Introduction to zk-SNARK. R1CS

Sep 12, 2024

Distributed Lab

Plan

What is zk-SNARK?

Boolean Circuits

Arithmetic Circuits

Linear Algebruh Preliminaries

Rank-1 Constraint System

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Definition

zk-SNARK

Zero-**K**nowledge **S**uccinct **N**on-interactive **AR**gument of **K**nowledge.

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- **Non-interactiveness** — to produce the proof, the prover does not need any interaction with the verifier.
- **Zero-Knowledge** — the verifier learns nothing about the data used to produce the proof, despite knowing that this data resolves the given problem and that the prover possesses it.

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Well... Let's take a look at some example.

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Imagine you're part of a treasure hunt...

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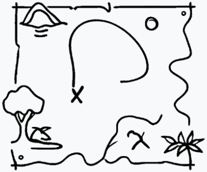
...but how to prove that without revealing the chest location?

Still don't get who is SNARK...

The Problem

You have found a hidden treasure chest, and you want to prove to the organizer that you know its location without actually revealing that.



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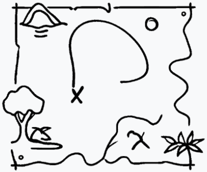

We can retrieve some information from that:

The Secret Data: the exact treasure location.

The Prover: you.

The Verifier: the treasure hunt organizer.

Ohh… Got it!

Here is how we can apply the zk-SNARK to our problem:

- **Argument of Knowledge**: You need to create a proof that demonstrates you know the chest is.

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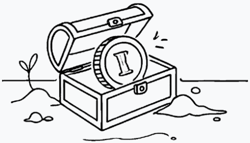

Well... The golden coin where the pirates' sign is engraved is our zk-SNARK proof!

The First Question To Resolve

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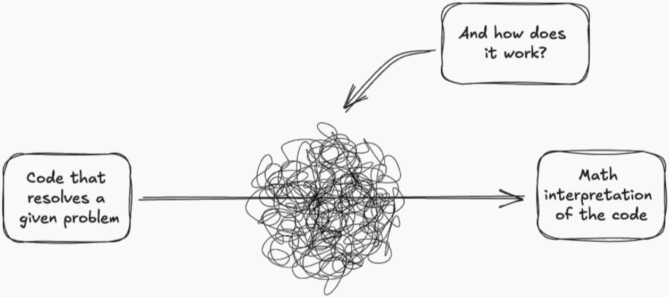
When we need to prove that some element is in a merkle tree, we can't come to a verifier and give them a "coin"...

> *Question?*
>
> How do we convert a program into a mathematical language?



Boolean Circuits

Boolean Circuits

We can do that in a way like the computer does it - boolean circuits.

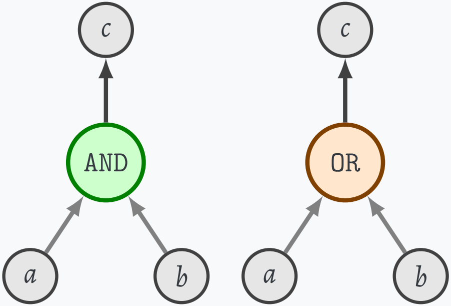


Figure: Boolean AND and OR Gates

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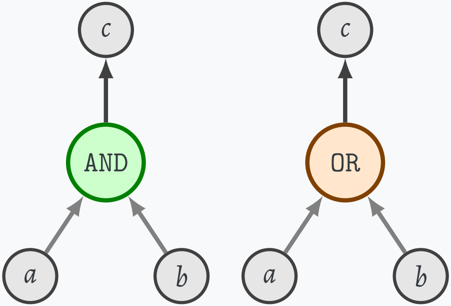

Figure: Boolean AND and OR Gates

A	B	A AND B
0	0	0
0	1	0
1	0	0
1	1	1

Figure: AND Gate Truth Table

Note

With any of {AND, NOT} or {OR, NOT} gates sets one can build any possible logical circuit, they are called **functionally complete** sets.

