Sigma Protocols

Distributed Lab

September 3, 2024

Plan

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   - Motivation for Σ-protocols
2. Schnorr Identification Protocol
   - Interactive Protocol
   - Non-interactive Schnorr's Identification Protocol
   - Schnorr's Signature Scheme
3. Sigma Protocols
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   - Okamoto Representation Protocol
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5. Coding Time!
   - Okamoto's Protocol

Introduction

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Announcement

Today, we will build and code our first non-interactive proof system using the Fiat-Shamir heuristic based on **Sigma protocols**!

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Note

Everything that has a natural "homomorphic"/discrete-log-like structure can be proven using Sigma (Σ) protocols!

Schnorr Identification Protocol

Problem Statement

Suppose $\mathbb{G}$ is a cyclic group of order q with a generator g. Then, the relation and language being considered are:

$$\mathcal{R} = \{(u, \alpha) \in \mathbb{G} \times \mathbb{Z}_q : u = g^\alpha\}, \ \mathcal{L}_\mathcal{R} = \{u \in \mathbb{G} : \exists \alpha \in \mathbb{Z}_q : u = g^\alpha\}$$

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Problem #2

Why cannot we simply send α? Because we do not want to reveal the witness! That is why we need a zero-knowledge non-interactive argument of knowledge (zk-NARK).

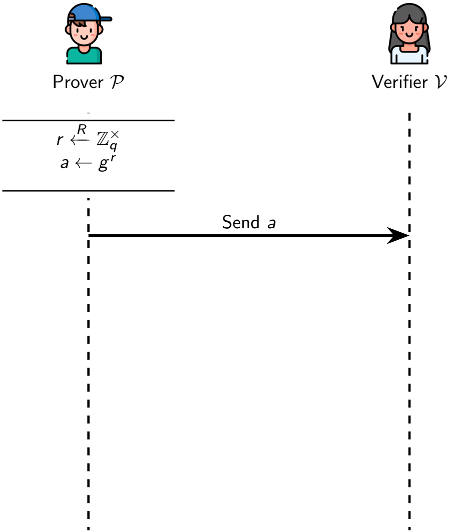
Protocol Flow

Prover $\mathcal{P}$

Verifier $\mathcal{V}$

$$r \xleftarrow{R} \mathbb{Z}_q^\times$$
$$a \leftarrow g^r$$

Protocol Flow



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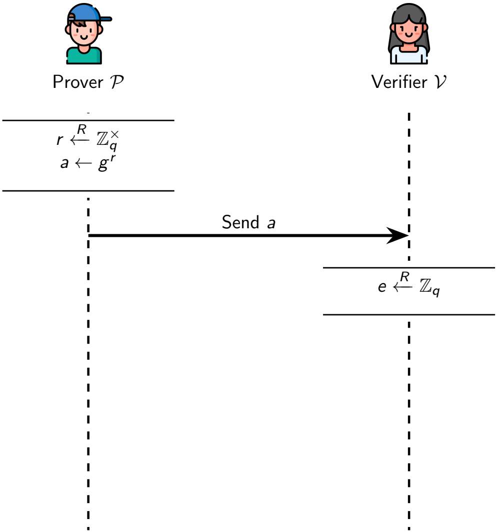

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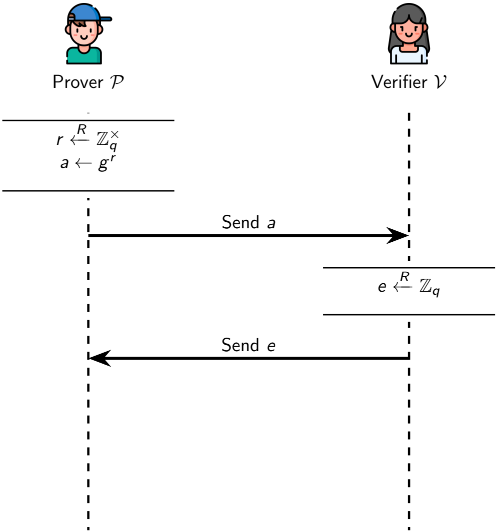

