Introduction to Zero-Knowledge Proofs

August 22, 2024

Distributed Lab

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Plan

Introduction

Classical Proofs

- First proofs you have probably encountered were **geometry proofs**.
- You were given **axioms** and you can prove certain **statements** x using them.
- The proof π is a sequence of logical steps that lead from axioms to the statement. Essentially, you have a witness w that proves the statement.
- Your teacher is the **verifier** $\mathcal{V}$ who checks your proof, while you are the **prover** $\mathcal{P}$.
- This is a **classical proof** and in a sense, it is a **non-interactive proof**.

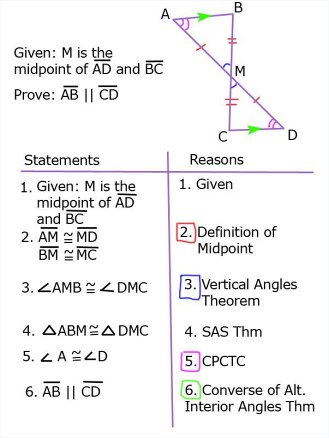


Figure: Geometry proof.

Motivation

Note

However, we cannot use such proofs in the digital world.

- Proofs must be verified by computers. Therefore, we need to develop **mathematic framework** to be able to program them.
- This leads to the question: what is **statement**? What is **proof**? What is **witness**? How to formally define them?
- We need to formalize these concepts.

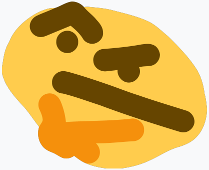


Figure: Hmm. . .

The most basic setting

- We have a **prover** $\mathcal{P}$ and a **verifier** $\mathcal{V}$.
- Prover $\mathcal{P}$ wants to prove some statement x to the verifier.
- Prover $\mathcal{P}$ has a **witness** w that contains all necessary information to prove the statement x. He sends π as a proof.
- Verifier $\mathcal{V}$ wants to be convinced that the statement x is true.

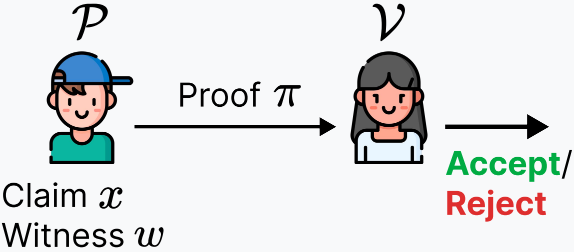


Figure: Typical setup for cryptographic proofs.

The Goal of SNARKs, STARKs etc.

We will try to solve the following problems:

- **Completeness:** If x is true, π proofs the statement.
- **Soundness:** If x is false, the prover $\mathcal{P}$ should not be able to convince the verifier $\mathcal{V}$ via any π^*.
- **Zero-knowledge:** π does not reveal anything about w.
- **Argument of knowledge:** $\mathcal{P}$ should convince $\mathcal{V}$ that besides x is true, he **knows** the witness w.
- **Succinctness:** The proof should be short, ideally polylogarithmic in $|x|$ ($|\pi| = \text{polylog}(|x|)$) + fast verification.
- **Arithmetization:** We need to convert the statement x into some algebraic form + make it relatively universal.

Note

SNARK, STARK, etc. will solve these problems!

Example to demonstrate the goal

Example

Given a hash function $H : \{0,1\}^* \to \{0,1\}^\ell$, $\mathcal{P}$ wants to convince $\mathcal{V}$ that he knows the preimage $x \in \{0,1\}^*$ such that $H(x) = y$.

- **Zero-knowledge:** The prover $\mathcal{P}$ does not want to reveal *anything* about the pre-image x to the verifier $\mathcal{V}$.
- **Argument of knowledge:** Proving y has a pre-image is useless. $\mathcal{P}$ must show he **knows** $x \in \{0,1\}^*$ s.t. $H(x) = y$.
- **Succinctness:** If the hash function takes n operations to compute, the proof should be **much** shorter than n operations. **State-of-art**: size is $\text{polylog}(n) = O((\log n)^c)$. Verification time is also typically polylogarithmic (or even $O(1)$ in some cases).

