Introduction to Zero-Knowledge Proofs

Distributed Lab

August 22, 2024

Plan

1. Introduction
   - Classical Proofs
   - Goal of the course
2. Relations. Languages. NP Statements.
   - Language of true statements. Examples.
   - P and NP Statements
3. Interactive Proofs
   - Quadratic Residue Interactive Proof
   - Completeness and Soundness
   - Zero-Knowledge and Honest-Verifier Zero-Knowledge
   - Proof of Knowledge
4. Fiat-Shamir Heuristic
   - Cryptographic Oracles
   - Fiat-Shamir Transformation

Introduction

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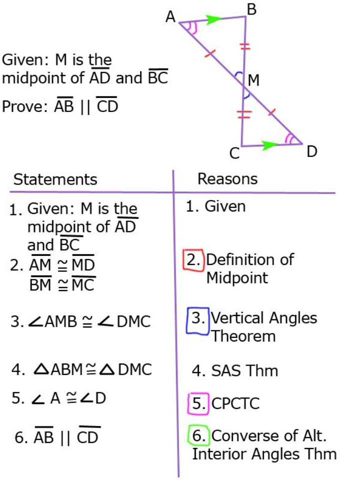

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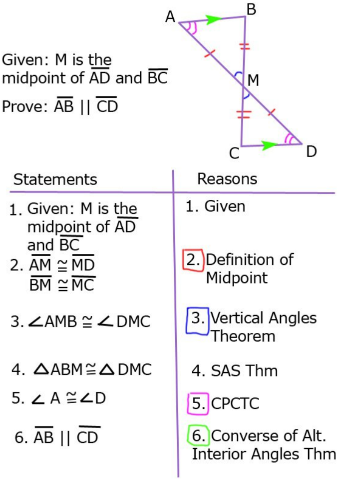

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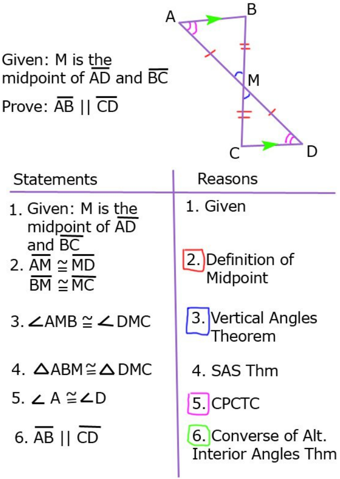

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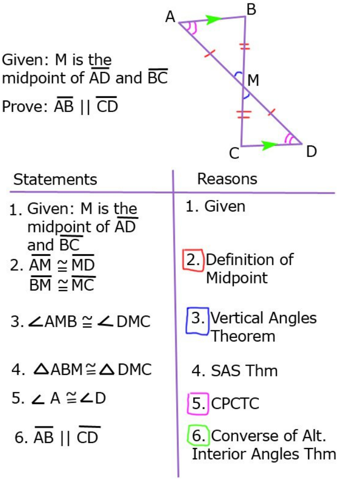

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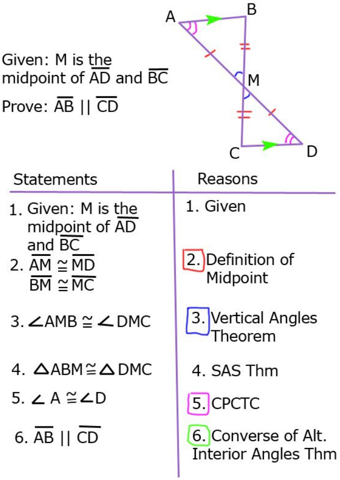
- This is a **classical proof** and in a sense, it is a **non-interactive proof**.



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- We need to formalize these concepts.