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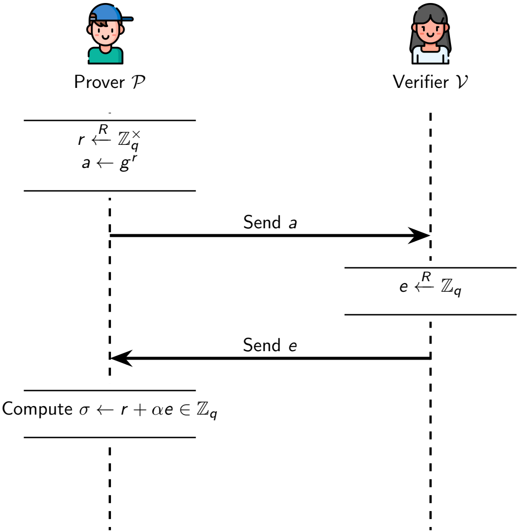

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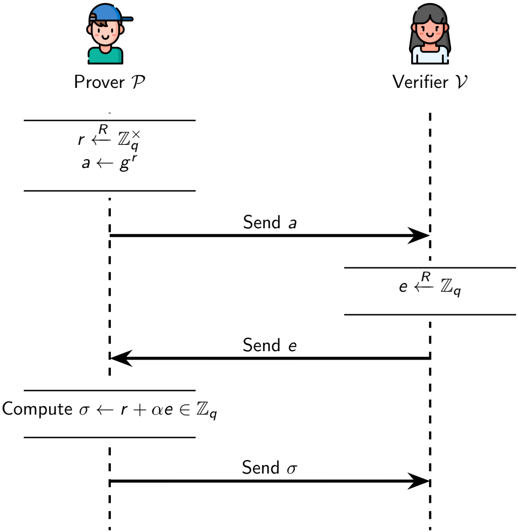

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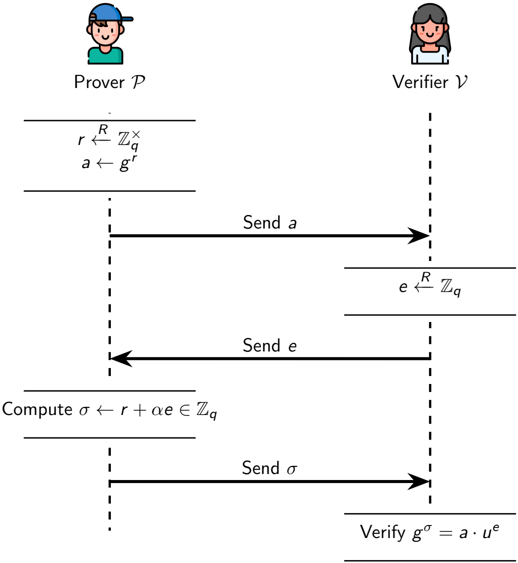

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Reminder

Suppose prover had messages $(m_1, m_2, \ldots, m_n)$ before verifier sends a challenge c. If x is a public statement, it suffices to choose $c \leftarrow H(x, m_1, \ldots, m_n)$ without any interaction.

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- $\mathsf{Verify}((a, \sigma), m, \mathsf{pk})$: The verifier checks if $g^\sigma = a \cdot u^e$ for $e \leftarrow H(u, m, a)$.

Note: In **green** we marked the only difference between the identification and signature protocols.

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- Upon receiving c, $\mathcal{P}$ computes the response z and sends it to $\mathcal{V}$.

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Let $\mathcal{R} \subset \mathcal{X} \times \mathcal{W}$ be an effective relation. A **Sigma protocol** for $\mathcal{R}$ is an interactive protocol $(\mathcal{P}, \mathcal{V})$ that satisfies the following properties:

- In the beginning, $\mathcal{P}$ computes a **commitment** a and sends it to $\mathcal{V}$.
- $\mathcal{V}$ chooses a random **challenge** $c \in \mathcal{C}$ from the challenge space $\mathcal{C}$ and sends it to $\mathcal{P}$.
- Upon receiving c, $\mathcal{P}$ computes the response z and sends it to $\mathcal{V}$.
- $\mathcal{V}$ outputs either accept or reject based on the **conversation** (a, c, z).

Generalization

Now, can we generalize the Schnorr protocol to any relation $\mathcal{R}$?
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Definition

(a, c, z) is an **accepting conversation** if $\mathcal{V}$ outputs accept on this tuple.