Note

But first, let us start with the basics.

Relations. Languages. NP Statements.

Language

Definition (Relation)

Given two sets $\mathcal{X}$ and $\mathcal{Y}$, the **relation** is $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{Y}$.

- $\mathcal{X}$ is typically a set of **statements**.
- $\mathcal{Y}$ is a set of **witnesses**.

Definition (Language of true statements)

Let $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{Y}$ be a relation. We say that a statement $x \in \mathcal{X}$ is a **true** statement if $(x, y) \in \mathcal{R}$ for some $y \in \mathcal{Y}$, otherwise the statement is called **false**. We define by $\mathcal{L}_\mathcal{R}$ (the language over relation $\mathcal{R}$) the set of all true statements, that is:

$$\mathcal{L}_\mathcal{R} = \{x \in \mathcal{X} : \exists y \in \mathcal{Y} \text{ such that } (x, y) \in \mathcal{R}\}.$$

Language Example #1: Semiprimes

Example (Product of Two Primes (Semiprimes))

Claim: number $n \in \mathbb{N}$ is the product of two prime numbers $w = (p, q) \in \mathbb{N} \times \mathbb{N}$. The **relation** is given by:

$$\mathcal{R} = \{(n, p, q) \in \mathbb{N}^3 : n = p \cdot q \text{ where } p, q \text{ are primes}\}$$

In this particular case, the **language of true statements** is defined as

$$\mathcal{L}_{\mathcal{R}} = \{n \in \mathbb{N} : \exists w = (p, q) \text{ are primes such that } n = p \cdot q\}$$

- **Valid witness #1:** $n = 15 \in \mathcal{L}_{\mathcal{R}}$. Witness: $w = (3, 5)$.
- **Invalid witness:** $n = 16 \notin \mathcal{L}_{\mathcal{R}}$. There is no valid witness.
- **Valid witness #2:** $n = 50252009 \in \mathcal{L}_{\mathcal{R}}$. Witness: $w = (5749, 8741)$.

Question: Is $n = 27$ a true statement? What about $n = 26$?

Language Example #2: Square Root

Reminder

$\mathbb{Z}_N^\times = \{x \in \mathbb{Z}_N : \gcd\{x, N\} = 1\}$. **Example:** $\mathbb{Z}_{10}^\times = \{1, 3, 7, 9\}$

Example

Claim: number $x \in \mathbb{Z}_N^\times$ is a **quadratic residue** modulo N: $(\exists w \in \mathbb{Z}_N^\times) : \{x \equiv w^2 \pmod N\}$ (w is **modular square root** of x).
Relation: $\mathcal{R} = \{(x, w) \in (\mathbb{Z}_N^\times)^2 : x \equiv w^2 \pmod N\}$
Language: $\mathcal{L}_\mathcal{R} = \{x \in \mathbb{Z}_N^\times : \exists w \in \mathbb{Z}_N^\times \text{ such that } x \equiv w^2 \pmod N\}$.
Examples for $N = 7$:

- $4 \in \mathcal{L}_\mathcal{R}$ since $5^2 \equiv 4 \pmod 7$.
- $3 \notin \mathcal{L}_\mathcal{R}$ since there is no valid witness for 3.

Question: Is $x = 1$ a true statement for $N = 5$? What about $x = 4$?

NP Statements: Demonstration

Well... We are simply going to send witness w to the verifier $\mathcal{V}$ and he will check if the statement is true (meaning, whether $x \in \mathcal{L}_{\mathcal{R}}$).

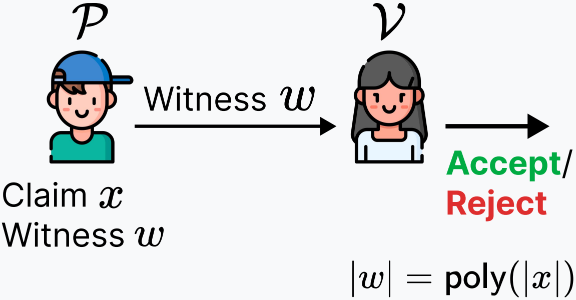


Figure: Typical setup for cryptographic proofs.

