Field Extensions and Elliptic Curves

Distributed Lab

August 1, 2024

Plan

1. Field Extensions
   - A bit of intuition
   - General Definition
   - Polynomial Fraction Rings
   - Finite Field Extensions
2. Algebraic Closure
   - Definition
3. Elliptic Curve
   - Definition
   - Group Structure

Field Extensions

$\mathbb{Q}$ vs $\mathbb{R}$

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$\mathbb{R}$ is sort of "an extended version of $\mathbb{Q}$".

What about $\mathbb{R}$?

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Complex numbers $\mathbb{C}$ are defined as the set of pairs $(x, y) \in \mathbb{R}^2$ where addition is defined as $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$, and the multiplication is:

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1).$$

A bit about complex numbers

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What is $(1+i)+(2+i)$? $i(1+i)$? $1/i$?

Field Extension

Conclusion + Question

$\mathbb{C}$ is sort of "an extended version of $\mathbb{R}$". Thus, we have

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}, \text{ where } \mathbb{Q}, \mathbb{R}, \mathbb{C} \text{ are fields}$$

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Example

$\mathbb{C}/\mathbb{R}$ is a field extension. So is $\mathbb{R}/\mathbb{Q}$.

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Similarly, $\mathbb{Q}(i) := \{p + qi : p, q \in \mathbb{Q}\}$ is a field extension of $\mathbb{Q}$.

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Define $\mathbb{Q}(\sqrt{2}, i) = \{\alpha + \beta\sqrt{2} : \alpha, \beta \in \mathbb{Q}(i)\}$. Typical element of $\mathbb{Q}(\sqrt{2}, i)$ can be written as:

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$$(a + bi) + (c + di)\sqrt{2} = a + c\sqrt{2} + b\sqrt{2}i + di\sqrt{2}$$

Notice

Each element of $\mathbb{Q}(\sqrt{2}, i)$ is a linear combination of $\{1, \sqrt{2}, i, \sqrt{2}i\}$. This is usually called a **basis**. Moreover, to denote the dimensionality of $\mathbb{Q}(\sqrt{2}, i)$ over $\mathbb{Q}$, we write $[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}] = 4$.

Real Polynomials modulo $x^2 + 1$

Definition… “Kinda”

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$$x^2 + 2x + 4 = (x^2 + 1) \cdot 1 + (2x + 3)$$

So in $\mathcal{P}$, we have $x^2 + 2x + 4 = 2x + 3$.

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Indeed,
$$\frac{1+x}{2} \cdot (1 - x) = \frac{1}{2} \cdot (1 - x^2) = \frac{1}{2}\left(-(x^2 + 1) + 2\right) = 1 \text{ (in } \mathcal{P})$$

Hold on a minute...

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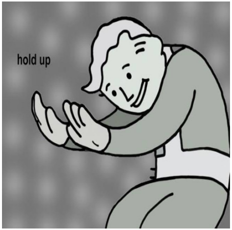

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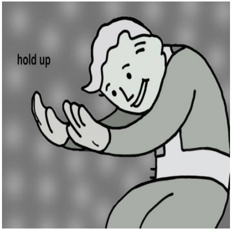

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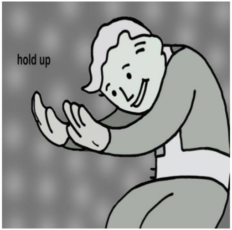

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Yes, any **irreducible** 2nd-degree polynomial $p(x)$ over $\mathbb{R}$ can be used. Typically, this is denoted as $\boxed{\mathbb{R}[x]/(p(x))}$.

Isomorphisms

Reminder

For two groups $(\mathbb{G}, +)$ and $(\mathbb{H}, \times)$ we defined homomorphism to be a function $\phi : \mathbb{G} \to \mathbb{H}$ such that

$$\phi(a + b) = \phi(a) \times \phi(b)$$

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But for now, “congruence” essentially means “exhibit the same structure”.

