Mathematics for Cryptography: Basic Notation and Groups

July 18, 2024

Distributed Lab

zkdl-camp.github.io

github.com/ZKDL-Camp

Plan

Some words about the course

About ZKDL

- ZKDL is an intensive course on low-level zero-knowledge cryptography.

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Note

This course is beneficial for everyone: even lecturers do not know all the material. Please, feel free to ask questions and provide feedback, and we will adjust the material accordingly.

Why ZKDL?

- Better Mathematics understanding.
- Skill of reading academic papers and writing your own ones.
- Public speech skills for lecturers on complex topics.
- Our knowledge structurization condensed in one course.
- Importance of ZK is quite obvious.
- And, of course, cryptography is fun!

Note

We are R&D experts in Cryptography, so we need to boost our skills in academic writing, lecturing, and understanding advanced topics.

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5. *Optionally*, we will conduct workshops on a separate day. We will discuss this later.

Contents

1. Mathematics Preliminaries: group and number theory, finite fields, polynomials, elliptic curves etc.
2. Building SNARKs from scratch.
3. Analysis of modern zero-knowledge proving systems: Groth16, Plonk, BulletProofs, STARK etc.
4. Specialization topics: low-level optimizations, advanced protocols such as folding schemes, Nova etc.

Notation

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- $\{1, 2, 2, 3\} = \{1, 2, 3\}$ — we do not count duplicates.

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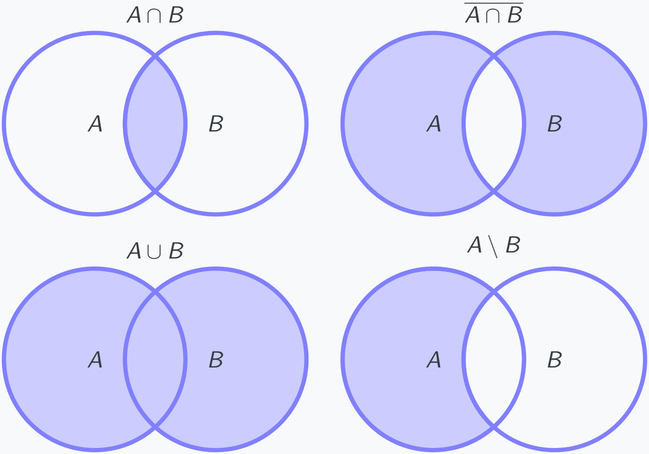
- $\{1, 2, 2, 3\} = \{1, 2, 3\}$ — we do not count duplicates.
- $\{1, 2, 3\} = \{2, 1, 3\}$ — order does not matter.
- $\{\{1, 2\}, \{3, 4\}, \{\sqrt{5}\}\}$ — elements can be sets themselves.

Operations on sets



Operations on sets: Examples

Question #1

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Question #3

What is the result of $\{1, 2, 3\} \cup \{3, 4, 5\}$?

Defining sets

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Question #2(*)

How to simplify the set $\{\sin \pi k : k \in \mathbb{Z}\}$?

Cartesian Product

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$A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$.

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Example

$\mathbb{R}^2$ is a set of all possible points in the Cartesian plane.

Cartesian Product Questions

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How to interpret the set $\{(x,y) \in \mathbb{N}^2 : x = y\}$?

Question #3(*)

How to interpret the set $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$?

Basic Logic

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- $\exists$ means "there exists", $\exists!$ means "there exists the only".
- $\wedge$ means "and".
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Is it true that $(\exists x \in \mathbb{N}) : \{x \geq 0 \wedge x < 1\}$?

Question #3

Is it true that $(\forall x \in \mathbb{Z}) (\exists y \in \mathbb{N}) : \{y > x\}$?

