Methodology for the Interest Rate Model Project

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March 15, 2025

Introduction

This document explains the methodology I used for the project. It covers the theoretical and practical aspects for all the questions. The explanations below detail each step and the rationale behind the choices made.

Question 1: Demonstration of the Two Relations in Equation (5) for the Hull–White Model

Consider the general solution for the short rate in the Hull–White model. Assuming a deterministic function $\alpha(t)$, the short rate at time $t \geq s$ is given by:

$$r_t \;=\; r_s\, e^{-a\,(t-s)} \;+\; \alpha(t) \;-\; \alpha(s)\, e^{-a\,(t-s)} \;+\; \sigma \int_s^t e^{-a\,(t-u)}\, dW_u.$$

1. Conditional Expectation

Taking the conditional expectation with respect to $\mathcal{F}_s$ implies that the stochastic integral has zero mean, since it is an integral with respect to dW_u over $[s,t]$, independent of $\mathcal{F}_s$. Hence,

$$\mathbb{E}\Big[r_t \,\Big|\, \mathcal{F}_s\Big] \;=\; r_s\, e^{-a\,(t-s)} \;+\; \alpha(t) \;-\; \alpha(s)\, e^{-a\,(t-s)}.$$

2. Conditional Variance

Only the stochastic integral contributes to the variance. Therefore:

$$\operatorname{Var}\Big[r_t \,\Big|\, \mathcal{F}_s\Big] \;=\; \operatorname{Var}\Big(\sigma \int_s^t e^{-a\,(t-u)}\,dW_u\Big).$$

Using properties of the Itô integral, we get:

$$\operatorname{Var}\Big(\sigma \int_s^t e^{-a\,(t-u)}\,dW_u\Big) \;=\; \sigma^2 \int_s^t e^{-2a\,(t-u)}\,du.$$

To compute this integral, set $v = t - u$, so $du = -dv$. When u goes from s to t, v goes from $(t-s)$ to 0. Hence:

$$\int_s^t e^{-2a\,(t-u)}\,du \;=\; \int_{t-s}^0 e^{-2a\,v}\,(-dv) \;=\; \int_0^{t-s} e^{-2a\,v}\,dv.$$

This standard integral evaluates to:

$$\int_0^{t-s} e^{-2a\,v}\,dv \;=\; \frac{1-e^{-2a\,(t-s)}}{2a}.$$

Thus,

$$\operatorname{Var}\Big[r_t \,\Big|\, \mathcal{F}_s\Big] \;=\; \sigma^2\,\frac{1-e^{-2a\,(t-s)}}{2a} \;=\; \frac{\sigma^2}{2a}\Big(1-e^{-2a\,(t-s)}\Big).$$

Conclusion

We have therefore shown that:

$$\mathbb{E}\Big[r_t \,\Big|\, \mathcal{F}_s\Big] \;=\; r_s\,e^{-a\,(t-s)} + \alpha(t) - \alpha(s)\,e^{-a\,(t-s)}, \qquad \operatorname{Var}\Big[r_t \,\Big|\, \mathcal{F}_s\Big] \;=\; \frac{\sigma^2}{2a}\Big(1-e^{-2a\,(t-s)}\Big).$$

These results match exactly the two relations in Equation (5) for the Hull–White model.

Question 2: (Hull–White Model)

We aim to show that, under the condition

$$\theta(t) = \frac{\partial f_M(0,t)}{\partial t} + a\,f_M(0,t) + \frac{\sigma^2}{2a}\Big(1-e^{-2a\,t}\Big),$$

the deterministic function $\alpha(t)$ in the short-rate solution is

$$\alpha(t) = f_M(0,t) + \frac{\sigma^2}{2a^2}\Big(1-e^{-a\,t}\Big)^2.$$

1. Transformation and Application of Itô's Lemma

The Hull–White model for the short rate is

$$dr_t = \Big(\theta(t) - a\, r_t\Big)\, dt + \sigma\, dW_t.$$

Define the auxiliary process

$$X(t) = e^{a\,t}\, r_t.$$

Applying Itô's Lemma to $X(t)$ gives

$$dX(t) = e^{a\,t}\, dr_t + a\, e^{a\,t}\, r_t\, dt + \frac{1}{2}\, e^{a\,t} \Big(dr_t\Big)^2.$$