Why Σ?

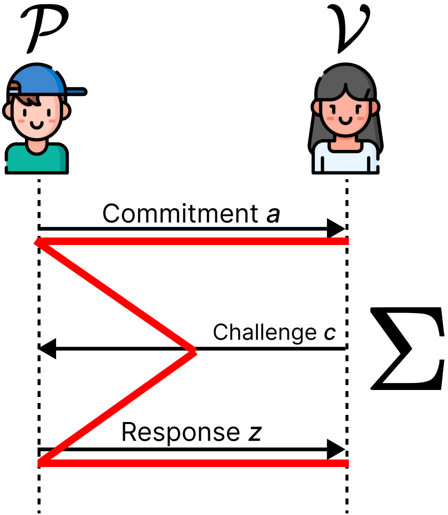

Figure: Why Σ-protocols are called so.

Special Soundness

Definition (Special Soundness)

Let $(\mathcal{P}, \mathcal{V})$ be a Σ-protocol for $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{Y}$. We that that $(\mathcal{P}, \mathcal{V})$ is **special sound** if there exists a witness extractor $\mathcal{E}$ such that, given statement $x \in \mathcal{X}$ and two accepting conversations (a, c, z) and (a, c', z') (where $c \neq c'$)[a], the extractor can always efficiently compute the witness w such that $(x, w) \in \mathcal{R}$.

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[a] Notice that initial commitments in both conversations are the same!

Example

The Schnorr protocol is special sound because, given two accepting conversations (a, e, σ) and (a, e', σ'), we can compute the witness α. You can verify that $\alpha = \Delta\sigma/\Delta e$ for $\Delta\sigma = \sigma' - \sigma$ and $\Delta e = e' - e$ suffices.

Sigma Protocols Examples

Okamoto's Protocol

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Notice that for the given u there are exactly q representations relative to g and h. Indeed, $\forall \beta \in \mathbb{Z}_q \, \exists! \alpha \in \mathbb{Z}_q : g^\alpha = uh^{-\beta}$.

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Question

How do we actually prove that $\mathcal{P}$ knows the representation of u?

$$\mathcal{R} = \left\{(u, (\alpha, \beta)) \in \mathbb{G} \times \mathbb{Z}_q^2 : u = g^\alpha h^\beta\right\}$$

Okamoto's Protocol Flow

Definition (Okamoto's Identification Protocol)

Okamoto's Protocol consists of two algorithms: $(\mathcal{P}, \mathcal{V})$, where the prover is assumed to know $(u, (\alpha, \beta)) \in \mathcal{R}$ defined above. The protocol is defined as follows:

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Announcement

We will code the non-interactive Okamoto's protocol in the next section! Stay tuned!

Okamoto's Protocol Correctness

Theorem

Okamoto's Protocol is a Σ-protocol for the relation $\mathcal{R}$ which is Honest-Verifier Zero-Knowledge (HVZK).

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Special Soundness. Suppose we are given two accepting conversations: $(u_r, c, (\alpha_z, \beta_z))$ and $(u_r, c', (\alpha'_z, \beta'_z))$ and we want to construct an extractor $\mathcal{E}$ which would give us a witness (α, β).

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from which the extractor $\mathcal{E}$ can efficiently compute witness as follows: $\alpha \leftarrow (\alpha_z - \alpha'_z)/(c - c')$ and $\beta \leftarrow (\beta_z - \beta'_z)/(c - c')$.

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Now, this makes it easier to define the relation $\mathcal{R}$ for the Chaum-Pedersen protocol:

$$\mathcal{R} = \left\{((u, v, w), \beta) \in \mathbb{G}^3 \times \mathbb{Z}_q : v = g^{\beta} \wedge w = u^{\beta}\right\}$$

Definition (Chaum-Pedersen Protocol)

Chaum-Pedersen Protocol consists of two algorithms: $(\mathcal{P}, \mathcal{V})$, where the prover is assumed to know $(\beta, (u, v, w)) \in \mathcal{R}$ defined above. The protocol is defined as follows:

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Note

If between input and output we have an easy-to-compute and hard-to-invert homomorphism, we can use Sigma protocols to prove pre-images of this homomorphism!