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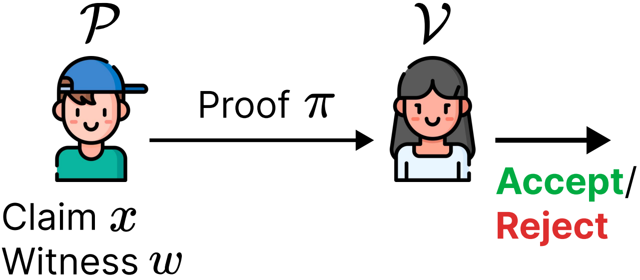

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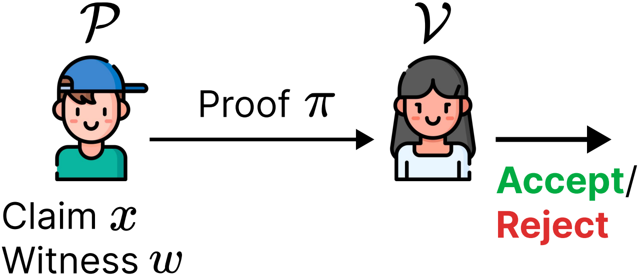

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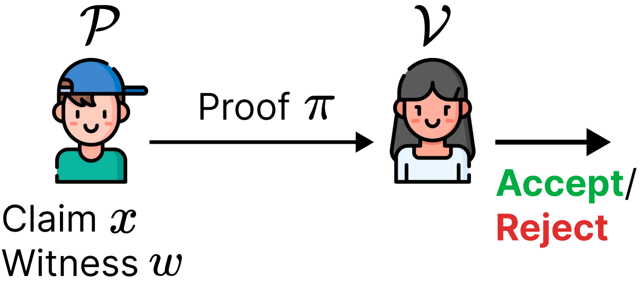

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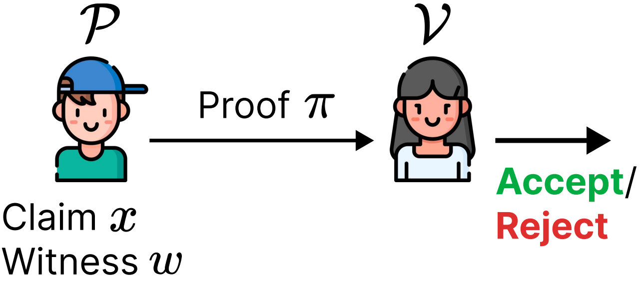
- Verifier $\mathcal{V}$ wants to be convinced that the statement x is true.



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SNARK, STARK, etc. will solve these problems!

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But first, let us start with the basics.

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Definition (Language of true statements)

Let $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{Y}$ be a relation. We say that a statement $x \in \mathcal{X}$ is a **true** statement if $(x, y) \in \mathcal{R}$ for some $y \in \mathcal{Y}$, otherwise the statement is called **false**. We define by $\mathcal{L}_{\mathcal{R}}$ (the language over relation $\mathcal{R}$) the set of all true statements, that is:

$$\mathcal{L}_{\mathcal{R}} = \{x \in \mathcal{X} : \exists y \in \mathcal{Y} \text{ such that } (x, y) \in \mathcal{R}\}.$$

Language Example #1: Semiprimes

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Question: Is $n = 27$ a true statement? What about $n = 26$?

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Question: Is $x = 1$ a true statement for $N = 5$? What about $x = 4$?

NP Statements: Demonstration

Well... We are simply going to send witness w to the verifier $\mathcal{V}$ and he will check if the statement is true (meaning, whether $x \in \mathcal{L}_{\mathcal{R}}$).

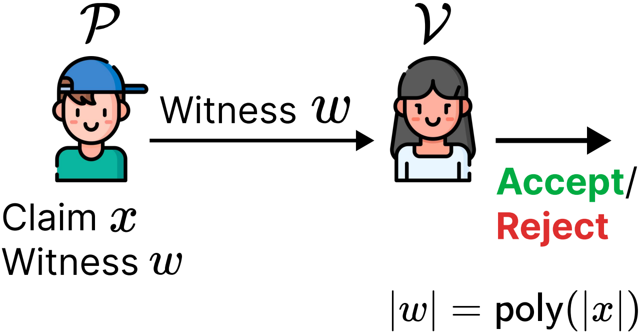

Figure: Typical setup for cryptographic proofs.