Key Theorems

Theorem

*Let $\mathbb{F}$ be a field and $\mu(x)$ — irreducible polynomial over $\mathbb{F}$ (**reduction polynomial**). Consider a set of polynomials over $\mathbb{F}[x]$ modulo $\mu(x) \in \mathbb{F}[x]$, formally denoted as $\mathbb{F}[x]/(\mu(x))$. Then, $\mathbb{F}[x]/(\mu(x))$ is a field.*

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Theorem

Let $\mathbb{F}$ be a field and $\mu \in \mathbb{F}[X]$ is an irreducible polynomial of degree n and let $\mathbb{K} := \mathbb{F}[X]/(\mu(X))$. Let $\theta \in \mathbb{K}$ be the root of μ over $\mathbb{K}$. Then,

$$\mathbb{K} = \{c_0 + c_1\theta + \cdots + c_{n-1}\theta^{n-1} : c_0, \ldots, c_{n-1} \in \mathbb{F}\}$$

Coming back to previous examples

Example

Again, consider $\mathbb{Q}(\sqrt{2}) = \{q + p\sqrt{2} : p, q \in \mathbb{Q}\}$. Then,

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Example

And $\mathbb{Q}(\sqrt{2}, i)$ is just a little bit more tricky. Notice that we can take

$$p(x) := (x^2 - 2)(x^2 + 1) = x^4 - x^2 - 2$$

So $\mathbb{Q}(\sqrt{2}, i) \cong \mathbb{Q}[x]/(x^4 - x^2 - 2)$.

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Definition

Suppose p is prime and $m \geq 2$. Let $\mu \in \mathbb{F}_p[X]$ be an irreducible polynomial of degree m. Then, elements of $\mathbb{F}_{p^m}$ are polynomials in $\mathbb{F}_p^{(\leq m)}[X]$ modulo $\mu(x)$. In other words,

$$\mathbb{F}_{p^m} = \{c_0 + c_1 X + \cdots + c_{m-1} X^{m-1} : c_0, \dots, c_{m-1} \in \mathbb{F}_p\},$$

where all operations are performed modulo $\mu(X)$.

Examples

It would be convenient to build $\mathbb{F}_{p^2}$ as $\mathbb{F}_p[i]/(i^2+1)$, but is it always possible? In other words, when $X^2 = -1$ has a solution in $\mathbb{F}_p$?

Examples

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- **Addition:** $(1+10i)+(18+15i) = 19+25i = 6i$.
- **Multiplication:** $(5+6i)(6+7i) = 30+71i+42i^2 = -12+71i = 7+14i$.

More Examples: Binary Extension Fields

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- **Inversion:** $(X^3 + X^2 + 1)^{-1} = X^2$ since $(X^3 + X^2 + 1) \cdot X^2 \bmod (X^4 + X + 1) = 1$.

More Examples: BN254

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Consider the **BN254 scalar field**, used in SNARKs:

$$p = \text{0x30644e72e131a029} \cdots \text{a8d3c208c16d87cfd47}$$

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Equivalently, we can write:

$$\mathbb{F}_{p^{12}} := \mathbb{F}_p[w]/(w^{12} - 18w^6 + 82)$$

Algebraic Closure

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$\mathbb{R}$ is not algebraically closed since $X^2 + 1$ has no roots in $\mathbb{R}$. However, $\mathbb{C}$ is algebraically closed, which follows from the fundamental theorem of algebra. Since $\mathbb{C}$ is a field extension of $\mathbb{R}$, it is also an algebraic closure of $\mathbb{R}$. This is commonly denoted as $\overline{\mathbb{R}} = \mathbb{C}$.

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A field $\mathbb{K}$ is called an **algebraic closure** of $\mathbb{F}$ if $\mathbb{K}/\mathbb{F}$ is algebraically closed. This is denoted as $\overline{\mathbb{F}} = \mathbb{K}$.