NP Statements

Definition (P Language)

Problem is in the P class if exists a polytime algorithm checking $x \in \mathcal{L}$.

Definition (NP Language)

A language $\mathcal{L}_{\mathcal{R}}$ belongs to the NP class if there exists a polynomial-time verifier $\mathcal{V}$ such that the following two properties hold:

- **Completeness:** If $x \in \mathcal{L}_{\mathcal{R}}$, then there is a witness w such that $\mathcal{V}(x, w) = 1$ with $|w| = \text{poly}(|x|)$. Essentially, it states that true claims have *short* proofs.
- **Soundness:** If $x \notin \mathcal{L}_{\mathcal{R}}$, then for any w it holds that $\mathcal{V}(x, w) = 0$. Essentially, it states that false claims have no proofs.

Theorem

Any NP *problem has a zero-knowledge proof.*

Question (aka Motivation)

But can we do better?

Sending witness is... Weird...

Figure: Hmm... #2

Interactive Proofs

Solution!

We add two more ingredients:

- **Interaction:** instead of **passively** receiving the proof, the verifier $\mathcal{V}$ can **interact** with the prover $\mathcal{P}$ by sending **challenges** and receiving **responses**.
- **Randomness:** $\mathcal{V}$ can send random coins (challenges) to the prover, which $\mathcal{P}$ can use to generate responses.

Prover $\mathcal{P}$
Comp. Unbounded

Verifier $\mathcal{V}$
Probabilistic Poly-Time (PPT)

Send m_1 →

← Toss coin r_1, send query q_1

Send m_2 →

← Toss coin r_2, send query q_2

Quadratic Residue Interactive Proof

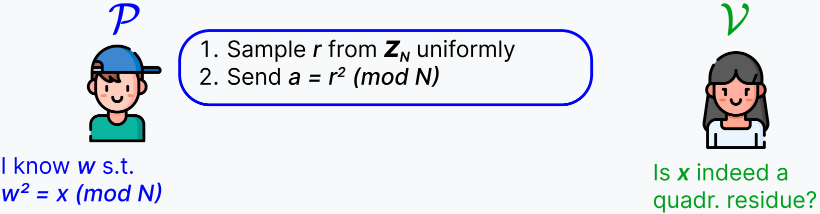
Problem Statement

- **Statement:** $x \in \mathcal{L}_{\mathcal{R}}$ where our **language** is defined as:

$$\mathcal{L}_{\mathcal{R}} = \{x \in \mathbb{Z}_N^\times : \exists w \in \mathbb{Z}_N^\times \text{ such that } x \equiv w^2 \pmod{N}\}$$

- **Witness:** $w =$ modular square root of x.

How does $\mathcal{P}$ and $\mathcal{V}$ interact? Consider the figure below.



Quadratic Residue Interactive Proof

$\mathcal{P}$

I know w s.t.
$w^2 = x \pmod N$

1. Sample r from $\mathbb{Z}_N$ uniformly
2. Send $a = r^2 \pmod N$

- If I gave you the square root of a and ax, you would be convinced that the claim is true, but you learn the witness w.
- Instead, I will send you either r or rw, but you are to choose!

$\mathcal{V}$

Is x indeed a quadr. residue?

Quadratic Residue Interactive Proof

$\mathcal{P}$

I know w s.t.
$w^2 = x \pmod N$

1. Sample r from $\mathbf{Z}_N$ uniformly
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- If I gave you the square root of a and ax, you would be convinced that the claim is true, but you learn the witness w.
- Instead, I will send you either r or rw, but you are to choose!

$\mathcal{V}$

Is x indeed a quadr. residue?

