# Pairing-Based SNARKs. Pinocchio And Groth16

*October 10, 2024*

**Distributed Lab**

zkdl-camp.github.io

github.com/ZKDL-Camp

# Plan

1 Recap

2 Encrypted Verification

3 Make It Sound

# Recap

# Recap. R1CS

Each **constraint** in the Rank-1 Constraint System must be in the form:

$$\langle \boldsymbol{a}, \boldsymbol{w} \rangle \times \langle \boldsymbol{b}, \boldsymbol{w} \rangle = \langle \boldsymbol{c}, \boldsymbol{w} \rangle$$

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Thus

$$\left( \sum_{i=1}^{n} a_i w_i \right) \times \left( \sum_{j=1}^{n} b_j w_j \right) = \sum_{k=1}^{n} c_k w_k$$

That is, actually, a quadratic equation with multiple variables.

# Recap. R1CS

Consider the simplest program:

```
def example(a: F, b: F, c: F) -> F:
    if a:
        return b * c
    else:
        return b + c
```

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$$x_2 \times x_3 = \text{mult} \tag{2}$$
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The coefficients vectors:

$$\begin{aligned}
\boldsymbol{a}_1 &= (0,0,1,0,0,0,0), & \boldsymbol{b}_1 &= (0,0,1,0,0,0,0), & \boldsymbol{c}_1 &= (0,0,1,0,0,0,0) \\
\boldsymbol{a}_2 &= (0,0,0,1,0,0,0), & \boldsymbol{b}_2 &= (0,0,0,0,1,0,0), & \boldsymbol{c}_2 &= (0,0,0,0,0,1,0) \\
\boldsymbol{a}_3 &= (0,0,1,0,0,0,0), & \boldsymbol{b}_3 &= (0,0,0,0,0,1,0), & \boldsymbol{c}_3 &= (0,0,0,0,0,0,1) \\
\boldsymbol{a}_4 &= (1,0,-1,0,0,0,0), & \boldsymbol{b}_4 &= (0,0,0,1,1,0,0), & \boldsymbol{c}_4 &= (0,1,0,0,0,0,-1)
\end{aligned}$$

# Recap. QAP

R1CS provides us with the following constraint vectors:

$$\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_m, \quad \boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_m, \quad \boldsymbol{c}_1, \boldsymbol{c}_2, \dots, \boldsymbol{c}_m,$$

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Of course, they form corresponding matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \text{ same goes for } B \text{ and } C$$

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An example of a single "if" statement:

$$\begin{aligned} \boldsymbol{a}_1 &= (0,0,1,0,0,0,0) \\ \boldsymbol{a}_2 &= (0,0,0,1,0,0,0) \\ \boldsymbol{a}_3 &= (0,0,1,0,0,0,0) \\ \boldsymbol{a}_4 &= (1,0,-1,0,0,0,0) \end{aligned}$$

$$\begin{array}{cc} & \begin{array}{ccccccc} \phantom{0} & \phantom{0} & 3 & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} \end{array} \\ \begin{array}{c} \\ \\ \\ 4 \end{array} & \begin{bmatrix} 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0 \\ 0&0&1&0&0&0&0\\ 1&0&\boxed{-1}&0&0&0&0 \end{bmatrix}\end{array}$$

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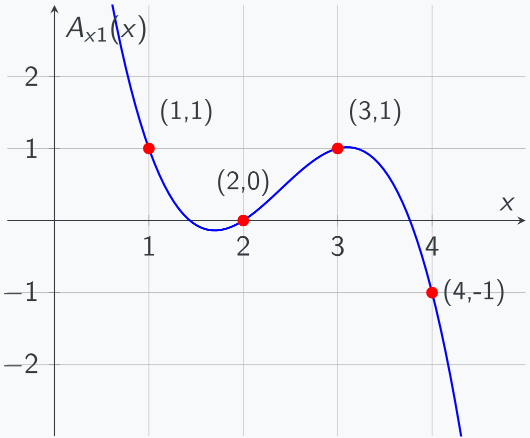


**Illustration:** The Lagrange inteprolation polynomial for points $\{(1,1),(2,0),(3,1),(4,-1)\}$ visualized over $\mathbb{R}$.

