Two-Colored Multiphase Model For Immiscible Flows

Antonio Figueroa-González

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1 LBGK - Single Fluid - Single Time relaxation

The LBGK method is based on the following equation

$$f_i(\boldsymbol{x}+\boldsymbol{e}_i,\, t+1) = f_i(\boldsymbol{x},t) + \Omega_i = f_i(\boldsymbol{x},t) - \frac{(f_i(\boldsymbol{x},t) - f_i^{eq}(\boldsymbol{x},t))}{\tau}. \tag{1}$$

In a code, the flowchart has the structure collision-streaming. The steps for a simulation are the following:

- The lattice information, the boundary conditions and the flow field parameters are set up.
- By using the initial conditions, the solver is initialized.
- Collision step (eq. 1):

$$f_i^{\alpha *} = f_i^{\alpha} - \frac{(f_i^{\alpha} - f_i^{\alpha,eq})}{\tau}, \tag{2}$$

where $f_i^{\alpha *}$ is the value resulting from the collision for the density distribution function (f) placed at the link "i" of the node "α".

- Streaming step. The calculated values in the previous step ($f_i^{\alpha *}$) are streamed to the associated neighbors.

$$f_i(\boldsymbol{x}_o + \boldsymbol{e}_i,\, t+1) = f_i^{\alpha *}, \tag{3}$$

where $\boldsymbol{x}_o$ is the position vector of the node "α". For example, in the model D3Q19, the new DDF in the link 2 of a given node comes from the $f_2^{\alpha *}$ stored in the neighbour node placed to its left.

- Apply boundary conditions.
- Calculate macroscopic fluid variables:

$$\rho(\boldsymbol{x}_o, t^n) = \sum_i f_i, \tag{4}$$

$$\rho \boldsymbol{u}(\boldsymbol{x}_o, t^n) = \sum_i \boldsymbol{e}_i f_i. \tag{5}$$

where t^n is the current time step. The procedure described above is repeated until the convergence is achieved.

2 Two-Colored Multiphase Model

In this model, the two immiscible flows are associated to two colors: red (r) and blue (b). As the fluids are immiscible, they can not rest in a given node for the same time step. However, they interact one to each other in the interface between them. Two distribution functions associated to each color are introduced: f_i^r and f_i^b. Following the LBGK schema, the density distribution functions are governed by

$$f_i^k(\boldsymbol{x} + \boldsymbol{e}_i, t+1) = f_i^k(\boldsymbol{x}, t) + \Omega_i^k, \tag{6}$$

where k can be r or b. These distribution function determine the macroscopic variables of the fluids:

$$\rho_r = \sum_i f_i^r, \quad \rho_b = \sum_i f_i^b, \quad \rho = \rho_r + \rho_b, \tag{7}$$

$$\rho \boldsymbol{u} = \sum_i \boldsymbol{e}_i f_i^r + \sum_i \boldsymbol{e}_i f_i^b. \tag{8}$$

In this model the collision operator Ω_i^k is modified to take into account the presence of the other fluid. It is given by

$$\Omega_i^k = (\Omega_i^k)^1 + (\Omega_i^k)^2, \tag{9}$$

where $(\Omega_i^k)^1$ is the usual collision operator for the single-time relaxation model:

$$(\Omega_i^k)^1 = -\frac{(f_i^k - f_i^{k,eq})}{\tau}, \tag{10}$$

and $(\Omega_i^k)^2$ is the perturbation introduced to take into account the surface tension between fluids:

$$(\Omega_i^k)^2 = \frac{A_k}{2} |\boldsymbol{F}| \left[\frac{\boldsymbol{e}_i \cdot \boldsymbol{F}}{|\boldsymbol{F}|^2} - \frac{1}{2} \right], \tag{11}$$

where A_k is a free parameter ($A_k \approx 0.001$) related to the surface tension and $\boldsymbol{F}$ is the density gradient, given by:

$$\boldsymbol{F} = \sum_i \boldsymbol{e}_i \left(\rho_r(\boldsymbol{x}_o + \boldsymbol{e}_i) - \rho_b(\boldsymbol{x}_o + \boldsymbol{e}_i) \right). \tag{12}$$

The approach is similar to the LBGK; there is only an additional f and an additional term in the collision operator. However, an intermediate step is required to "recolor" the domain in order to guarantee the immiscibility of both fluids.

Recolor step It is possible to define f_i as $f_i = f_i^r + f_i^b$. Therefore, f_i satisfies:

$$f_i(\boldsymbol{x} + \boldsymbol{e}_i, t+1) = f_i^r(\boldsymbol{x}, t) + \Omega_i^r + f_i^b(\boldsymbol{x}, t) + \Omega_i^b. \tag{13}$$

Now, one can associated the result of the collision to f_i^*. Similarly to $f_i^{\alpha *}$, f_i^* is obtained by computing:
[1]

$$f_i^* = f_i^r + f_i^b + \Omega_i^r + \Omega_i^b. \tag{14}$$

Once f_i^* is obtained, the recolor step is doing as follows:

- If $i \neq 0$ (links which are not the resting node)

$$f_i^{r*} = \frac{\rho_r}{\rho_r + \rho_b} f_i^* + \beta \frac{\rho_r \rho_b}{(\rho_r + \rho_b)^2} \cos(\phi_i), \tag{15}$$

$$f_i^{b*} = \frac{\rho_b}{\rho_r + \rho_b} f_i^* - \beta \frac{\rho_r \rho_b}{(\rho_r + \rho_b)^2} \cos(\phi_i), \tag{16}$$

- If $i = 0$

$$f_0^{r*} = \frac{\rho_r}{\rho_r + \rho_b} f_0^*, \tag{17}$$

$$f_0^{b*} = \frac{\rho_b}{\rho_r + \rho_b} f_0^*. \tag{18}$$

In the last four equations, ϕ_i is the angle between $\boldsymbol{e}_i$ and $\boldsymbol{F}$ ($\cos(\phi_i) = \boldsymbol{e}_i \cdot \boldsymbol{F} / |\boldsymbol{e}_i| \, |\boldsymbol{F}|$) and β is another parameter which controls the interfacial width ($\beta \approx 0.2$).

Finally, the streaming step can be conducted:

$$f_i^r(\boldsymbol{x}_o + \boldsymbol{e}_i, \, t+1) = f_i^{r*}, \tag{19}$$

$$f_i^b(\boldsymbol{x}_o + \boldsymbol{e}_i, \, t+1) = f_i^{b*}. \tag{20}$$

The steps for a simulation would be then:

- The lattice information, the boundary conditions and the flow field parameters are set up (ρ_r, ρ_b and $\boldsymbol{u}$ are initially indicated).
- By using the initial conditions, the solver is initialized.
- Collision step: $f_i^* = f_i^r + f_i^b + \Omega_i^r + \Omega_i^b$. Thus, one should compute or have available $f_i^{r,eq}$, $f_i^{b,eq}$ $\boldsymbol{F}$, ρ_r and ρ_b for each node.
- Recolor step: compute f_i^{r*} and f_i^{b*}.
- Streaming: $f_i^r(\boldsymbol{x}_o + \boldsymbol{e}_i, \, t+1) = f_i^{r*}$, $f_i^b(\boldsymbol{x}_o + \boldsymbol{e}_i, \, t+1) = f_i^{b*}$.
- Apply boundary conditions.
- Calculate macroscopic fluid variables: ρ_r, ρ_b and $\boldsymbol{u}$.