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- **Soundness:** If $x \notin \mathcal{L}_{\mathcal{R}}$, then for any w it holds that $\mathcal{V}(x, w) = 0$. Essentially, it states that false claims have no proofs.

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- **Soundness:** If $x \notin \mathcal{L}_{\mathcal{R}}$, then for any w it holds that $\mathcal{V}(x, w) = 0$. Essentially, it states that false claims have no proofs.

Theorem

Any **NP** *problem has a zero-knowledge proof (GMW86).*

Question (aka Motivation)

But can we do better?

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But can we do better?
Sending witness is. . . Weird. . .

Figure: Hmm. . . #2

Interactive Proofs

Solution!

We add two more ingredients:

- **Interaction:** instead of **passively** receiving the proof, the verifier $\mathcal{V}$ can **interact** with the prover $\mathcal{P}$ by sending **challenges** and receiving **responses**.

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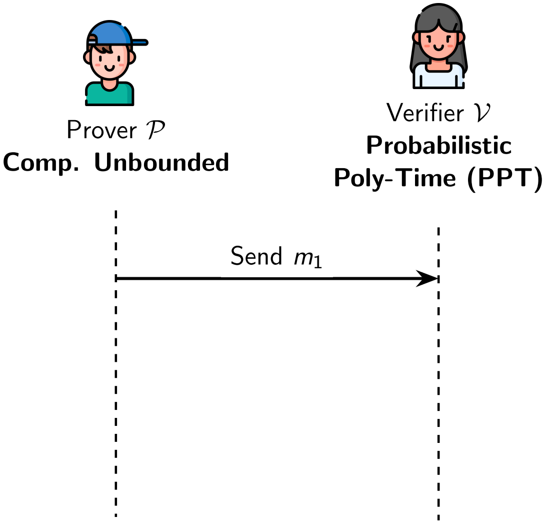

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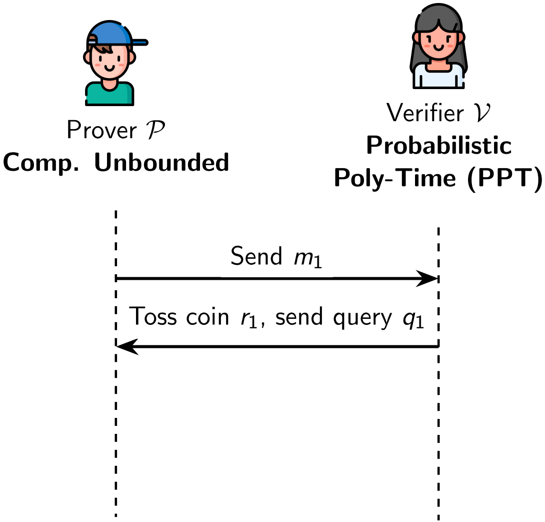

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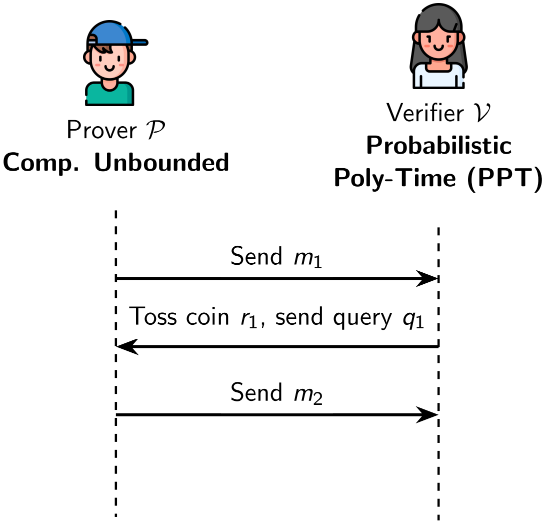