# Recap. QAP

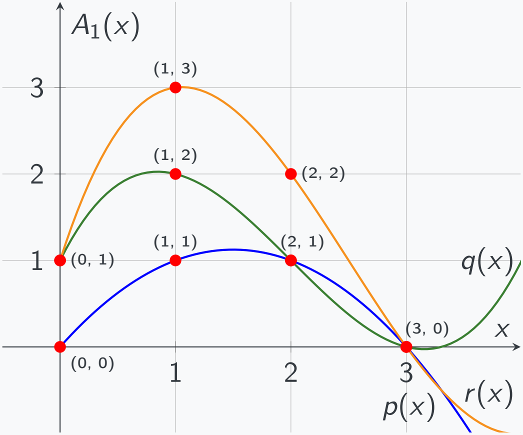


**Figure:** Addition of two polynomials

Now, using coefficients encoded with polynomials, we can build a constraint number $X \in \{1, \dots\ m\}$ in the next way:

$$
\begin{aligned}
&(w_1 A_1(X) + w_2 A_2(X) + \cdots + w_n A_n(X)) \times \\
\times &(w_1 B_1(X) + w_2 B_2(X) + \cdots + w_n B_n(X)) = \\
= &(w_1 C_1(X) + w_2 C_2(X) + \cdots + w_n C_n(X))
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\end{aligned}
$$

Or written more concisely:

$$
\left(\sum_{i=1}^{n} w_i A_i(X)\right) \times \left(\sum_{i=1}^{n} w_i B_i(X)\right) = \left(\sum_{i=1}^{n} w_i C_i(X)\right)
$$

$$
A(X) \times B(X) = C(X)
$$

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Now, we can define a polynomial $M(X)$, that has zeros at all elements from the set $\Omega = \{1, \ldots, m\}$

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It means, that $M(X)$ can be divided by **vanishing polynomial** $Z_\Omega(X)$ without a remainder!

$$Z_\Omega(X) = \prod_{i=1}^{m}(X - i), \qquad H(X) = \frac{M(X)}{Z_\Omega(X)} \text{ is a polynomial}$$

# Encrypted Verification

# Current Point

We've managed to encode into **a single polynomial** an entire computation (a program), of any size, independent of how much data it consumes.

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Where the knowledge of the correct witness is a knowledge of the quotient polynomial $H(X)$.

$$M(X) = H(X) \times Z_{\Omega}(X)$$

# Notation Preliminaries

In this section we'll use a group of points on elliptic curve denoted as $\mathbb{G}$ of prime order $q$ with a generator $g$.

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The symmetric pairing function $e : \mathbb{G} \times \mathbb{G} \to \mathbb{G}_T$, where $(\mathbb{G}_T, \times)$ is a target group.

# Naive Proof

Suppose, we are given a circuit $\mathcal{C}$ with a maximum degree $d$ of polynomials used underneath.

Thus, all parties additionally know the target polynomial $Z(x)$ and QAP polynomials $\{L_i(x)\}_{i\in[n]}, \{R_i(x)\}_{i\in[n]}, \{O_i(x)\}_{i\in[n]}$, where $n$ is number of witness elements.

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✓ Provides witness $\boldsymbol{w}$ to a Verifier.

**Verifier**

✓ Checks $(\sum_{i=1}^n w_i A_i(X)) \times (\sum_{i=1}^n w_i B_i(X)) = (\sum_{i=1}^n w_i C_i(X))$

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We, definitely, need to encrypt the witness data $\boldsymbol{w}$ somehow...

Let's define the *encryption* operation as follows:

$$\mathsf{Enc} : \mathbb{F} \to \mathbb{G}, \quad \mathsf{Enc}(x) := g^x$$

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Essentially, $\mathrm{Enc}(p(\tau))$ is the **KZG Commitment**.

### *Example*

Consider the polynomial: $p(x) = x^2 - 5x + 2$, the encryption of $p(\tau)$:

$$\mathrm{Enc}(p(\tau)) = g^{p(\tau)} = g^{\left(\tau^2 - 5\tau + 2\right)} = \left(g^{\tau^2}\right)^1 \cdot \left(g^{\tau^1}\right)^{-5} \cdot \left(g^{\tau^0}\right)^2$$

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## *Question*

KZG Commitment requires encrypted powers of $\tau$: $\{g^{\tau^i}\}_{i \in [d]}$. But where the prover can take them?