Randomness and Sequences

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Notation

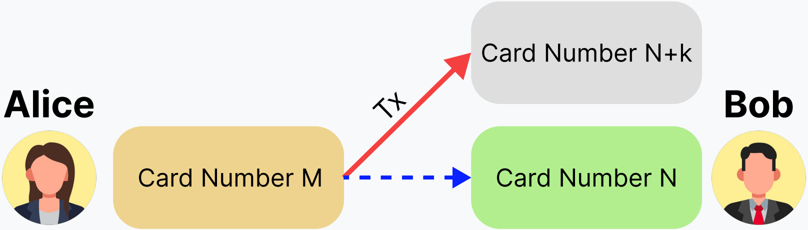
To denote an infinite sequence $x_1, x_2, \cdots$, we use $\{x_i\}_{i \in \mathbb{N}}$. To denote a finite sequence $x_1, x_2, \cdots, x_n$, we use $\{x_i\}_{i=1}^{n}$. To enumerate through a list of indeces $\mathcal{I} \subset \mathbb{N}$, we use notation $\{x_i\}_{i \in \mathcal{I}}$.

Basic Group Theory

Why Groups?!

Well, first of all, we want to work with integers...

Imagine that Alice pays to Bob with a card number N, but instead of paying to a number N, the system pays to another card number $N+k$, $k \ll N$, which is only by 0.001% different. Bob would not be 99.999% happy...



Why Groups?!

But integers on their own are not enough. We need to define a structure that allows us to perform operations on them.

This is very similar to interfaces: we abstract from the implementation, just merely stating we have “some” operation.

Example

Consider set $\mathbb{G} := \{\text{Dmytro}, \text{Dan}, \text{Friendship}\}$. We can safely define an operation $\oplus$ as:

$$\text{Dmytro} \oplus \text{Dan} = \text{Friendship}$$
$$\text{Dan} \oplus \text{Friendship} = \text{Dmytro}$$
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$$\begin{aligned}\text{Dmytro} \oplus \text{Dan} &= \text{Friendship} \\ \text{Dan} \oplus \text{Friendship} &= \text{Dmytro} \\ \text{Friendship} \oplus \text{Dmytro} &= \text{Dan}\end{aligned}$$

Rhetorical question

What makes $(\mathbb{G}, \oplus)$ a group?

Group Definition

Definition

Group $(\mathbb{G}, \oplus)$, is a set with a binary operation $\oplus$ with following rules:

1. **Closure:** Binary operations always outputs an element from $\mathbb{G}$, that is $\forall a, b \in \mathbb{G} : a \oplus b \in \mathbb{G}$.

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Definition

A group is called **abelian** if it satisfies the additional rule called **commutativity**: $\forall a, b \in \mathbb{G} : a \oplus b = b \oplus a$.

Explanation for Developers: Trait

```rust
/// Trait that represents a group.
pub trait Group: Sized {
    /// Checks whether the two elements are equal.
    fn eq(&self, other: &Self) -> bool;
    /// Returns the identity element of the group.
    fn identity() -> Self;
    /// Adds two elements of the group.
    fn add(&self, a: &Self) -> Self;
    /// Returns the negative of the element.
    fn negate(&self) -> Self;
    /// Subtracts two elements of the group.
    fn sub(&self, a: &Self) -> Self {
        self.add(&a.negate())
    }
}
```

More on that: https://github.com/ZKDL-Camp/lecture-1-math.

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Question #2

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Small Note on Notation

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Rule of thumb

We use additive notation when we imply that the group $\mathbb{G}$ is the set of points on the elliptic curve, while multiplicative is typically used in the rest of the cases.

Abelian Groups Examples and Non-Examples

Question #3

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Question #4

Set V is a set of tuples (v_1, v_2, v_3) where each $v_i \in \mathbb{R} \setminus \{0\}$. Define the operation $\odot$ as

$$(v_1, v_2, v_3) \odot (u_1, u_2, u_3) = (v_1 u_1, v_2 u_2, v_3 u_3)$$

Is $(V, \odot)$ a group? If no, what is missing?

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Is $(V, \odot)$ a group? If no, what is missing?

Conclusion

Group is just a fancy name for a set with a binary operation that behaves nicely.