Boolean Circuit Example

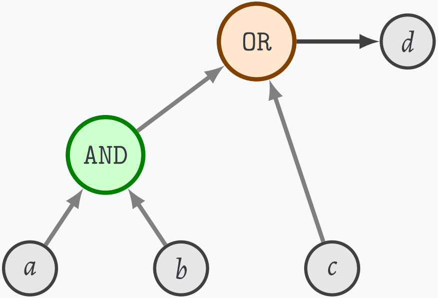

Figure: Example of a circuit evaluating $d = (a \ \texttt{AND} \ b) \ \texttt{OR} \ c$.

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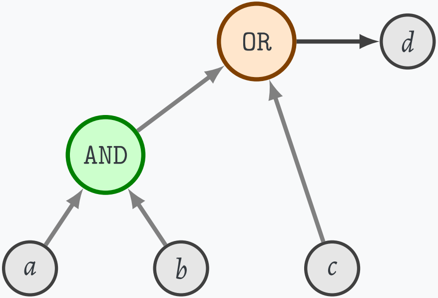

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Boolean circuits receive an input vector of 0, 1 and resolve to true (1) or false (0);

The above circuit can be satisfied with the next values:

$$a = 1, \quad b = 1, \quad c = 0, \quad d = 1$$

SHA-256 Boolean circuit

File	No. ANDs	No. XORs	No. INVs
sha256Final.txt	22,272	91,780	2,194

Figure: Stats of a SHA256 boolean circuit implementation.

More than 100000 gates. Impressive, doesn't it?

But it also shows how inconvenient the boolean circuits are.

Arithmetic Circuits

Arithmetic Circuits

Similar to Boolean Circuits, the **Arithmetic circuits** consist of gates and wires.

- Wires: elements of some finite field $\mathbb{F}_p$.
- Gates: addition ($\oplus$) and multiplication ($\odot$) corresponding to the field.

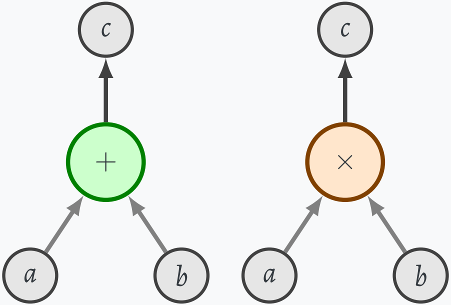

Figure: Addition and Multiplication Gates

Arithmetic Circuits Example I

Example

```
def multiply(a: F, b: F) -> F:
    return a * b
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The witness vector (essentially, our solution vector) is $\mathbf{w} = (r, a, b)$, for example: $(6, 2, 3)$.

We assume that the a and b are input values.

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Note

We can think of the "=" in the gate as an assertion.

Arithmetic Circuits Example II

Example

Now, suppose we want to implement the evaluation of the polynomial $Q(x_1, x_2) = x_1^3 + x_2^2 \in \mathbb{F}[x_1, x_2]$ using arithmetic circuits.

```
def evaluate(x1: F, x2: F) -> F:
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Looks easy, right?

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Looks easy, right? But the circuit is now much less trivial.

$$
\begin{aligned}
x_1^2 &= x_1 \times x_1 \\
x_1^3 &= x_1^2 \times x_1 \\
x_2^2 &= x_2 \times x_2 \\
Q &= x_1^3 + x_2^2
\end{aligned}
\quad \text{or} \quad
\begin{aligned}
r_1 &= x_1 \times x_1 \\
r_2 &= r_1 \times x_1 \\
r_3 &= x_2 \times x_2 \\
Q &= r_2 + r_3
\end{aligned}
$$

Arithmetic Circuits Example II

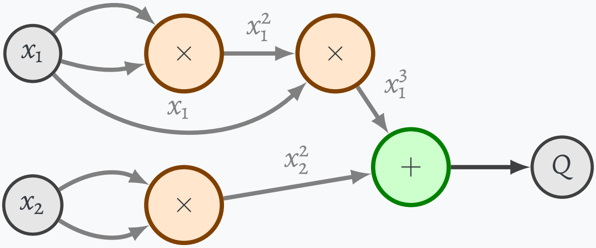

Figure: Example of a circuit evaluating $x_1^3 + x_2^2$.

