Mathematics for Cryptographers. Preliminaries.

ZKDL Camp

July 18, 2024

Plan

Some words about the course

About ZKDL

- ZKDL Camp is a series of lectures and workshops on zero-knowledge proofs and cryptography.
- Here, we will learn state-of-the-art zero-knowledge systems: what are SNARKs, how they work under the hood from total scratch.
- If possible, we will conduct workshops, where we will show practical implementations of the theoretical material.
- Primary audience: cryptographers, R&D Engineers, ZK developers, and everyone wanting to boost their understanding of cryptography.

Note

This is not a regular course: we require a lot of commitment and the material is fairly complex. However, we will try to make it as simple as possible.

Approximate Camp Structure

1. Mathematics Preliminaries (3-4 lectures): group and number theory, finite fields, polynomials, elliptic curves etc.
2. Deep Delve into SNARKs: General definition, arithmetic circuits, commitment schemes, encryption etc.
3. Analysis of modern zero-knowledge proving systems: Groth16, Plonk, BulletProofs, STARK etc.
4. Specialization topics: low-level optimizations, advanced protocols such as folding schemes, Nova etc.

Notation

Sets

Definition

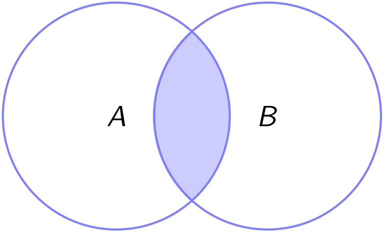
Set is a collection of distinct objects, considered as an object in its own right.

Example

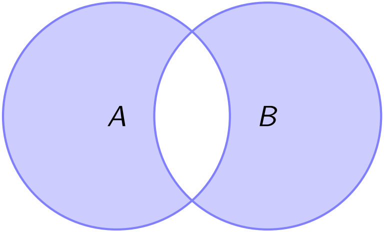
- $\mathbb{N}$ is a set of natural numbers.
- $\mathbb{Z}$ is a set of integers.
- $\mathbb{R}$ is a set of real numbers.
- $\mathbb{C}$ is a set of complex numbers.
- $\{1, 2, 5, 10\}$ is a set of four elements.
- $\{1, 2, 2, 3\}$ simply equals to $\{1, 2, 3\}$ – we do not count duplicates.

Operations on sets

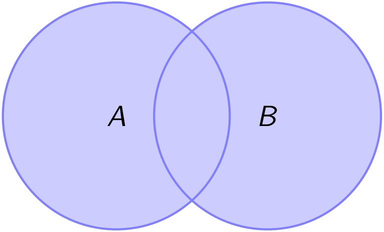
$A \cap B$



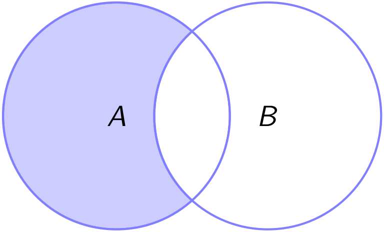
$\overline{A \cap B}$



$A \cup B$



$A \setminus B$



Defining sets

Example

- $\{x \in \mathbb{R} : x^2 = 1\}$ – a set of real numbers that satisfy the equation $x^2 = 1$.
- $\{x \in \mathbb{Z} : x \text{ is even}\}$ – a set of even integers.
- $\{x^2 : x \in \mathbb{R}, x^3 = 1\}$ – a set of squares of real numbers that satisfy the equation $x^3 = 1$.
- $\{x \in \mathbb{N} : x \text{ is prime}\}$ – a set of prime natural numbers.

Question #1

How to simplify the set $\{x \in \mathbb{N} : x^2 = 2\}$?

Question #2(*)

How to simplify the set $\{\sin \pi k : k \in \mathbb{Z}\}$?

Basic Logic

- $\forall$ means "for all".
- $\exists$ means "there exists".
- $\wedge$ means "and".
- $\vee$ means "or".

Question #1

Is it true that $(\forall x \in \mathbb{N}) : \{x > 0\}$?

Question #2

Is it true that $(\exists x \in \mathbb{N}) : \{x \geq 0 \wedge x < 1\}$?

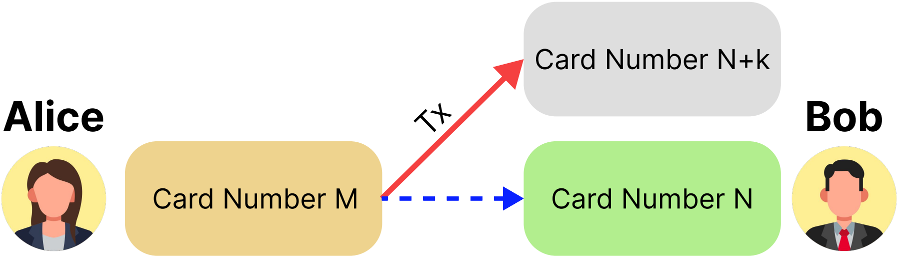
Question #3

Is it true that $(\forall x \in \mathbb{Z})\,(\exists y \in \mathbb{N}) : \{y > x\}$?

