# Introduction to Zero-Knowledge Proofs

Distributed Lab

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# Plan

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# Introduction

# Classical Proofs

- First proofs you have probably encountered were **geometry proofs**.
- You were given **axioms** and you can prove certain **statements** $x$ using them.
- The proof $\pi$ is a sequence of logical steps that lead from axioms to the statement. Essentially, you have a witness $w$ that proves the statement.
- Your teacher is the **verifier** $\mathcal{V}$ who checks your proof, while you are the **prover** $\mathcal{P}$.
- This is a **classical proof** and in a sense, it is a **non-interactive proof**.

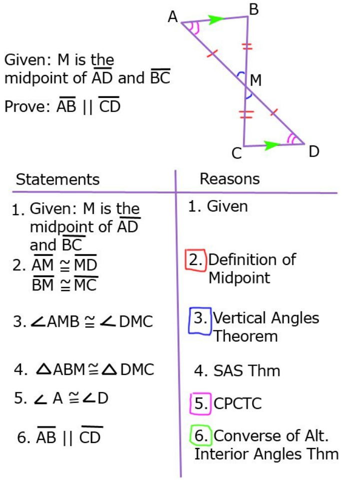



Figure: Geometry proof.

# Motivation

## Note

However, we cannot use such proofs in the digital world.

- Proofs must be verified by computers. Therefore, we need to develop **mathematic framework** to be able to program them.
- This leads to the question: what is **statement**? What is **proof**? What is **witness**? How to formally define them?
- We need to formalize these concepts.

Figure: Hmm...

# The most basic setting

- We have a **prover** $\mathcal{P}$ and a **verifier** $\mathcal{V}$.
- Prover $\mathcal{P}$ wants to prove some statement $x$ to the verifier.
- Prover $\mathcal{P}$ has a **witness** $w$ that contains all necessary information to prove the statement $x$. He sends $\pi$ as a proof.
- Verifier $\mathcal{V}$ wants to be convinced that the statement $x$ is true.

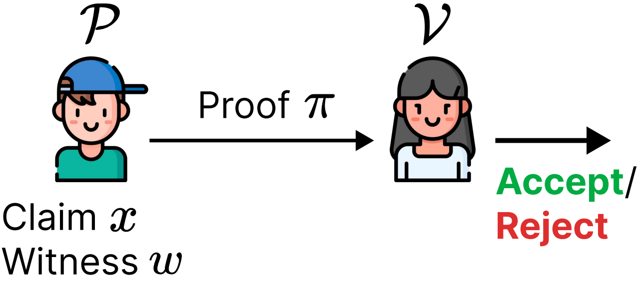



Figure: Typical setup for cryptographic proofs.

# The Goal of SNARKs, STARKs etc.

We will try to solve the following problems:

- **Completeness:** If $x$ is true, $\pi$ proofs the statement.
- **Soundness:** If $x$ is false, the prover $\mathcal{P}$ should not be able to convince the verifier $\mathcal{V}$ via any $\pi^*$.
- **Zero-knowledge:** $\pi$ does not reveal anything about $w$.
- **Argument of knowledge:** Sometimes, the prover $\mathcal{P}$ should convince the verifier $\mathcal{V}$ that besides $x$ is true, he **knows** the witness $w$.
- **Succinctness:** The proof should be short, ideally polylogarithmic in the size of the statement ($\pi = \text{polylog}(|x|)$) + fast verification.
- **Arithmetization:** We need to convert the statement $x$ into some algebraic form + make it relatively universal.

**Note**

SNARK, STARK, etc. will solve these problems!

# Example to demonstrate the goal

## Example

Given a hash function $H : \{0,1\}^* \to \{0,1\}^\ell$, $\mathcal{P}$ wants to convince $\mathcal{V}$ that he knows the preimage $x \in \{0,1\}^*$ such that $H(x) = y$.

- **Zero-knowledge:** The prover $\mathcal{P}$ does not want to reveal *anything* about the pre-image $x$ to the verifier $\mathcal{V}$.
- **Argument of knowledge:** Proving $y$ has a pre-image is useless. $\mathcal{P}$ must show he **knows** $x \in \{0,1\}^*$ s.t. $H(x) = y$.
- **Succinctness:** If the hash function takes $n$ operations to compute, the proof should be **much** shorter than $n$ operations. **State-of-art**: size is $\text{polylog}(n) = O((\log n)^c)$. Verification time is also typically polylogarithmic (or even $O(1)$ in some cases).