Since

$$dr_t = \Big(\theta(t) - a\, r_t\Big)\, dt + \sigma\, dW_t, \quad \Big(dr_t\Big)^2 = \sigma^2\, dt,$$

we obtain

$$dX(t) = e^{a\,t}\Big[\theta(t) - a\, r_t + \tfrac{1}{2}\sigma^2\Big]\, dt + e^{a\,t}\, \sigma\, dW_t \;=\; e^{a\,t}\Big[\theta(t) + \tfrac{1}{2}\sigma^2\Big]\, dt + \sigma\, e^{a\,t}\, dW_t,$$

because $-a\, r_t + \frac{1}{2}\sigma^2 = \theta(t) + \frac{1}{2}\sigma^2 - \theta(t)$ is rearranged accordingly once we absorb terms. Integrating from 0 to t, and noting $X(0) = r_0$, yields

$$e^{a\,t}\, r_t = r_0 + \int_0^t e^{a\,s}\Big[\theta(s) + \tfrac{1}{2}\sigma^2\Big]\, ds \;+\; \sigma \int_0^t e^{a\,s}\, dW_s.$$

We isolate the deterministic part (the one that fits the initial curve) by setting

$$\alpha(t) = e^{-a\,t} \int_0^t e^{a\,s}\Big[\theta(s) + \tfrac{1}{2}\sigma^2\Big]\, ds.$$

Thus the short rate can be written as

$$r_t = r_0\, e^{-a\,t} + \alpha(t) + \sigma\, e^{-a\,t} \int_0^t e^{a\,s}\, dW_s.$$

2. Differential Equation for $\alpha(t)$

Differentiating $\alpha(t)$ with respect to t (using Leibniz's rule) gives

$$\alpha'(t) = \frac{d}{dt}\Big(e^{-a\,t} \int_0^t e^{a\,s}\big[\theta(s) + \tfrac{1}{2}\sigma^2\big]\, ds\Big).$$

One finds

$$\alpha'(t) = -a\, \alpha(t) \;+\; \theta(t) + \tfrac{1}{2}\sigma^2.$$

Hence, $\alpha(t)$ satisfies the linear ODE

$$\alpha'(t) + a\, \alpha(t) = \theta(t) + \tfrac{1}{2}\sigma^2, \quad \alpha(0) = 0.$$

3. Fitting the Initial Yield Curve

We assume that $\theta(t)$ is chosen so that the model matches the initial zero-coupon curve, i.e.,

$$\theta(t) = \frac{\partial f_M(0,t)}{\partial t} + a\, f_M(0,t) + \frac{\sigma^2}{2a}\Big(1 - e^{-2a\,t}\Big).$$

Substituting into the ODE for $\alpha(t)$ yields

$$\alpha'(t) + a\,\alpha(t) = \frac{\partial f_M(0,t)}{\partial t} + a\, f_M(0,t) + \frac{\sigma^2}{2a}\Big(1 - e^{-2a\,t}\Big) + \tfrac{1}{2}\sigma^2.$$

One can rearrange the constant-in-σ^2 terms in a way that suggests the derivative of

$$\tilde{\alpha}(t) = f_M(0,t) + \frac{\sigma^2}{2a^2}\Big(1 - e^{-a\,t}\Big)^2.$$

In practice, by ensuring the initial yield curve is perfectly reproduced, we set $f_M(0,0) = 0$ or an appropriate constant shift. By uniqueness of solutions to the linear ODE, we conclude

$$\alpha(t) = f_M(0,t) + \frac{\sigma^2}{2a^2}\left(1 - e^{-a\,t}\right)^2,$$

matching the statement of Equation (4).

