Stereo Rectification Errors Caused by Calibration

Method proposed by Fusiello et al. [1] is widely used to rectify stereo image pairs, which can be expressed as follow.

$$s\tilde{m}_2^{'} = A_s R_2 R_t A_s^{-1} \tilde{m}_2; \qquad (1)$$
$$s\tilde{m}_1^{'} = A_s R_1 R_t A_s^{-1} \tilde{m}_1;$$

The relative rotation R can be divided into *Roll*, *Pitch* and *Yaw*, their rotation matrices can be written as follow.

$$R_{roll} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(roll) & -\sin(roll) \\ 0 & \sin(roll) & \cos(roll) \end{bmatrix} \quad R_{pitch} = \begin{bmatrix} \cos(pitch) & 0 & \sin(pitch) \\ 0 & 1 & 0 \\ -\sin(pitch) & 0 & \cos(pitch) \end{bmatrix} \quad R_{yaw} = \begin{bmatrix} \cos(yaw) & -\sin(yaw) & 0 \\ \sin(yaw) & \cos(yaw) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (2)$$

Then we apply R_{roll}, R_{Pitch} and R_{Yaw} to the equation (1) respectively, for example, applying R_{roll} as follow.

$$\begin{aligned} & s\tilde{m}_2^{'} = A_s R_{roll} A_s^{-1} \tilde{m}_2; \\ & s\tilde{m}_1^{'} = A_s I_3 A_s^{-1} \tilde{m}_1; \\ & \triangle u = u_1' - u_2' \\ & \triangle v = v_1' - v_2' \end{aligned} \qquad (3)$$

Then

$$\begin{cases} \triangle u = u_1 - c_x + \dfrac{f_x \cdot TH_2}{\cos(roll) - TV_2 \cdot \sin(roll)} \\ \triangle v = v_1 - c_y + \dfrac{f_y(\sin(roll) + TV_2\cos(roll))}{\cos(roll) - TV_2 \cdot \sin(roll)} \end{cases} \quad TV_2 = \frac{c_y - v_2}{f_y}, TH_2 = \frac{c_x - u_2}{f_x}; \qquad (4)$$

Then

$$\begin{cases} d_r(\triangle u) = fx \cdot \dfrac{TH_2 \cdot (\sin(roll) + TV_2\cos(roll))}{(\cos(roll) - TV_2\sin(roll))^2} d(roll) \\ d_r(\triangle v) = fy \cdot \dfrac{1 + TV_2}{(\cos(roll) - TV_2\sin(roll))^2} d(roll) \end{cases} \qquad (5)$$

Similarly, applying the R_{Pitch} and R_{Yaw} to the equation (1) and get equations as follow

$$\begin{cases} d_p(\triangle u) = -f_x \cdot \dfrac{1 + TH_2}{(\cos(pitch) + TH_2\sin(pitch))^2} d(pitch) \\ d_p(\triangle v) = f_x \cdot \dfrac{TV_2(\sin(pitch) - TH_2\cos(pitch))}{(\cos(pitch) + TH_2\sin(pitch))^2} d(pitch) \end{cases} \qquad (6)$$

$$\begin{cases} d_w(\triangle u) = -f_x \cdot (TV_2\cos(yaw) + TH_2\sin(yaw)) d(yaw) \\ d_w(\triangle v) = f_y \cdot (TH_2\cos(yaw) - TV_2\sin(yaw)) d(yaw) \end{cases}$$

Since *Roll*, *Pitch* and *Yaw* are all very small, thus *cos(angle) = 1*, *sin(angle)=angle*, *(cos(angle) ±THsin(angle))²=1* and *(cos(angle) ±TVsin(angle))²=1*. Thus, the equations above can be simplified to as follow

$$\begin{cases} d_r(\triangle u) = f_x \cdot TH_2(roll + TV_2) d(roll); \\ d_r(\triangle v) = f_y \cdot (1 + TV_2) d(roll); \end{cases} \qquad (7)$$

$$\begin{cases} d_p(\triangle u) = -f_x \cdot (1 + TH_2) d(pitch) \\ d_p(\triangle v) = f_y \cdot TV_2(pitch - TH_2) d(pitch) \end{cases}$$

$$\begin{cases} d_w(\triangle u) = -f_x \cdot (TV_2 + TH_2 \cdot yaw) d(yaw) \\ d_w(\triangle v) = f_y \cdot (TH_2 - TV_2 \cdot yaw) d(yaw) \end{cases}$$

When TH and TV are much larger than the *Roll, Pitch* and *Yaw*, the three angles can be ignored; When they are close, the $d(\triangle u)$ and $d(\triangle v)$ are too small to take into account, ignoring the three angles has few influence to the result. Thus, we can simplify the equations as follow;

$$\begin{cases} d_r(\triangle u) = f_x \cdot TH_2TV_2 d(roll) \\ d_r(\triangle v) = f_y \cdot (1+TV_2) d(roll) \end{cases} \quad \begin{cases} d_p(\triangle u) = -f_x \cdot (1+TH_2) d(pitch) \\ d_p(\triangle v) = f_y \cdot TV_2TH_2 d(pitch) \end{cases} \quad \begin{cases} d_w(\triangle u) = -f_x TV_2 d(yaw) \\ d_w(\triangle v) = f_y TH_2 d(yaw) \end{cases} \tag{8}$$

