

Introduction to zk-SNARK. R1CS

Distributed Lab

Sep 12, 2024



Plan

- 1 What is zk-SNARK?
- 2 Boolean Circuits
- 3 Arithmetic Circuits
- 4 Linear Algebra Preliminaries
- 5 Rank-1 Constraint System

What is zk-SNARK?

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- **Non-interactiveness** — to produce the proof, the prover does not need any interaction with the verifier.
- **Zero-Knowledge** — the verifier learns nothing about the data used to produce the proof, despite knowing that this data resolves the given problem and that the prover possesses it.

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Well... Let's take a look at some example.

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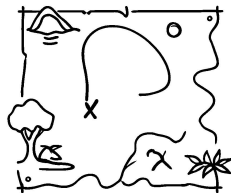
...and you've found a hidden treasure chest...



...but how to prove that without revealing the chest location?

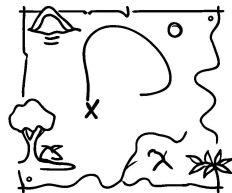
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We can retrieve some information from that:

The Secret Data: the exact treasure location.

The Prover: you.

The Verifier: the treasure hunt organizer.

Ohh... Got it!

Here is how we can apply the zk-SNARK to our problem:

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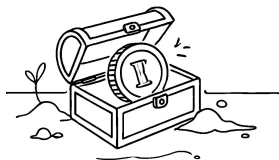
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Well... The golden coin where the pirates' sign is engraved is our zk-SNARK proof!

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When we need to prove that some element is in a merkle tree, we can't come to a verifier and give them a "coin"...

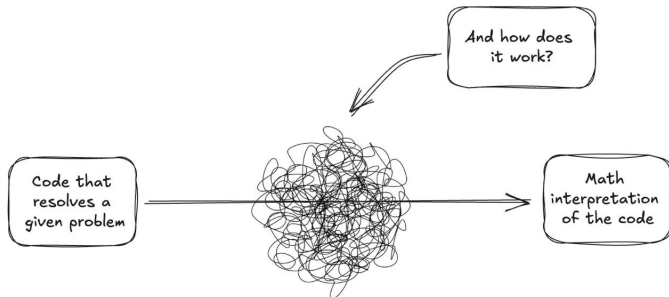
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Question?

How do we convert a program into a mathematical language?



Boolean Circuits

Boolean Circuits

We can do that in a way like the computer does it - boolean circuits.

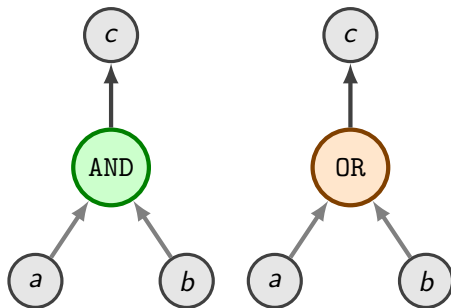


Figure: Boolean AND and OR Gates

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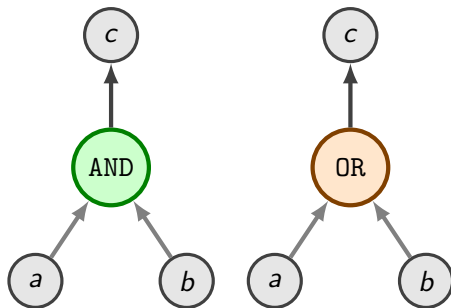


Figure: Boolean AND and OR Gates

A	B	A AND B
0	0	0
0	1	0
1	0	0
1	1	1

Figure: AND Gate Truth Table

Note

With any of $\{\text{AND}, \text{NOT}\}$ or $\{\text{OR}, \text{NOT}\}$ gates sets one can build any possible logical circuit, they are called **functionally complete** sets.

Boolean Circuit Example

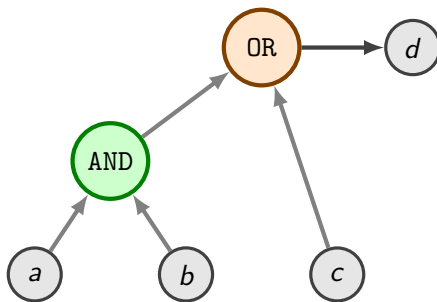


Figure: Example of a circuit evaluating $d = (a \text{ AND } b) \text{ OR } c$.