Algebraic Closure for Finite Fields

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Proof. Suppose $f_1, f_2, \dots, f_n \in \mathbb{F}$ are all elements of $\mathbb{F}$. Consider the following polynomial:

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Clearly, $p(x)$ is a non-constant polynomial and has no roots in $\mathbb{F}$, since for any $f \in \mathbb{F}$, one has $p(f) = 1$. ■

So what?

But what form does the $\overline{\mathbb{F}}_p$ have? Well, it is a union of all $\mathbb{F}_{p^k}$ for $k \geq 1$. This is formally written as:

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Remark

But this definition is super counter-intuitive! So here how we usually interpret it. Suppose I tell you that polynomial $q(x)$ has a root in $\overline{\mathbb{F}}_p$. What that means is that there exists some extension $\mathbb{F}_{p^m}$ such that for some $\alpha \in \mathbb{F}_{p^m}$, $q(\alpha) = 0$. We do not know how large this m is, but we know that it exists. For that reason, $\overline{\mathbb{F}}_p$ is defined as an infinite union of all possible field extensions.

Elliptic Curve

Definition

Definition

Suppose that $\mathbb{K}$ is a field. An **elliptic curve** E over $\mathbb{K}$ is defined as a set of points $(x, y) \in \mathbb{K}^2$:

$$y^2 = x^3 + ax + b,$$

called a **Short Weierstrass equation**, where $a, b \in \mathbb{K}$ and $4a^3 + 27b^2 \neq 0$. We denote $E/\mathbb{K}$ to denote the elliptic curve over field $\mathbb{K}$.

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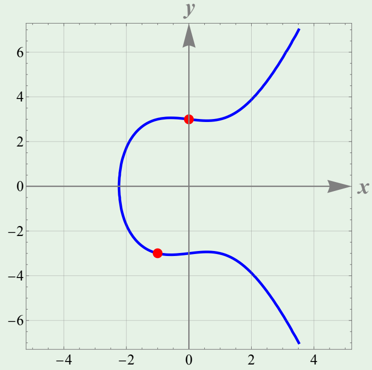
Definition

We say that $P = (x_P, y_P) \in \mathbb{A}^2(\mathbb{K})$ is the **affine representation** of the point on the elliptic curve $E/\mathbb{K}$ if it satisfies the equation $y_P^2 = x_P^3 + ax_P + b$.

Examples

Example

Consider $E/\mathbb{Q} : y^2 = x^3 - x + 9$. Valid affine points on $E/\mathbb{Q}$ are, for example, $P = (0, 3), Q = (-1, -3) \in \mathbb{A}^2(\mathbb{Q})$.



More Examples

Some more examples[1]:

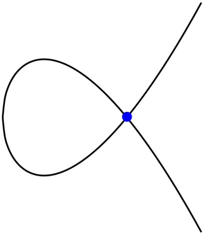


Figure 2.1: Singular curve $y^2 = x^3 - 3x + 2$ over $\mathbb{R}$.

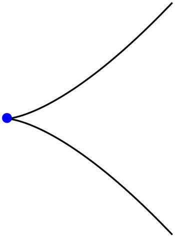


Figure 2.2: Singular curve $y^2 = x^3$ over $\mathbb{R}$.

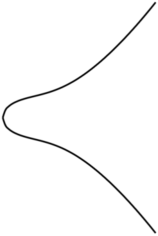


Figure 2.3: Smooth curve $y^2 = x^3 + x + 1$ over $\mathbb{R}$.

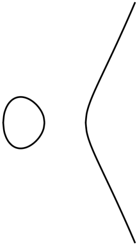


Figure 2.4: Smooth curve $y^2 = x^3 - x$ over $\mathbb{R}$.