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Consider $(\mathbb{Z}, +)$. Then, $3\mathbb{Z} = \{3k : k \in \mathbb{Z}\} \subset \mathbb{Z}$ is a subgroup.

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*Question #2**

Let $\mathrm{GL}(\mathbb{R}, 2)$ be a mutliplicative group of invertable matrices, while $\mathrm{SL}(\mathbb{R}, 2)$ be a multiplicative group of matrices with determinant 1. Is $\mathrm{SL}(\mathbb{R}, 2) \leq \mathrm{GL}(\mathbb{R}, 2)$?

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Answer. Yes. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(\mathbb{R}, 2)$ the inverse is $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Also, $\det(AB) = \det A \cdot \det B$, so the product of two matrices with determinant 1 has determinant 1, so the operation in closed.

Cyclic Subgroup.

Definition

Given a group $\mathbb{G}$ and $g \in \mathbb{G}$ the cyclic subgroup generated by g is

$$\langle g \rangle = \{g^n : n \in \mathbb{Z}\} = \{\ldots, g^{-3}, g^{-2}, g^{-1}, e, g, g^2, g^3, \ldots\}.$$

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Example

Consider the group of integers modulo 12, denoted by $\mathbb{Z}_{12}$. Consider $2 \in \mathbb{Z}_{12}$, the subgroup generated by 2 is then

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Definition

We say that a group $\mathbb{G}$ is **cyclic** if there exists an element $g \in \mathbb{G}$ such that $\mathbb{G}$ is generated by g, that is, $\mathbb{G} = \langle g \rangle$.

Cyclic Subgroup Examples.

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Question #2

What is the generator of
$7\mathbb{Z} = \{7k : k \in \mathbb{Z}\} = \{\ldots, -14, -7, 0, 7, 14, \ldots\}$?

Homomorphism

Definition

A **homomorphism** is a function $\phi : \mathbb{G} \to \mathbb{H}$ between two groups $(\mathbb{G}, \oplus)$ and $(\mathbb{H}, \odot)$ that preserves the group structure, i.e.,

$$\forall a, b \in \mathbb{G} : \phi(a \oplus b) = \phi(a) \odot \phi(b)$$

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$$\forall a, b \in \mathbb{G} : \phi(a \oplus b) = \phi(a) \odot \phi(b)$$

Example

Consider $(\mathbb{Z}, +)$ and $(\mathbb{R}_{>0}, \times)$. Then, the function $\phi : \mathbb{Z} \to \mathbb{R}_{>0}$ defined as $\phi(k) = 2^k$ is a homomorphism.

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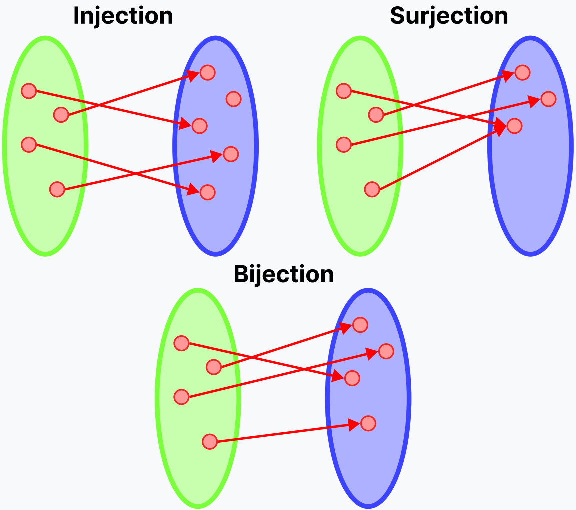
Proof. Take any $n, m \in \mathbb{Z}$ and consider $\phi(n + m)$:

$$\phi(n + m) = 2^{n+m} = 2^n \times 2^m = \phi(n) \times \phi(m)$$

Mapping types



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$\phi : k \mapsto 2^k$ from the previous example is a homomorphism between $(\mathbb{Z}, +)$ and $(\mathbb{R}_{>0}, \times)$, but not an isomorphism. Indeed, there is no $x \in \mathbb{Z}$ such that $2^x = 3 \in \mathbb{R}_{>0}$.