## Note

But first, let us start with the basics.

# Relations. Languages. NP Statements.

# Language

## Definition (Relation)

Given two sets $\mathcal{X}$ and $\mathcal{Y}$, the **relation** is $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{Y}$.

- $\mathcal{X}$ is typically a set of **statements**.
- $\mathcal{Y}$ is a set of **witnesses**.

## Definition (Language of true statements)

Let $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{Y}$ be a relation. We say that a statement $x \in \mathcal{X}$ is a **true** statement if $(x, y) \in \mathcal{R}$ for some $y \in \mathcal{Y}$, otherwise the statement is called **false**. We define by $\mathcal{L}_{\mathcal{R}}$ (the language over relation $\mathcal{R}$) the set of all true statements, that is:

$$\mathcal{L}_{\mathcal{R}} = \{x \in \mathcal{X} : \exists y \in \mathcal{Y} \text{ such that } (x, y) \in \mathcal{R}\}.$$

# Language Example #1: Semiprimes

## Example (Product of Two Primes (Semiprimes))

**Claim:** number $n \in \mathbb{N}$ is the product of two prime numbers $w = (p, q) \in \mathbb{N} \times \mathbb{N}$. The **relation** is given by:

$$\mathcal{R} = \{(n, p, q) \in \mathbb{N}^3 : n = p \cdot q \text{ where } p, q \text{ are primes}\}$$

In this particular case, the **language of true statements** is defined as

$$\mathcal{L}_{\mathcal{R}} = \{n \in \mathbb{N} : \exists w = (p, q) \text{ are primes such that } n = p \cdot q\}$$

- **Valid witness #1:** $n = 15 \in \mathcal{L}_{\mathcal{R}}$. Witness: $w = (3, 5)$.
- **Invalid witness:** $n = 16 \notin \mathcal{L}_{\mathcal{R}}$. There is no valid witness.
- **Valid witness #2:** $n = 50252009 \in \mathcal{L}_{\mathcal{R}}$. Witness: $w = (5749, 8741)$.

**Question:** Is $n = 27$ a true statement? What about $n = 26$?

# Language Example #2: Square Root

## Reminder

$\mathbb{Z}_N^\times = \{x \in \mathbb{Z}_N : \gcd\{x, N\} = 1\}$. **Example:** $\mathbb{Z}_{10}^\times = \{1, 3, 7, 9\}$

## Example

**Claim**: number $x \in \mathbb{Z}_N^\times$ is a **quadratic residue** modulo $N$:
$(\exists w \in \mathbb{Z}_N^\times) : \{x \equiv w^2 \pmod{N}\}$ ($w$ is **modular square root** of $x$).
**Relation**: $\mathcal{R} = \{(x, w) \in (\mathbb{Z}_N^\times)^2 : x \equiv w^2 \pmod{N}\}$
**Language:** $\mathcal{L}_\mathcal{R} = \{x \in \mathbb{Z}_N^\times : \exists w \in \mathbb{Z}_N^\times$ such that $x \equiv w^2 \pmod{N}\}$.
**Examples** for $N = 7$:

- $4 \in \mathcal{L}_\mathcal{R}$ since $5^2 \equiv 4 \pmod{7}$.
- $3 \notin \mathcal{L}_\mathcal{R}$ since there is no valid witness for 3.

**Question:** Is $x = 1$ a true statement for $N = 5$? What about $x = 4$?

# NP Statements: Demonstration

Well. . . We are simply going to send witness $w$ to the verifier $\mathcal{V}$ and he will check if the statement is true (meaning, whether $x \in \mathcal{L}_{\mathcal{R}}$).

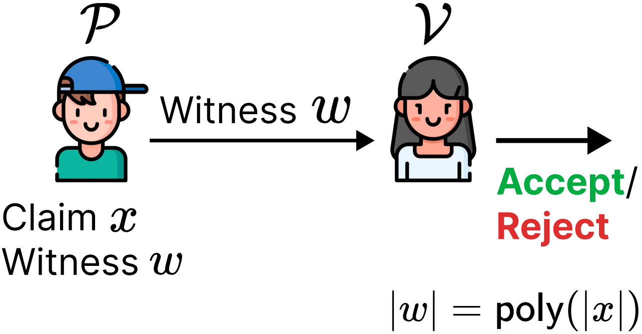



Figure: Typical setup for cryptographic proofs.