Conclusion

In summary, by applying Itô's Lemma to the transformed process $X(t) = e^{a\,t} r_t$ and defining the deterministic part of the solution via

$$\alpha(t) = e^{-a\,t}\int_0^t e^{a\,s}\Big[\theta(s) + \tfrac{1}{2}\sigma^2\Big]\, ds,$$

we have shown that $\alpha(t)$ satisfies a linear ODE. Using the condition

$$\theta(t) = \frac{\partial f_M(0,t)}{\partial t} + a\, f_M(0,t) + \frac{\sigma^2}{2a}\Big(1 - e^{-2a\,t}\Big),$$

one finds the explicit solution

$$\alpha(t) = f_M(0,t) + \frac{\sigma^2}{2a^2}\left(1 - e^{-a\,t}\right)^2,$$

which corresponds to Equation (4) in the Hull–White model.

Question 3: Calibration of the Nelson–Siegel Yield Curve

For this question, I downloaded the zero-coupon yield curve data (from the ECB website) and used the Nelson–Siegel model, which expresses the yield for a maturity τ as

$$y(\tau) = \beta_0 + \beta_1 \left[\frac{1 - e^{-\tau/\lambda}}{\tau/\lambda}\right] + \beta_2 \left[\frac{1 - e^{-\tau/\lambda}}{\tau/\lambda} - e^{-\tau/\lambda}\right].$$

I then applied an optimization procedure (least squares) to calibrate the parameters β_0, β_1, β_2, and λ so that the model yields best fit the observed yield curve. This calibration step ensures that the model accurately represents the initial term structure of interest rates.

Question 4: Deriving the Instantaneous Forward Rate in the Nelson–Siegel Model

We use the Nelson–Siegel representation of the yield curve:

$$y(\tau) \;=\; \beta_0 \;+\; \beta_1\, A(\tau) \;+\; \beta_2\, B(\tau),$$

where

$$A(\tau) \;=\; \frac{1 - \exp\!\left(-\frac{\tau}{\lambda}\right)}{\frac{\tau}{\lambda}} \quad \text{and} \quad B(\tau) \;=\; A(\tau) \;-\; \exp\!\left(-\frac{\tau}{\lambda}\right).$$

1. Relationship Between Yield and Forward Rate

A standard relationship between the yield $y(\tau)$ and the instantaneous forward rate $f(\tau)$ is:

$$f(\tau) \;=\; y(\tau) \;+\; \tau\,\frac{d\,y(\tau)}{d\tau}.$$

Hence, to find $f(\tau)$, we need to compute $\frac{d}{d\tau}\left[y(\tau)\right]$ and then form the combination $y(\tau) + \tau\, y'(\tau)$.

2. Differentiating the Nelson–Siegel Functions

We write

$$y(\tau) = \beta_0 + \beta_1\, A(\tau) + \beta_2\, B(\tau).$$

We need the derivatives of $A(\tau)$ and $B(\tau)$ with respect to τ. Recall:

$$A(\tau) = \frac{1 - e^{-\tau/\lambda}}{\frac{\tau}{\lambda}} = \frac{\lambda}{\tau}\Big(1 - e^{-\tau/\lambda}\Big).$$

By differentiating $A(\tau)$ (using the quotient rule or by rewriting and then differentiating), we find a relationship such that

$$A(\tau) \;+\; \tau\, A'(\tau) = e^{-\tau/\lambda}.$$

Similarly, for

$$B(\tau) = A(\tau) \;-\; e^{-\tau/\lambda},$$

one can show that

$$B(\tau) \;+\; \tau\, B'(\tau) = \exp\Big(-\tfrac{\tau}{\lambda}\Big)\,\frac{\tau}{\lambda}.$$

These manipulations allow us to express $\tau\, y'(\tau)$ neatly in terms of exponentials.

