Degradation modeling under multiple failure modes

This is a statistical method.

④ Model degradation status of each sensor signal $x_{lm}(t)$

$$x_{lm}(t) = g_{lmk}(t) + \varepsilon_{lmk}(t)$$
$$m=1,\dots,M \qquad = \phi(t)\Gamma_{lmk} + \varepsilon_{lmk}(t)$$
$$\Gamma_{lmk} \equiv \Gamma_{lm} \mid Z_l = k \sim N(u_{mk}, \Sigma_{mk})$$
$$\varepsilon_{lmk}(t) \equiv \varepsilon_{lm}(t) \mid Z_l = k \sim N(0, \sigma^2_{mk})$$

1. Model Framework

We assume there are K failure modes. Under failure mode k, with $k=1,\dots,K$, we define the failure time T_l of unit l as the time when the underlying degradation status $g_l(t)$ exceeds a predefined failure threshold D_k.

$$T_l = \arg\min_t \; g_l(t) > D_k \qquad \text{Eq. (1)}$$

① Modeling health index (HI): $y_{lk}(t) = \psi(x_l(t))\, w_k$ — a fusion model

$y_{lk}(t)$: HI of unit l at time t under failure mode k

$x_l(t)$: the vector of sensor signals $x_l(t) = [x_{l1}(t), \dots, x_{lM}(t)] \in R^{1\times M}$

$\psi(\cdot)$: a basis function, e.g. linear, kernel, etc.

w_k : fusion coefficient under failure mode k, $w_k \in R^{m\times 1}$

② Modeling degradation status of HI: $g_{lk}(t) = \phi(t)\Gamma_{lk}$

$g_{lk}(t)$: degradation status of unit l at time t under failure mode k

$\phi(t)$: basis function in terms of time t, e.g., quadratic $\phi(t) = [1, t, t^2]$, $p=3$
$\phi(t) \in R^{1\times p}$

Γ_{lk} : degradation parameter of unit l under failure mode k, $\Gamma_{lk} \in R^{p\times 1}$

Failure mode: $Z_l \sim \text{Multinomial}(\pi_1, \dots, \pi_K)$

Degradation parameter: $\forall l,\ \Gamma_{lk} \equiv \Gamma_l \mid Z_l = k \sim N(u_k, \Sigma_k)$, $k=1,\dots,K$

Noise term: $\varepsilon_{lk}(t) \equiv \varepsilon_l(t) \mid Z_l = k \sim N(0, \sigma_k^2)$

③ The relationship between HI and degradation status

$$y_{lk}(t) = \psi(x_l(t))\, w_k = g_{lk}(t) + \varepsilon_{lk}(t)$$
$$= \phi(t)\Gamma_{lk} + \varepsilon_{lk}(t)$$

2. Parameter Estimation

Matrix form: $y_{lk} = \psi(x_l) w_k = \Phi_l \Gamma_{lyk} + \varepsilon_{lk}$

Historical units: sensor signals until failure. (totally n_l time points)

(known data) failure time t_{n_l}

Note: ① the failure modes of historical units are unknown

② The failure threshold D_k is unknown, but we set arbitrary value of D_k, e.g. $D_k = 1$

EM Algorithm:

Complete data likelihood:

$$L(\cancel{y}, Z, \Gamma) = p(\cancel{y} \mid \Gamma, Z)\, p(\Gamma \mid Z)\, p(Z)$$

$$= \prod_{l=1}^{L} p(\cancel{y_l}\, x_l \mid \Gamma_l, Z_l)\, p(\Gamma_l \mid Z_l)\, p(Z_l)$$

Define: $\rho_{lk} = \begin{cases} 1, & \text{if } Z_l = k \\ 0 & \text{if } Z_l \neq k \end{cases}$

① $p(Z_l)$: $\quad p(Z_l; \tau) = \prod_{k=1}^{K} (\tau_k)^{\rho_{lk}}$

② $p(\Gamma_l \mid Z_l)$: $p(\Gamma_{l1}, \dots, \Gamma_{lm}, \Gamma_{ly} \mid Z_l) = \prod_{k=1}^{K} \left(\prod_{m=1}^{M} p(\Gamma_{lm} \mid Z_l = k; \mu_{mk}, \Sigma_{mk}) \cdot p(\Gamma_{ly} \mid Z_l = k; \mu_{yk}, \Sigma_{yk}) \right)$

$$= \prod_{k=1}^{K} \left(\prod_{m=1}^{M} N(\Gamma_{lm} \mid \mu_{mk}, \Sigma_{mk}) \cdot N(\Gamma_{ly} \mid \mu_{mk}, \Sigma_{mk}) \right)^{\rho_{lk}}$$