Basic Group Theory

Why Groups?!

Well, first of all, we want to work with integers...
Imagine that Alice pays to Bob with a card number N, but instead of paying to a number N, the system pays to another card number $N + k, k \ll N$, which is only by 0.001% different. Bob would not be 99.999% happy...



Why Groups?!

But integers on their own are not enough. We need to define a structure that allows us to perform operations on them.
This is very similar to interfaces: we abstract from the implementation, just merely stating we have "some" addition/multiplication.

Example

Consider set $\mathbb{G} := \{\text{Dmytro}, \text{Dan}, \text{Friendship}\}$. We can safely define an operation $\oplus$ as:

$$\text{Dmytro} \oplus \text{Dan} = \text{Friendship}$$
$$\text{Dan} \oplus \text{Friendship} = \text{Dmytro}$$
$$\text{Friendship} \oplus \text{Dmytro} = \text{Dan}$$

Rethorical question

What makes $(\mathbb{G}, \oplus)$ a group?

Group Definition

Definition

Group $(\mathbb{G}, \oplus)$, is a set with a binary operation $\oplus$ with following rules:

1. **Closure:** Binary operations always outputs an element from $\mathbb{G}$, that is $\forall a, b \in \mathbb{G} : a \oplus b \in \mathbb{G}$.
2. **Associativity:** $\forall a, b, c \in \mathbb{G} : (a \oplus b) \oplus c = a \oplus (b \oplus c)$.
3. **Identity element:** There exists a so-called identity element $e \in \mathbb{G}$ such that $\forall a \in \mathbb{G} : e \oplus a = a \oplus e = a$.
4. **Inverse element:** $\forall a \in \mathbb{G}\ \exists b \in \mathbb{G} : a \oplus b = b \oplus a = e$. We commonly denote the inverse element as $(\ominus a)$.

Definition

A group is called **abelian** if it satisfies the additional rule called **commutativity**: $\forall a, b \in \mathbb{G} : a \oplus b = b \oplus a$.

Explanation for Developers: Trait

```rust
/// Trait that represents a group.
pub trait Group: Sized {
    /// Checks whether the two elements are equal.
    fn eq(&self, other: &Self) -> bool;
    /// Returns the identity element of the group.
    fn identity() -> Self;
    /// Adds two elements of the group.
    fn add(&self, a: &Self) -> Self;
    /// Returns the negative of the element.
    fn negate(&self) -> Self;
    /// Subtracts two elements of the group.
    fn sub(&self, a: &Self) -> Self {
        self.add(&a.negate())
    }
}
```

More on that: `https://github.com/ZKDL-Camp/lecture-1-math`.

Group Examples

Example

A group of integers with the regular addition $(\mathbb{Z}, +)$ (also called the *additive* group of integers) is a group.

Example

The multiplicative group of positive real numbers $(\mathbb{R}_{>0}, \cdot)$ is a group for similar reasons.

Question #1

Is $(\mathbb{R}, \cdot)$ a group? If no, what is missing?

Question #2

Is $(\mathbb{Z}, \cdot)$ a group? If no, what is missing?

Abelian Groups Examples and Non-Examples

Question #3

Is $(\mathbb{R}, -)$ a group? If no, what is missing?

Question #4

Set V is a set of tuples (v_1, v_2, v_3) where each $v_i \in \mathbb{R} \setminus \{0\}$. Define the operation $\odot$ as

$$(v_1, v_2, v_3) \odot (u_1, u_2, u_3) = (v_1 u_1, v_2 u_2, v_3 u_3)$$

Is $(V, \odot)$ a group? If no, what is missing?

Conclusion

Group is just a fancy name for a set with a binary operation that behaves nicely.

Polynomials

Polynomial

Definition

Field K is a set equipped with appropriate **addition** and **multiplication** operations with the corresponding well-defined inverses, where you can perform the basic arithmetic.

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Definition

Field K is a set equipped with appropriate **addition** and **multiplication** operations with the corresponding well-defined inverses, where you can perform the basic arithmetic.

- $\mathbb{R}$ (real numbers) is a field.
- $\mathbb{Q}$ (rational numbers) is a field.
- $\mathbb{C}$ (complex numbers) is a field.
- $\mathbb{N}$ (natural numbers) is not a field: there is no additive inverse for 2 (-2 is not in $\mathbb{N}$).
- $\mathbb{Z}$ (integers) is not a field: additive inverse is defined, but the multiplicative is not (2^{-1} is not defined).

Thanks for your attention!