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Question

What can we do to make ϕ an isomorphism?

Field

Informal Definition

Field $\mathbb{F}$ is a set equipped with appropriate **addition** and **multiplication** operations with the corresponding well-defined inverses, where you can perform the basic arithmetic.

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Definition

A **field** is a set $\mathbb{F}$ with two operations $\oplus$ and $\odot$ such that:

1. $(\mathbb{F}, \oplus)$ is an abelian group with identity $e_\oplus$.
2. $(\mathbb{F} \setminus \{e_\oplus\}, \odot)$ is an abelian group.
3. The **distributive law** holds:
   $\forall a, b, c \in \mathbb{F} : a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$.

Field Examples

Example

The set of real numbers $(\mathbb{R}, +, \times)$ is obviously a field. So is $(\mathbb{Q}, +, \times)$.

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Definition

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Example

The set $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ with operations modulo 5 is a field.
Operation examples:

- $3 + 4 = 2$.
- $3 \times 2 = 1$.
- $4^{-1} = 4$ since $4 \times 4 = 1$.

Polynomials

Definition

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A **polynomial** $f(x)$ is a function of the form

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = \sum_{k=0}^{n} c_k x^k,$$

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Definition

A set of polynomials depending on x with coefficients in a field $\mathbb{F}$ is denoted as $\mathbb{F}[x]$, that is

$$\mathbb{F}[x] = \left\{ p(x) = \sum_{k=0}^{n} c_k x^k : c_k \in \mathbb{F},\ k = 0, \ldots, n \right\}.$$

Examples of Polynomials

Example

Consider the finite field $\mathbb{F}_3$. Then, some examples of polynomials from $\mathbb{F}_3[x]$ are listed below:

1. $p(x) = 1 + x + 2x^2$.
2. $q(x) = 1 + x^2 + x^3$.
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If we were to evaluate these polynomials at $1 \in \mathbb{F}_3$, we would get:

1. $p(1) = 1 + 1 + 2 \cdot 1 \bmod 3 = 1$.
2. $q(1) = 1 + 1 + 1 \bmod 3 = 0$.
3. $r(1) = 2 \cdot 1 = 2$.

More about polynomials

Definition

The **degree** of a polynomial $p(x) = c_0 + c_1 x + c_2 x^2 + \dots$ is the largest $k \in \mathbb{Z}_{\geq 0}$ such that $c_k \neq 0$. We denote the degree of a polynomial as $\deg p$. We also denote by $\mathbb{F}^{(\leq m)}[x]$ a set of polynomials of degree at most m.

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Theorem

For any two polynomials $p, q \in \mathbb{F}[x]$ and $n = \deg p, m = \deg q$, the following two statements are true:

1. $\deg(pq) = n + m$.
2. $\deg(p + q) = \max\{n, m\}, n \neq m$ *and* $\deg(p + q) \leq m, m = n$.

Roots of Polynomials

Definition

Let $p(x) \in \mathbb{F}[x]$ be a polynomial of degree $\deg p \geq 1$. A field element $x_0 \in \mathbb{F}$ is called a root of $p(x)$ if $p(x_0) = 0$.

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Theorem

Let $p(x) \in \mathbb{F}[x], \deg p \geq 1$. Then, $x_0 \in \mathbb{F}$ is a root of $p(x)$ if and only if there exists a polynomial $q(x)$ (with $\deg q = n - 1$) such that

$$p(x) = (x - x_0)q(x)$$

Polynomial Division

Theorem

Given $f, g \in \mathbb{F}[x]$ *with* $g \neq 0$, *there are unique polynomials* $p, q \in \mathbb{F}[x]$ *such that*

$$f = q \cdot g + r,\ 0 \leq \deg r < \deg g$$

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Example

Consider $f(x) = x^3 + 2$ and $g(x) = x + 1$ over $\mathbb{R}$. Then, we can write $f(x) = (x^2 - x + 1)g(x) + 1$, so the remainder of the division is $r \equiv 1$. Typically, we denote this as:

$$f \operatorname{div} g = x^2 - x + 1, \quad f \bmod g = 1.$$

The notation is pretty similar to one used in integer division.