Arithmetic Circuits Example III

Example

Well, it is quite clear how to represent any polynomial-like expressions. But how can we translate `if` statements?

```
def example(a: bool, b: F, c: F) -> F:
    if a:
        return b * c
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Corresponding equations for the circuit are:

$$r_1 = b \times c, \quad r_3 = 1 - a, \quad r_5 = r_3 \times r_2$$
$$r_2 = b + c, \quad r_4 = a \times r_1, \quad r = r_4 + r_5$$

Arithmetic Circuits Example III

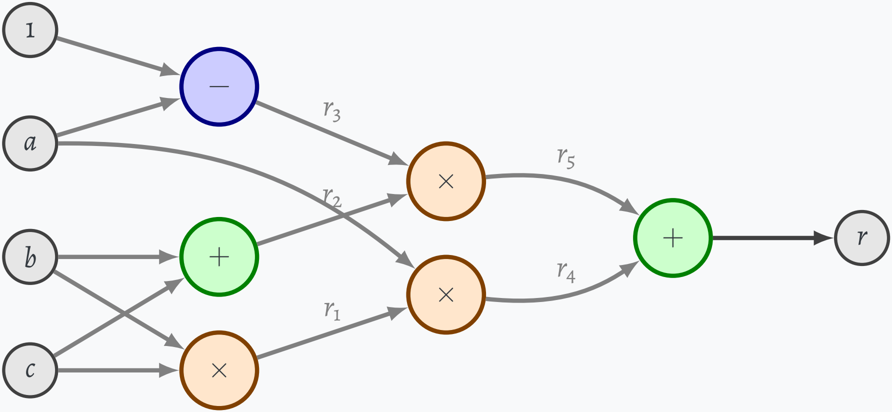

Figure: Example of a circuit evaluating the `if` statement logic.

Linear Algebruh Preliminaries

Vector Space

Definition

A **vector space** V over the field $\mathbb{F}$ is an abelian group for addition "+" together with a scalar multiplication operation "·" from $\mathbb{F} \times V$ to V, sending $(\lambda, x) \mapsto \lambda x$ and such that for any $\mathbf{v}, \mathbf{u} \in V$ and $\lambda, \mu \in \mathbb{F}$ we have:

- $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$
- $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$
- $(\lambda\mu)\mathbf{v} = \lambda(\mu\mathbf{v})$
- $1\mathbf{v} = \mathbf{v}$

Any element $\mathbf{v} \in V$ is called a **vector**, and any element $\lambda \in \mathbb{F}$ is called a **scalar**. We also mark vector elements in boldface.

Matrix

The matrix is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns. For example, the matrix A with m rows and n columns, consisting of elements from the finite field $\mathbb{F}$ is denoted as $A \in \mathbb{F}^{m \times n}$.

Definition

Let A, B be two matrices over the field $\mathbb{F}$. The following operations are defined:

- **Matrix addition/subtraction**: $A \pm B = \{a_{i,j} \pm b_{i,j}\}_{i,j=1}^{m \times n}$. The matrices A and B must have the same size $m \times n$.
- **Scalar multiplication**: $\lambda A = \{\lambda a_{i,j}\}_{1 \leqslant i,j \leqslant n}$ for any $\lambda \in \mathbb{F}$.
- **Matrix multiplication**: $C = AB$ is a matrix $C \in \mathbb{F}^{m \times p}$ with elements $c_{i,j} = \sum_{\ell=1}^{n} a_{i,\ell} b_{\ell,j}$. The number of columns in A must be equal to the number of rows in B, that is $A \in \mathbb{F}^{m \times n}$ and $B \in \mathbb{F}^{n \times p}$.

Matrix Multiplication

Example

Consider

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 3}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 2}$$

We cannot add A and B since they have different sizes. However, we can multiply them:

$$AB = \begin{bmatrix} 5 & 6 \\ 7 & 9 \end{bmatrix}, \quad BA = \begin{bmatrix} 4 & 4 & 5 \\ 7 & 7 & 5 \\ 3 & 3 & 3 \end{bmatrix}$$

To see why, for example, the upper left element of AB is 5, we can calculate it as $\sum_{\ell=1}^{3} a_{1,\ell} b_{\ell,1} = 1 \times 2 + 1 \times 1 + 2 \times 1 = 5$.

Vector As A Matrix

Note