# NP Statements

**Definition (P Language)**

Problem is in the **P** class if exists a polytime algorithm checking $x \in \mathcal{L}$.

**Definition (NP Language)**

A language $\mathcal{L}_{\mathcal{R}}$ belongs to the **NP** class if there exists a polynomial-time verifier $\mathcal{V}$ such that the following two properties hold:

- **Completeness:** If $x \in \mathcal{L}_{\mathcal{R}}$, then there is a witness $w$ such that $\mathcal{V}(x, w) = 1$ with $|w| = \text{poly}(|x|)$. Essentially, it states that true claims have *short* proofs.
- **Soundness:** If $x \notin \mathcal{L}_{\mathcal{R}}$, then for any $w$ it holds that $\mathcal{V}(x, w) = 0$. Essentially, it states that false claims have no proofs.

**Theorem**

*Any* **NP** *problem has a zero-knowledge proof.*

# Question (aka Motivation)

But can we do better?
Sending witness is... Weird...

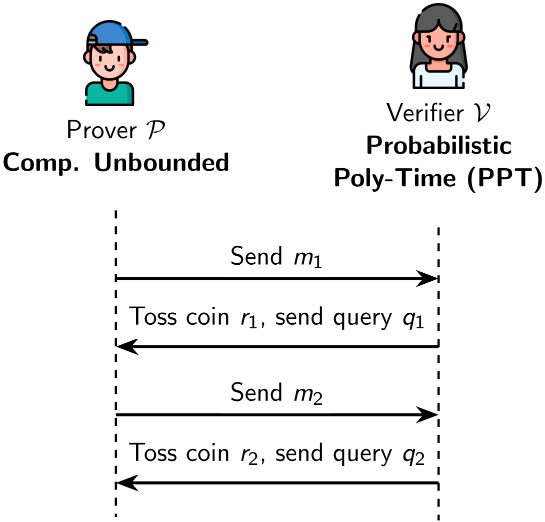
Figure: Hmm... #2

# Interactive Proofs

# Solution!

We add two more ingredients:

- **Interaction:** instead of **passively** receiving the proof, the verifier $\mathcal{V}$ can **interact** with the prover $\mathcal{P}$ by sending **challenges** and receiving **responses**.
- **Randomness:** $\mathcal{V}$ can send random coins (challenges) to the prover, which $\mathcal{P}$ can use to generate responses.



# Quadratic Residue Interactive Proof

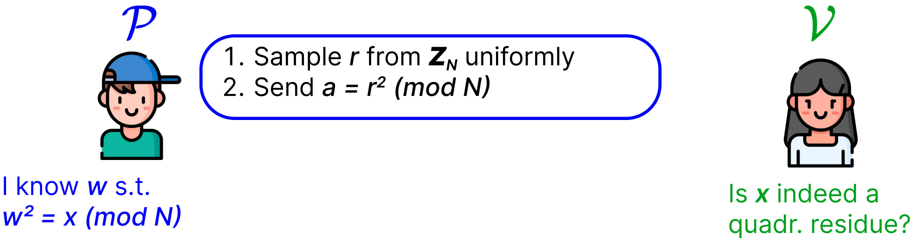
## Problem Statement

- **Statement:** $x \in \mathcal{L}_{\mathcal{R}}$ where our **language** is defined as:

$$\mathcal{L}_{\mathcal{R}} = \{x \in \mathbb{Z}_N^{\times} : \exists w \in \mathbb{Z}_N^{\times} \text{ such that } x \equiv w^2 \pmod{N}\}$$

- **Witness:** $w =$ modular square root of $x$.

How does $\mathcal{P}$ and $\mathcal{V}$ interact? Consider the figure below.



# Quadratic Residue Interactive Proof

$\mathcal{P}$

I know *w* s.t.
$w^2 = x \pmod{N}$

1. Sample $r$ from $\mathbb{Z}_N$ uniformly
2. Send $a = r^2 \pmod{N}$

- If I gave you the square root of $a$ and $ax$, you would be convinced that the claim is true, but you learn the witness $w$.
- Instead, I will send you either $r$ or $rw$, but you are to choose!

$\mathcal{V}$

Is $x$ indeed a quadr. residue?

# Quadratic Residue Interactive Proof

$\mathcal{P}$

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$\mathcal{V}$

Is $x$ indeed a quadr. residue?

