Modelling Transportation Systems

Stochastic Process

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Learning Objectives

At the completion of this topic, you should be able to

- understand the concepts of random process;
- become familiar with one discrete random process, Bernoulli process.

Outline

1. Definition of Bernoulli process
2. Random processes
3. Basic properties of Bernoulli process
4. Distribution of inter arrival times
5. The time of the kth success
6. Merging and splitting

The Bernoulli process

- A sequence of independent Bernoulli trials
- At each trial, i:
  - P(success)= P (X_i=1)= p
  - P(failure)= P (X_i=0)= 1-p
- Examples
  - Sequence of lottery wins/losses
  - Arrivals (each second) to a bank

Random processes

- First view: sequence of random variables $X_1, X_2, \ldots$
- $E[X_t] =$ p
- $Var[X_t] =$ p(1-p)
- Second view: What is the right sample space?
- $P[X_t = 1 \; for \; all \; t] =$
- Random processes we will study:

✓-Bernoulli process (memoryless, discrete time)

✓-Poisson process (memoryless, continuous time)

✓-Markov chains (with memory/dependence across time)

Bernoulli process

- 1) for a given amount of time, how many jobs have arrived
- Number of succsses X in n time slots

$$\mathrm{P(X = k)} = C_n^k p^k (1-p)^{n-k}, \qquad k = 0,1,2,\ldots,n$$

- $E[X] = np$
- $Var\,(X) = np(1-p)$
- 2) For a given number of jobs, how much time did it take to arrive

Bernoulli process

- 2) For a given number of jobs, how much time did it take to arrive
- T_1: number of trials until first success
  - $\mathrm{P}(T_1 = \mathrm{t}) = (1-p)^{t-1}p$ t=1,2,..
  - $\mathrm{E}(T_1) = 1/p$
  - $\mathrm{E}(T_1) = (1-p)/p^2$
  - Memoryless property
- If you buy a lottery ticket every day, what is the distribution of the length of the first string of losing days

Bernoulli process

- 2) For a given number of jobs, how much time did it take to arrive
- T_k**:** number of trials between the $k-1\ th$ to the kth success

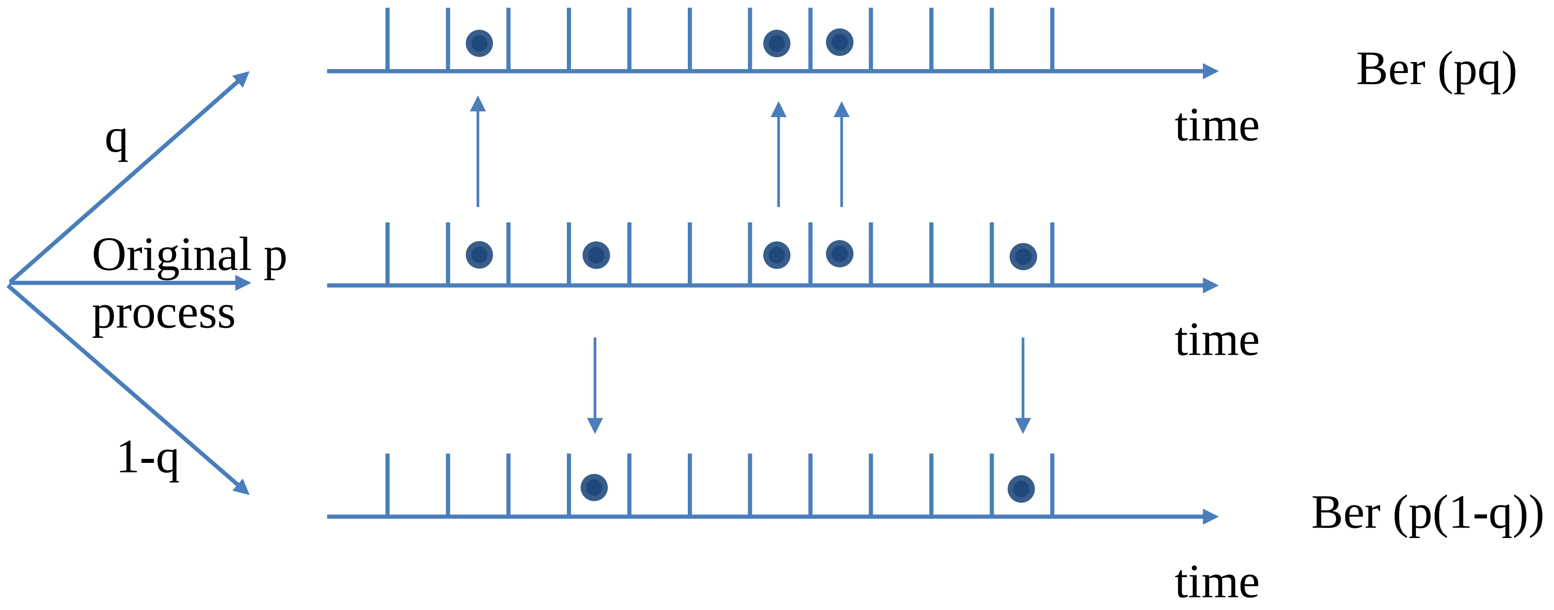
- Y_k**:** number of trials until the kth success
- $Y_k = T_1 + T_2 + T_3 + \cdots T_k$
- $P_{Y_k} = P(Y_k = t) = P(k-1\ arrivals\ in\ [1, t-1]\ \ and\ 1\ arrrival\ at\ t$

