_\ell \neq k \end{cases}$

$p(z_\ell)$: $\quad p(z_\ell; \tau) = \prod_{k=1}^{K} (\tau_k)^{\rho_{\ell k}}$

$p(\Gamma_\ell \mid z_\ell)$: $\quad p(\Gamma_\ell \mid z_\ell; \mu_k, \Sigma_k) = \prod_{k=1}^{K} \left[(2\pi)^{-\frac{p}{2}} |\Sigma_k|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} (\Gamma_\ell - \mu_k)^T \Sigma_k^{-1} (\Gamma_\ell - \mu_k) \right\} \right]^{\rho_{\ell k}}$

$p(y_\ell \mid \Gamma_\ell, z_\ell)$: $\quad p(y_\ell \mid \Gamma_\ell, z_\ell; \sigma_k^2) = \prod_{k=1}^{K} \left[(2\pi\sigma_k^2)^{-\frac{n_\ell}{2}} \exp\left\{ -\frac{1}{2\sigma_k^2} (y_\ell - \Phi_\ell \Gamma_\ell)^T (y_\ell - \Phi_\ell \Gamma_\ell) \right\} \right]^{\rho_{\ell k}}$

Parameters to be estimated: $\Theta = \begin{cases} w_k \\ \lambda_k \\ u_k, \Sigma_k, \sigma_k^2 \end{cases}$

z_l, Γ_l are also unknown. Θ cannot be estimated directly. Therefore, EM algorithm is used, where z_l, Γ_l are taken as latent variables.

E step: Given x_l, and $w_k, \lambda_k, u_k, \Sigma_k, \sigma_k^2$ at last iteration. Then we have $y_l = \varphi(x_l) w_k$.

We calculate the posterior distribution of z_l and Γ_l.

$$p(z_l = k \mid y_l) = \frac{p(y_l \mid z_l = k)\, p(z_l = k)}{\sum_{k=1}^{K} p(y_l \mid z_l = k)\, p(z_l = k)}$$

here: $p(z_l = k) = \lambda_k$

$$y_l \mid z_l = k \sim \mathcal{N}\left(\phi_l u_k,\ \phi_l u_k \phi_l^T + \sigma_k^2 I\right)$$

Thus: $\hat{\rho}_{lk} = p(z_l = k \mid y_l) = \dfrac{\lambda_k \mathcal{N}(y_l \mid \phi_l u_k, \phi_l u_k \phi_l^T + \sigma_k^2 I)}{\sum_{k=1}^{K} \lambda_k \mathcal{N}(y_l \mid \phi_l u_k, \phi_l u_k \phi_l^T + \sigma_k^2 I)}$

$$p(\Gamma_l \mid z_l = k, y_l) = \frac{p(y_l \mid \Gamma_l, z_l = k)\, p(\Gamma_l \mid z_l = k)}{\sum_{k=1}^{K} p(y_l \mid \Gamma_l, z_l = k)\, p(\Gamma_l \mid z_l = k)}$$

$$y_{lk} = \Phi_l \Gamma_{lk} + \varepsilon_{lk}$$

$\Gamma_{lk} \sim N_3(\mu_k, \Sigma_k)$ $\quad$ $\varepsilon_{lk} \sim N(0, \sigma_k^2)$

Some prob:

$$\begin{array}{c} 1\times 3 \quad 3\times 1 \qquad 1\times 3 \quad 3\times 3 \quad 3\times 1 \end{array}$$

$$Y_l(t) \mid Z_l = k \sim N\left(\Phi_l(t)\cdot\mu_k,\ \Phi_l(t)\Sigma_k\Phi_l(t) + \sigma_k^2\right)$$

$$P(Y_l \mid Z_l = k) = \prod_{t=1}^{n_l} P(Y_l(t) \mid Z_l = k)$$

$$=$$

$$Y_l \mid \Gamma_l, Z_l = k \sim N(\Phi_l \Gamma_{lk} + \sigma_k^2 I)$$

here: $p(\Gamma_l | z_l = k)$: $\Gamma_{lk} \equiv \Gamma_l | z_l = k \sim N(\mu_k, \Sigma_k)$

$p(y_l | \Gamma_l, z_l = k)$ $\quad y_l | \Gamma_l, z_l = k \sim N(\phi_l \Gamma_{lk}, \sigma_k^2 I)$