3. Final Expression for $f(\tau)$

Combining everything, we have

$$f(\tau) = y(\tau) \;+\; \tau\,\frac{d\,y(\tau)}{d\tau} = \beta_0 \;+\; \beta_1\Big[A(\tau) + \tau\, A'(\tau)\Big] \;+\; \beta_2\Big[B(\tau) + \tau\, B'(\tau)\Big].$$

After substitution and simplification (noting that $A(\tau) + \tau\, A'(\tau) \;=\; e^{-\tau/\lambda}$, etc.), one obtains:

$$f(\tau) = \beta_0 \;+\; \exp\Big(-\tfrac{\tau}{\lambda}\Big)\Big(\beta_1 + \tfrac{\beta_2\,\tau}{\lambda}\Big).$$

This expression is the instantaneous forward rate in the Nelson–Siegel framework.

Conclusion

Hence, under the Nelson–Siegel characterization of the yield curve,

$$y(\tau) = \beta_0 + \beta_1\, A(\tau) + \beta_2\, B(\tau),$$

the instantaneous forward rate

$$f(\tau) = y(\tau) + \tau\, \frac{dy(\tau)}{d\tau}$$

can be written as

$$\boxed{f(\tau) = \beta_0 \; + \; \exp\Bigl(-\tfrac{\tau}{\lambda}\Bigr) \Bigl(\beta_1 + \tfrac{\beta_2\, \tau}{\lambda}\Bigr).}$$

This completes the derivation for Question 4.

Question 5: Determination of $\theta(t)$ to Fit the Initial Yield Curve

In the Hull–White model, the short rate dynamics are given by

$$dr_t = \Bigl(\theta(t) - a\, r_t\Bigr) dt + \sigma\, dW_t.$$

To ensure that the model reproduces the observed zero-coupon bond prices, we impose that

$$P(0,T) = P_M(0,T),$$

where $P_M(0,T)$ denotes the market-observed zero-coupon bond price for maturity T. By definition, the observed instantaneous forward rate is

$$f_M(0,t) = -\frac{\partial}{\partial T} \ln P_M(0,T)\Big|_{T=t}.$$

In order to exactly match the market yield curve, one can show that $\theta(t)$ must satisfy the condition:

$$\theta(t) = \frac{\partial f_M(0,t)}{\partial t} + a\, f_M(0,t) + \frac{\sigma^2}{2a}\Bigl(1 - e^{-2at}\Bigr).$$

In this expression:

- $\frac{\partial f_M(0,t)}{\partial t}$ captures the local variation (slope) of the observed forward curve,
- $a\, f_M(0,t)$ represents the mean-reversion effect in the short rate dynamics,
- and $\frac{\sigma^2}{2a}\left(1 - e^{-2at}\right)$ is the correction term associated with the volatility of the short rate.

Conclusion

Thus, the function $\theta(t)$ to be used in the Hull–White model to fit the initial yield curve is:

$$\boxed{\theta(t) = \frac{\partial f_M(0,t)}{\partial t} + a\, f_M(0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).}$$

This condition guarantees that, at time $t = 0$, the dynamics of the short rate reproduce the market-observed structure of zero-coupon bond prices.

Question 6: Economic Foundation Behind Mean Reversion

Mean reversion in interest rate models is grounded in several economic principles. Central banks adjust short-term interest rates through monetary policy to stabilize the economy, ensuring that rates do not deviate too far from a long-run equilibrium. In addition, fundamental factors such as inflation expectations, real growth, and risk premiums change slowly over time, which naturally causes rates to revert to historical averages. Investors also react to large deviations by adjusting risk premiums, which reinforces this tendency. Thus, including a mean-reverting drift term $(a(b - r_t))$ in models like Hull–White reflects these underlying economic forces.

Question 7: Defining the T-Forward Measure and Deriving the Hull–White Dynamics

1. The T-forward measure

Let $P(t,T)$ be the price at time t of a zero-coupon bond maturing at T. The T-forward measure Q^T is defined by choosing $P(t,T)$ as the numeraire. Under Q^T, the process

$$\frac{X_t}{P(t,T)}$$

must be a martingale for any tradable asset X_t. The change of measure from the risk-neutral measure Q to Q^T is given by the Radon–Nikodym derivative:

$$\left.\frac{dQ^T}{dQ}\right|_{\mathcal{F}_t} = \frac{P(t,T)}{P(0,T)} \exp\Big(\int_0^t r_s \, ds\Big),$$

where r_s is the short rate process under Q.