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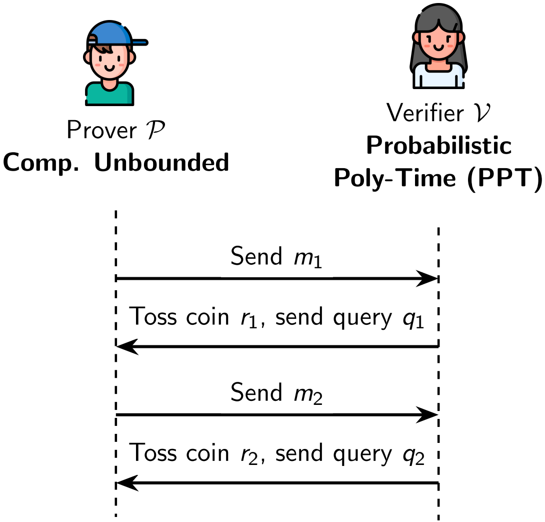

Quadratic Residue Interactive Proof

Problem Statement

- **Statement:** $x \in \mathcal{L}_{\mathcal{R}}$ where our **language** is defined as:

$$\mathcal{L}_{\mathcal{R}} = \{x \in \mathbb{Z}_N^{\times} : \exists w \in \mathbb{Z}_N^{\times} \text{ such that } x \equiv w^2 \pmod{N}\}$$

- **Witness:** $w =$ modular square root of x.

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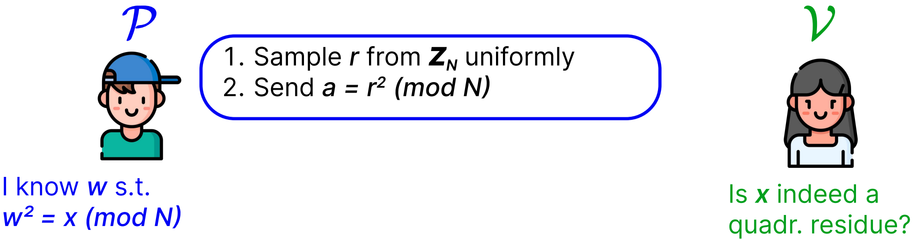
How does $\mathcal{P}$ and $\mathcal{V}$ interact? Consider the figure below.



Quadratic Residue Interactive Proof

$\mathcal{P}$

I know *w* s.t.
$w^2 = x \pmod N$

1. Sample r from $\mathbb{Z}_N$ uniformly
2. Send $a = r^2 \pmod N$

- If I gave you the square root of *a* and *ax*, you would be convinced that the claim is true, but you learn the witness *w*.
- Instead, I will send you either *r* or *rw*, but you are to choose!

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Is ***x*** indeed a quadr. residue?

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- If $b=0$, send $z = r$
- If $b=1$, send $z = rw \pmod N$

Check if $z^2 = ax^b$

Quadratic Residue Interactive Proof: Analysis

Interactive Protocol

1. $\mathcal{P}$ samples $r \xleftarrow{R} \mathbb{Z}_N^\times$ and sends $a = r^2$ to $\mathcal{V}$.

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Completeness. If $b = 0$, then $z = r$ and thus $z^2 = r^2 = a$, check passes. If $b = 1$, then $z = rw$ and thus $z^2 = r^2w^2 = ax$, check passes.

Quadratic Residue Interactive Proof: Analysis

Soundness. The main reason why the protocol is sound is insribed in the theorem below.

Theorem

For any prover $\mathcal{P}^*$ with $x \notin \mathcal{L}_{\mathcal{R}}$, the probability of $\mathcal{V}$ accepting the proof is at most $1/2$.

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For any prover $\mathcal{P}^*$ with $x \notin \mathcal{L}_{\mathcal{R}}$, the probability of $\mathcal{V}$ accepting the proof is at most $1/2$.

Corollary. After repeating the protocol λ times, we have

$$\Pr[\mathcal{V} \text{ accepts after } \lambda \text{ rounds}] \leq \frac{1}{2^\lambda} = \mathsf{negl}(\lambda).$$

Thus, we showed both **completeness** and **soundness** of the protocol.

Interactive Protocol Definition

$\langle \mathcal{P}, \mathcal{V} \rangle (x)$ reads as "interaction between $\mathcal{P}$ and $\mathcal{V}$ on the statement x".