Thus: $\Gamma_l | z_l = k, y_l \sim N\left(\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l + \Sigma_k^{-1}\right)^{-1}\left(\frac{1}{\sigma_k^2}\phi_l^T y_l + \Sigma_k^{-1}\mu_k\right), \left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l + \Sigma_k^{-1}\right)^{-1} \right)$

We obtain: $\hat{\Gamma}_{lk} = \left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l + \Sigma_k^{-1}\right)^{-1}\left(\frac{1}{\sigma_k^2}\phi_l^T y_l + \Sigma_k^{-1}\mu_k\right)$

$$\hat{\Sigma}_{lk} = \left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l + \Sigma_k^{-1}\right)^{-1}$$

M step: maximize $Q(\Theta | \Theta_j)$, update parameters

$$\hat{\pi}_k = \frac{N_k}{N}, \quad N_k = \sum_{l=1}^{L} \hat{\rho}_{lk} \quad N = \sum_{k=1}^{K} N_k$$

$$\hat{\mu}_k = \frac{1}{N_k}\sum_{l=1}^{L} \hat{\rho}_{lk} E(\Gamma_l | z_l = k, y_l)$$

$$= \frac{1}{N_k}\sum_{l=1}^{L} \hat{\rho}_{lk} \hat{\Gamma}_{lk}$$

$$\hat{\Sigma}_k = \frac{1}{N_k}\sum_{l=1}^{L} \hat{\rho}_{lk} E\left[(\Gamma_l - \mu_k)(\Gamma_l - \mu_k)^T \middle| z_l = k, y_l\right]$$

$$= \frac{1}{N_k}\sum_{l=1}^{L} \hat{\rho}_{lk}\left[\hat{\Sigma}_{lk} + (\hat{\Gamma}_{lk} - \hat{\mu}_k)^T(\hat{\Gamma}_{lk} - \hat{\mu}_k)\right]$$

$$\hat{\sigma}_k^2 = \frac{1}{N_k}\sum_{l=1}^{L} \hat{\rho}_{lk} E\left[(y_l - \phi_l\Gamma_l)^T(y_l - \phi_l\Gamma_l) \middle| z_l = k, y_l\right]$$

$$= \frac{1}{N_k}\sum_{l=1}^{L} \hat{\rho}_{lk}\left[\|x_l\hat{w}_k - \phi_l\hat{\Gamma}_{lk}\|^2 + Tr(\phi_l \hat{\Sigma}_{lk} \phi_l)\right]$$

$$\Longrightarrow \mathbb{E}\left(\underline{\psi}(x_i)\, w \mid z_i = k\right)$$

To update w_k:

At first, we regard w_k as known, and obtain the estimate of Γ_ℓ under failure mode k from its posterior distribution.

$$\hat{\Gamma}_{\ell k} = \left(\frac{1}{\sigma_k^2}\phi_\ell^T\phi_\ell + \Sigma_k^{-1}\right)^{-1}\left(\frac{1}{\sigma_k^2}\phi_\ell^T\psi(x_\ell)w_k + \Sigma_k^{-1}u_k\right)$$

Since $\psi(x_\ell)w_k$ is normally distributed given $z_\ell = k$, i.e.

$$\psi(x_\ell)w \mid z_\ell = k \sim N\left(\phi_\ell u_k,\ \phi_\ell\Sigma_k\phi_\ell^T + \sigma_k^2 I\right),$$

we have $\hat{\Gamma}_{\ell k} \mid z_\ell = k \sim N\left(\underline{E(\hat{\Gamma}_{\ell k} \mid z_\ell = k),\ Var(\hat{\Gamma}_{\ell k} \mid z_\ell = k)}\right)$

$$E(\hat{\Gamma}_{\ell k} \mid z_\ell = k) = \left(\frac{1}{\sigma_k^2}\phi_\ell^T\phi_\ell + \Sigma_k^{-1}\right)^{-1}\left(\frac{1}{\sigma_k^2}\phi_\ell^T \underline{E(\psi(x_\ell)w|_{z_\ell=k})} + \Sigma_k^{-1}u_k\right)$$

$$= \left(\frac{1}{\sigma_k^2}\phi_\ell^T\phi_\ell + \Sigma_k^{-1}\right)^{-1}\left(\frac{1}{\sigma_k^2}\phi_\ell^T\underline{\phi_\ell u_k} + \Sigma_k^{-1}u_k\right) = u_k$$