[1] Figure taken from "Pairings for Beginners"

But real elliptic curves are not that simple. Here how they look like[2]:

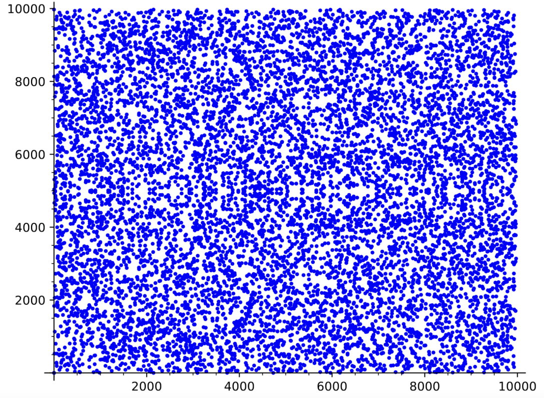


Figure: Curve $E/\mathbb{F}_{9973} : y^2 = x^3 - 2x + 1$ over the finite field

[2] Figure taken from "Moonmath"

Defining a Group Structure: A Few Words

Definition

The set of points on the curve, denoted as $E_{a,b}(\mathbb{K})$, is defined as:

$$E_{a,b}(\mathbb{K}) = \{(x, y) \in \mathbb{A}^2(\mathbb{K}) : y^2 = x^3 + ax + b\} \cup \{\mathcal{O}\},$$

where $\mathcal{O}$ is the so-called **point at infinity**.

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If $(x_P, y_P) \in E(\mathbb{K})$ then $(x_P, -y_P) \in E(\mathbb{K})$.

Remark #2

Typically, $\mathbb{K} = \overline{\mathbb{F}}_p$: we do not conretize over which finite field we define the elliptic curve.

Defining a Group Structure: Chord Method

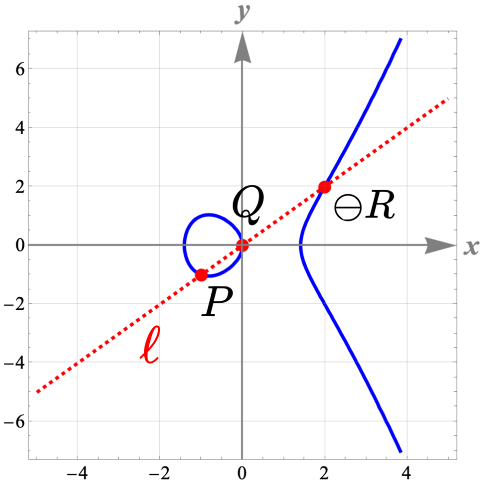

Figure: Chord method for adding two points

Defining a Group Structure: Tangent Method

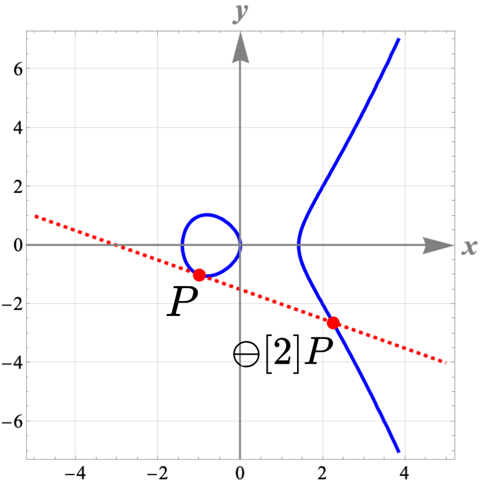

Figure: Tangent method for the point doubling

Idea of Derivation

Line equation through $P = (x_P, y_P), Q = (x_Q, y_Q)$:

$$\ell : y = \lambda(x - x_P) + y_P, \ \lambda = \frac{y_Q - y_P}{x_Q - x_P}$$

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Substituting y from the second equation to the first one, we get a cubic equation. Using Vieta's formula, one can derive

$$x_P + x_Q + x_R = \lambda^2$$

The rest is easy to finish.