Boolean Circuit Example

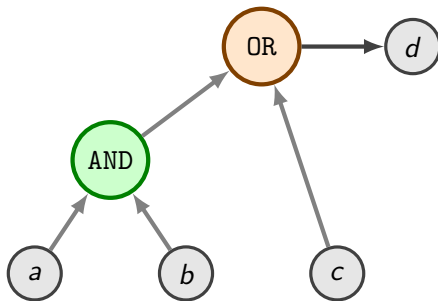


Figure: Example of a circuit evaluating $d = (a \text{ AND } b) \text{ OR } c$.

Boolean circuits receive an input vector of 0, 1 and resolve to true (1) or false (0);

The above circuit can be satisfied with the next values:

$$a = 1, \quad b = 1, \quad c = 0, \quad d = 1$$

SHA-256 Boolean circuit

File	No. ANDs	No. XORs	No. INVs
sha256Final.txt	22,272	91,780	2,194

Figure: Stats of a SHA256 boolean circuit implementation.

More than 100000 gates. Impressive, doesn't it?

But it also shows how inconvenient the boolean circuits are.

Arithmetic Circuits

Arithmetic Circuits

Similar to Boolean Circuits, the **Arithmetic circuits** consist of gates and wires.

- Wires: elements of some finite field \mathbb{F}_p .
- Gates: addition (\oplus) and multiplication (\odot) corresponding to the field.

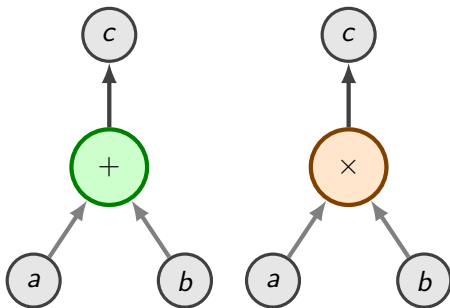


Figure: Addition and Multiplication Gates

Arithmetic Circuits Example I

Example

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def multiply(a: F, b: F)  $\rightarrow$  F:  
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The witness vector (essentially, our solution vector) is $\mathbf{w} = (r, a, b)$, for example: $(6, 2, 3)$.

We assume that the a and b are input values.

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Note

We can think of the “=” in the gate as an assertion.