Ok, I choose random bit $b$

# Quadratic Residue Interactive Proof

$\mathcal{P}$

I know $w$ s.t.
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1. Sample $r$ from $\mathbb{Z}_N$ uniformly
2. Send $a = r^2 \pmod N$

- If I gave you the square root of $a$ and $ax$, you would be convinced that the claim is true, but you learn the witness $w$.
- Instead, I will send you either $r$ or $rw$, but you are to choose!

$\mathcal{V}$

Is $x$ indeed a quadr. residue?

Ok, I choose random bit $b$

- If $b=0$, send $z = r$
- If $b=1$, send $z = rw \pmod N$

Check if $z^2 = ax^b$

# Quadratic Residue Interactive Proof: Analysis

## Interactive Protocol

1. $\mathcal{P}$ samples $r \xleftarrow{R} \mathbb{Z}_N^\times$ and sends $a = r^2$ to $\mathcal{V}$.
2. $\mathcal{V}$ sends a random bit $b \in \{0, 1\}$ to $\mathcal{P}$.
3. $\mathcal{P}$ sends $z = r \cdot w^b$ to $\mathcal{V}$.
4. $\mathcal{V}$ accepts if $z^2 = a \cdot x^b$, otherwise it rejects.
5. Repeat $\lambda \in \mathbb{N}$ times.

## Lemma

*The aforementioned protocol is* **complete** *and* **sound***.*

**Completeness.** If $b = 0$, then $z = r$ and thus $z^2 = r^2 = a$, check passes. If $b = 1$, then $z = rw$ and thus $z^2 = r^2 w^2 = ax$, check passes.

# Quadratic Residue Interactive Proof: Analysis

**Soundness.** The main reason why the protocol is sound is insribed in the theorem below.

**Theorem**

*For any prover $\mathcal{P}^*$ with $x \notin \mathcal{L}_{\mathcal{R}}$, the probability of $\mathcal{V}$ accepting the proof is at most $1/2$.*

**Corollary.** After repeating the protocol $\lambda$ times, we have

$$\Pr[\mathcal{V} \text{ accepts after } \lambda \text{ rounds}] \leq \frac{1}{2^\lambda} = \mathsf{negl}(\lambda).$$

Thus, we showed both **completeness** and **soundness** of the protocol.

# Interactive Protocol Definition

$\langle \mathcal{P}, \mathcal{V} \rangle(x)$ reads as "interaction between $\mathcal{P}$ and $\mathcal{V}$ on the statement $x$".

## Definition

A pair of algorithms $(\mathcal{P}, \mathcal{V})$ is called an **interactive proof** for a language $\mathcal{L}_{\mathcal{R}}$ if $\mathcal{V}$ is a polynomial-time verifier and the following two properties hold:

- **Completeness:** For any $x \in \mathcal{L}_{\mathcal{R}}$, $\Pr[\langle \mathcal{P}, \mathcal{V} \rangle(x) = \text{accept}] = 1$.
- **Soundness:** For any $x \notin \mathcal{L}_{\mathcal{R}}$ and for any prover $\mathcal{P}^*$, we have

$$\Pr[\langle \mathcal{P}^*, \mathcal{V} \rangle(x) = \text{accept}] \leq \text{negl}(\lambda)$$

## Definition

**The class of interactive proofs (IP)** is defined as:

$$\textbf{IP} = \{\mathcal{L} : \text{there is an interactive proof } (\mathcal{P}, \mathcal{V}) \text{ for } \mathcal{L}\}.$$

# Zero-Knowledge Informal Definition

## Definition

An interactive proof system $(\mathcal{P}, \mathcal{V})$ is called **zero-knowledge** if for any polynomial-time verifier $\mathcal{V}^*$ and any $x \in \mathcal{L}_{\mathcal{R}}$, the interaction $\langle \mathcal{P}, \mathcal{V}^* \rangle(x)$ gives nothing new about the witness $w$.

## Definition

The pair of algorithms $(\mathcal{P}, \mathcal{V})$ is called a **zero-knowledge interactive protocol** if it is *complete*, *sound*, and *zero-knowledge*.

I know witness,
but I will not show
you it!


Well, the claim is true,
but what was the witness
anyway?!

# Verifier's View

**Question #1**

What has the verifier learned during the interaction?