Ok, I choose random bit b

Quadratic Residue Interactive Proof

$\mathcal{P}$

I know w s.t.
$w^2 = x \ (mod\ N)$

1. Sample r from $\mathbb{Z}_N$ uniformly
2. Send $a = r^2 \ (mod\ N)$

- If I gave you the square root of a and ax, you would be convinced that the claim is true, but you learn the witness w.
- Instead, I will send you either r or rw, but you are to choose!

$\mathcal{V}$

Is x indeed a quadr. residue?

Ok, I choose random bit b

- If b=0, send $z = r$
- If b=1, send $z = rw \ (mod\ N)$

Check if $z^2 = ax^b$

Quadratic Residue Interactive Proof: Analysis

Interactive Protocol

1. $\mathcal{P}$ samples $r \xleftarrow{R} \mathbb{Z}_N^\times$ and sends $a = r^2$ to $\mathcal{V}$.
2. $\mathcal{V}$ sends a random bit $b \in \{0, 1\}$ to $\mathcal{P}$.
3. $\mathcal{P}$ sends $z = r \cdot w^b$ to $\mathcal{V}$.
4. $\mathcal{V}$ accepts if $z^2 = a \cdot x^b$, otherwise it rejects.
5. Repeat $\lambda \in \mathbb{N}$ times.

Lemma

*The aforementioned protocol is **complete** and **sound**.*

Completeness. If $b = 0$, then $z = r$ and thus $z^2 = r^2 = a$, check passes. If $b = 1$, then $z = rw$ and thus $z^2 = r^2w^2 = ax$, check passes.

Quadratic Residue Interactive Proof: Analysis

Soundness. The main reason why the protocol is sound is insribed in the theorem below.

Theorem

For any prover $\mathcal{P}^*$ with $x \notin \mathcal{L}_{\mathcal{R}}$, the probability of $\mathcal{V}$ accepting the proof is at most $1/2$.

Corollary. After repeating the protocol λ times, we have

$$\Pr[\mathcal{V} \text{ accepts after } \lambda \text{ rounds}] \leq \frac{1}{2^\lambda} = \mathsf{negl}(\lambda).$$

Thus, we showed both **completeness** and **soundness** of the protocol.

Interactive Protocol Definition

$\langle \mathcal{P}, \mathcal{V} \rangle(x)$ reads as "interaction between $\mathcal{P}$ and $\mathcal{V}$ on statement x".

Definition

A pair of algorithms $(\mathcal{P}, \mathcal{V})$ is called an **interactive proof** for a language $\mathcal{L}_{\mathcal{R}}$ if $\mathcal{V}$ is a polynomial-time verifier and the following two properties hold:

- **Completeness:** For any $x \in \mathcal{L}_{\mathcal{R}}$, $\Pr[\langle \mathcal{P}, \mathcal{V} \rangle(x) = \text{accept}] = 1$.
- **Soundness:** For any $x \notin \mathcal{L}_{\mathcal{R}}$ and for any prover $\mathcal{P}^*$, we have

$$\Pr[\langle \mathcal{P}^*, \mathcal{V} \rangle(x) = \text{accept}] \leq \text{negl}(\lambda)$$

Definition

The class of interactive proofs (IP) is defined as:

$$\text{IP} = \{\mathcal{L} : \text{there is an interactive proof } (\mathcal{P}, \mathcal{V}) \text{ for } \mathcal{L}\}.$$

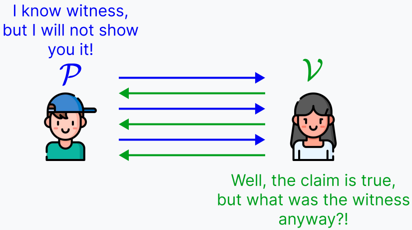
Zero-Knowledge Informal Definition

Definition

An interactive proof system $(\mathcal{P}, \mathcal{V})$ is called **zero-knowledge** if for any polynomial-time verifier $\mathcal{V}^*$ and any $x \in \mathcal{L}_{\mathcal{R}}$, the interaction $\langle \mathcal{P}, \mathcal{V}^* \rangle(x)$ gives nothing new about the witness w.

