Summary and Contrast of “Managing Smile Risk”

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Introduction

Option pricing has evolved significantly from the elegant simplicity of the Black–Scholes model to more complex approaches that can capture market imperfections such as the volatility smile (or skew). In the classical Black–Scholes framework, the assumption of constant volatility leads to a unique mapping between an option’s price and its implied volatility. However, market data reveal that implied volatility varies with strike and maturity. Managing this smile is a critical challenge for pricing and hedging large portfolios of options. This document summarizes the main ideas and results from the article *Managing Smile Risk* and contrasts them with the models and concepts covered in our course.

Summary of the Article

The paper primarily addresses the discrepancies between theoretical models and observed market behavior in the context of implied volatility.

The Volatility Smile

The authors begin by highlighting the problem: whereas the Black–Scholes model assumes constant volatility, empirical evidence shows that implied volatility varies with the strike and maturity of the options, producing a smile-like (or skewed) shape. This variation challenges the conventional one-to-one pricing provided by Black's model.

Local Volatility Models

To overcome the limitations of constant volatility, practitioners introduced local volatility models (notably those by Dupire and Derman–Kani). These models use a state-dependent volatility function, denoted by

$$\sigma_{\text{loc}}(t, \hat{F}),$$

which is calibrated to exactly match observed market prices for vanilla options. Under these models, the dynamics of the forward price are given by

$$d\hat{F}(t) = \sigma_{\text{loc}}(t, \hat{F}(t))\, \hat{F}(t)\, dW.$$

In a static sense, local volatility models can replicate the smile. However, a major drawback is that they predict smile dynamics that are contrary to market experience. In particular, if the underlying asset's forward price decreases, these models indicate that the smile shifts to higher strikes, and vice versa. Such counterintuitive predictions lead to unstable hedging strategies, as the delta and vega computed under these models do not behave as expected.

The SABR Model

To remedy the shortcomings of local volatility models, the article introduces the SABR model—a two-factor stochastic volatility model. In SABR, both the forward price and its volatility evolve stochastically and are correlated:

$$d\hat{F} = \hat{\alpha}\,\hat{F}^{\beta}\,dW_1, \quad d\hat{\alpha} = \nu\,\hat{\alpha}\,dW_2, \quad \text{with } dW_1\,dW_2 = \rho\,dt.$$

Here:

- $\hat{F}$ is the forward price,
- $\hat{\alpha}$ is the stochastic volatility,
- β is an elasticity parameter,
- ν represents the volatility of volatility,
- ρ is the correlation coefficient between the forward and its volatility.

Using singular perturbation techniques, the authors derive explicit closed-form approximations for the implied volatility under the SABR framework. A simplified version of Hagan's SABR formula is given by

$$\sigma_{\text{imp}}(K, f) = \frac{\alpha}{f^{1-\beta}}\frac{z}{x(z)}\left[1 + \text{correction terms}\right],$$

with

$$z = \frac{\nu}{\alpha}\,f^{1-\beta}\ln\left(\frac{f}{K}\right), \quad x(z) = \ln\left(\frac{\sqrt{1-2\rho\,z+z^2}+z-\rho}{1-\rho}\right).$$

Note: The original paper denotes the implied volatility by $\sigma_B(K, f)$. Here, we use $\sigma_{\text{imp}}(K, f)$ to emphasize that it is the volatility implied by calibrating Black's formula to market prices. This formula not only accounts for the basic level of volatility but also incorporates adjustments for vanna, volga, and other higher-order risk factors.

Capturing Market Dynamics

A major strength of the SABR model is its ability to predict that the smile shifts in the same direction as the underlying asset's price movement—a behavior that is observed in actual markets. When the forward price increases, the entire implied volatility curve moves accordingly, resulting in more stable delta and vega hedging. This property contrasts sharply with local volatility models and significantly enhances risk management for large option books, especially in markets with single exercise dates (e.g., swaptions, caplets/floorlets).

Contrast with Class Concepts

The article's approach can be directly contrasted with the traditional models studied in our class:

1. Black–Scholes Model

In our classes, we primarily focused on the Black–Scholes model, which assumes constant volatility. While this assumption simplifies option pricing and leads to the well-known Black–Scholes formula, it fails to account for the volatility smile. The model does not explain why different strikes require different volatilities in order to match observed market prices.

2. Local Volatility Models

We also examined local volatility models, which allow the volatility to depend on time and the underlying asset price, formally written as $\sigma_{\text{loc}}(t, \hat{F})$. These models are capable of fitting the market smile exactly at a given time. However, as noted in the article, the dynamic properties predicted by local volatility models (i.e., the smile shifting in a direction opposite to the underlying asset movement) do not match market observations. This misalignment results in hedging strategies that often underperform compared to those based on simpler Black–Scholes hedges.

3. Stochastic Volatility and the SABR Model

The SABR model extends beyond both Black–Scholes and local volatility approaches by introducing stochastic volatility, which evolves alongside the forward price. In our class discussions, we touched upon stochastic volatility models but did not dive deeply into models that accurately capture smile dynamics. The SABR model fills this gap by correlating the random evolution of both the asset price and its volatility. This correlation allows the model to produce smile dynamics that move in concert with the underlying asset's price—improving the stability and effectiveness of hedging strategies.

Conclusion

The article *Managing Smile Risk* clearly demonstrates that while local volatility models can be calibrated to match market prices at a specific moment, they fail to predict the evolution of the volatility smile and yield unstable hedges. By incorporating stochastic volatility and modeling the correlation between the underlying asset price and its volatility, the SABR model provides a significantly improved framework for option pricing and risk management. Its closed-form approximations for implied volatility capture important market features, such as vanna and volga risks, and most importantly, align the dynamics of the smile with actual market behavior.

In contrast to the constant volatility assumption in the Black–Scholes model and the static calibration of local volatility models using $\sigma_{\text{loc}}(t, \hat{F})$, the SABR model's dynamic approach supports more robust risk management and hedging strategies. This advancement underscores the necessity of adopting more sophisticated models—such as SABR—in order to effectively manage market smile risk, a topic that is central to both academic research and practical trading environments.