# Trusted Setup

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This way, we can find the KZG commitment for each polynomial. For example:

$$\mathsf{com}(L) \triangleq g^{L(\tau)} = g^{\sum_{i=0}^{d} L_i \tau^i} = \prod_{i=0}^{d} (g^{\tau^i})^{L_i},$$

Now, we can calculate:

$$g^{L(\tau)}, g^{R(\tau)}, g^{O(\tau)}, g^{H(\tau)}, g^{Z(\tau)}$$

But how can we verify $H(x)Z(x) = L(x)R(x) - O(x)$ in the ecnrypted space?

Well, first notice that the check is equivalent to:

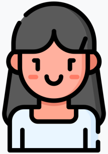
$$L(\tau)R(\tau) = Z(\tau)H(\tau) + O(\tau).$$

So, we can check this equality as follows:

$$e(\mathrm{com}(L), \mathrm{com}(R)) = e(\mathrm{com}(Z), \mathrm{com}(H)) \cdot e(\mathrm{com}(O), g),$$

**Trusted Setup:** $\tau \xleftarrow{R} \mathbb{F}$, $\{g^{\tau^i}\}_{i\in[d]}$, **delete**$(\tau)$.

Prover $\mathcal{P}$

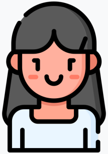


Verifier $\mathcal{V}$

**Trusted Setup:** $\tau \xleftarrow{R} \mathbb{F}$, $\{g^{\tau^i}\}_{i\in[d]}$, **delete**$(\tau)$.

✓ $H(x) = \frac{L(x)\times R(x) - O(x)}{Z(x)}$.

Prover $\mathcal{P}$



Verifier $\mathcal{V}$

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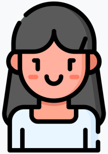
✓ KZG commitments:

$\pi_L \leftarrow \text{com}(L)$, $\pi_R \leftarrow \text{com}(R)$,

$\pi_O \leftarrow \text{com}(O)$, $\pi_H \leftarrow \text{com}(H)$,

Prover $\mathcal{P}$



Verifier $\mathcal{V}$

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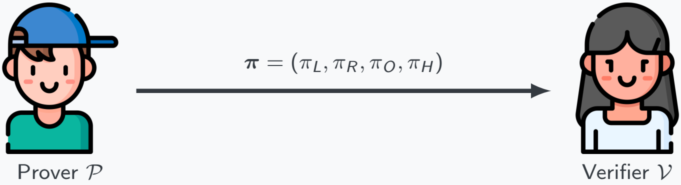

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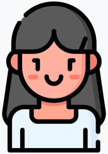
Prover $\mathcal{P}$

$\boldsymbol{\pi} = (\pi_L, \pi_R, \pi_O, \pi_H)$ $\longrightarrow$



Verifier $\mathcal{V}$

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- ✗ Does it work?

# Why it doesn't work??

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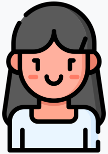

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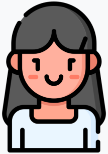

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***Problem***

Prover isn't forced to use the values from the trusted setup.

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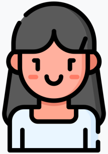

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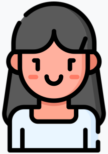

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  $L'(x) \times R'(x) - O'(x) = M'(x)(x)$
- ✓ KZG commitments:
  $\pi_{L'(x)} \leftarrow \mathrm{com}(L'(x))$, $\pi_{R'(x)} \leftarrow \mathrm{com}(R'(x))$,
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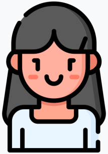

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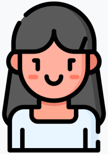
  $L'(x) \times R'(x) - O'(x) = M'(x)(x)$
- ✓ KZG commitments:
  $\pi_{L'(x)} \leftarrow \text{com}(L'(x))$, $\pi_{R'(x)} \leftarrow \text{com}(R'(x))$,
  $\pi_{O'(x)} \leftarrow \text{com}(O'(x))$, $\pi_{H'(x)} \leftarrow \text{com}(H'(x))$,

$$\boldsymbol{\pi} = (\pi_{L'(x)}, \pi_{R'(x)}, \pi_{O'(x)}, \pi_{H'(x)}) \longrightarrow$$



Prover $\mathcal{P}$ — Verifier $\mathcal{V}$

**Problem**

Prover isn't forced to use the values from the trusted setup.

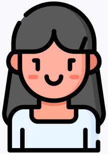
# Why it doesn't work??