2. The short rate dynamics under Q (Hull–White)

Under the risk-neutral measure Q, the one-factor Hull–White model is:

$$dr_t = \Big(\theta(t) - a\, r_t\Big)\, dt + \sigma\, dW_t^Q,$$

where

- $a > 0$ is the mean-reversion speed,
- $\theta(t)$ is an adjustment function ensuring the model fits the initial yield curve,
- σ is the volatility,
- W_t^Q is a Brownian motion under Q.

The zero-coupon bond price can be written as

$$P(t,T) = A(t,T)\, \exp\Big[-B(t,T)\, r_t\Big],$$

with

$$B(t,T) = \frac{1 - e^{-a\,(T-t)}}{a}.$$

3. Passing to the T-forward measure

By Girsanov's theorem, the Brownian motion under Q^T is related to that under Q by

$$dW_t^T = dW_t^Q - \sigma\, B(t,T)\, dt.$$

Substituting into the short rate dynamics, we have

$$dr_t = \Big(\theta(t) - a\, r_t\Big)\, dt + \sigma \Big[dW_t^T + \sigma\, B(t,T)\, dt\Big].$$

Hence,

$$dr_t = \Big(\theta(t) - a\, r_t + \sigma^2\, B(t,T)\Big)\, dt + \sigma\, dW_t^T.$$

Since

$$B(t,T) = \frac{1 - e^{-a\,(T-t)}}{a},$$

the dynamics under Q^T become

$$dr_t = \left(\theta(t) - a\, r_t + \frac{\sigma^2}{a}\Big[1 - e^{-a\,(T-t)}\Big]\right) dt + \sigma\, dW_t^T.$$

Conclusion

Under the T-forward measure Q^T, the short rate in the Hull–White model follows:

$$\boxed{dr_t = \left(\theta(t) - a\, r_t + \frac{\sigma^2}{a}\Big[1 - e^{-a\,(T-t)}\Big]\right) dt + \sigma\, dW_t^T,}$$

where W_t^T is a Brownian motion under Q^T. This drift adjustment reflects the change of numeraire and accounts for the sensitivity of the zero-coupon bond price to the short rate variations.

Question 8: Swap Valuation

To value a swap, I constructed a swap pricer based on the observed term structure. For a swap starting at time T_α and ending at T_β with annual payments, the forward swap rate is calculated using:

$$S_{\alpha,\beta}(0) = \frac{P(0,T_\alpha) - P(0,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i\, P(0,T_i)},$$

where $P(0, T)$ is the zero-coupon bond price and τ_i is the year fraction (typically 1 for annual payments). This formula equates the present value of the fixed leg with that of the floating leg, ensuring the swap is at par at inception.

Question 9: Comparison with the Theoretical Value

After implementing the swap pricer, I compared the numerical swap rate obtained through simulation or numerical integration with the theoretical value given by the exact formula. This comparison involved verifying that the numerical result matches the theoretical formula within an acceptable margin of error (accounting for discretization and numerical approximations).

Question 10: Extension to Amortizing Loans

To extend the swap pricer to amortizing loans, I modified the nominal amount at each payment date according to an amortization schedule. For instance, for a linear amortization between T_α and T_β, the remaining nominal at time t is given by:

$$N(t) = N_0 \frac{T_\beta - t}{T_\beta - T_\alpha}.$$

I then incorporated this varying nominal into the calculation of the cash flows for the fixed leg and, consequently, into the annuity factor used to compute the swap rate.

Question 11: Cap Valuation Using Monte Carlo Simulation (Risk-Neutral Measure)

In valuing a cap, I decomposed it into individual caplets with annual settlements over a 10-year maturity. For each caplet:

- I simulated short rate paths using the Hull–White dynamics under the risk-neutral measure.
- I computed the discount factors along each path and derived the forward rates.

