

# QAP, PCP, POE

*Oct 1, 2024*

**Distributed Lab**

# Plan

Recap

Quadratic Arithmetic Program

Probabilistically Checkable Proofs

QAP as a Linear PCP

# Recap

# Recap. ZK-SNARK

## *Definition*

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- **Non-interactiveness** — to produce the proof, the prover does not need any interaction with the verifier.
- **Zero-Knowledge** — the verifier learns nothing about the data used to produce the proof, despite knowing that this data resolves the given problem and that the prover possesses it.



# Recap. Arbitrary Program To Circuits

We can do that in a way like the computer does it - boolean circuits.

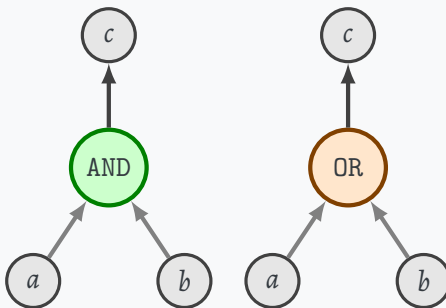


Figure: Boolean AND and OR Gates

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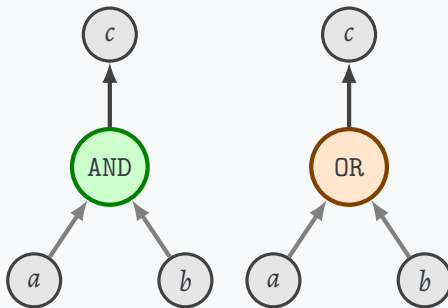


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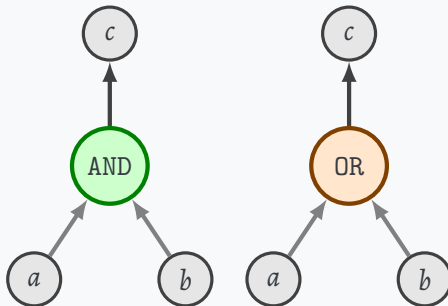


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> 100000 gates just for SHA256... But nothing stops us from using something more powerful instead of boolean values, gates.

# Recap. Arbitrary Program To Circuits

Similar to Boolean Circuits, the **Arithmetic circuits** consist of gates and wires.

- Wires: elements of some finite field  $\mathbb{F}_p$ .
- Gates: addition ( $\oplus$ ) and multiplication ( $\odot$ ) corresponding to the field.