As for R_t in equation (1), it can be regard as rotations around axis y and z by $\theta_z = \left|\frac{t_z}{t}\right|$ and $\theta_y = \left|\frac{t_y}{t}\right|$. We apply R_{θ_z} and R_{θ_y} into equation (1) respectively, an example is shown as follow.

$$\begin{aligned} s\tilde{m}_2' &= A_s R_{\theta_z} A_s^{-1} \tilde{m}_2; \\ s\tilde{m}_1' &= A_s R_{\theta_z} A_s^{-1} \tilde{m}_1; \\ \triangle u &= u_1' - u_2' \\ \triangle v &= v_1' - v_2' \end{aligned} \tag{9}$$

Similar to the derivation above, we can get

$$\begin{cases} d_z(\triangle u) = fx \cdot \left(\frac{1+TH_1}{(\cos(\theta_z)+TH_1\sin(\theta_z))^2} - \frac{1+TH_2}{(\cos(\theta_z)+TH_2\sin(\theta_z))^2}\right) d(\theta_z) \\ d_z(\triangle v) = fy \cdot \left(\frac{TV_2(\sin(\theta_z)-TH_2\cos(\theta_z))}{(\cos(\theta_z)+TH_2\sin(\theta_z))^2} - \frac{TV_1(\sin(\theta_z)-TH_1\cos(\theta_z))}{(\cos(\theta_z)+TH_1\sin(\theta_z))^2}\right) d(\theta_z) \end{cases}$$
$$\begin{cases} d_y(\triangle u) = fx \cdot ((TV_2-TV_1)\cos(\theta_y) + (TH_2-TH_1)\sin(\theta_y)) d(\theta_y) \\ d_y(\triangle v) = fy \cdot ((TH_1-TH_2)\cos(\theta_y) + (TV_1-TV_2)\sin(\theta_y)) d(\theta_y) \end{cases} \tag{10}$$

Then they can be simplified as follow

$$\begin{cases} d_z(\triangle u) = \triangle u_{org} d(\theta_z) \\ d_z(\triangle v) = fy \cdot (TH_1TV_1 - TH_2TV_2) d(\theta_z) = f_y\left(TH_2 \frac{\triangle v_{org}}{f_y} + TV_1 \frac{\triangle u_{org}}{f_x}\right) d(\theta_z) \end{cases}$$
$$\begin{cases} d_y(\triangle u) = \triangle v_{org} d(\theta_y) \\ d_y(\triangle v) = \triangle u_{org} d(\theta_y) \end{cases} \tag{11}$$

Accumulating all the results above, we get the rectification error equations as follow

$$\begin{aligned} d(\triangle u) &= | f_x TH_2TV_2 d(roll)| + | f_x(1+TH_2) d(pitch)| + | f_x TV_2 d(yaw)| + |\triangle u_{org} d(\theta_z)| + |\triangle v_{org} d(\theta_y)|; \\ d(\triangle v) &= | f_y(1+TV_2) d(roll)| + | f_y TH_2TV_2 d(pitch)| + | f_y TH_2 d(yaw)| + \left| f_y\left(TH_2 \frac{\triangle v_{org}}{f_y} + TV_1 \frac{\triangle u_{org}}{f_x}\right) d(\theta_z)\right| + |\triangle u_{org} d(\theta_y)|; \end{aligned} \tag{12}$$

Where

$$\triangle u_{org} = u_1 - u_2; \triangle v_{org} = v_1 - v_2 \tag{13}$$

To get the relationship between $\triangle u_{org}$ and $\triangle u$, we rewrite equation (1) as follow

$$\begin{aligned} \tilde{m}_1 &= s_1 A_s^{-1} R_t^{-1} A_s \tilde{m}_1' \\ \tilde{m}_2 &= s_2 A_s^{-1} R_t^{-1} R^{-1} A_s \tilde{m}_2' \end{aligned} \quad \begin{aligned} H_1 &= \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} = A_s^{-1} R_t^{-1} A_s; \\ H_2 &= \begin{bmatrix} h_{11}' & h_{12}' & h_{13}' \\ h_{21}' & h_{22}' & h_{23}' \\ h_{31}' & h_{32}' & h_{33}' \end{bmatrix} = A_s^{-1} R^{-1} R_t^{-1} A_s; \end{aligned} \tag{14}$$

Then we get

$$\triangle u_{org} = u_1 - u_2 = \frac{h_{11}u_1' + h_{12}v_1' + h_{13}}{s_1} - \frac{h_{11}'u_2' + h_{12}'v_2' + h_{13}'}{s_2} \tag{15}$$

s_1 and s_2 are scale factors, since the rotations are all slight, s_1 and s_2 can be set to 1 to simplify the derivation. After the perfect transformation, $\triangle v$ should be zero. To simplify the derivation, the transformation based on the baseline results is regard as perfect one, thus $v_1' = v_2'$ and the equation above can be simplified as follow.

$$\triangle u_{org} = h_{11} \triangle u + (h_{11} - h_{11}') u_2' + (h_{12} - h_{12}') v_2' + (h_{13} - h_{13}') \tag{16}$$

1. A. Fusiello, E. Trucco, and A. Verri, "A compact algorithm for rectification of stereo pairs," Machine Vision and Applications **12**, 16-22 (2000).