Definition

The pair of algorithms $(\mathcal{P}, \mathcal{V})$ is called a **zero-knowledge interactive protocol** if it is *complete*, *sound*, and *zero-knowledge*.



Verifier's View

Question #1

What has the verifier learned during the interaction?

- First things first, he learned that the statement x is true.
- He also knows queries $(q_1, \dots, q_\ell)$ and random coins $(r_1, \dots, r_\ell)$ he tossed (since he is the one who has sent them).
- Moreover, he knows the prover's messages $(m_1, m_2, \dots, m_\ell)$.

Definition

All the conversation that verifier has witnessed is called **verifier's view** and is denoted as

$$\mathsf{view}_{\mathcal{V}}(\mathcal{P}, \mathcal{V}) = (m_1, r_1, q_1, m_2, r_2, q_2, \dots, m_\ell, r_\ell, q_\ell).$$

Fact: $\mathsf{view}_{\mathcal{V}}(\mathcal{P}, \mathcal{V})$ is a **random variable**.

Verifier's View: Example

Example

For QN test, set $N := 3 \times 2^{30} + 1$ (prime number), and $\mathcal{P}$ wants to convince that $1286091780 \in \mathcal{L}_R$. Conversation is the following:

1. During the first round, $\mathcal{P}$ sends 672192003 to $\mathcal{V}$.
2. $\mathcal{V}$ sends $b = 0$ to $\mathcal{P}$.
3. $\mathcal{P}$ sends 2606437826 to $\mathcal{V}$.
4. $\mathcal{V}$ verifies that indeed $2606437826^2 \equiv 672192003 \pmod{N}$.
5. During the second round, $\mathcal{P}$ sends 2619047580 to $\mathcal{V}$.
6. $\mathcal{V}$ chooses $b = 1$ and sends to $\mathcal{P}$.
7. $\mathcal{P}$ sends 1768388249 to $\mathcal{V}$.
8. $\mathcal{V}$ verifies that $1768388249^2 \equiv 2619047580 \times 1286091780 \pmod{N}$.
9. Conversation ends.

Verifier's View: Example

Example

The **view of the verifier** $\mathcal{V}$ is the following:

$$\mathrm{view}_{\mathcal{V}}(\mathcal{V}, \mathcal{P})[1286091780]$$
$$= (672192003, 0, 2606437826, 2619047580, 1, 1768388249)$$

- Essentially, this view is the same as you have witnessed.
- You have not learned anything about w that prover $\mathcal{P}$ knows.
- The witness was $w = 3042517305$ and two randomnesses were $r_1 = 2606437826$ and $r_2 = 3023142760$.
- This is a random variable: conversation could be different.

Zero-Knowledge Formally: Simulation Paradigm

Question #2

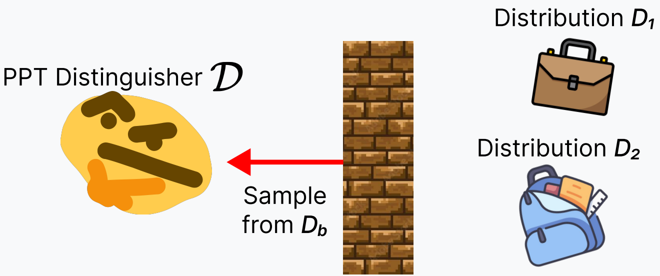
What does it mean that the protocol is zero-knowledge?

- Protocol is zero-knowledge if, given the verifier's $\text{view}_{\mathcal{V}}(\mathcal{P}, \mathcal{V})$, verifier cannot infer any information about the witness w.
- What does it mean that verifier $\mathcal{V}$ learns nothing new? It means that this view could have been simulated by $\mathcal{V}$ *without even running an interaction*.
- Call the view after the real interaction as **real view**, while the view after the simulation as **simulated view**.

Note

Such idea of defining the zero-knowledge is called **simulation paradigm** and currently the most widely used way to prove zero-knowledge.