Group Law

Definition

1. Point at infinity $\mathcal{O}$ is an identity element.

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$$x_R := \lambda^2 - x_P - x_Q, \quad y_R := \lambda(x_P - x_R) - y_P.$$

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3. If $x_P = x_Q$ and $y_P = y_Q$ (that is, $P = Q$), use the **tangent method**. Define the slope of the tangent at P as $\lambda := \frac{3x_P^2 + a}{2y_P}$ and set
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4. Otherwise, define $P \oplus Q := \mathcal{O}$.

One more Illustration

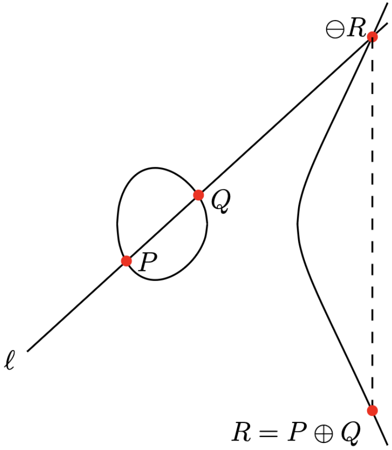


Figure 2.5: Elliptic curve addition.

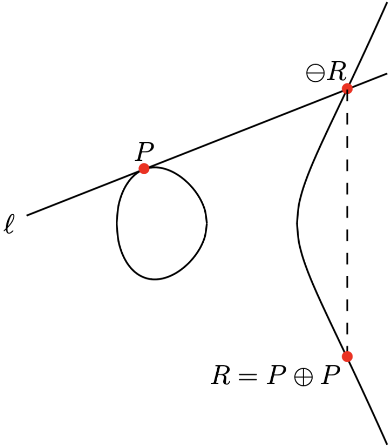


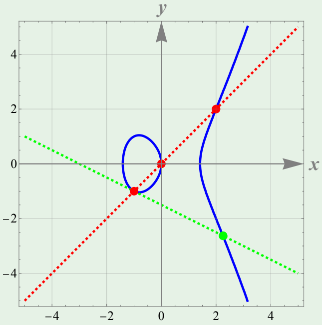
Figure 2.6: Elliptic curve doubling.

Example

Example

Consider $E/\mathbb{R} : y^2 = x^3 - 2x$.

- **Addition:** $(-1,1) \oplus (0,0) = (2,-2)$, $(2,2) \oplus (-1,-1) = (0,0)$.
- **Doubling:** $[2](-1,-1) = \left(\frac{9}{4}, -\frac{21}{8}\right)$.



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Hasse's Theorem on Elliptic Curves. $r = p^m + 1 - t$ *for some integer* $|t| \leq 2\sqrt{p^m}$. *A bit more intuitive explanation: the number of points on the curve is close to* $p^m + 1$. *The value* t *is called the* **trace of Frobenius**.

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Remark

In fact, $r = |E(\mathbb{F}_{p^m})|$ can be computed in $O(\log(p^m))$, so the number of points can be computed efficiently even for fairly large primes p.

Discrete Logarithm

Definition

Let $P \in E(\overline{\mathbb{F}}_p)$ and $\alpha \in \mathbb{Z}_r$. Define the scalar multiplication $[\alpha]P$ as adding P to itself $\alpha - 1$ times (also set $[0]P := \mathcal{O}$).

Discrete Logarithm

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Suppose E is cyclic, meaning, $\langle G \rangle = E$ for some $G \in E$. The **discrete logarithm problem** on E consists in the following: suppose $P = [\alpha]G$ for some $\alpha \in \mathbb{Z}_r$. Find α based on P.

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Suppose E is cyclic, meaning, $\langle G \rangle = E$ for some $G \in E$. The **discrete logarithm problem** on E consists in the following: suppose $P = [\alpha]G$ for some $\alpha \in \mathbb{Z}_r$. Find α based on P.

Remark

If r is a product of primes $p_1, p_2, \ldots, p_k$ such that $p_1 < p_2 < \cdots < p_k$, then the best-known algorithm to solve the discrete logarithm problem is no significantly better than $O(\sqrt{p_1})$.

Thank you for your attention!