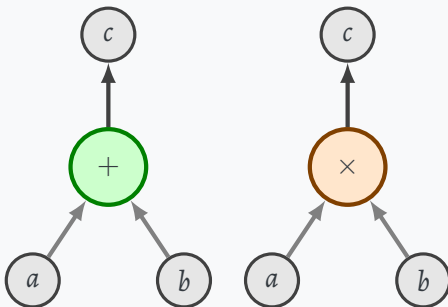


Figure: Addition and Multiplication Gates

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## Example

How can we translate if statements?

```
def example(a: bool, b: F, c: F) -> F:  
    if a:  
        return b * c  
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Corresponding equations for the circuit are:

$$\begin{aligned} r_1 &= b \times c, & r_3 &= 1 - a, & r_5 &= r_3 \times r_2 \\ r_2 &= b + c, & r_4 &= a \times r_1, & r &= r_4 + r_5 \end{aligned}$$

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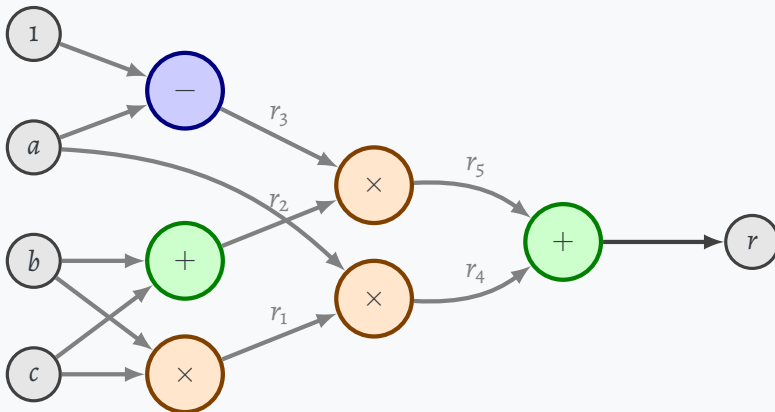


Figure: Example of a circuit evaluating the if statement logic.



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# Recap. R1CS

Each **constraint** in the Rank-1 Constraint System must be in the form:

$$\langle \mathbf{a}, \mathbf{w} \rangle \times \langle \mathbf{b}, \mathbf{w} \rangle = \langle \mathbf{c}, \mathbf{w} \rangle$$

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Thus

$$\left( \sum_{i=1}^n a_i w_i \right) \times \left( \sum_{j=1}^n b_j w_j \right) = \sum_{k=1}^n c_k w_k$$

That is, actually, a quadratic equation with multiple variables.

# Recap. R1CS

## Example

Consider the most basic circuit with one multiplication gate:

$x_1 \times x_2 = r$ . The witness vector  $\mathbf{w} = (r, x_1, x_2)$ . So

$$w_2 \times w_3 = w_1$$

$$(0 + w_2 + 0) \times (0 + 0 + w_3) = w_1 + 0 + 0$$

$$(0w_1 + 1w_2 + 0w_3) \times (0w_1 + 0w_2 + 1w_3) = 1w_1 + 0w_2 + 0w_3$$

Therefore the coefficients vectors are:

$$\mathbf{a} = (0, 1, 0), \quad \mathbf{b} = (0, 0, 1), \quad \mathbf{c} = (1, 0, 0).$$

The general form of our constraint is:

$$(a_1w_1 + a_2w_2 + a_3w_3)(b_1w_1 + b_2w_2 + b_3w_3) = c_1w_1 + c_2w_2 + c_3w_3$$

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$$x_1 \times x_1 = x_1 \quad (\text{binary check}) \quad (1)$$

$$x_2 \times x_3 = \text{mult} \quad (2)$$

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The coefficients vectors:

$$\mathbf{a}_1 = (0, 0, 1, 0, 0, 0, 0), \quad \mathbf{b}_1 = (0, 0, 1, 0, 0, 0, 0), \quad \mathbf{c}_1 = (0, 0, 1, 0, 0, 0, 0)$$

$$\mathbf{a}_2 = (0, 0, 0, 1, 0, 0, 0), \quad \mathbf{b}_2 = (0, 0, 0, 0, 1, 0, 0), \quad \mathbf{c}_2 = (0, 0, 0, 0, 0, 1, 0)$$

$$\mathbf{a}_3 = (0, 0, 1, 0, 0, 0, 0), \quad \mathbf{b}_3 = (0, 0, 0, 0, 0, 1, 0), \quad \mathbf{c}_3 = (0, 0, 0, 0, 0, 0, 1)$$

$$\mathbf{a}_4 = (1, 0, -1, 0, 0, 0, 0), \quad \mathbf{b}_4 = (0, 0, 0, 1, 1, 0, 0), \quad \mathbf{c}_4 = (0, 1, 0, 0, 0, 0, -1)$$

# Quadratic Arithmetic Program

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- Although Rank-1 Constraint Systems provide a powerful method for representing computations, they are not succinct.