It just so happens that when working with vectors, we usually assume that they are **column vectors**. This means that the vector $v = (v_1, v_2, \dots, v_n)$ is represented as a matrix:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

This is a common convention in linear algebra, and we will use it in the following sections.

Matrix Transpose

Definition (Transposition)

Given a matrix $A \in \mathbb{F}^{m \times n}$, the **transpose** of A is a matrix $A^\top \in \mathbb{F}^{n \times m}$ with elements $A^\top_{ij} = A_{ji}$.

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A^\top = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B^\top = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}^\top = [1, 2, 3]$$

Inner Product

Definition

Consider the vector space $\mathbb{V}$ over the finite field $\mathbb{F}_p$. The **inner product** is a function $\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{F}_p$ satisfying the following conditions for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}$:

- $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$.
- $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$.
- $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ for all $\mathbf{u} \in \mathbb{V}$ iff $\mathbf{v} = \mathbf{0}$.
- $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ for all $\mathbf{v} \in \mathbb{V}$ iff $\mathbf{u} = \mathbf{0}$.

Plenty of functions can be built that satisfy the inner product definition, we'll use the one that is usually called **dot product**.

Dot Product

Definition

Consider the vector space $\mathbb{V}$ over the finite field $\mathbb{F}_p$. The **dot product** on $\mathbb{V}$ is a function $\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{F}$, defined for every $\mathbf{u}, \mathbf{v} \in \mathbb{V}$ as follows:

$$\langle \mathbf{u}, \mathbf{v} \rangle := \mathbf{u}^\top \mathbf{v} = \sum_{i=1}^{n} u_i v_i$$

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Note

The dot product can also be denoted using the dot notation as:

$$\mathbf{u} \cdot \mathbf{v}$$

That is why it's called the "dot" product.

Dot Product

Example

Let $\mathbf{u}, \mathbf{v}$ are vectors over the real number $\mathbb{R}$, where

$$\mathbf{u} = (1, 2, 3), \quad \mathbf{v} = (2, 4, 3)$$

Then:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{3} u_i v_i = 2 \cdot 1 + 2 \cdot 4 + 3 \cdot 3 = 2 + 8 + 9 = 19$$

Hadamard Product

Definition

Suppose $A, B \in \mathbb{F}^{m \times n}$. The **Hadamard product** $A \odot B$ gives a matrix C such that $C_{i,j} = A_{i,j} B_{i,j}$. Essentially, we multiply elements elementwise.

Example

Consider $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$. Then, the Hadamard product:

$$A \odot B = \begin{bmatrix} 1 \cdot 3 & 1 \cdot 2 & 2 \cdot 1 \\ 3 \cdot 0 & 0 \cdot 2 & 3 \cdot 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Outer Product

Definition