③ $P(X_l \mid \Gamma_l, Z_l) = P(x_{l1}, \dots, x_{lm}, \psi(x_l) w \mid \Gamma_l, Z_l)$

$$= \prod_{k=1}^{K} \left(\prod_{m=1}^{M} p(x_{lm} \mid \Gamma_{lmk}, Z_l = k; \sigma_{mk}^2) \cdot p(\psi(x_l) w_k \mid \Gamma_{lyk}, Z_l = k; \sigma_{yk}^2, w_k) \right)$$

这项对吗？

$p(x_{l1}, \dots, x_{lm}, \psi(x_l) w \mid \Gamma_l, Z_l)$

$= p(x_{l1}, \dots, x_{lm} \mid \Gamma_l, Z_l)\, p(\psi(x_l) w \mid x_{l1}, \dots, x_{lm}, \Gamma_l, Z_l)$

$\downarrow$ independent $\qquad \downarrow$? 成立吗？

$= \prod_{m=1}^{M} p(x_{lm} \mid \Gamma_l, Z_l) \qquad = p(\psi(x_l) w \mid \Gamma_l, Z_l)$

$$= \prod_{k=1}^{K} \left(\prod_{m=1}^{M} N(X_{lm} \mid \phi_l \Gamma_{lmk}, \sigma_{mk}^2 I) \cdot N(\underbrace{\psi(x_l) w_k}_{y_{lk}} \mid \phi_l \Gamma_{lyk}, \sigma_{yk}^2 I) \right)^{\rho_{lk}}$$

~~$p(\Gamma_l \mid Z_l)$: $p(\Gamma_l \mid Z_l; \mu_k, \Sigma_k) = \prod_{k=1}^{K} \left[(2\pi)^{-\frac{p}{2}} |\Sigma_k|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} (\Gamma_l - \mu_k)^T \Sigma_k^{-1} (\Gamma_l - \mu_k) \right\} \right]^{\rho_{lk}}$~~

~~$p(y_l \mid \Gamma_l, Z_l)$: $p(y_l \mid \Gamma_l, Z_l; \sigma_k^2) = \prod_{k=1}^{K} \left[(2\pi\sigma_k^2)^{-\frac{n_l}{2}} \exp\left\{ -\frac{1}{2\sigma_k^2} (y_l - \phi_l \Gamma_l)^T (y_l - \phi_l \Gamma_l) \right\} \right]^{\rho_{lk}}$~~

Parameters to be estimated: $\Theta = \begin{cases} w_k \\ \lambda_k \\ u_{mk}, \Sigma_{mk}, \sigma_{mk}^2, \; m=1,\dots,M \\ \cancel{u_k, \Sigma_k, \sigma_k^2} \; u_{yk}, \Sigma_{yk}, \sigma_{yk}^2 \end{cases}$

z_l, T_l are also unknown. Θ cannot be estimated directly. Therefore, EM algorithm is used, where z_l, T_l are taken as latent variables.

E step: Given x_l, and $w_k, \lambda_k, \cancel{u_k, \Sigma_k, \sigma_k^2}$ $u_{mk}, \Sigma_{mk}, \sigma_{mk}^2, m=1,\dots,K$; $u_{yk}, \Sigma_{yk}, \sigma_{yk}^2$ at last iteration. Then we have $y_l = \psi(x_l) w_k$.

We calculate the posterior distribution of z_l and T_l.

$$p(z_l = k \mid \cancel{y_l}\, x_l) = \frac{p(\cancel{y_l}\, x_l \mid z_l = k)\, p(z_l = k)}{\sum_{k=1}^{K} p(\cancel{y_l}\, x_l \mid z_l = k)\, p(z_l = k)}$$

③ $p(x_l \mid z_l = k) = \prod_{m=1}^{M} p(x_{lm} \mid z_l = k)\, p(\psi(x_l) w_k \mid z_l = k)$

$x_{lm} \mid z_l = k \sim N(\phi_l u_{mk}, \phi_l \Sigma_{mk} \phi_l^T + \sigma_{mk}^2 I)$

$\psi(x_l) w_k \mid z_l = k \sim N(\phi_l u_{yk}, \phi_l \Sigma_{yk} \phi_l^T + \sigma_{yk}^2 I)$

here: ① $p(z_l = k) = \lambda_k$

$\cancel{y_l \mid z_l = k \sim N(\phi_l u_k, \phi_l u_k \phi_l^T + \sigma_k^2 I)}$