Polynomial Divisibility

Definition

A polynomial $f(x) \in \mathbb{F}[x]$ is called **divisible** by $g(x) \in \mathbb{F}[x]$ (or, g **divides** f, written as $g \mid f$) if there exists a polynomial $h(x) \in \mathbb{F}[x]$ such that $f = gh$.

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Definition

A polynomial $f(x) \in \mathbb{F}[x]$ is said to be **irreducible** in $\mathbb{F}$ if there are no polynomials $g, h \in \mathbb{F}[x]$ both of degree more than 1 such that $f = gh$.

Polynomial Divisibility

Example

A polynomial $f(x) = x^2 + 16$ is irreducible in $\mathbb{R}$. Also $f(x) = x^2 - 2$ is irreducible over $\mathbb{Q}$, yet it is reducible over $\mathbb{R}$:
$f(x) = (x - \sqrt{2})(x + \sqrt{2})$.

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$f(x) = (x - \sqrt{2})(x + \sqrt{2})$.

Example

There are no polynomials over complex numbers $\mathbb{C}$ with degree more than 2 that are irreducible. This follows from the *fundamental theorem of algebra*. For example, $x^2 + 16 = (x - 4i)(x + 4i)$.

Interpolation

Question

How can we define the polynomial?

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The most obvious way is to specify coefficients $(c_0, c_1, \ldots, c_n)$. Can we do it in a different way?

Theorem

Given $n+1$ distinct points $(x_0, y_0), \ldots, (x_n, y_n)$, there exists a unique polynomial $p(x)$ of degree at most n such that $p(x_i) = y_i$ for all $i = 0, \ldots, n$.

Illustration with two points

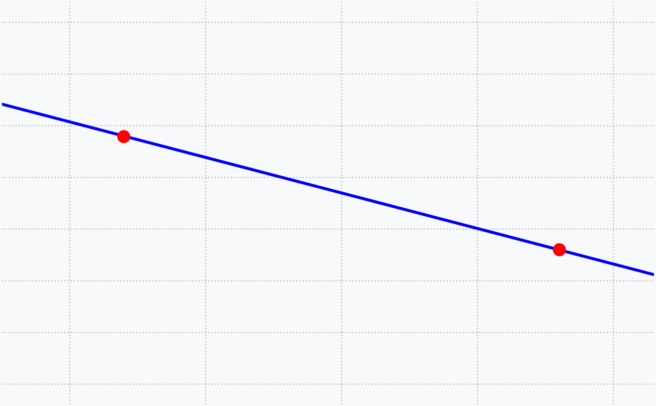

Figure: 2 points on the plane uniquely define the polynomial of degree 1 (linear function).

Illustration with five points

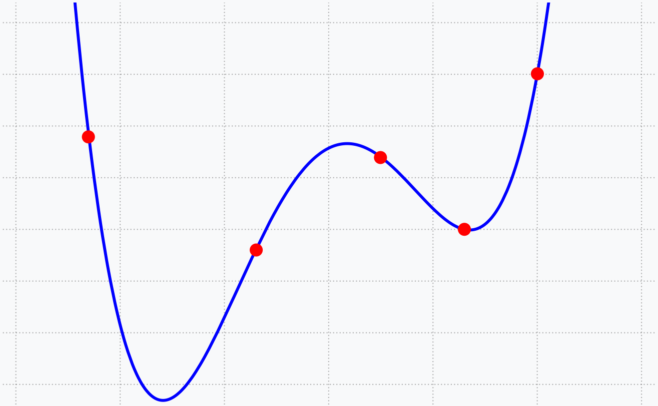

Figure: 5 points on the plane uniquely define the polynomial of degree 4.