- First things first, he learned that the statement $x$ is true.
- He also knows queries $(q_1, \dots, q_\ell)$ and random coins $(r_1, \dots, r_\ell)$ he tossed (since he is the one who has sent them).
- Moreover, he knows the prover's messages $(m_1, m_2, \dots, m_\ell)$.

**Definition**

All the conversation that verifier has witnessed is called **verifier's view** and is denoted as

$$\text{view}_{\mathcal{V}}(\mathcal{P}, \mathcal{V}) = (m_1, r_1, q_1, m_2, r_2, q_2, \dots, m_\ell, r_\ell, q_\ell).$$

**Fact:** $\text{view}_{\mathcal{V}}(\mathcal{P}, \mathcal{V})$ is a **random variable**.

# Verifier's View: Example

## Example

For QN test, set $N := 3 \times 2^{30} + 1$ (prime number), and $\mathcal{P}$ wants to convince that $1286091780 \in \mathcal{L}_R$. Conversation is the following:

1. During the first round, $\mathcal{P}$ sends 672192003 to $\mathcal{V}$.
2. $\mathcal{V}$ sends $b = 0$ to $\mathcal{P}$.
3. $\mathcal{P}$ sends 2606437826 to $\mathcal{V}$.
4. $\mathcal{V}$ verifies that indeed $2606437826^2 \equiv 672192003 \pmod{N}$.
5. During the second round, $\mathcal{P}$ sends 2619047580 to $\mathcal{V}$.
6. $\mathcal{V}$ chooses $b = 1$ and sends to $\mathcal{P}$.
7. $\mathcal{P}$ sends 1768388249 to $\mathcal{V}$.
8. $\mathcal{V}$ verifies that $1768388249^2 \equiv 2619047580 \times 1286091780 \pmod{N}$.
9. Conversation ends.

# Verifier's View: Example

## Example

The **view of the verifier** $\mathcal{V}$ is the following:

$$\begin{aligned} &\text{view}_{\mathcal{V}}(\mathcal{V},\mathcal{P})[1286091780] \\ &= (672192003, 0, 2606437826, 2619047580, 1, 1768388249) \end{aligned}$$

- Essentially, this view is the same as you have witnessed.
- You have not learned anything about $w$ that prover $\mathcal{P}$ knows.
- The witness was $w = 3042517305$ and two randomnesses were $r_1 = 2606437826$ and $r_2 = 3023142760$.
- This is a random variable: conversation could be different.

# Zero-Knowledge Formally: Simulation Paradigm

**Question #2**

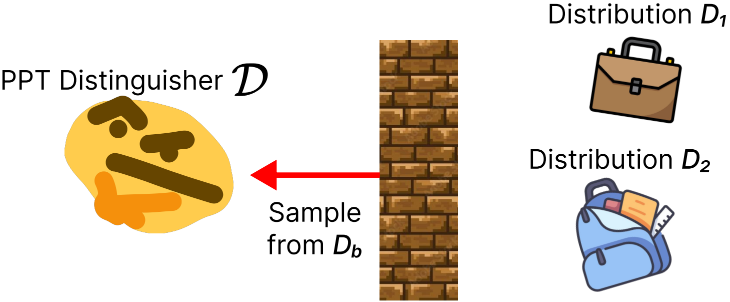
What does it mean that the protocol is zero-knowledge?

- Protocol is zero-knowledge if, given the verifier's $\text{view}_{\mathcal{V}}(\mathcal{P}, \mathcal{V})$, verifier cannot infer any information about the witness $w$.
- What does it mean that verifier $\mathcal{V}$ learns nothing new? It means that this view could have been simulated by $\mathcal{V}$ *without even running an interaction*.
- Call the view after the real interaction as **real view**, while the view after the simulation as **simulated view**.

**Note**

Such idea of defining the zero-knowledge is called **simulation paradigm** and currently the most widely used way to prove zero-knowledge.

# Computational Indistinguishability




**Definition (Informal Computational Indistinguisability)**

$D_1$ and $D_2$ are **computationally indistinguishable** (denoted by $D_1 \approx D_2$) if for any PPT distinguisher $\mathcal{D}$, even after polynomial number $k$ of samples from $D_b$ (where $b \xleftarrow{R} \{0,1\}$), for prediction $\hat{b}$: $\Pr[\hat{b} = b] < \frac{1}{2} + \mathsf{negl}(k)$.