$$Var(\hat{\Gamma}_{\ell k} \mid z_\ell = k) = \boxed{\left(\frac{1}{\sigma_k^2}\phi_\ell^T\phi_\ell + \Sigma_k^{-1}\right)^{-1}\frac{1}{\sigma_k^2}\phi_\ell^T}\ \underline{Var(\psi(x_\ell)w \mid z_\ell = k)}\left(\left(\frac{1}{\sigma_k^2}\phi_\ell^T\phi_\ell + \Sigma_k^{-1}\right)^{-1}\frac{1}{\sigma_k^2}\phi_\ell^T\right)^T$$

$$= \left(\frac{1}{\sigma_k^2}\phi_\ell^T\phi_\ell + \Sigma_k^{-1}\right)^{-1}\frac{1}{\sigma_k^2}\phi_\ell^T\underline{\left(\phi_\ell\Sigma_k\phi_\ell^T + \sigma_k^2 I\right)}\frac{1}{\sigma_k^2}\phi_\ell\left(\frac{1}{\sigma_k^2}\phi_\ell^T\phi_\ell + \Sigma_k^{-1}\right)^{-1}$$

$$= \left(\frac{1}{\sigma_k^2}\phi_\ell^T\phi_\ell + \Sigma_k^{-1}\right)^{-1}\left(\frac{1}{\sigma_k^2}\phi_\ell^T\phi_\ell\Sigma_k\phi_\ell^T\phi_\ell + \phi_\ell^T\phi_\ell\right)\frac{1}{\sigma_k^2}\left(\frac{1}{\sigma_k^2}\phi_\ell^T\phi_\ell + \Sigma_k^{-1}\right)^{-1}$$

$$= \frac{1}{\sigma_k^2}\left(\frac{1}{\sigma_k^2}\phi_\ell^T\phi_\ell + \Sigma_k^{-1}\right)^{-1}\left(\frac{1}{\sigma_k^2}\phi_\ell^T\phi_\ell + \Sigma_k^{-1}\right)\Sigma_k\phi_\ell^T\phi_\ell\left(\frac{1}{\sigma_k^2}\phi_\ell^T\phi_\ell + \Sigma_k^{-1}\right)$$

$$= \frac{1}{\sigma_k^2}\Sigma_k\phi_\ell^T\phi_\ell\left(\frac{1}{\sigma_k^2}\phi_\ell^T\phi_\ell + \Sigma_k^{-1}\right)^{-1}$$

At failure time: we have $\phi(T_\ell)u_k = D_k$ according to Eq. (1).

$T_\ell = \arg\min_t g_\ell(t) > D_k$

$E(g_\ell(t) - D_k) = 0$

$\phi(T_\ell)\hat{\Gamma}_{\ell k}|_{z_\ell = k}$ also follows a multivariate normal distribution

$$E\left(\phi(T_\ell)\hat{\Gamma}_{\ell k}|_{z_\ell=k}\right) = \phi(T_\ell)E(\hat{\Gamma}_{\ell k}) = \phi(T_\ell)u_k = D_k$$

$$E(\hat{T}_{1k} \mid Z_1 = k) = \mu_k$$

$$\sum \phi(\tau_1) \underbrace{E(\hat{T}_{1k} \mid Z_1 = k)}_{\mu_k} = D_k$$

$$Var(\phi(\tau_l)\hat{\Gamma}_{lk} \mid z_l=k) = \phi(\tau_l)\, Var(\phi(\tau_l)\hat{\Gamma}_{lk} \mid z_l=k)\, \phi^T(\tau_l)$$

$$= \frac{1}{\sigma_k^2}\phi(\tau_l)\left(\Sigma_k \phi_l^T\phi_l\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l + \Sigma_k^{-1}\right)^{-1}\right)\phi^T(\tau_l)$$

$$\phi(\tau_l)\hat{\Gamma}_{lk} \mid z_l=k \sim \mathcal{N}\Big(D_k,\ \underbrace{\frac{1}{\sigma_k^2}\phi(\tau_l)\left(\Sigma_k\phi_l^T\phi_l\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l+\Sigma_k^{-1}\right)^{-1}\right)\phi^T(\tau_l)}_{\sigma_{\tau_l}^2}\Big)$$