Illustration with three points

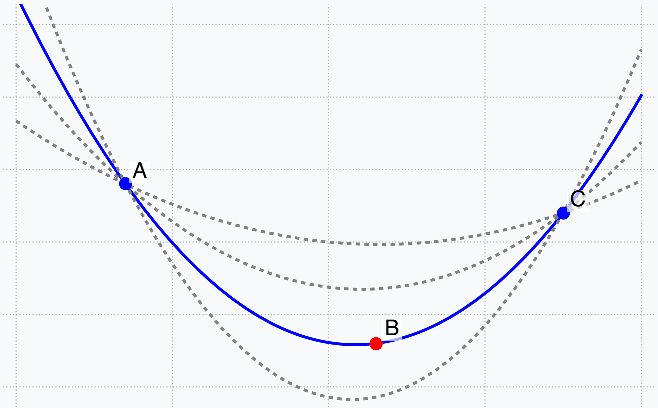

Figure: 2 points are not enough to define the quadratic polynomial $(c_2x^2 + c_1x + c_0)$.

Lagrange Interpolation

One of the ways to interpolate the polynomial is to use the Lagrange interpolation.

Theorem

Given $n+1$ distinct points $(x_0, y_0), \ldots, (x_n, y_n)$, the polynomial $p(x)$ that passes through these points is given by

$$p(x) = \sum_{i=0}^{n} y_i \ell_i(x), \quad \ell_i(x) = \prod_{i=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}.$$

Application: Shamir Secret Sharing

Motivation

How to share a secret α among n people in such a way that any t of them can reconstruct the secret, but any $t-1$ cannot?

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Secret Sharing scheme is a pair of efficient algorithms $(\mathrm{Gen}, \mathrm{Comb})$ which work as follows:

- $\mathrm{Gen}(\alpha, t, n)$: probabilistic sharing algorithm that yields n shards $(\alpha_1, \ldots, \alpha_t)$ for which t shards are needed to reconstruct the secret α.

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- $\mathsf{Comb}(\mathcal{I}, \{\alpha_i\}_{i\in\mathcal{I}})$: deterministic reconstruction algorithm that reconstructs the secret α from the shards $\mathcal{I} \subset \{1, \ldots, n\}$ of size t.

Shamir's Protocol

Note

Here, we require the **correctness**: for every $\alpha \in F$, for every possible output $(\alpha_1, \ldots, \alpha_n) \leftarrow \mathrm{Gen}(\alpha, t, n)$, and any t-size subset $\mathcal{I}$ of $\{1, \ldots, n\}$ we have

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Definition

Now, **Shamir's protocol** works as follows: $F = \mathbb{F}_q$ and

- $\mathrm{Gen}(\alpha, k, n)$: choose random $k_1, \ldots, k_{t-1} \xleftarrow{R} \mathbb{F}_q$ and define the polynomial

  $$\omega(x) := \alpha + k_1 x + k_2 x^2 + \cdots + k_{t-1} x^{t-1} \in \mathbb{F}_q^{\leq (t-1)}[x],$$

  and then compute $\alpha_i \leftarrow \omega(i) \in \mathbb{F}_q,\ i = 1, \ldots, n.$

Shamir's Protocol

Definition

- $\mathrm{Comb}(\mathcal{I}, \{\alpha_i\}_{i\in\mathcal{I}})$: interpolate the polynomial $\omega(x)$ using the Lagrange interpolation and output $\omega(0) = \alpha$.

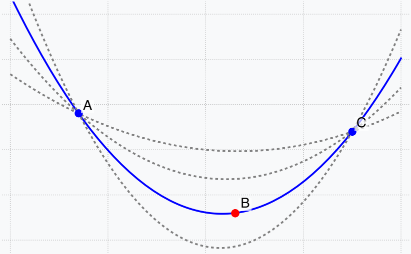

Figure: There are infinitely many quadratic polynomials passing through two blue points (gray dashed lines). However, knowing the red point allows us to uniquely determine the polynomial and thus get its value at 0.

Thanks for your attention!