# Zero-Knowledge Formally (Kind of)

Finally, we are ready to define the **zero-knowledge**.

**Definition (Honest-Verifier Zero-Knowledge (HVZK))**

An interactive protocol $(\mathcal{P}, \mathcal{V})$ is **honest-verifier zero-knowledge (HVZK)** for a language $\mathcal{L}_{\mathcal{R}}$ there exists a poly-time simulator Sim such that for any valid statement $x \in \mathcal{L}_{\mathcal{R}}$:

$$\mathsf{view}_{\mathcal{V}}(\mathcal{P}, \mathcal{V})[x] \approx \mathsf{Sim}(x, 1^{\lambda})$$

**Definition (Zero-Knowledge (ZK))**

An interactive protocol $(\mathcal{P}, \mathcal{V})$ is **zero-knowledge (ZK)** for a language $\mathcal{L}_{\mathcal{R}}$ if *for every poly-time verifier* $\mathcal{V}^*$ there *exists a poly-time simulator* Sim such that for any valid statement $x \in \mathcal{L}_{\mathcal{R}}$:

$$\mathsf{view}_{\mathcal{V}^*}(\mathcal{P}, \mathcal{V}^*)[x] \approx \mathsf{Sim}(x, 1^{\lambda})$$

# Proof of Knowledge: Why?

Now, the main issue with the above definition is that *we have proven the statement correctness, but we have not proven that the prover* **knows** *the witness*. These are completely two distinct things!

## Example

Consider the **discrete logarithm relation and language** for a cyclic group $E(\mathbb{F}_p)$ of order $r$:

$$\mathcal{R} = \{(P, \alpha) \in E(\mathbb{F}_p) \times \mathbb{Z}_r : P = [\alpha]G\},$$
$$\mathcal{L}_{\mathcal{R}} = \{P \in E(\mathbb{F}_p) : \exists \alpha \in \mathbb{Z}_r \text{ such that } P = [\alpha]G\}$$

## Question

What does it mean that $X \in \mathcal{L}_{\mathcal{R}}$?

Turns out $\mathcal{L}_{\mathcal{R}} = E(\mathbb{F}_p)$, so the proof $X \in \mathcal{L}_{\mathcal{R}}$ itself is useless.

# Proof of Knowledge: Definition

1. The knowledge of witness means that we can **extract** the witness while interacting with the prover.
2. Thus, there should be an algorithm called **extractor** $\mathcal{E}$ which can extract the witness $w$.
3. $\mathcal{E}$ is given more power than $\mathcal{V}$ (otherwise, if the protocol is zero-knowledge, we cannot extract $w$). $\mathcal{E}$ can **rewind** and **call** prover $\mathcal{P}$ multiple times.
4. Sometimes, this is referred to as "extractor $\mathcal{E}$ uses $\mathcal{P}$ as an oracle".

**Definition (Proof of Knowledge)**

The interactive protocol $(\mathcal{P}, \mathcal{V})$ is a **proof of knowledge** for $\mathcal{L}_{\mathcal{R}}$ if exists a poly-time extractor algorithm $\mathcal{E}$ such that for any valid statement $x \in \mathcal{L}_{\mathcal{R}}$, in expected poly-time $\mathcal{E}^{\mathcal{P}}(x)$ outputs $w$ such that $(x, w) \in \mathcal{R}$.

# Proof of Knowledge: Example

**Lemma**

*The quadratic residue interactive protocol is a proof of knowledge.*

**Proof.** Let us define the extractor $\mathcal{E}$ for the statement $x$ as follows:

1. Run the prover to receive $a \equiv r^2 \pmod{N}$ ($r$ is chosen randomly from $\mathbb{Z}_N^*$).
2. Set verifier's message to $b = 0$ to get $z_1 \leftarrow r$.
3. **Rewind** and set verifier's message to $b = 1$ to get $z_2 \leftarrow rw \pmod{N}$.
4. Output $z_2/z_1 \pmod{N}$

The extractor $\mathcal{E}$ will always output $w$ if $x \in \mathcal{L}_{\mathcal{R}}$. $\square$

# Fiat-Shamir Heuristic

# Fiat-Shamir Transformation

### Statement

**Any** interactive public-coin protocol can be converted into a non-interactive public-coin protocol with preserving completeness, soundness, and zero-knowledge.

One of such transformations is called **Fiat-Shamir heuristic**. Idea:

1.

*Thanks for your attention!*