Computational Indistinguishability



Definition (Informal Computational Indistinguishability)

D_1 and D_2 are **computationally indistinguishable** (denoted by $D_1 \approx D_2$) if for any PPT distinguisher $\mathcal{D}$, even after polynomial number k of samples from D_b (where $b \xleftarrow{R} \{0, 1\}$), for prediction $\hat{b}$: $\Pr[\hat{b} = b] < \frac{1}{2} + \mathrm{negl}(k)$.

Zero-Knowledge Formally (Kind of)

Finally, we are ready to define the **zero-knowledge**.

Definition (Honest-Verifier Zero-Knowledge (HVZK))

An interactive protocol $(\mathcal{P}, \mathcal{V})$ is **honest-verifier zero-knowledge (HVZK)** for a language $\mathcal{L}_{\mathcal{R}}$ there exists a poly-time simulator Sim such that for any valid statement $x \in \mathcal{L}_{\mathcal{R}}$:

$$\mathsf{view}_{\mathcal{V}}(\mathcal{P}, \mathcal{V})[x] \approx \mathsf{Sim}(x, 1^{\lambda})$$

Definition (Zero-Knowledge (ZK))

An interactive protocol $(\mathcal{P}, \mathcal{V})$ is **zero-knowledge (ZK)** for a language $\mathcal{L}_{\mathcal{R}}$ if *for every poly-time verifier* $\mathcal{V}^*$ there *exists a poly-time simulator* Sim such that for any valid statement $x \in \mathcal{L}_{\mathcal{R}}$:

$$\mathsf{view}_{\mathcal{V}^*}(\mathcal{P}, \mathcal{V}^*)[x] \approx \mathsf{Sim}(x, 1^{\lambda})$$

Proof of Knowledge: Why?

Now, the main issue with the above definition is that *we have proven the statement correctness, but we have not proven that the prover* ***knows*** *the witness*. These are completely two distinct things!

Example

Consider the **discrete logarithm relation and language** for a cyclic group $E(\mathbb{F}_p)$ of order r:

$$\mathcal{R} = \{(P, \alpha) \in E(\mathbb{F}_p) \times \mathbb{Z}_r : P = [\alpha]G\},$$
$$\mathcal{L}_{\mathcal{R}} = \{P \in E(\mathbb{F}_p) : \exists \alpha \in \mathbb{Z}_r \text{ such that } P = [\alpha]G\}$$

Question

What does it mean that $X \in \mathcal{L}_{\mathcal{R}}$?

Turns out $\mathcal{L}_{\mathcal{R}} = E(\mathbb{F}_p)$, so the proof $X \in \mathcal{L}_{\mathcal{R}}$ itself is useless.

Proof of Knowledge: Definition

1. The knowledge of witness means that we can **extract** the witness while interacting with the prover.
2. Thus, there should be an algorithm called **extractor** $\mathcal{E}$ which can extract the witness w.
3. $\mathcal{E}$ is given more power than $\mathcal{V}$ (otherwise, if the protocol is zero-knowledge, we cannot extract w). $\mathcal{E}$ can **rewind** and **call** prover $\mathcal{P}$ multiple times.
4. Sometimes, this is referred to as "extractor $\mathcal{E}$ uses $\mathcal{P}$ as an oracle".

Definition (Proof of Knowledge)

The interactive protocol $(\mathcal{P}, \mathcal{V})$ is a **proof of knowledge** for $\mathcal{L}_{\mathcal{R}}$ if exists a poly-time extractor algorithm $\mathcal{E}$ such that for any valid statement $x \in \mathcal{L}_{\mathcal{R}}$, in expected poly-time $\mathcal{E}^{\mathcal{P}}(x)$ outputs w such that $(x, w) \in \mathcal{R}$.

Proof of Knowledge: Example

Lemma

The quadratic residue interactive protocol is a proof of knowledge.