Thus:

$$\hat{\rho}_{lk} = p(z_l = k \mid \cancel{y_l}\, x_l) = \frac{\lambda_k \prod_{m=1}^{M} N(x_{lm} \mid \phi_l u_{mk}, \phi_l \Sigma_{mk} \phi_l^T + \sigma_{mk}^2 I)\, N(\psi(x_l) w_k \mid \phi_l u_{yk}, \phi_l \Sigma_{yk} \phi_l^T + \sigma_{yk}^2 I)}{\sum_{k=1}^{K} \lambda_k \prod_{m=1}^{M} N(x_{lm} \mid \phi_l u_{mk}, \phi_l \Sigma_{mk} \phi_l^T + \sigma_{mk}^2 I)\, N(\psi(x_l) w_k \mid \phi_l u_{yk}, \phi_l \Sigma_{yk} \phi_l^T + \sigma_{yk}^2 I)}$$

$$p(T_l \mid z_l = k, \cancel{y_l}\, x_l) = \frac{p(\cancel{y_l}\, x_l \mid T_l, z_l = k)\, p(T_l \mid z_l = k)}{\sum_{k=1}^{K} p(\cancel{y_l}\, x_l \mid T_l, z_l = k)\, p(T_l \mid z_l = k)}$$

For $\Gamma_{\ell m}$, $m=1,\cdots K$

here: $p(\Gamma_{\ell m} \mid z_\ell = k)$: $\Gamma_{\ell k m} \equiv \Gamma_\ell \mid z_\ell = k \sim N(u_{km}, \Sigma_{km})$

$x_{\ell m}$ $\Gamma_{\ell m}$

$p(\cancel{y_\ell} \mid \cancel{\Gamma_\ell}, z_\ell = k)$ $\quad x_{\ell m} \cancel{y_\ell} \mid \Gamma_{\ell m}, z_\ell = k \sim N(\phi_\ell \Gamma_{\ell k m}, \sigma_{mk}^2 I)$

For $\Gamma_{\ell j}$, $\cdots$

Thus: $\Gamma_\ell \mid z_\ell = k, y_\ell \sim N\left(\left(\frac{1}{\sigma_k^2}\phi_\ell^T\phi_\ell + \Sigma_k^{-1}\right)^{-1}\left(\frac{1}{\sigma_k^2}\phi_\ell^T y_\ell + \Sigma_k^{-1} u_k\right), \right.$

$$\left. \left(\frac{1}{\sigma_k^2}\phi_\ell^T\phi_\ell + \Sigma_k^{-1}\right)^{-1} \right)$$

We obtain: $\hat{\Gamma}_{\ell k} = \left(\frac{1}{\sigma_k^2}\phi_\ell^T\phi_\ell + \Sigma_k^{-1}\right)^{-1}\left(\frac{1}{\sigma_k^2}\phi_\ell^T y_\ell + \Sigma_k^{-1} u_k\right)$

$$\hat{\Sigma}_{\ell k} = \left(\frac{1}{\sigma_k^2}\phi_\ell^T\phi_\ell + \Sigma_k^{-1}\right)^{-1}$$

modify this part by yourself.

M step: maximize $Q(\Theta \mid \Theta_j)$, update parameters

$$\hat{\pi}_k = \frac{N_k}{N}, \quad N_k = \sum_{\ell=1}^{L} \hat{\rho}_{\ell k} \quad N = \sum_{k=1}^{K} N_k$$

$$\hat{u}_k = \frac{1}{N_k} \sum_{\ell=1}^{L} \hat{\rho}_{\ell k} E(\Gamma_\ell \mid z_\ell = k, y_\ell)$$

$$= \frac{1}{N_k} \sum_{\ell=1}^{L} \hat{\rho}_{\ell k} \hat{\Gamma}_{\ell k}$$

$$\hat{\Sigma}_k = \frac{1}{N_k} \sum_{\ell=1}^{L} \hat{\rho}_{\ell k} E\left[(\Gamma_\ell - u_k)(\Gamma_\ell - u_k)^T \mid z_\ell = k, y_\ell\right]$$

$$= \frac{1}{N_k} \sum_{\ell=1}^{L} \hat{\rho}_{\ell k}\left[\hat{\Sigma}_{\ell k} + (\hat{\Gamma}_{\ell k} - \hat{u}_k)^T(\hat{\Gamma}_{\ell k} - \hat{u}_k)\right]$$