**Trusted Setup:** $\tau \xleftarrow{R} \mathbb{F}$, $\{g^{\tau^i}\}_{i \in [d]}$, **delete**$(\tau)$.

- ✓ $H'(x) \xleftarrow{R} \mathbb{F}[x]$, $M'(x) = Z(x) \times H'(x)$.
- ✓ Finds $L'(x), R'(x), O'(x)$ such that: $L'(x) \times R'(x) - O'(x) = M'(x)(x)$
- ✓ KZG commitments:
  $\pi_{L'(x)} \leftarrow \text{com}(L'(x))$, $\pi_{R'(x)} \leftarrow \text{com}(R'(x))$,
  $\pi_{O'(x)} \leftarrow \text{com}(O'(x))$, $\pi_{H'(x)} \leftarrow \text{com}(H'(x))$,

✓ $e(\pi_{L'(x)}, \pi_{R'(x)}) ==$ $e(\text{com}(Z), \pi_H) \cdot e(\pi_{O'(x)}, g)$.

$\boldsymbol{\pi} = (\pi_{L'(x)}, \pi_{R'(x)}, \pi_{O'(x)}, \pi_{H'(x)})$ $\longrightarrow$



Prover $\mathcal{P}$

Verifier $\mathcal{V}$

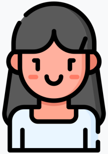
***Problem***

Prover isn't forced to use the values from the trusted setup.

# Proof Of Exponent

**Trusted Setup:** $\tau, \alpha \xleftarrow{R} \mathbb{F}$, $\{\{g^{\tau^i}\}_{i \in [d]}$, $\{g^{\alpha\tau^i}\}_{i \in [d]}\}$, **delete**$(\tau, \alpha)$.

Prover $\mathcal{P}$



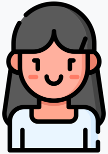
Verifier $\mathcal{V}$

# Proof Of Exponent

**Trusted Setup:** $\tau, \alpha \xleftarrow{R} \mathbb{F}$, $\{\{g^{\tau^i}\}_{i\in[d]}$, $\{g^{\alpha\tau^i}\}_{i\in[d]}\}$, **delete**$(\tau, \alpha)$.

✓ $H(x) = \frac{L(x)\times R(x) - O(x)}{Z(x)}$.

Prover $\mathcal{P}$



Verifier $\mathcal{V}$

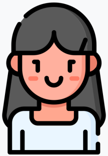
# Proof Of Exponent

**Trusted Setup:** $\tau, \alpha \xleftarrow{R} \mathbb{F}$, $\{\{g^{\tau^i}\}_{i\in[d]}, \{g^{\alpha\tau^i}\}_{i\in[d]}\}$, **delete**$(\tau, \alpha)$.

- ✓ $H(x) = \frac{L(x)\times R(x) - O(x)}{Z(x)}$.
- ✓ KZG commitments:

$$\begin{aligned} \pi_L &\leftarrow g^{L(\tau)}, & \pi'_L &\leftarrow g^{\alpha L(\tau)}, \\ \pi_R &\leftarrow g^{R(\tau)}, & \pi'_R &\leftarrow g^{\alpha R(\tau)}, \\ \pi_O &\leftarrow g^{O(\tau)}, & \pi'_O &\leftarrow g^{\alpha O(\tau)}, \\ \pi_H &\leftarrow g^{H(\tau)}, & \pi'_H &\leftarrow g^{\alpha H(\tau)}. \end{aligned}$$

Prover $\mathcal{P}$



Verifier $\mathcal{V}$

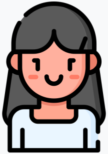
# Proof Of Exponent

**Trusted Setup:** $\tau, \alpha \xleftarrow{R} \mathbb{F}$, $\{\{g^{\tau^i}\}_{i\in[d]}, \{g^{\alpha\tau^i}\}_{i\in[d]}\}$, **delete**$(\tau, \alpha)$.