Proof. Let us define the extractor $\mathcal{E}$ for the statement x as follows:

1. Run the prover to receive $a \equiv r^2 \pmod{N}$ (r is chosen randomly from $\mathbb{Z}_N^*$).
2. Set verifier's message to $b = 0$ to get $z_1 \leftarrow r$.
3. **Rewind** and set verifier's message to $b = 1$ to get $z_2 \leftarrow rw \pmod{N}$.
4. Output $z_2/z_1 \pmod{N}$

The extractor $\mathcal{E}$ will always output w if $x \in \mathcal{L}_{\mathcal{R}}$. $\square$

Fiat-Shamir Heuristic

Cryptographic Oracle

Definition (Cryptographic Oracle)

Informally, *cryptographic oracle* is simply a function $\mathcal{O}$ that gives in $O(1)$ an answer to some typically very hard problem.

Example (CDH Problem)

Consider the **Computational Diffie-Hellman (CDH)** problem on the cyclic elliptic curve $E(\mathbb{F}_p)$ of prime order r with a generator G.
Hard Problem: $[\alpha\beta]G$ given $[\alpha]G$ and $[\beta]G$ where $\alpha, \beta \in \mathbb{Z}_r$.
Oracle: However, we *could* assume that such problem can be solved in $O(1)$ by a cryptographic oracle $\mathcal{O}_{\text{CDH}} : ([\alpha]G, [\beta]G) \mapsto [\alpha\beta]G$.
This way, we can rigorously prove the security of some cryptographic protocols *even* if the Diffie-Hellman problem is suddenly solved.

Random Oracle (RO)

One popular cryptographic oracles is a **random oracle** $\mathcal{O}_{\mathrm{R}}$.

Definition (Informal definition of RO)

Suppose someone is inputting x to the random oracle $\mathcal{O}_{\mathrm{R}} : \mathcal{X} \to \mathcal{Y}$[a]. The oracle $\mathcal{O}_{\mathrm{R}}$ does the following:

1. If x has been queried before, the oracle returns the same value as it returned before.
2. If x has not been queried before, the oracle returns a randomly uniformly sampled value from the output space $\mathcal{Y}$.

[a] Typically, RO works with a family of functions $f : \mathcal{X} \to \mathcal{Y}$, but we are not going too deep into the details.

Question

Which very well-known cryptographic object can "serve" as a random oracle?

Fiat-Shamir Transformation

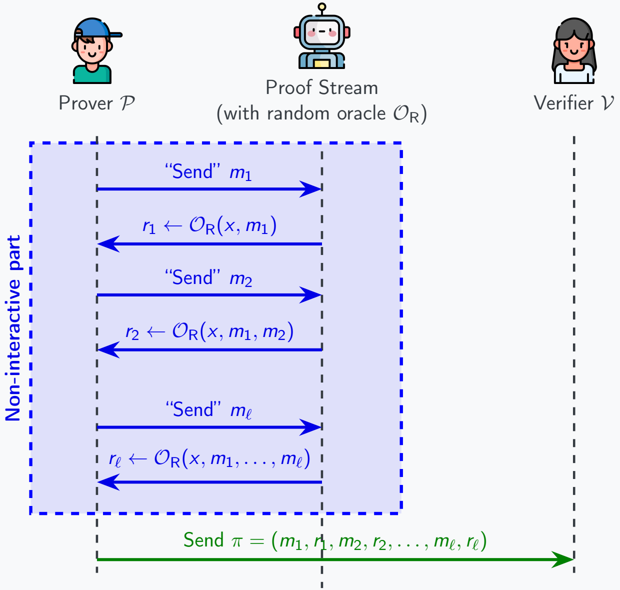
Statement

Any interactive public-coin protocol can be converted into a non-interactive public-coin protocol with preserving completeness, soundness, and zero-knowledge using the random oracle.

One of such transformations is called **Fiat-Shamir heuristic**. Idea:

1. If all what $\mathcal{V}$ does is sending uniformly random values, this is an overkill.
2. Instead of $\mathcal{V}$ sending random values, prover should be able to generate it himself, but he should not know the randomness in advance.
3. Thus, we can replace the verifier's messages with the hash (random oracle) of all the previous conversation.

Fiat-Shamir Heuristic Illustration



Thank you for your attention

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