$$\hat{\sigma}_k^2 = \frac{1}{N_k} \sum_{\ell=1}^{L} \hat{\rho}_{\ell k} E\left[(y_\ell - \phi_\ell\Gamma_\ell)^T(y_\ell - \phi_\ell\Gamma_\ell) \mid z_\ell = k, y_\ell\right]$$

$$= \frac{1}{N_k} \sum_{\ell=1}^{L} \hat{\rho}_{\ell k}\left[\|x_\ell \hat{w}_k - \phi_\ell \hat{\Gamma}_{\ell k}\|^2 + Tr(\phi_\ell \hat{\Sigma}_{\ell k} \phi_\ell)\right]$$

modify this part by yourself

~~After~~ M step, To update w_k:

At first, we regard w_k as known, and obtain the estimate of Γ_l under failure mode k from its posterior distribution.

$$\hat{\Gamma}_{lk} = \left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l + \Sigma_k^{-1}\right)^{-1}\left(\frac{1}{\sigma_k^2}\phi_l^T\psi(x_l)w_k + \Sigma_k^{-1}u_k\right)$$

Since $\psi(x_l)w_k$ is normally distributed given $z_l=k$, i.e.

$$\psi(x_l)w \mid z_l=k \sim N(\phi_l u_k, \phi_l\Sigma_k\phi_l^T + \sigma_k^2 I),$$

we have $\hat{\Gamma}_{lk} \mid z_l=k \sim N(E(\hat{\Gamma}_{lk}\mid z_l=k), Var(\hat{\Gamma}_{lk}\mid z_l=k))$

$$E(\hat{\Gamma}_{lk}\mid z_l=k) = \left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l + \Sigma_k^{-1}\right)^{-1}\left(\frac{1}{\sigma_k^2}\phi_l^T E(\psi(x_l)w|_{z_l=k}) + \Sigma_k^{-1}u_k\right)$$

$$= \left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l + \Sigma_k^{-1}\right)^{-1}\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l u_k + \Sigma_k^{-1}u_k\right) = u_k$$

$$Var(\hat{\Gamma}_{lk}\mid z_l=k) = \left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l + \Sigma_k^{-1}\right)^{-1}\frac{1}{\sigma_k^2}\phi_l^T\, Var(\psi(x_l)w\mid z_l=k)\left(\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l + \Sigma_k^{-1}\right)^{-1}\frac{1}{\sigma_k^2}\phi_l^T\right)^T$$

$$= \left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l + \Sigma_k^{-1}\right)^{-1}\frac{1}{\sigma_k^2}\phi_l^T\left(\phi_l\Sigma_k\phi_l^T + \sigma_k^2 I\right)\frac{1}{\sigma_k^2}\phi_l\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l + \Sigma_k^{-1}\right)^{-1}$$

$$= \left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l + \Sigma_k^{-1}\right)^{-1}\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l\Sigma_k\phi_l^T\phi_l + \phi_l^T\phi_l\right)\frac{1}{\sigma_k^2}\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l + \Sigma_k^{-1}\right)^{-1}$$

$$= \frac{1}{\sigma_k^2}\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l + \Sigma_k^{-1}\right)^{-1}\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l + \Sigma_k^{-1}\right)\Sigma_k\phi_l^T\phi_l\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l + \Sigma_k^{-1}\right)$$

$$= \frac{1}{\sigma_k^2}\Sigma_k\phi_l^T\phi_l\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l + \Sigma_k^{-1}\right)^{-1}$$

At failure time: we have $\phi(\tau_l)u_k = D_k$ according to Eq. (1).

$\tau_l = \arg\min_t g_l(t) > D_k$

$E(g_l(t) - D_k) = 0$

$\phi(\tau_l)\hat{\Gamma}_{lk}|_{z_l=k}$ also follows a multivariate normal distribution

$$E(\phi(\tau_l)\hat{\Gamma}_{lk}|_{z_l=k}) = \phi(\tau_l)E(\hat{\Gamma}_{lk}) = \phi(\tau_l)u_k = D_k$$

$$Var(\phi(\tau_l)\hat{\Gamma}_{lk} \mid z_l=k) = \phi(\tau_l)\, Var(\phi(\tau_l)\hat{\Gamma}_{lk} \mid z_l=k)\, \phi^T(\tau_l)$$

$$= \frac{1}{\sigma_k^2}\phi(\tau_l)\left(\Sigma_k \phi_l^T \phi_l \left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l + \Sigma_k^{-1}\right)^{-1}\right)\phi^T(\tau_l)$$

$$\phi(\tau_l)\hat{\Gamma}_{lk} \mid z_l=k \sim \mathcal{N}\Big(D_k,\ \underbrace{\frac{1}{\sigma_k^2}\phi(\tau_l)\left(\Sigma_k\phi_l^T\phi_l\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l+\Sigma_k^{-1}\right)^{-1}\right)\phi^T(\tau_l)}_{\sigma_{\tau_l}^2}\Big)$$

