Lecture #1 Exercises

Distributed Lab

July 18, 2024

Exercise 1. Which of the following statements is **false**?

1. $(\forall a, b \in \mathbb{Q}, a \neq b)\ (\exists q \in \mathbb{R}) : \{a < q < b\}$.
2. $(\forall \varepsilon > 0)\,(\exists n_\varepsilon \in \mathbb{N})\,(\forall n \geq n_\varepsilon) : \{1/n < \varepsilon\}$.
3. $(\forall k \in \mathbb{Z})\ (\exists n \in \mathbb{N}) : \{n < k\}$.
4. $(\forall x \in \mathbb{Z} \setminus \{-1\})\ (\exists! y \in \mathbb{Q}) : \{(x+1)y = 2\}$.

Exercise 2. Denote $X := \{(x, y) \in \mathbb{Q}^2 : xy = 1\}$. Oleksandr claims the following:

1. $X \cap \mathbb{N}^2 = \{(1, 1)\}$.
2. $|X \cap \mathbb{Z}^2| = 2|X \cap \mathbb{N}^2|$.
3. X is a group under the operation $(x_1, y_1) \oplus (x_2, y_2) = (x_1x_2, y_1y_2)$.

Which statements are **true**?

a) Only 1.
b) Only 1 and 2.
c) Only 1 and 3.
d) Only 2 and 3.
e) All statements are correct.

Exercise 3. Does a tuple $(\mathbb{Z}, \oplus)$ with operation $a \oplus b = a + b - 1$ define a group?

a) Yes, and this group is abelian.
b) Yes, but this group is not abelian.
c) No, since the associativity property does not hold.
d) No, since there is no identity element in this group.
e) No, since there is no inverse element in this group.

Exercise 4. Consider the Cartesian plane $\mathbb{R}^2$, where two coordinates are real numbers. For two points A, B define the operation $\oplus$ as follows: $A \oplus B$ is the midpoint on segment AB. Does $(\mathbb{R}^2, \oplus)$ define a group?

a) Yes, and this group is abelian.
b) Yes, but this group is not abelian.
c) No, since the associativity property does not hold and there is no identity element in this group.
d) No, since the associativity property does not hold, but we might define an identity element nonetheless.

Exercise 5. Find the inverse of 4 in $\mathbb{F}_{11}$.

a) 8
b) 5
c) 3
d) 7

Exercise 6. Suppose for three polynomials $p, q, r \in \mathbb{F}[x]$ we have $\deg p = 3, \deg q = 4, \deg r = 5$. Which of the following is true for $n := \deg\{(p-q)r\}$?

a) $n = 9$.
b) n might be less than 9.
c) $n = 20$.
d) n is less than $\deg\{qr\}$.

Exercise 7. Define the polynomial over $\mathbb{F}_5$: $f(x) := 4x^2 + 7$. Which of the following is the root of $f(x)$?

a) 2
b) 3
c) 4
d) This polynomial has no roots over $\mathbb{F}_5$.

Exercise 8. Quadratic polynomial $p(x) = ax^2 + bx + c \in \mathbb{R}[x]$ has zeros at 1 and 2 and $p(0) = 2$. Find the value of $a + b + c$.

a) 0
b) -1
c) 1
d) Not enough information to determine.

Exercise 9. Which of the following is a **valid** endomorphism $f : X \to X$?

a) $X = [0, 1]$, $f : x \mapsto x^2$.
b) $X = [0, 1]$, $f : x \mapsto x + 1$.
c) $X = \mathbb{R}_{>0}$, $f : x \mapsto (x-1)^3$.
d) $X = \mathbb{Q}_{>0}$, $f : x \mapsto \sqrt{x}$.

Exercise 10*. Denote by $\text{GL}(2, \mathbb{R})$ a set of 2×2 invertable matrices with real entries. Define two functions $\varphi : \text{GL}(2, \mathbb{R}) \to \mathbb{R}$:

$$\varphi_1 \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = ad - bc, \ \varphi_2 \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = a + d \tag{1}$$

Den claims the following:

1. φ_1 is a group homomorphism between multiplicative groups $(\text{GL}(2, \mathbb{R}), \times)$ and $(\mathbb{R}, \times)$.
2. φ_2 is a group homomorphism between additive groups $(\text{GL}(2, \mathbb{R}), +)$ and $(\mathbb{R}, +)$.

Which of the following is **true**?

a) Only statement 1 is correct.
b) Only statement 2 is correct.
c) Both statements 1 and 2 are correct.
d) None of the statements is correct.