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Definition

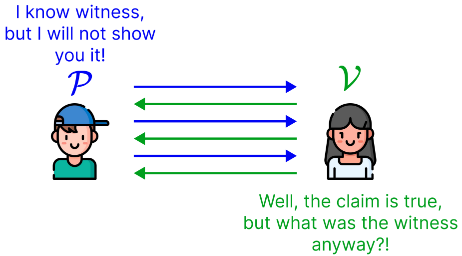
The class of interactive proofs (IP) is defined as:

$$\textbf{IP} = \{\mathcal{L} : \text{there is an interactive proof } (\mathcal{P}, \mathcal{V}) \text{ for } \mathcal{L}\}.$$

Zero-Knowledge Informal Definition

Definition

An interactive proof system $(\mathcal{P}, \mathcal{V})$ is called **zero-knowledge** if for any polynomial-time verifier $\mathcal{V}^*$ and any $x \in \mathcal{L}_{\mathcal{R}}$, the interaction $\langle \mathcal{P}, \mathcal{V}^* \rangle(x)$ gives nothing new about the witness w.



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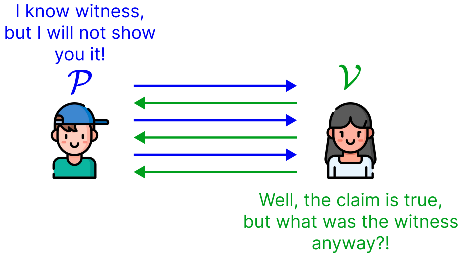
The pair of algorithms $(\mathcal{P}, \mathcal{V})$ is called a **zero-knowledge interactive protocol** if it is *complete*, *sound*, and *zero-knowledge*.



Verifier's View

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All the conversation that verifier has witnessed is called **verifier's view** and is denoted as

$$\text{view}_{\mathcal{V}}(\mathcal{P}, \mathcal{V}) = (m_1, r_1, q_1, m_2, r_2, q_2, \ldots, m_\ell, r_\ell, q_\ell).$$

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Fact: $\text{view}_{\mathcal{V}}(\mathcal{P}, \mathcal{V})$ is a **random variable**.

Verifier's View: Example

Example

For QN test, set $N := 3 \times 2^{30} + 1$ (prime number), and $\mathcal{P}$ wants to convince that $1286091780 \in \mathcal{L}_R$. Conversation is the following:

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9. Conversation ends.

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The **view of the verifier** $\mathcal{V}$ is the following:

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- The witness was $w = 3042517305$ and two randomnesses were $r_1 = 2606437826$ and $r_2 = 3023142760$.
- This is a random variable: conversation could be different.

Zero-Knowledge Formally: Simulation Paradigm

Question #2

What does it mean that the protocol is zero-knowledge?

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- What does it mean that verifier $\mathcal{V}$ learns nothing new? It means that this view could have been simulated by $\mathcal{V}$ *without even running an interaction*.

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- What does it mean that verifier $\mathcal{V}$ learns nothing new? It means that this view could have been simulated by $\mathcal{V}$ *without even running an interaction*.
- Call the view after the real interaction as **real view**, while the view after the simulation as **simulated view**.

Zero-Knowledge Formally: Simulation Paradigm

Question #2

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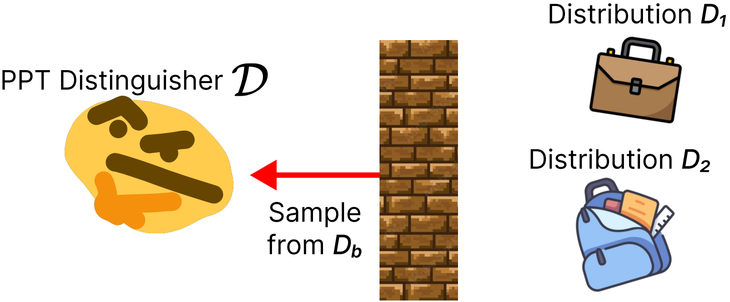
Note

Such idea of defining the zero-knowledge is called **simulation paradigm** and currently the most widely used way to prove zero-knowledge.