Therefore, the likelihood function

$$\mathcal{L}_w = \prod_{l=1}^{L} p(\phi(\tau_l)\hat{\Gamma}_{lk} \mid z_l=k) = (2\pi\sigma_{\tau_l}^2)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma_{\tau_l}^2}\left(\phi(\tau_l)\hat{\Gamma}_{lk} - \mu_k\right)^2\right\}$$

$\rightarrow$ units

The log-likelihood function:

$$\log \mathcal{L}_w = \sum_{l=1}^{L} -\frac{1}{2}\log(2\pi\sigma_{\tau_l}^2) - \frac{\Phi(\tau_l)}{2\sigma_{\tau_l}^2}\Big(\underbrace{\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l+\Sigma_k^{-1}\right)^{-1}\left(\frac{1}{\sigma_k^2}\phi_l^T\psi(x_l)w_k + \Sigma_k^{-1}\mu_k\right) - D_k}_{\Phi}\Big)^2.$$

Since $\log(2\pi\sigma_{\tau_l}^2)$ is not related to w, we can estimate w_k by minimizing $-2\log\mathcal{L}_w - \sum_{l=1}^{L}\log(2\pi\sigma_{\tau_l}^2)$, i.e.

$$\hat{w}_k = \arg\min_{w_k} \sum_{l=1}^{L} \frac{\left(\phi(\tau_l)\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l+\Sigma_k^{-1}\right)^{-1}\left(\frac{1}{\sigma_k^2}\phi_l^T\psi(x_l)w_k+\Sigma_k^{-1}\mu_k\right)-D_k\right)^2}{\frac{1}{\sigma_k^2}\phi(\tau_l)\left(\Sigma_k\phi_l^T\phi_l\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l+\Sigma_k^{-1}\right)^{-1}\right)\phi^T(\tau_l)}$$

$$= \arg\min_{w_k} \mathcal{L}_w$$

$$\frac{\partial \mathcal{L}_w}{\partial w_k} = \sum_{l=1}^{L}\frac{2}{\sigma_{\tau_l}^2}\left(\phi(\tau_l)\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l+\Sigma_k^{-1}\right)^{-1}\left(\frac{1}{\sigma_k^2}\phi_l^T\psi(x_l)w_k+\Sigma_k^{-1}\mu_k\right)-D_k\right)$$

$$\cdot\underbrace{\left[\phi(\tau_l)\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l+\Sigma_k^{-1}\right)^{-1}\frac{1}{\sigma_k^2}\phi_l^T\psi(x_l)\right]}_{a_l} = 0$$

Let $a_1 = \phi(\tau_l)\left(\frac{1}{\sigma_K^2}\phi_l^T\phi_l + \Sigma_K^{-1}\right)^{-1}\frac{1}{\sigma_K^2}\phi_l^T\psi(x_l)$

$$\sum_{l=1}^{L}\frac{1}{\sigma_{\tau_l}^2}\, a_1^T\left(a_1 w_K + \phi(\tau_l)\left(\frac{1}{\sigma_K^2}\phi_l^T\phi_l + \Sigma_K^{-1}\right)^{-1}\Sigma_K^{-1}u_K - D_K\right) = 0$$

Let $b_1 = D_K - \phi(\tau_l)\left(\frac{1}{\sigma_K^2}\phi_l^T\phi_l + \Sigma_K^{-1}\right)^{-1}\Sigma_K^{-1}u_K$

$$\sum_{l=1}^{L}\frac{1}{\sigma_{\tau_l}^2}\, a_1^T\left(a_1 w_K - b_1\right) = 0$$

Let $a_2 = \dfrac{a_1}{\frac{1}{\sigma_K}\sqrt{\sigma_{\tau_l}^2}} = \dfrac{\phi(\tau_l)\left(\frac{1}{\sigma_K^2}\phi_l^T\phi_l + \Sigma_K^{-1}\right)^{-1}\frac{1}{\sigma_K^2}\phi_l^T\psi(x_l)}{\frac{1}{\sigma_K}\sqrt{\frac{1}{\sigma_K^2}\phi(\tau_l)\,\Sigma_K\phi_l^T\phi_l\left(\frac{1}{\sigma_K^2}\phi_l^T\phi_l + \Sigma_K^{-1}\right)^{-1}\phi^T(\tau_l)}}$