Arithmetic Circuits Example II

Example

Now, suppose we want to implement the evaluation of the polynomial $Q(x_1, x_2) = x_1^3 + x_2^2 \in \mathbb{F}[x_1, x_2]$ using arithmetic circuits.

```
def evaluate(x1: F, x2: F) -> F:  
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Looks easy, right?

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Looks easy, right? But the circuit is now much less trivial.

$$\begin{array}{ll} x_1^2 = x_1 \times x_1 & r_1 = x_1 \times x_1 \\ x_1^3 = x_1^2 \times x_1 & r_2 = r_1 \times x_1 \\ x_2^2 = x_2 \times x_2 & r_3 = x_2 \times x_2 \\ Q = x_1^3 + x_2^2 & Q = r_2 + r_3 \end{array} \quad \text{or}$$

Arithmetic Circuits Example II

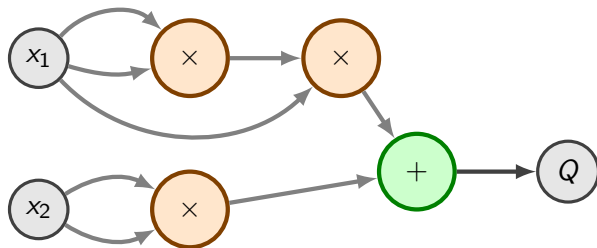


Figure: Example of a circuit evaluating $x_1^3 + x_2^2$.

Arithmetic Circuits Example III

Example

Well, it is quite clear how to represent any polynomial-like expressions. But how can we translate `if` statements?

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def example(a: bool, b: F, c: F)  $\rightarrow$  F:  
    if a:  
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Corresponding equations for the circuit are:

$$\begin{array}{lll} r_1 = b \times c, & r_3 = 1 - a, & r_5 = r_3 \times r_2 \\ r_2 = b + c, & r_4 = a \times r_1, & r = r_4 + r_5 \end{array}$$

Arithmetic Circuits Example III

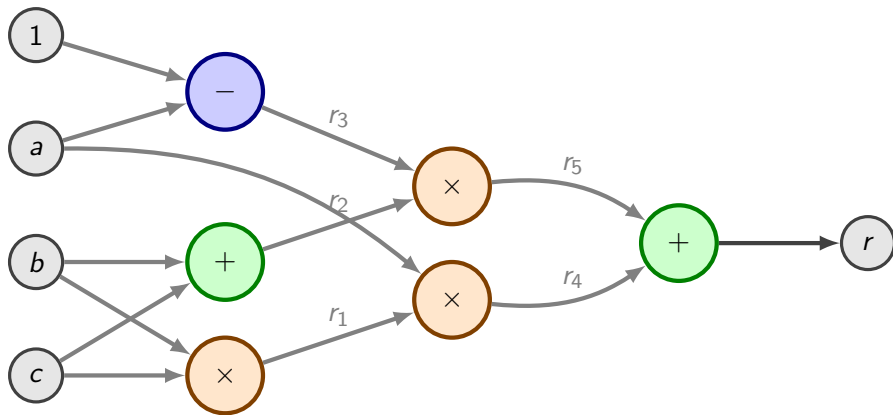


Figure: Example of a circuit evaluating the if statement logic.

Linear Algebra Preliminaries

Definition

A **vector space** V over the field \mathbb{F} is an abelian group for addition “+” together with a scalar multiplication operation “ \cdot ” from $\mathbb{F} \times V$ to V , sending $(\lambda, x) \mapsto \lambda x$ and such that for any $\mathbf{v}, \mathbf{u} \in V$ and $\lambda, \mu \in \mathbb{F}$ we have:

- $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$
- $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$
- $(\lambda\mu)\mathbf{v} = \lambda(\mu\mathbf{v})$
- $1\mathbf{v} = \mathbf{v}$

Any element $\mathbf{v} \in V$ is called a **vector**, and any element $\lambda \in \mathbb{F}$ is called a **scalar**. We also mark vector elements in boldface.

Matrix

The matrix is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns. For example, the matrix A with m rows and n columns, consisting of elements from the finite field \mathbb{F} is denoted as $A \in \mathbb{F}^{m \times n}$.

Definition

Let A, B be two matrices over the field \mathbb{F} . The following operations are defined:

- **Matrix addition/subtraction:** $A \pm B = \{a_{i,j} \pm b_{i,j}\}_{i,j=1}^{m \times n}$. The matrices A and B must have the same size $m \times n$.
- **Scalar multiplication:** $\lambda A = \{\lambda a_{i,j}\}_{1 \leq i,j \leq n}$ for any $\lambda \in \mathbb{F}$.
- **Matrix multiplication:** $C = AB$ is a matrix $C \in \mathbb{F}^{m \times p}$ with elements $c_{i,j} = \sum_{\ell=1}^n a_{i,\ell} b_{\ell,j}$. The number of columns in A must be equal to the number of rows in B , that is $A \in \mathbb{F}^{m \times n}$ and $B \in \mathbb{F}^{n \times p}$.

Matrix Multiplication

Example

Consider

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \in \mathbb{R}^{2 \times 3}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \in \mathbb{R}^{3 \times 2}$$

We cannot add A and B since they have different sizes. However, we can multiply them:

$$AB = \begin{bmatrix} 5 & 6 \\ 7 & 9 \end{bmatrix}, \quad BA = \begin{bmatrix} 4 & 4 & 5 \\ 7 & 7 & 5 \\ 3 & 3 & 3 \end{bmatrix}$$

To see why, for example, the upper left element of AB is 5, we can calculate it as $\sum_{\ell=1}^3 a_{1,\ell} b_{\ell,1} = 1 \times 2 + 1 \times 1 + 2 \times 1 = 5$.

Vector As A Matrix

Note

It just so happens that when working with vectors, we usually assume that they are **column vectors**. This means that the vector $v = (v_1, v_2, \dots, v_n)$ is represented as a matrix:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

This is a common convention in linear algebra, and we will use it in the following sections.

Matrix Transpose

Definition (Transposition)

Given a matrix $A \in \mathbb{F}^{m \times n}$, the **transpose** of A is a matrix $A^T \in \mathbb{F}^{n \times m}$ with elements $A_{ij}^T = A_{ji}$.

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}^T = [1, 2, 3]$$

Inner Product

Definition

Consider the vector space \mathbb{V} over the finite field \mathbb{F}_p . The **inner product** is a function $\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{F}_p$ satisfying the following conditions for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}$:

- $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$.
- $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$.
- $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ for all $\mathbf{u} \in \mathbb{V}$ iff $\mathbf{v} = \mathbf{0}$.
- $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ for all $\mathbf{v} \in \mathbb{V}$ iff $\mathbf{u} = \mathbf{0}$.

Plenty of functions can be built that satisfy the inner product definition, we'll use the one that is usually called **dot product**.

Dot Product

Definition

Consider the vector space \mathbb{V} over the finite field \mathbb{F}_p . The **dot product** on \mathbb{V} is a function $\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{F}$, defined for every $\mathbf{u}, \mathbf{v} \in \mathbb{V}$ as follows:

$$\langle \mathbf{u}, \mathbf{v} \rangle := \mathbf{u}^\top \mathbf{v} = \sum_{i=1}^n u_i v_i$$

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Note

The dot product can also be denoted using the dot notation as:

$$\mathbf{u} \cdot \mathbf{v}$$

That is why it's called the “dot” product.

Dot Product

Example

Let \mathbf{u}, \mathbf{v} are vectors over the real number \mathbb{R} , where

$$\mathbf{u} = (1, 2, 3), \quad \mathbf{v} = (2, 4, 3)$$

Then:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^3 u_i v_i = 2 \cdot 1 + 2 \cdot 4 + 3 \cdot 3 = 2 + 8 + 9 = 19$$

Hadamard Product

Definition

Suppose $A, B \in \mathbb{F}^{m \times n}$. The **Hadamard product** $A \odot B$ gives a matrix C such that $C_{i,j} = A_{i,j}B_{i,j}$. Essentially, we multiply elements elementwise.

Example

Consider $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$. Then, the Hadamard product:

$$A \odot B = \begin{bmatrix} 1 \cdot 3 & 1 \cdot 2 & 2 \cdot 1 \\ 3 \cdot 0 & 0 \cdot 2 & 3 \cdot 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Outer Product

Definition

Given two vectors $\mathbf{u} \in \mathbb{F}^n$, $\mathbf{v} \in \mathbb{F}^m$ the **outer product** is a the matrix whose entries are all products of an element in the first vector with an element in the second vector:

$$\mathbf{u} \otimes \mathbf{v} := \mathbf{u}\mathbf{v}^T = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \cdots & u_2 v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_m v_1 & u_m v_2 & \cdots & u_m v_n \end{bmatrix}$$

Lemma (Properties of outer product)

For any scalar $c \in \mathbb{F}$ and $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{F}^n \times \mathbb{F}^m \times \mathbb{F}^p$:

- *Transpose:* $(\mathbf{u} \otimes \mathbf{v}) = (\mathbf{v} \otimes \mathbf{u})^T$
- *Distributivity:* $\mathbf{u} \otimes (\mathbf{v} + \mathbf{w}) = \mathbf{u} \otimes \mathbf{v} + \mathbf{u} \otimes \mathbf{w}$
- *Scalar Multiplication:* $c(\mathbf{v} \otimes \mathbf{u}) = (c\mathbf{v}) \otimes \mathbf{u} = \mathbf{v} \otimes (c\mathbf{u})$
- *Rank:* the outer product $\mathbf{u} \otimes \mathbf{v}$ is a rank-1 matrix if \mathbf{u} and \mathbf{v} are non-zero vectors

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The rows/columns number 2 and 3 in the result matrix can be represented as a linear combination of the first row/column, specifically by multiplying it by 2 and 3, respectively.

Rank-1 Constraint System

Constraint Definition

Definition

Each **constraint** in the Rank-1 Constraint System must be in the form:

$$\langle \mathbf{a}, \mathbf{w} \rangle \times \langle \mathbf{b}, \mathbf{w} \rangle = \langle \mathbf{c}, \mathbf{w} \rangle$$

Where \mathbf{w} is a vector containing all the *input*, *output*, and *intermediate* variables involved in the computation. The vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} are vectors of coefficients corresponding to these variables, and they define the relationship between the linear combinations of \mathbf{w} on the left-hand side and the right-hand side of the equation.

Constraint Example

Example

Consider the most basic circuit with one multiplication gate: $x_1 \times x_2 = r$. The witness vector $\mathbf{w} = (r, x_1, x_2)$. So

$$w_2 \times w_3 = w_1$$

$$(0 + w_2 + 0) \times (0 + 0 + w_3) = w_1 + 0 + 0$$

$$(0w_1 + 1w_2 + 0w_3) \times (0w_1 + 0w_2 + 1w_3) = 1w_1 + 0w_2 + 0w_3$$

Therefore the coefficients vectors are:

$$\mathbf{a} = (0, 1, 0), \quad \mathbf{b} = (0, 0, 1), \quad \mathbf{c} = (1, 0, 0).$$

The general form of our constraint is:

$$(a_1 w_1 + a_2 w_2 + a_3 w_3)(b_1 w_1 + b_2 w_2 + b_3 w_3) = c_1 w_1 + c_2 w_2 + c_3 w_3$$

Constraint System Example

Now, let us consider a more complex example.