- ✓ $H(x) = \frac{L(x)\times R(x) - O(x)}{Z(x)}$.
- ✓ KZG commitments:

$$\begin{aligned} \pi_L &\leftarrow g^{L(\tau)}, & \pi'_L &\leftarrow g^{\alpha L(\tau)}, \\ \pi_R &\leftarrow g^{R(\tau)}, & \pi'_R &\leftarrow g^{\alpha R(\tau)}, \\ \pi_O &\leftarrow g^{O(\tau)}, & \pi'_O &\leftarrow g^{\alpha O(\tau)}, \\ \pi_H &\leftarrow g^{H(\tau)}, & \pi'_H &\leftarrow g^{\alpha H(\tau)}. \end{aligned}$$

$\boldsymbol{\pi} = (\pi_L, \pi_R, \pi_O, \pi_H)$



Prover $\mathcal{P}$ Verifier $\mathcal{V}$

# Proof Of Exponent

**Trusted Setup:** $\tau, \alpha \xleftarrow{R} \mathbb{F}$, $\{\{g^{\tau^i}\}_{i\in[d]}$, $\{g^{\alpha\tau^i}\}_{i\in[d]}\}$, **delete**$(\tau, \alpha)$.

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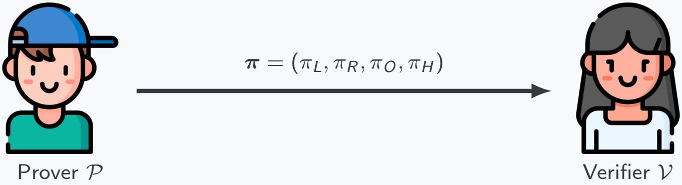
$$\begin{aligned} \pi_L &\leftarrow g^{L(\tau)}, & \pi'_L &\leftarrow g^{\alpha L(\tau)}, \\ \pi_R &\leftarrow g^{R(\tau)}, & \pi'_R &\leftarrow g^{\alpha R(\tau)}, \\ \pi_O &\leftarrow g^{O(\tau)}, & \pi'_O &\leftarrow g^{\alpha O(\tau)}, \\ \pi_H &\leftarrow g^{H(\tau)}, & \pi'_H &\leftarrow g^{\alpha H(\tau)}. \end{aligned}$$

✓ $e(\pi_L, \pi_R) ==$ $e(\mathsf{com}(Z), \pi_H) \cdot e(\pi_O, g)$.



# Proof Of Exponent

**Trusted Setup:** $\tau, {\color{green}\alpha} \xleftarrow{R} \mathbb{F}$, $\{\{g^{\tau^i}\}_{i\in[d]}$, ${\color{green}\{g^{\alpha\tau^i}\}_{i\in[d]}}\}$, **delete**$(\tau, {\color{green}\alpha})$.

✓ $H(x) = \frac{L(x)\times R(x) - O(x)}{Z(x)}$.

✓ KZG commitments:

$$
\begin{aligned}
\pi_L &\leftarrow g^{L(\tau)}, & \pi'_L &\leftarrow g^{\alpha L(\tau)},\\
\pi_R &\leftarrow g^{R(\tau)}, & \pi'_R &\leftarrow g^{\alpha R(\tau)},\\
\pi_O &\leftarrow g^{O(\tau)}, & \pi'_O &\leftarrow g^{\alpha O(\tau)},\\
\pi_H &\leftarrow g^{H(\tau)}, & \pi'_H &\leftarrow g^{\alpha H(\tau)}.
\end{aligned}
$$

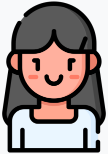
✓ $e(\pi_L, \pi_R) == e(\mathsf{com}(Z), \pi_H)\cdot e(\pi_O, g)$.

✓ Proof of Exponent:

$$
\begin{aligned}
e(\pi_L, g^{\alpha}) &= e(\pi'_L, g),\\
e(\pi_R, g^{\alpha}) &= e(\pi'_R, g),\\
e(\pi_O, g^{\alpha}) &= e(\pi'_O, g),\\
e(\pi_H, g^{\alpha}) &= e(\pi'_H, g).
\end{aligned}
$$

Prover $\mathcal{P}$

$$\boldsymbol{\pi} = (\pi_L, \pi_R, \pi_O, \pi_H)$$



Verifier $\mathcal{V}$

# Including PoE

- ✓ Succint
- ✓ Non-Interactive
- ✓ Zero-Knowledge

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- ✓ Succint
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- ✗ Sound

**_Problem_**

There is no guarantee that the same witness w was used to calculate all the commitments $\pi_L, \pi_R, \pi_O, \pi_H$.