- We need to transform our computations into a form that is more convenient for proving statements about them.

We finished with:

$$\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m, \quad \mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m, \quad \mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_m,$$



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$$\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m, \quad \mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m, \quad \mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_m,$$

Of course, they form corresponding matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} \quad C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}$$

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An example of a single “if” statement:

$$\mathbf{a}_1 = (0, 0, 1, 0, 0, 0, 0)$$

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$$\mathbf{a}_3 = (0, 0, 1, 0, 0, 0, 0)$$

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Pleeeenty of zeroes, doesn't it? And this is just one out of 3 matrices...

The previous witness vector:

$$\mathbf{w} = (1, r, x_1, x_2, x_3, \text{mult}, \text{selectMult})$$

Let's take a closer look at the matrix columns:

$$\begin{bmatrix} \circ & \circ & \overset{3}{\boxed{1}} & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & 1 & \circ & \circ & \circ \\ \circ & \circ & 1 & \circ & \circ & \circ & \circ \\ 1 & \circ & \boxed{-1} & \circ & \circ & \circ & \circ \end{bmatrix}$$

Consider 4th constraint:  $(1 - x_1) \times (x_2 + x_3) = r - \text{selectMult}$

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So, every column is a mapping of constraint number to a coefficient for the witness element.

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As we know, such a mapping can be built using Lagrange interpolation polynomial with the following formula:

$$L(x) = \sum_{i=0}^n y_i \ell_i(x), \quad \ell_i(x) = \prod_{j=0, j \neq i}^n \frac{x - x_j}{x_i - x_j}.$$

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There are  $n$  columns and  $m$  constraints. So, it results in  $n$  polynomials such that:

$$A_j(i) = a_{i,j}, \quad i \in \{1, 2, \dots, m\}, \quad j \in \{1, 2, \dots, n\}$$



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The same is true for matrices  $B$  and  $C$ , with  $3n$  polynomials in total,  $n$  for each of the coefficients matrices:

$$A_1(x), A_2(x), \dots, A_n(x), B_1(x), B_2(x), \dots, B_n(x), C_1(x), C_2(x), \dots, C_n(x)$$

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### Note