Given two vectors $\mathbf{u} \in \mathbb{F}^n, \mathbf{v} \in \mathbb{F}^m$ the **outer product** is a the matrix whose entries are all products of an element in the first vector with an element in the second vector:

$$\mathbf{u} \otimes \mathbf{v} := \mathbf{u}\mathbf{v}^\top = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \cdots & u_2 v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_m v_1 & u_m v_2 & \cdots & u_m v_n \end{bmatrix}$$

Outer Product

Lemma (Properties of outer product)

For any scalar $c \in \mathbb{F}$ *and* $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{F}^n \times \mathbb{F}^m \times \mathbb{F}^p$:

- *Transpose:* $(\mathbf{u} \otimes \mathbf{v}) = (\mathbf{v} \otimes \mathbf{u})^T$
- *Distributivity:* $\mathbf{u} \otimes (\mathbf{v} + \mathbf{w}) = \mathbf{u} \otimes \mathbf{v} + \mathbf{u} \otimes \mathbf{w}$
- *Scalar Multiplication:* $c(\mathbf{v} \otimes \mathbf{u}) = (c\mathbf{v}) \otimes \mathbf{u} = \mathbf{v} \otimes (c\mathbf{u})$
- *Rank: the outer product* $\mathbf{u} \otimes \mathbf{v}$ *is a rank-1 matrix if* $\mathbf{u}$ *and* $\mathbf{v}$ *are non-zero vectors*

Outer Product

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Let $\mathbf{u}, \mathbf{v}$ are vectors over the real number $\mathbb{R}$, where

$$\mathbf{u} = (1, 2, 3), \quad \mathbf{v} = (2, 4, 3)$$

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The rows/columns number 2 and 3 in the result matrix can be represented as a linear combination of the first row/column, specifically by multiplying it by 2 and 3, respectively.

Rank-1 Constraint System

Constraint Definition

Definition

Each **constraint** in the Rank-1 Constraint System must be in the form:

$$\langle \mathbf{a}, \mathbf{w} \rangle \times \langle \mathbf{b}, \mathbf{w} \rangle = \langle \mathbf{c}, \mathbf{w} \rangle$$

Where $\mathbf{w}$ is a vector containing all the *input*, *output*, and *intermediate* variables involved in the computation. The vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors of coefficients corresponding to these variables, and they define the relationship between the linear combinations of $\mathbf{w}$ on the left-hand side and the right-hand side of the equation.

Constraint Example

Example

Consider the most basic circuit with one multiplication gate: $x_1 \times x_2 = r$. The witnes vector $\mathbf{w} = (r, x_1, x_2)$. So

$$w_2 \times w_3 = w_1$$
$$(0 + w_2 + 0) \times (0 + 0 + w_3) = w1 + 0 + 0$$
$$(0w_1 + 1w_2 + 0w_3) \times (0w_1 + 0w_2 + 1w_3) = 1w_1 + 0w_2 + 0w_3$$

Therefore the coefficients vectors are:

$$\mathbf{a} = (0, 1, 0), \quad \mathbf{b} = (0, 0, 1), \quad \mathbf{c} = (1, 0, 0).$$

The general form of our constraint is:

$$(a_1w_1 + a_2w_2 + a_3w_3)(b_1w_1 + b_2w_2 + b_3w_3) = c_1w_1 + c_2w_2 + c_3w_3$$

Constraint System Example

Now, let us consider a more complex example.