Computational Indistinguishability



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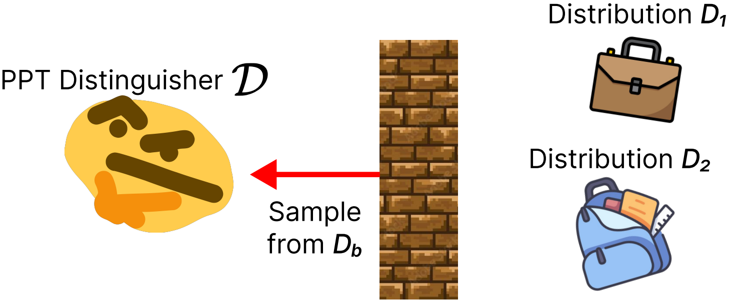

Definition (Informal Computational Indistinguisability)

D_1 and D_2 are **computationally indistinguishable** (denoted by $D_1 \approx D_2$) if for any PPT distinguisher $\mathcal{D}$, even after polynomial number k of samples from D_b (where $b \xleftarrow{R} \{0, 1\}$), for prediction $\hat{b}$: $\Pr[\hat{b} = b] < \frac{1}{2} + \text{negl}(k)$.

Zero-Knowledge Formally (Kind of)

Finally, we are ready to define the **zero-knowledge**.

Definition (Honest-Verifier Zero-Knowledge (HVZK))

An interactive protocol $(\mathcal{P}, \mathcal{V})$ is **honest-verifier zero-knowledge (HVZK)** for a language $\mathcal{L}_{\mathcal{R}}$ there exists a poly-time simulator Sim such that for any valid statement $x \in \mathcal{L}_{\mathcal{R}}$:

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An interactive protocol $(\mathcal{P}, \mathcal{V})$ is **zero-knowledge (ZK)** for a language $\mathcal{L}_{\mathcal{R}}$ if *for every poly-time verifier* $\mathcal{V}^*$ there *exists a poly-time simulator* Sim such that for any valid statement $x \in \mathcal{L}_{\mathcal{R}}$:

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Consider the **discrete logarithm relation and language** for a cyclic group $E(\mathbb{F}_p)$ of order r:

$$\mathcal{R} = \{(P, \alpha) \in E(\mathbb{F}_p) \times \mathbb{Z}_r : P = [\alpha]G\},$$
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Turns out $\mathcal{L}_{\mathcal{R}} = E(\mathbb{F}_p)$, so the proof $X \in \mathcal{L}_{\mathcal{R}}$ itself is useless.

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Definition (Proof of Knowledge)

The interactive protocol $(\mathcal{P}, \mathcal{V})$ is a **proof of knowledge** for $\mathcal{L}_{\mathcal{R}}$ if exists a poly-time extractor algorithm $\mathcal{E}$ such that for any valid statement $x \in \mathcal{L}_{\mathcal{R}}$, in expected poly-time $\mathcal{E}^{\mathcal{P}}(x)$ outputs w such that $(x, w) \in \mathcal{R}$.

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The extractor $\mathcal{E}$ will always output w if $x \in \mathcal{L}_{\mathcal{R}}$. $\square$

Fiat-Shamir Heuristic

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This way, we can rigorously prove the security of some cryptographic protocols *even* if the Diffie-Hellman problem is suddenly solved.

Random Oracle (RO)

One of the most popular cryptographic oracles is the **random oracle** $\mathcal{O}_{\mathsf{R}}$.

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Suppose someone is inputting x to the random oracle $\mathcal{O}_{\mathsf{R}} : \mathcal{X} \to \mathcal{Y}^{a}$. The oracle $\mathcal{O}_{\mathsf{R}}$ does the following:

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[a] Typically, RO works with a family of functions $f : \mathcal{X} \to \mathcal{Y}$, but we are not going too deep into the details.

Question

Which very well-known cryptographic object can "serve" as a random oracle?