We could have assigned any *unique* index from  $\mathbb{F}$  to each constraint (say,  $t_i$  for each  $i \in \{1, \dots, m\}$ ) and interpolate through these points:

$$A_j(t_i) = a_{i,j}, \quad i \in \{1, 2, \dots, m\}, \quad j \in \{1, 2, \dots, n\}$$

## Example

Considering the witness vector  $\mathbf{w}$  and matrix  $A$  from the previous example, for the variable  $x_1$ , the next set of points can be derived:

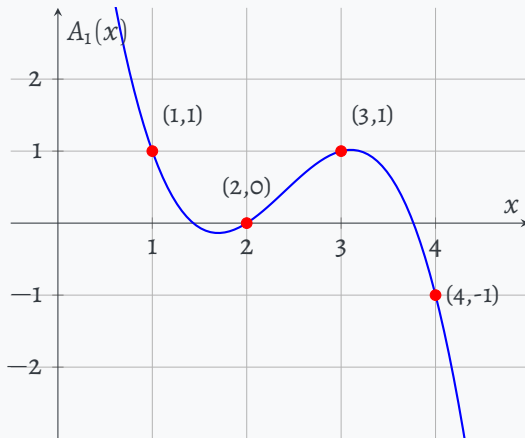
$$\{(1, 1), (2, 0), (3, 1), (4, -1)\}$$

The Lagrange interpolation polynomial for this set of points:

$$\begin{aligned} \ell_1(x) &= -\frac{(x-2)(x-3)(x-4)}{6}, & \ell_2(x) &= \frac{(x-1)(x-3)(x-4)}{2}, \\ \ell_3(x) &= -\frac{(x-1)(x-2)(x-4)}{2}, & \ell_4(x) &= \frac{(x-1)(x-2)(x-3)}{6}. \end{aligned}$$

Thus, the polynomial is given by:

$$\begin{aligned} A_{x_1}(x) &= 1 \cdot \ell_1(x) + 0 \cdot \ell_2(x) + 1 \cdot \ell_3(x) + (-1) \cdot \ell_4(x) \\ &= -\frac{5}{6}x^3 + 6x^2 - \frac{79}{6}x + 9 \end{aligned}$$



**Illustration:** The Lagrange interpolation polynomial for points  $\{(1, 1), (2, 0), (3, 1), (4, -1)\}$  visualized over  $\mathbb{R}$ .

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Consider two polynomials  $p(x)$  and  $q(x)$ :

$$p(x) = -\frac{1}{2}x^2 + \frac{3}{2}x, \quad q(x) = \frac{1}{3}x^3 - 2x^2 + \frac{8}{3}x + 1.$$

With corresponding sets of points:

$$\{(0, 0), (1, 1), (2, 1), (3, 0)\}, \quad \{(0, 1), (1, 2), (2, 1), (3, 0)\}$$

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With corresponding sets of points:

$$\{(0, 0), (1, 1), (2, 1), (3, 0)\}, \quad \{(0, 1), (1, 2), (2, 1), (3, 0)\}$$

The sum of these polynomials can be calculated as:

$$r(x) = \frac{1}{3}x^3 - 2\frac{1}{2}x^2 + 4\frac{1}{6}x + 1$$

The resulting polynomial  $r(x)$  corresponds to the set of points:

$$\{(0, 1), (1, 3), (2, 2), (3, 0)\}$$

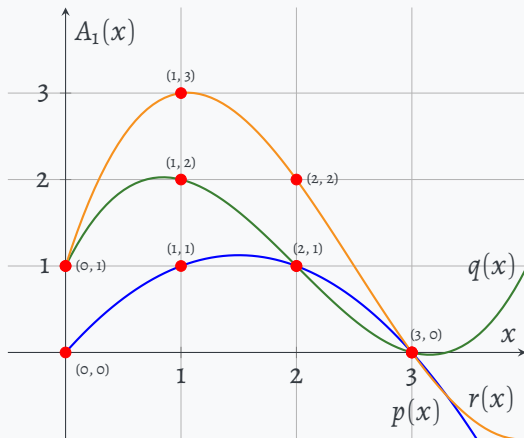


Figure: Addition of two polynomials



Now, using coefficients encoded with polynomials, we can build a constraint number  $X \in \{1, \dots, m\}$  in the next way:

$$\begin{aligned} & (w_1A_1(X) + w_2A_2(X) + \dots + w_nA_n(X)) \times \\ & \times (w_1B_1(X) + w_2B_2(X) + \dots + w_nB_n(X)) = \\ & = (w_1C_1(X) + w_2C_2(X) + \dots + w_nC_n(X)) \end{aligned}$$

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Or written more concisely:

$$\left( \sum_{i=1}^n w_i A_i(X) \right) \times \left( \sum_{i=1}^n w_i B_i(X) \right) = \left( \sum_{i=1}^n w_i C_i(X) \right)$$

Hold on, but why does it hold? Let us substitute any  $X = j$  into this equation:

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Recall that we interpolated polynomials to have  $A_i(j) = a_{j,i}$ . Therefore, the equation above can be reduced to:

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But hold on again! Notice that  $\sum_{i=1}^n w_i a_{j,i} = \langle \mathbf{w}, \mathbf{a}_j \rangle$  and therefore we have:

$$\langle \mathbf{w}, \mathbf{a}_j \rangle \times \langle \mathbf{w}, \mathbf{b}_j \rangle = \langle \mathbf{w}, \mathbf{c}_j \rangle \quad \forall j \in \{1, \dots, m\},$$

so we ended up with the initial  $m$  constraint equations!

Now let us define polynomials  $A(X)$ ,  $B(X)$ ,  $C(X)$  for easier notation:

$$A(X) = \sum_{i=1}^n w_i A_i(X), \quad B(X) = \sum_{i=1}^n w_i B_i(X), \quad C(X) = \sum_{i=1}^n w_i C_i(X)$$

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Therefore:

$$A(X) \times B(X) = C(X)$$

Now, we can define a polynomial  $M(X)$ , that has zeros at all elements from the set  $\Omega = \{1, \dots, m\}$

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It means, that  $M(X)$  can be divide by **vanishing polynomial**  $Z_\Omega(X)$  without a remainder!

$$Z_\Omega(X) = \prod_{i=1}^m (X - i), \quad H(X) = \frac{M(X)}{Z_\Omega(X)}$$



## Definition (Quadratic Arithmetic Program)

Suppose that  $m$  R1CS constraints with a witness of size  $n$  are written in a form

$$A\mathbf{w} \odot B\mathbf{w} = C\mathbf{w}, \quad A, B, C \in \mathbb{F}^{m \times n}$$

Then, the **Quadratic Arithmetic Program** consists of  $3n$  polynomials  $A_1, \dots, A_n, B_1, \dots, B_n, C_1, \dots, C_n$  such that:

$$A_j(i) = a_{ij}, \quad B_j(i) = b_{ij}, \quad C_j(i) = c_{ij}, \quad \forall i \in \{1, \dots, m\} \quad \forall j \in \{1, \dots, n\}$$

Then,  $\mathbf{w} \in \mathbb{F}^n$  is a valid assignment for the given QAP and **target polynomial**  $Z(X) = \prod_{i=1}^m (X - i)$  if and only if there exists such a polynomial  $H(X)$  such that

$$\left( \sum_{i=1}^n w_i A_i(X) \right) \left( \sum_{i=1}^n w_i B_i(X) \right) - \left( \sum_{i=1}^n w_i C_i(X) \right) = Z(X)H(X)$$

# Probabilistically Checkable Proofs

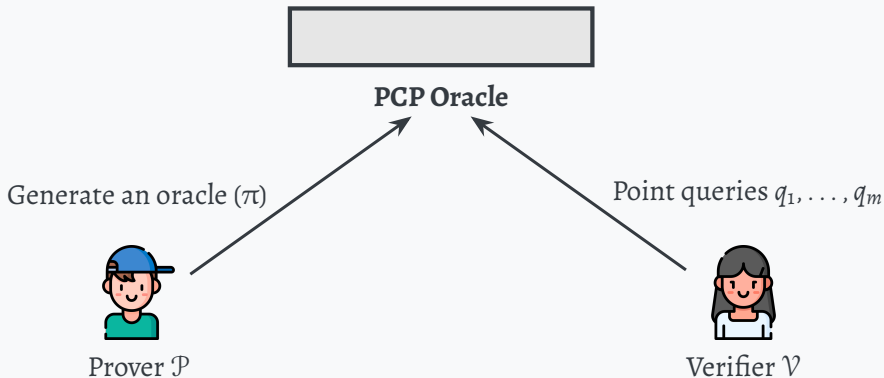


Figure: Illustration of a Probabilistically Checkable Proof (PCP) system. The prover  $\mathcal{P}$  generates a PCP oracle  $\pi$  that is queried by the verifier  $\mathcal{V}$  at specific points  $q_1, \dots, q_m$ .

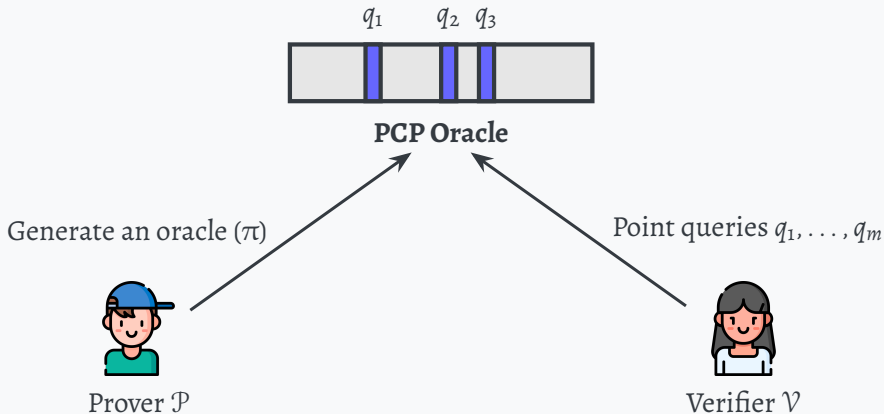


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Three main extensions of PCPs that are frequently used in SNARKs are:

- **IPCP (Interactive PCP)**: The prover commits to the PCP oracle and then, based on the interaction between the prover and verifier, the verifier queries the oracle and decides whether to accept the proof.