```
def r(x1: F, x2: F, x3: F) -> F:
    return x2 * x3 if x1 else x2 + x3
```

That can be expressed as:

$$r = x_1 \times (x_2 \times x_3) + (1 - x_1) \times (x_2 + x_3)$$

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```

That can be expressed as:

$$r = x_1 \times (x_2 \times x_3) + (1 - x_1) \times (x_2 + x_3)$$

We need a boolean restriction for x_1:

$$x_1 \times (1 - x_1) = 0$$

Constraint System Example

Now, let us consider a more complex example.

```
def r(x1: F, x2: F, x3: F) -> F:
    return x2 * x3 if x1 else x2 + x3
```

That can be expressed as:

$$r = x_1 \times (x_2 \times x_3) + (1 - x_1) \times (x_2 + x_3)$$

We need a boolean restriction for x_1:

$$x_1 \times (1 - x_1) = 0$$

Thus, the next constraints can be build:

$$x_1 \times x_1 = x_1 \quad \text{(binary check)} \tag{1}$$

$$x_2 \times x_3 = \mathsf{mult} \tag{2}$$

$$x_1 \times \mathsf{mult} = \mathsf{selectMult} \tag{3}$$

$$(1 - x_1) \times (x_2 + x_3) = r - \mathsf{selectMult} \tag{4}$$

Constraint System Example

The witness vector: $\mathbf{w} = (1, r, x_1, x_2, x_3, \text{mult}, \text{selectMult})$. The coefficients vectors:

$$
\begin{aligned}
\mathbf{a}_1 &= (0,0,1,0,0,0,0), & \mathbf{b}_1 &= (0,0,1,0,0,0,0), & \mathbf{c}_1 &= (0,0,1,0,0,0,0) \\
\mathbf{a}_2 &= (0,0,0,1,0,0,0), & \mathbf{b}_2 &= (0,0,0,0,1,0,0), & \mathbf{c}_2 &= (0,0,0,0,0,1,0) \\
\mathbf{a}_3 &= (0,0,1,0,0,0,0), & \mathbf{b}_3 &= (0,0,0,0,0,1,0), & \mathbf{c}_3 &= (0,0,0,0,0,0,1) \\
\mathbf{a}_4 &= (1,0,-1,0,0,0,0), & \mathbf{b}_4 &= (0,0,0,1,1,0,0), & \mathbf{c}_4 &= (0,1,0,0,0,0,-1)
\end{aligned}
$$

Using the arithmetic in a large $\mathbb{F}_p$, consider the following values:

$$x_1 = 1, \quad x_2 = 3, \quad x_3 = 4$$

Verifying the constraints:

1. $x_1 \times x_1 = x_1 \quad (1 \times 1 = 1)$
2. $x_2 \times x_3 = \text{mult} \quad (3 \times 4 = 12)$
3. $x_1 \times \text{mult} = \text{selectMult} \quad (1 \times 12 = 12)$
4. $(1 - x_1) \times (x_2 + x_3) = r - \text{selectMult} \quad (0 \times 7 = 12 - 12)$

R1CS In Matrix Form

Theorem

Consider a Rank-1 Constraint System (R1CS) defined by m constraints. Each constraint is associated with coefficient vectors $\mathbf{a}_i$, $\mathbf{b}_i$, and $\mathbf{c}_i$, where $i \in \{1, 2, \ldots, m\}$ and also a witness vector $\mathbf{w}$ consisting of n elements. Then this system can also be represented using the corresponding matrices A, B, and C.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \quad C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

such that all constraints can be reduced to the single equation:

$$A\mathbf{w} \odot B\mathbf{w} = C\mathbf{w}$$

R1CS In Matrix Form

Proof. Matrices defined this way can be expressed as

$$A = \begin{bmatrix} \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \\ \vdots \\ \mathbf{a}_m^\top \end{bmatrix}, \quad B = \begin{bmatrix} \mathbf{b}_1^\top \\ \mathbf{b}_2^\top \\ \vdots \\ \mathbf{b}_m^\top \end{bmatrix}, \quad C = \begin{bmatrix} \mathbf{c}_1^\top \\ \mathbf{c}_2^\top \\ \vdots \\ \mathbf{c}_m^\top \end{bmatrix}$$

R1CS In Matrix Form

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Consider an expression $A\mathbf{w}$:

$$A\mathbf{w} = \begin{bmatrix} \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \\ \vdots \\ \mathbf{a}_m^\top \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^\top \mathbf{w} \\ \mathbf{a}_2^\top \mathbf{w} \\ \vdots \\ \mathbf{a}_m^\top \mathbf{w} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n a_{1i} w_i \\ \sum_{i=1}^n a_{2i} w_i \\ \vdots \\ \sum_{i=1}^n a_{ni} w_i \end{bmatrix} = \begin{bmatrix} \langle \mathbf{a}_1, \mathbf{w} \rangle \\ \langle \mathbf{a}_2, \mathbf{w} \rangle \\ \vdots \\ \langle \mathbf{a}_m, \mathbf{w} \rangle \end{bmatrix}$$

R1CS In Matrix Form

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Therefore, we have:

$$A\mathbf{w} = \begin{bmatrix} \langle \mathbf{a}_1, \mathbf{w} \rangle \\ \langle \mathbf{a}_2, \mathbf{w} \rangle \\ \vdots \\ \langle \mathbf{a}_m, \mathbf{w} \rangle \end{bmatrix}, \quad B\mathbf{w} = \begin{bmatrix} \langle \mathbf{b}_1, \mathbf{w} \rangle \\ \langle \mathbf{b}_2, \mathbf{w} \rangle \\ \vdots \\ \langle \mathbf{b}_m, \mathbf{w} \rangle \end{bmatrix}, \quad C\mathbf{w} = \begin{bmatrix} \langle \mathbf{c}_1, \mathbf{w} \rangle \\ \langle \mathbf{c}_2, \mathbf{w} \rangle \\ \vdots \\ \langle \mathbf{c}_m, \mathbf{w} \rangle \end{bmatrix}$$

R1CS In Matrix Form

Proof. Matrices defined this way can be expressed as

$$A = \begin{bmatrix} \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \\ \vdots \\ \mathbf{a}_m^\top \end{bmatrix}, \quad B = \begin{bmatrix} \mathbf{b}_1^\top \\ \mathbf{b}_2^\top \\ \vdots \\ \mathbf{b}_m^\top \end{bmatrix}, \quad C = \begin{bmatrix} \mathbf{c}_1^\top \\ \mathbf{c}_2^\top \\ \vdots \\ \mathbf{c}_m^\top \end{bmatrix}$$

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Thus, $A\mathbf{w} \odot B\mathbf{w} = C\mathbf{w}$ is equivalent to the system of m constraints:

$$\langle \mathbf{a}_j, \mathbf{w} \rangle \times \langle \mathbf{b}_j, \mathbf{w} \rangle = \langle \mathbf{c}_j, \mathbf{w} \rangle, \; j \in \{1, \dots, m\}.$$

R1CS In Matrix Form

Example

The vectors $\mathbf{a}_i$ from the previous examples have the form:

$$\begin{aligned}
\mathbf{a}_1 &= (0, 0, 1, 0, 0, 0, 0) \\
\mathbf{a}_2 &= (0, 0, 0, 1, 0, 0, 0) \\
\mathbf{a}_3 &= (0, 0, 1, 0, 0, 0, 0) \\
\mathbf{a}_4 &= (1, 0, -1, 0, 0, 0, 0)
\end{aligned}$$

This corresponds to $n = 7, m = 4$

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} & a_{2,7} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} & a_{3,7} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Why Rank-1?

Lemma

Suppose we have a constraint $\langle \mathbf{a}, \mathbf{w} \rangle \times \langle \mathbf{b}, \mathbf{w} \rangle = \langle \mathbf{c}, \mathbf{w} \rangle$ with coefficient vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and witness vector $\mathbf{w}$ (all from $\mathbb{F}^n$). Then it can be expressed in the form:

$$\mathbf{w}^\top A\mathbf{w} + \mathbf{c}^\top \mathbf{w} = 0$$

*Where A is the outer product of vectors $\mathbf{a}, \mathbf{b}$, so a **rank-1** matrix.*

Why Rank-1?

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Lemma proof. Consider $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{w} \in \mathbb{F}^n$.

$$\left(\sum_{i=1}^{n} a_i w_i\right) \times \left(\sum_{j=1}^{n} b_j w_j\right) = \sum_{k=1}^{n} c_k w_k$$

Combine the products into a double sum on the left side:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j w_i w_j = \mathbf{w}^\top (\mathbf{a} \otimes \mathbf{b})\mathbf{w} = \mathbf{w}^\top A\mathbf{w}$$

Thus, the constraint can be written as:

$$\mathbf{w}^\top A\mathbf{w} + \mathbf{c}^\top \mathbf{w} = 0$$

Thanks for your attention!