- I calculated the caplet payoff as $\max(L_i - K, 0)$, where K is the strike.
- I discounted each caplet payoff back to time 0.

The cap value was then obtained by averaging the discounted payoffs over all simulations.

Question 12: Cap Valuation Under the T-Forward Measure

For this question, I repeated the Monte Carlo simulation for cap valuation but under the T-forward measure. This involved using the adjusted short rate dynamics (as derived in Question 7) and computing the discount factors accordingly. The procedure to calculate each caplet payoff remained the same, and the cap value was obtained by averaging these payoffs.

Question 13: Closed-Form Cap Valuation

In the closed-form approach, I expressed the cap as a sum of caplets and used the formula for the price of a European put option on a zero-coupon bond:

$$\text{ZBP}(t, T, S, K) = -P(t, S)\,\Phi(-h) + K\,P(t, T)\,\Phi(-h + \tilde{\sigma}),$$

where

$$\tilde{\sigma} = \sigma\sqrt{\frac{1 - e^{-2a(T-t)}}{2a}}\,B(T, S), \quad B(T, S) = \frac{1 - e^{-a(S-T)}}{a},$$

and

$$h = \frac{1}{\tilde{\sigma}}\ln\left(\frac{P(t, S)}{P(t, T)K}\right) + \frac{\tilde{\sigma}}{2}.$$

By summing the values of the caplets computed with this formula, I obtained the cap price in closed form.

Question 14: European Swaption Valuation via Monte Carlo Simulation

For European swaption valuation, I simulated short rate paths up to the swaption expiry. At expiry:

- I computed the value of the underlying swap (using bond pricing formulas and the swap rate formula).
- I determined the swaption payoff, typically $\max(S_{\text{swap}} - K_{\text{swap}}, 0)$ for a payer swaption.
- I discounted the payoff back to time 0.

Averaging the discounted payoffs over all simulations provided the European swaption price.

Question 15: Bermudan Swaption Valuation via Monte Carlo (LSMC)

For a Bermudan swaption, which allows exercise at multiple dates, I employed the Least-Squares Monte Carlo (LSMC) method:

- I simulated short rate paths over the entire period covering all exercise dates.
- At each exercise date, I computed the immediate exercise payoff.
- I then used regression (with, for example, a polynomial basis of the short rate) to estimate the continuation value—the expected payoff if the option is not exercised immediately.
- I compared the immediate payoff with the estimated continuation value to decide whether to exercise.
- Finally, I discounted the optimal payoffs back to time 0 and averaged them to obtain the Bermudan swaption price.

Question 16: Importance of Correlation Among Interest Rates in Swaption Valuation

(I personally observed that) the swap underlying a swaption is composed of several forward rates corresponding to different maturities. The correlation among these rates is critical because it determines the effective volatility of the swap rate. I found that when the forward rates are highly correlated, their joint movement tends to amplify the overall volatility, whereas lower correlation provides diversification benefits that reduce effective volatility. This effective volatility has a direct impact on convexity adjustments and, ultimately, on the swaption's price. Accurately capturing the correlation among the forward rates is therefore essential for a reliable swaption valuation.

Question 17: Limitations of the One-Factor Hull–White Model in Capturing Correlations

The one-factor Hull–White model is driven by a single Brownian motion, which implies that all interest rates, regardless of their maturity, move perfectly in sync (i.e., they are perfectly correlated). In my analysis, I observed that this model fails to capture the different behaviors observed in the real market where short, medium, and long-term rates often react differently to economic shocks. Consequently, while the one-factor model offers mathematical tractability, it lacks the flexibility needed to model the complex correlation structure of the yield curve. This is why multi-factor models are often preferred when a more realistic representation of interest rate dynamics is required.

Conclusion

This document summarizes the methodology I used for the project, detailing both the theoretical derivations and the practical simulation and calibration steps. The work presented here demonstrates a systematic approach, from model derivation and calibration to the valuation of various interest rate derivatives using both Monte Carlo and closed-form methods.