- **IOP (Interactive Oracle Proof)**: The prover and verifier interact and on each round, the prover commits to a new oracle. The verifier queries the oracle and decides whether to accept the proof.
- **LPCP (Linear PCP)**: The prover commits to a linear function and the verifier queries the function at specific points.

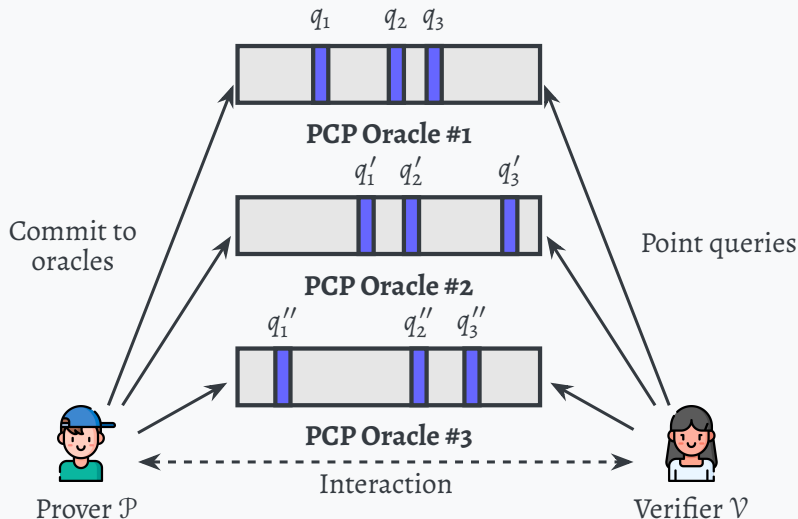


Figure: Illustration of an Interactive Oracle Proof (IOP). On each round  $i$  ( $1 \leq i \leq r$ ),  $\mathcal{V}$  sends a message  $m_i$ , and  $\mathcal{P}$  commits to a new oracle  $\pi_i$ , which  $\mathcal{V}$  can query at  $\mathbf{q}_i = (q_{i,1}, \dots, q_{i,m})$ .

## QAP as a Linear PCP

### *Definition (Linear PCP)*

A **Linear PCP** is a PCP where the prover commits to a linear function  $\pi = (\pi_1, \dots, \pi_k)$  and the verifier queries the function at specific points  $q_1, \dots, q_r$ . Then, the prover responds with the values of the function at these points:

$$\langle \pi_1, q_1 \rangle, \langle \pi_2, q_2 \rangle, \dots, \langle \pi_r, q_r \rangle.$$



## Example (QAP as a Linear PCP)

Recall that key QAP equation is:

$$L(x) \times R(x) - O(x) = Z(x)H(x).$$

Now, consider the following **linear PCP for QAP**:

1.  $\mathcal{P}$  commits to an extended witness  $\mathbf{w}$  and coefficients  $\mathbf{h} = (h_1, \dots, h_n)$  of  $H(x)$ .
2.  $\mathcal{V}$  samples  $\gamma \xleftarrow{R} \mathbb{F}$  and sends query  $\gamma = (\gamma, \gamma^2, \dots, \gamma^n)$  to  $\mathcal{P}$ .
3.  $\mathcal{P}$  reveals the following values:

$$\pi_1 \leftarrow \langle \mathbf{w}, \mathbf{L}(\gamma) \rangle,$$

$$\pi_2 \leftarrow \langle \mathbf{w}, \mathbf{R}(\gamma) \rangle,$$

$$\pi_3 \leftarrow \langle \mathbf{w}, \mathbf{O}(\gamma) \rangle,$$

$$\pi_4 \leftarrow Z(\gamma) \cdot \langle \mathbf{h}, \gamma \rangle.$$

4.  $\mathcal{V}$  checks whether  $\pi_1\pi_2 - \pi_3 = \pi_4$ .

## Question

Why is it safe to use such a check?

The polynomials  $L(x)$ ,  $R(x)$  and  $O(x)$  are interpolated polynomials using  $|C|$  (number of gates) points, so:

$$\deg(L) \leq |C|, \quad \deg(R) \leq |C|, \quad \deg(O) \leq |C|$$

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Thus, we can estimate the degree of polynomial

$$M(x) = L(x)R(x) - O(x).$$

$$\deg(M) \leq \max\{\deg(L) + \deg(R), \deg(O)\} \leq 2|C|$$

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If an adversary  $\mathcal{A}$  does not know a valid witness  $\mathbf{w}$ , he can compute a polynomial  $\tilde{M}(x) \leftarrow \mathcal{A}(\cdot)$  that satisfies a verifier  $\mathcal{V}$ :

$$\Pr_{s \xleftarrow{R} \mathbb{F}} [\tilde{M}(s) = M(s)] \leq \frac{2|C|}{|\mathbb{F}|}$$

*Thanks for your attention!*