```
def r(x1: bool, x2: F, x3: F)  $\rightarrow$  F:  
    return x2 * x3 if x1 else x2 + x3
```

That can be expressed as:

$$r = x_1 \times (x_2 \times x_3) + (1 - x_1) \times (x_2 + x_3)$$

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Thus, the next constraints can be build:

$$x_1 \times x_1 = x_1 \quad (\text{binary check}) \quad (1)$$

$$x_2 \times x_3 = \text{mult} \quad (2)$$

$$x_1 \times \text{mult} = \text{selectMult} \quad (3)$$

$$(1 - x_1) \times (x_2 + x_3) = r - \text{selectMult} \quad (4)$$

Constraint System Example

The witness vector: $\mathbf{w} = (1, r, x_1, x_2, x_3, \text{mult}, \text{selectMult})$.

The coefficients vectors:

$$\begin{aligned} \mathbf{a}_1 &= (0, 0, 1, 0, 0, 0, 0), & \mathbf{b}_1 &= (0, 0, 1, 0, 0, 0, 0), & \mathbf{c}_1 &= (0, 0, 1, 0, 0, 0, 0) \\ \mathbf{a}_2 &= (0, 0, 0, 1, 0, 0, 0), & \mathbf{b}_2 &= (0, 0, 0, 0, 1, 0, 0), & \mathbf{c}_2 &= (0, 0, 0, 0, 0, 1, 0) \\ \mathbf{a}_3 &= (0, 0, 1, 0, 0, 0, 0), & \mathbf{b}_3 &= (0, 0, 0, 0, 0, 1, 0), & \mathbf{c}_3 &= (0, 0, 0, 0, 0, 0, 1) \\ \mathbf{a}_4 &= (1, 0, -1, 0, 0, 0, 0), & \mathbf{b}_4 &= (0, 0, 0, 1, 1, 0, 0), & \mathbf{c}_4 &= (0, 1, 0, 0, 0, 0, -1) \end{aligned}$$

Using the arithmetic in a large finite field \mathbb{F}_p , consider the following values:

$$x_1 = 1, \quad x_2 = 3, \quad x_3 = 4$$

Verifying the constraints:

- ① $x_1 \times x_1 = x_1 \quad (1 \times 1 = 1)$
- ② $x_2 \times x_3 = \text{mult} \quad (3 \times 4 = 12)$
- ③ $x_1 \times \text{mult} = \text{selectMult} \quad (1 \times 12 = 12)$
- ④ $(1 - x_1) \times (x_2 + x_3) = r - \text{selectMult} \quad (0 \times 7 = 12 - 12)$

R1CS In Matrix Form

Theorem

Consider a Rank-1 Constraint System (R1CS) defined by m constraints. Each constraint is associated with coefficient vectors \mathbf{a}_i , \mathbf{b}_i , and \mathbf{c}_i , where $i \in \{1, 2, \dots, m\}$ and also a witness vector \mathbf{w} consisting of n elements. Then this system can also be represented using the corresponding matrices A , B , and C .

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} \quad C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}$$

such that all constraints can be reduced to the single equation:

$$A\mathbf{w} \odot B\mathbf{w} = C\mathbf{w}$$

R1CS In Matrix Form

Proof. Matrices defined this way can be expressed as

$$A = \begin{bmatrix} \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \\ \vdots \\ \mathbf{a}_m^\top \end{bmatrix}, \quad B = \begin{bmatrix} \mathbf{b}_1^\top \\ \mathbf{b}_2^\top \\ \vdots \\ \mathbf{b}_m^\top \end{bmatrix}, \quad C = \begin{bmatrix} \mathbf{c}_1^\top \\ \mathbf{c}_2^\top \\ \vdots \\ \mathbf{c}_m^\top \end{bmatrix}$$