?

$$= \frac{\phi(\tau_l)\left(\frac{1}{\sigma_K^2}\phi_l^T\phi_l + \Sigma_K^{-1}\right)^{-1}\phi_l^T\psi(x_l)}{\sqrt{\phi(\tau_l)\,\Sigma_K\phi_l^T\phi_l\left(\frac{1}{\sigma_K^2}\phi_l^T\phi_l + \Sigma_K^{-1}\right)^{-1}\phi^T(\tau_l)}}$$

$$b_2 = \frac{b_1}{\frac{1}{\sigma_K}\sqrt{\sigma_{\tau_l}^2}} = \frac{\sigma_K^2\left(D_K - \phi(\tau_l)\left(\frac{1}{\sigma_K^2}\phi_l^T\phi_l + \Sigma_K^{-1}\right)^{-1}\Sigma_K^{-1}u_K\right)}{\sqrt{\phi(\tau_l)\,\Sigma_K\phi_l^T\phi_l\left(\frac{1}{\sigma_K^2}\phi_l^T\phi_l + \Sigma_K^{-1}\right)^{-1}\phi^T(\tau_l)}}$$

We have $\sum_{l=1}^{L} a_2^T a_2 w_K - a_2^T b_2 = 0$

Then $\hat{w}_K = \left(\sum_{l=1}^{L} a_2^T a_2\right)^{-1}\sum_{l=1}^{L} a_2^T b_2$

3. Failure Mode Recognition & RUL Prediction

① Failure mode:

$$p(z_l = k \mid y_l) = \frac{p(y_l \mid z_l = k)\, p(z_l = k)}{\sum_{k=1}^{K} p(y_l \mid z_l = k)\, p(z_l = k)}$$

$$y_l = \psi(x_l) w_k$$

$$= \frac{\pi_k \mathcal{N}(y_l \mid \phi_l u_k, \phi_l u_k \phi_l^T + \sigma_k^2 I)}{\sum_{k=1}^{K} \pi_k \mathcal{N}(y_l \mid \phi_l u_k, \phi_l u_k \phi_l^T + \sigma_k^2 I)}$$

② RUL:

$$\Gamma_l \mid z_l = k, y_l \sim \mathcal{N}\left(\left(\frac{1}{\sigma_k^2} \phi_l^T \phi_l + \Sigma_k^{-1} \right)^{-1} \left(\frac{1}{\sigma_k^2} \phi_l^T y_l + \Sigma_k^{-1} u_k \right), \left(\frac{1}{\sigma_k^2} \phi_l^T \phi_l + \Sigma_k^{-1} \right)^{-1} \right)$$

For an in-service unit l,

$$g_l(t_{nl} + t) = \phi(t_{nl} + t) \Gamma_l$$

$$g_l(t_{nl} + t) \mid_{z_l = k, y_l} \sim \mathcal{N}(u_{t_{nl}+t}, \Sigma_{t_{nl}+t})$$

$$u_{t_{nl}+t} = \phi(t_{nl}+t) \left(\frac{1}{\sigma_k^2} \phi_l^T \phi_l + \Sigma_k^{-1} \right)^{-1} \left(\frac{1}{\sigma_k^2} \phi_l^T \psi(x_l) w_k + \Sigma_k^{-1} u_k \right)$$

$$\Sigma_{t_{nl}+t} = \phi(t_{nl}+t) \left(\frac{1}{\sigma_k^2} \phi_l^T \phi_l + \Sigma_k^{-1} \right)^{-1} \phi^T(t_{nl}+t)$$

The probability that the degradation status exceeds the failure threshold at time t after t_{nl} is

$$F_{T_l|y_l}(t) = P(T_l \le t \mid y_l) = P(g_l(t_{nl}+t) > D \mid y_l)$$

$$= \sum_{k=1}^{K} P(g_l(t_{nl}+t) > D_k \mid y_l, z_l = k)\, P(z_l = k \mid y_l)$$

$$P(T_l \le t \mid T_l \ge 0, y_l) = \frac{P(0 \le T_l \le t \mid y_l)}{P(T_l \ge 0 \mid y_l)}$$

$$= \frac{F_{T_l|y_l}(t) - F_{T_l|y_l}(0)}{1 - F_{T_l|y_l}(0)}$$