Problem Statement

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- **Okamoto Protocol:** Here we have $\mathbb{H} = \mathbb{Z}_q^2$, $\mathbb{T} = \mathbb{G}$, and $\psi : \mathbb{Z}_q^2 \to \mathbb{G}$ is defined as $\psi(\alpha, \beta) = g^\alpha h^\beta$.

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- **Okamoto Protocol:** Here we have $\mathbb{H} = \mathbb{Z}_q^2$, $\mathbb{T} = \mathbb{G}$, and $\psi : \mathbb{Z}_q^2 \to \mathbb{G}$ is defined as $\psi(\alpha, \beta) = g^\alpha h^\beta$.
- **Chaum-Pedersen Protocol:** Here we have $\mathbb{H} = \mathbb{Z}_q$, $\mathbb{T} = \mathbb{G}^2$, and $\psi : \mathbb{Z}_q \to \mathbb{G}^2$ is defined as $\psi(\beta) = (g^\beta, u^\beta)$.

Sigma Protocol

Definition (Sigma Protocol for the pre-image of a homomorphism)

The protocol consists of two algorithms: $(\mathcal{P}, \mathcal{V})$, where the prover is assumed to know the witness $h \in \mathbb{H}$ defined above. The protocol is defined as follows:

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Theorem

Such protocol is a Σ-protocol for the relation $\mathcal{R}$ which is Honest-Verifier Zero-Knowledge (HVZK).

Combining Σ-Protocols

One of the features (which we are not going to delve into) is the ability to combine Σ-protocols to prove more complex statements.

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Example

$\mathcal{P}$ can prove that he either knows the discrete log of u or the representation of u relative to g and h. Moreover, $\mathcal{V}$ does not know which of the two statements $\mathcal{P}$ is proving.

Coding Time!

Methodology

Reminder

Suppose prover had messages $(m_1, m_2, \ldots, m_n)$ before verifier sends a challenge c. If x is a public statement, it suffices to choose $c \leftarrow H(x, m_1, \ldots, m_n)$ without any interaction.

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Let us turn **Okamoto's Protocol** into a non-interactive proof using the Fiat-Shamir heuristic!

Reminder: Okamoto's Identification Protocol

1. $\mathcal{P}$ computes $\alpha_r \xleftarrow{R} \mathbb{Z}_q,\ \beta_r \xleftarrow{R} \mathbb{Z}_q,\ u_r \leftarrow g^{\alpha_r} h^{\beta_r}$ and sends commitment u_r to $\mathcal{V}$.
2. $\mathcal{V}$ samples the challenge $c \xleftarrow{R} \mathbb{Z}_q$ and sends c to $\mathcal{P}$.
3. $\mathcal{P}$ computes $\alpha_z \leftarrow \alpha_r + \alpha c, \beta_z \leftarrow \beta_r + \beta c$ and sends $\mathbf{z} = (\alpha_z, \beta_z)$.
4. $\mathcal{V}$ checks whether $g^{\alpha_z} h^{\beta_z} = u_r u^c$ and accepts or rejects the proof.

Non-Interactive Okamoto Protocol

Okamoto's Non-Interactive Identification Protocol

Now, we apply the *Fiat-Shamir Transformation*.

- Prove(1^λ): On input $(u, (\alpha, \beta)) \in \mathbb{G} \times \mathbb{Z}_q^2$,
  1. Sample $\alpha_r, \beta_r \xleftarrow{R} \mathbb{Z}_q$ and compute $u_r \leftarrow g^{\alpha_r} h^{\beta_r}$.
  2. Using the hash function $H : \mathbb{G} \times \mathbb{G} \to \mathcal{C}$, compute $c \leftarrow H(u, u_r)$.
  3. Compute $\alpha_z \leftarrow \alpha_r + \alpha c, \beta_z \leftarrow \beta_r + \beta c$ and publish (u_r, α_z, β_z) as a proof π.
- Verify: Upon receiving statement u and a proof $\pi = (u_r, \alpha_z, \beta_z)$, the verifier:
  1. Recomputes the challenge c using the hash function.
  2. Accepts if and only if $g^{\alpha_z} h^{\beta_z} = u_r u^c$.

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Thank you for your attention!