Fiat-Shamir Transformation

Statement

Any interactive public-coin protocol can be converted into a non-interactive public-coin protocol with preserving completeness, soundness, and zero-knowledge using the random oracle.

One of such transformations is called **Fiat-Shamir heuristic**. Idea:

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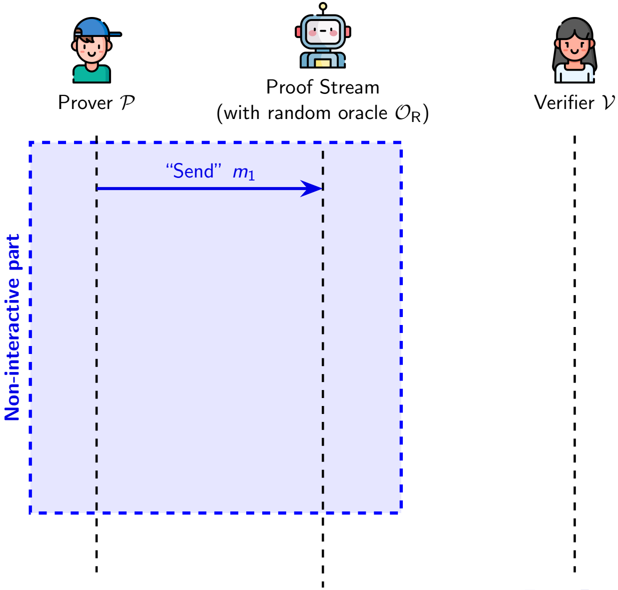
3. Thus, we can replace the verifier's messages with the hash (random oracle) of all the previous conversation.

Fiat-Shamir Heuristic Illustration



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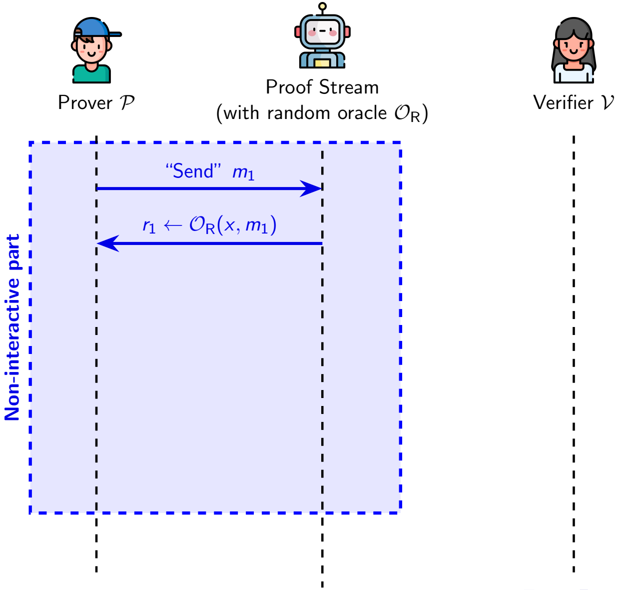

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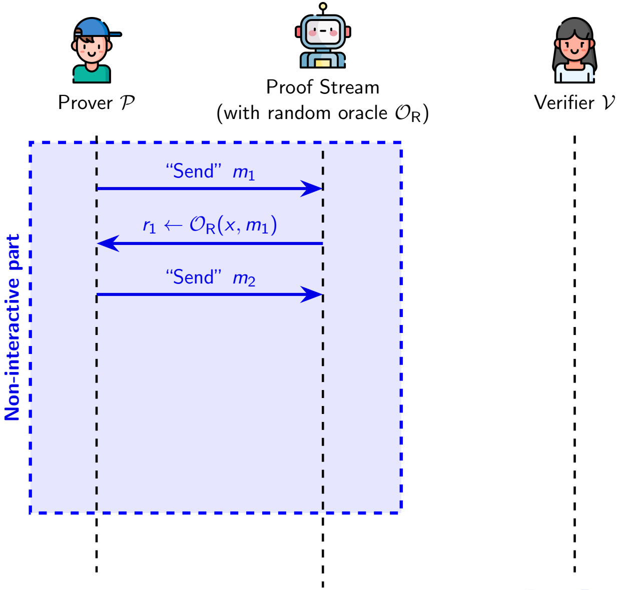