Therefore, the likelihood function

$$L_w = \prod_{l=1}^{L} p(\phi(\tau_l)\hat{\Gamma}_{lk} \mid z_l=k) = (2\pi\sigma_{\tau_l}^2)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma_{\tau_l}^2}\left(\phi(\tau_l)\hat{\Gamma}_{lk} - \mu_k\right)^2\right\}$$

The log-likelihood function:

$$\log L_w = \sum_{l=1}^{L} -\frac{1}{2}\log(2\pi\sigma_{\tau_l}^2) - \frac{1}{2\sigma_{\tau_l}^2}\left(\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l+\Sigma_k^{-1}\right)^{-1}\left(\frac{1}{\sigma_k^2}\phi_l^T\psi(x_l)w_k+\Sigma_k^{-1}\mu_k\right)-D_k\right)^2.$$

Since $\log(2\pi\sigma_{\tau_l}^2)$ is not related to w, we can estimate w_k by minimizing $-2\log L_w - \sum_{l=1}^{L}\log(2\pi\sigma_{\tau_l}^2)$, i.e.

$$\hat{w}_k = \arg\min_{w_k} \sum_{l=1}^{L} \frac{\left(\phi(\tau_l)\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l+\Sigma_k^{-1}\right)^{-1}\left(\frac{1}{\sigma_k^2}\phi_l^T\psi(x_l)w_k+\Sigma_k^{-1}\mu_k\right)-D_k\right)^2}{\frac{1}{\sigma_k^2}\phi(\tau_l)\left(\Sigma_k\phi_l^T\phi_l\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l+\Sigma_k^{-1}\right)^{-1}\right)\phi^T(\tau_l)}$$

$$= \arg\min_{w_k} \mathcal{L}_w$$

$$\frac{\partial \mathcal{L}_w}{\partial w_k} = \sum_{l=1}^{L}\frac{2}{\sigma_{\tau_l}^2}\left(\phi(\tau_l)\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l+\Sigma_k^{-1}\right)^{-1}\left(\frac{1}{\sigma_k^2}\phi_l^T\psi(x_l)w_k+\Sigma_k^{-1}\mu_k\right)-D_k\right)$$

$$\cdot\left[\phi(\tau_l)\left(\frac{1}{\sigma_k^2}\phi_l^T\phi_l+\Sigma_k^{-1}\right)^{-1}\frac{1}{\sigma_k^2}\phi_l^T\psi(x_l)\right] = 0$$

Let $a_1 = \phi(\tau_l)\left(\frac{1}{\sigma_K^2}\phi_l^T\phi_l + \Sigma_K^{-1}\right)^{-1}\frac{1}{\sigma_K^2}\phi_l^T\psi(x_l)$

$$\sum_{l=1}^{L} \frac{1}{\sigma_{\tau l}^2} a_1^T\left(a_1 w_K + \phi(\tau_l)\left(\frac{1}{\sigma_K^2}\phi_l^T\phi_l + \Sigma_K^{-1}\right)^{-1}\Sigma_K^{-1}u_K - D_K\right) = 0$$

Let $b_1 = D_K - \phi(\tau_l)\left(\frac{1}{\sigma_K^2}\phi_l^T\phi_l + \Sigma_K^{-1}\right)^{-1}\Sigma_K^{-1}u_K$

$$\sum_{l=1}^{L} \frac{1}{\sigma_{\tau l}^2} a_1^T(a_1 w_K - b_1) = 0$$

Let $a_2 = \frac{a_1}{\sigma_K\sqrt{\sigma_{\tau l}^2}} = \frac{\phi(\tau_l)\left(\frac{1}{\sigma_K^2}\phi_l^T\phi_l + \Sigma_K^{-1}\right)^{-1}\frac{1}{\sigma_K^2}\phi_l^T\psi(x_l)}{\frac{1}{\sigma_K}\sqrt{\frac{1}{\sigma_K^2}\phi(\tau_l)\,\Sigma_K\phi_l^T\phi_l\left(\frac{1}{\sigma_K^2}\phi_l^T\phi_l + \Sigma_K^{-1}\right)^{-1}\phi^T(\tau_l)}}$