# Make It Sound

# Additional Optimization

Recal that:

$$L(x) = \sum_{i=0}^{n} w_i L_i(x), \quad R(x) = \sum_{i=0}^{n} w_i R_i(x), \quad O(x) = \sum_{i=0}^{n} w_i O_i(x).$$

# Additional Optimization

Recal that:

$$L(x) = \sum_{i=0}^{n} w_i L_i(x), \quad R(x) = \sum_{i=0}^{n} w_i R_i(x), \quad O(x) = \sum_{i=0}^{n} w_i O_i(x).$$

Here public data is:

$$\{L_i(x)\}_{i \in [n]}, \{R_i(x)\}_{i \in [n]}, \{O_i(x)\}_{i \in [n]}$$

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Recal that:

$$L(x) = \sum_{i=0}^{n} w_i L_i(x), \quad R(x) = \sum_{i=0}^{n} w_i R_i(x), \quad O(x) = \sum_{i=0}^{n} w_i O_i(x).$$

Here public data is:

$$\{L_i(x)\}_{i\in[n]}, \{R_i(x)\}_{i\in[n]}, \{O_i(x)\}_{i\in[n]}$$

Moreover, it's defined only by the circuit and trusted setup, thus, it can calculated before proof generation as a part of the trusted setup.

# Additional Optimization

Updated Trusted Setup:

$$
\begin{aligned}
\{g^{\tau^i}\}_{i\in[d]}, &\quad \{g^{\alpha\tau^i}\}_{i\in[d]},\\
\{g^{L_i(\tau)}\}_{i\in[n]}, &\quad \{g^{\alpha L_i(\tau)}\}_{i\in[n]},\\
\{g^{R_i(\tau)}\}_{i\in[n]}, &\quad \{g^{\alpha R_i(\tau)}\}_{i\in[n]},\\
\{g^{O_i(\tau)}\}_{i\in[n]}, &\quad \{g^{\alpha O_i(\tau)}\}_{i\in[n]}
\end{aligned}
$$

# Additional Optimization

Updated Trusted Setup:

$$\begin{aligned} &\{g^{\tau^i}\}_{i\in[d]}, \quad \{g^{\alpha\tau^i}\}_{i\in[d]}, \\ &\{g^{L_i(\tau)}\}_{i\in[n]}, \quad \{g^{\alpha L_i(\tau)}\}_{i\in[n]}, \\ &\{g^{R_i(\tau)}\}_{i\in[n]}, \quad \{g^{\alpha R_i(\tau)}\}_{i\in[n]}, \\ &\{g^{O_i(\tau)}\}_{i\in[n]}, \quad \{g^{\alpha O_i(\tau)}\}_{i\in[n]} \end{aligned}$$

Consider the polynomial $L(x) = \sum_{i=0}^{n} w_i L_i(x)$.

# Additional Optimization

Updated Trusted Setup:

$$
\begin{aligned}
&\{g^{\tau^i}\}_{i\in[d]}, && \{g^{\alpha\tau^i}\}_{i\in[d]},\\
&\{g^{L_i(\tau)}\}_{i\in[n]}, && \{g^{\alpha L_i(\tau)}\}_{i\in[n]},\\
&\{g^{R_i(\tau)}\}_{i\in[n]}, && \{g^{\alpha R_i(\tau)}\}_{i\in[n]},\\
&\{g^{O_i(\tau)}\}_{i\in[n]}, && \{g^{\alpha O_i(\tau)}\}_{i\in[n]}
\end{aligned}
$$

Consider the polynomial $L(x) = \sum_{i=0}^{n} w_i L_i(x)$.

$\mathcal{P}$ can compute the KZG commitment $\pi_L$ and its PoE $\pi'_L$ as follows:

$$
\pi_L \triangleq g^{L(\tau)} = g^{\sum_{i=0}^{n} w_i L_i(\tau)} = \prod_{i=0}^{n} (g^{L_i(\tau)})^{w_i},
$$

$$
\pi'_L \triangleq g^{\alpha L(\tau)} = g^{\alpha \sum_{i=0}^{n} w_i L_i(\tau)} = \prod_{i=0}^{n} (g^{\alpha L_i(\tau)})^{w_i}.
$$

# Witness Consistency Check

**Problem**

Prover isn't forced to use the same witness while calculating commitments.

To prove that the same w is used in all commitments, we need some "checksum" term that will somehow combine all polynomials $L(x)$, $R(x)$, and $O(x)$ with the witness w.

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**Problem**

Prover isn't forced to use the same witness while calculating commitments.

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Hmm... Let's introduce one more coefficient:

$$\beta \xleftarrow{R} \mathbb{F}.$$

# Thank you for your attention

♥

zkdl-camp.github.io

github.com/ZKDL-Camp