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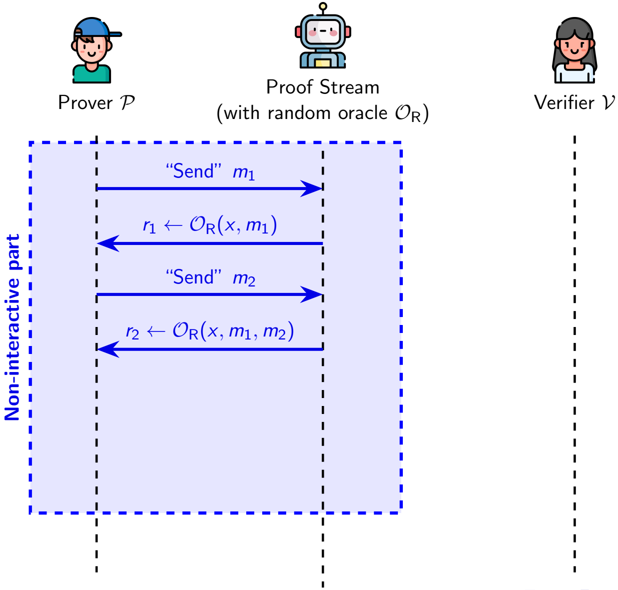

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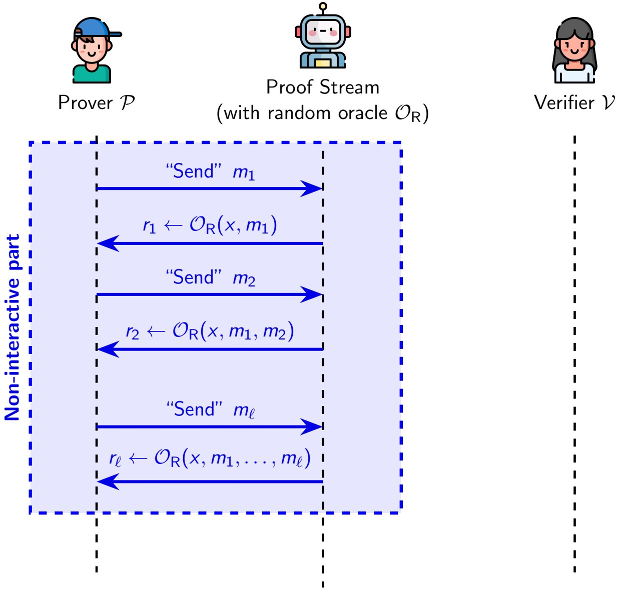

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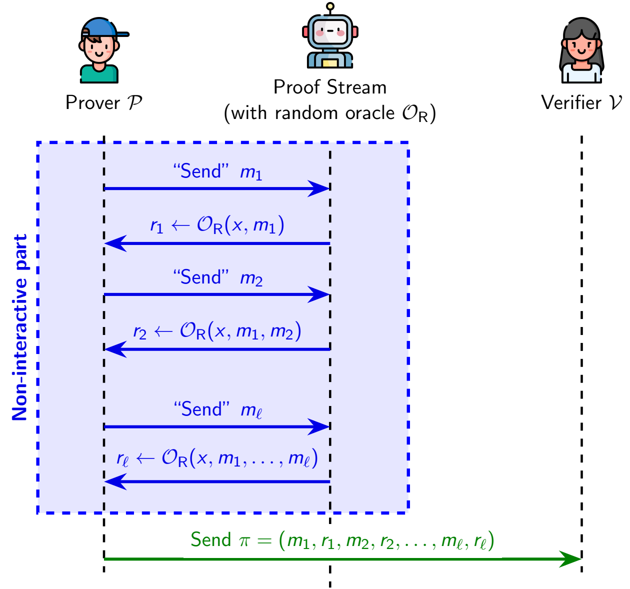

Thank you for your attention!