$$= \frac{\phi(\tau_l)\left(\frac{1}{\sigma_K^2}\phi_l^T\phi_l + \Sigma_K^{-1}\right)^{-1}\phi_l^T\psi(x_l)}{\sqrt{\phi(\tau_l)\Sigma_K\phi_l^T\phi_l\left(\frac{1}{\sigma_K^2}\phi_l^T\phi_l + \Sigma_K^{-1}\right)^{-1}\phi^T(\tau_l)}}$$

$$b_2 = \frac{b_1}{\frac{1}{\sigma_K}\sqrt{\sigma_{\tau l}^2}} = \frac{\sigma_K^2\left(D_K - \phi(\tau_l)\left(\frac{1}{\sigma_K^2}\phi_l^T\phi_l + \Sigma_K^{-1}\right)^{-1}\Sigma_K^{-1}u_K\right)}{\sqrt{\phi(\tau_l)\Sigma_K\phi_l^T\phi_l\left(\frac{1}{\sigma_K^2}\phi_l^T\phi_l + \Sigma_K^{-1}\right)^{-1}\phi^T(\tau_l)}}$$

We have $\sum_{l=1}^{L} a_2^T a_2 w_K - a_2^T b_2 = 0$

Then $\hat{w}_K = \left(\sum_{l=1}^{L} a_2^T a_2\right)^{-1}\sum_{l=1}^{L} a_2^T b_2$

3. Failure Mode Recognition & RUL Prediction

① Failure mode:

$$p(z_l = k \mid y_l) = \frac{p(y_l \mid z_l = k)\, p(z_l = k)}{\sum_{k=1}^{K} p(y_l \mid z_l = k)\, p(z_l = k)}$$

Modify this part by yourself

$$y_l = \psi(x_l) w_k$$

$$= \frac{\pi_k \mathcal{N}(y_l \mid \phi_l u_k, \phi_l u_k \phi_l^T + \sigma_k^2 I)}{\sum_{k=1}^{K} \pi_k \mathcal{N}(y_l \mid \phi_l u_k, \phi_l u_k \phi_l^T + \sigma_k^2 I)}$$

② RUL:

$$\Gamma_l \mid z_l = k, y_l \sim \mathcal{N}\left(\left(\frac{1}{\sigma_k^2} \phi_l^T \phi_l + \Sigma_k^{-1}\right)^{-1} \left(\frac{1}{\sigma_k^2} \phi_l^T y_l + \Sigma_k^{-1} u_k\right), \left(\frac{1}{\sigma_k^2} \phi_l^T \phi_l + \Sigma_k^{-1}\right)^{-1} \right)$$

For an in-service unit l,

$$g_l(t_{nl} + t) = \phi(t_{nl} + t)\Gamma_l$$

$$g_l(t_{nl} + t) \mid_{z_l = k, y_l} \sim \mathcal{N}(u_{t_{nl}+t}, \Sigma_{t_{nl}+t})$$

$$u_{t_{nl}+t} = \phi(t_{nl} + t)\left(\frac{1}{\sigma_k^2} \phi_l^T \phi_l + \Sigma_k^{-1}\right)^{-1} \left(\frac{1}{\sigma_k^2} \phi_l^T \psi(x_l) w_k + \Sigma_k^{-1} u_k\right)$$

$$\Sigma_{t_{nl}+t} = \phi(t_{nl} + t)\left(\frac{1}{\sigma_k^2} \phi_l^T \phi_l + \Sigma_k^{-1}\right)^{-1} \phi^T(t_{nl} + t)$$

The probability that the degradation status exceeds the failure threshold at time t after t_{nl} is

$$F_{T_l|y_l}(t) = P(T_l \leq t \mid y_l) = P(g_l(t_{nl}+t) > D \mid y_l)$$

$$= \sum_{k=1}^{K} P(g_l(t_{nl}+t) > D_k \mid y_l, z_l = k)\, P(z_l = k \mid y_l)$$

$$P(T_l \leq t \mid T_l \geq 0, y_l) = \frac{P(0 \leq T_l \leq t \mid y_l)}{P(T_l \geq 0 \mid y_l)}$$

$$= \frac{F_{T_l|y_l}(t) - F_{T_l|y_l}(0)}{1 - F_{T_l|y_l}(0)}$$