Consider an expression $A\mathbf{w}$:

$$A\mathbf{w} = \begin{bmatrix} \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \\ \vdots \\ \mathbf{a}_m^\top \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^\top \mathbf{w} \\ \mathbf{a}_2^\top \mathbf{w} \\ \vdots \\ \mathbf{a}_m^\top \mathbf{w} \end{bmatrix}$$

Therefore, we have:

$$A\mathbf{w} = \begin{bmatrix} \langle \mathbf{a}_1, \mathbf{w} \rangle \\ \langle \mathbf{a}_2, \mathbf{w} \rangle \\ \vdots \\ \langle \mathbf{a}_m, \mathbf{w} \rangle \end{bmatrix}, \quad B\mathbf{w} = \begin{bmatrix} \langle \mathbf{b}_1, \mathbf{w} \rangle \\ \langle \mathbf{b}_2, \mathbf{w} \rangle \\ \vdots \\ \langle \mathbf{b}_m, \mathbf{w} \rangle \end{bmatrix}, \quad C\mathbf{w} = \begin{bmatrix} \langle \mathbf{c}_1, \mathbf{w} \rangle \\ \langle \mathbf{c}_2, \mathbf{w} \rangle \\ \vdots \\ \langle \mathbf{c}_m, \mathbf{w} \rangle \end{bmatrix}$$

Thus, $A\mathbf{w} \odot B\mathbf{w} = C\mathbf{w}$ is equivalent to the system of m constraints:

$$\langle \mathbf{a}_j, \mathbf{w} \rangle \times \langle \mathbf{b}_j, \mathbf{w} \rangle = \langle \mathbf{c}_j, \mathbf{w} \rangle, \quad j \in \{1, \dots, m\}.$$

Example

The vectors \mathbf{a}_i from the previous examples have the form:

$$\mathbf{a}_1 = (0, 0, 1, 0, 0, 0, 0)$$

$$\mathbf{a}_2 = (0, 0, 0, 1, 0, 0, 0)$$

$$\mathbf{a}_3 = (0, 0, 1, 0, 0, 0, 0)$$

$$\mathbf{a}_4 = (1, 0, -1, 0, 0, 0, 0)$$

This corresponds to $n = 7, m = 4$

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} & a_{2,7} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} & a_{3,7} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Why Rank-1?

Lemma

Suppose we have a constraint $\langle \mathbf{a}, \mathbf{w} \rangle \times \langle \mathbf{b}, \mathbf{w} \rangle = \langle \mathbf{c}, \mathbf{w} \rangle$ with coefficient vectors \mathbf{a} , \mathbf{b} , \mathbf{c} and witness vector \mathbf{w} (all from \mathbb{F}^n). Then it can be expressed in the form:

$$\mathbf{w}^\top A \mathbf{w} + \mathbf{c}^\top \mathbf{w} = 0$$

Where A is the outer product of vectors \mathbf{a} , \mathbf{b} , so a **rank-1** matrix.

Lemma proof. Consider $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{w} \in \mathbb{F}^n$.

$$\left(\sum_{i=1}^n a_i w_i \right) \times \left(\sum_{j=1}^n b_j w_j \right) = \sum_{k=1}^n c_k w_k$$

Combine the products into a double sum on the left side:

$$\sum_{i=1}^n \sum_{j=1}^n a_i b_j w_i w_j = \mathbf{w}^\top (\mathbf{a} \otimes \mathbf{b}) \mathbf{w} = \mathbf{w}^\top A \mathbf{w}$$

Thus, the constraint can be written as:

$$\mathbf{w}^\top A \mathbf{w} + \mathbf{c}^\top \mathbf{w} = 0$$

Thanks for your attention!