$$= \binom{t-1}{k-1} p^{k-1}(1-p)^{t-k}p, \qquad for\ t \geq k$$

- $\mathrm{E}(Y_k) = k/p$
- $\mathrm{Var}(Y_k) =?$

Splitting of a Bernoulli process

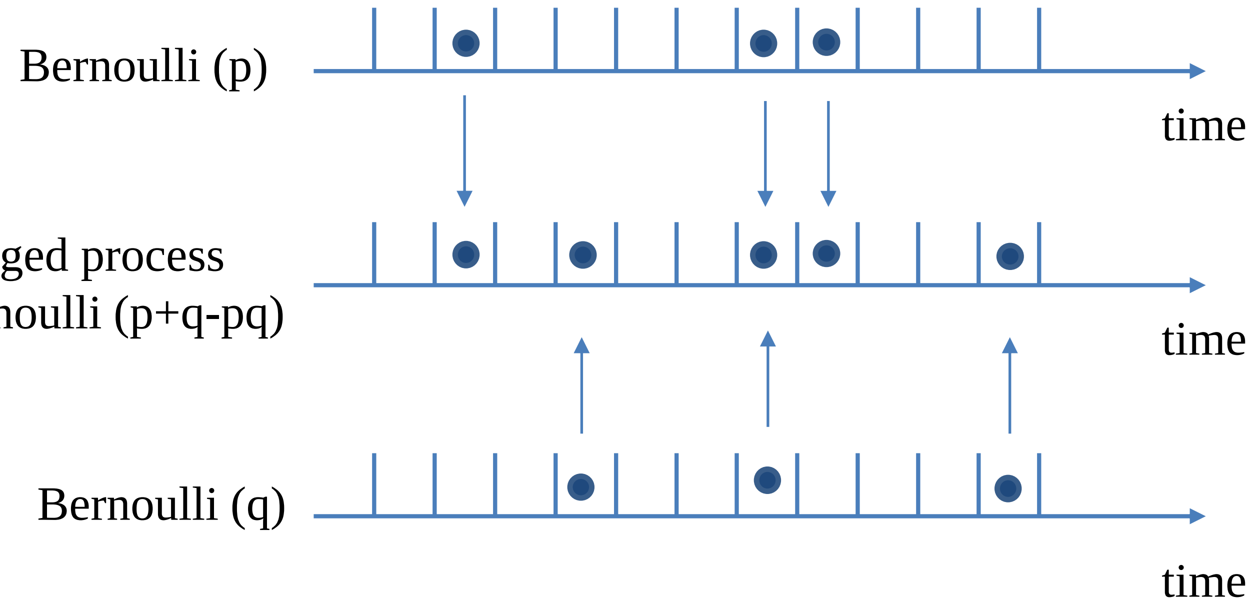
- Using independent coin flips



- Yields Bernoulli processes

Merging of Independent Bernoulli processes

- Using independent coin flips



- Yields Bernoulli processes (collisions are counted as one arrival)

Poisson process

Outline

1. Definition of Poisson process

2. Distribution of number of arrivals

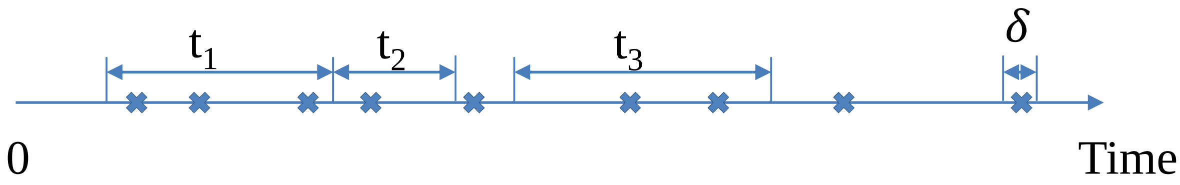
4. Distribution of inter arrival times

5. Other properties of the Poisson process

Bernoulli process review

- Discrete time; success probability p
- Number of arrivals in n time slots: binomial pmf
- Inter arrival times: geometric pmf
- Time to k arrivals: Pascal pmf
- Memorylessness

Definition of Poisson process

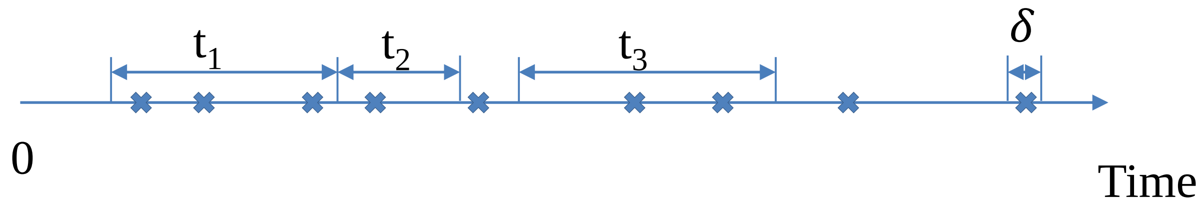


- Time homogeneity:
- $P(k, \tau)$ = Prob. of k arrivals in interval of duration τ
- Numbers of arrivals in disjoint time intervals are **independent**
- Small interval probabilities:
- For very small δ

$$P(k, \delta) \approx \begin{cases} 1 - \lambda\delta, & if\ k = 0 \\ \lambda\delta, & if\ k = 1 \\ 0, & if\ k > 1 \end{cases}$$

- λ: $arrival\ rate$

Definition of Poisson process



- Finely discretize [0,t]: approximately Bernoulli
- N_t (of discrete approximation): binomial
- Taking $\delta \to 0$ （$or\ n \to \infty$） gives:

$$\mathrm{P}(\mathrm{k}, t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, k = 0,1, \ldots$$

- $\mathrm{E}(N_t) = \lambda t$
- $\mathrm{Var}(N_t) = \lambda t$

Interarrival Times

- Y_k: time of the kth arrival
- *Erlang distribution*
- $f_{Y_k}(\mathrm{y}) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, k = 0,1, \ldots$
- Time of first arrival (k=1)

  exponential: $f_{Y_1}(\mathrm{y}) = \lambda e^{-\lambda y}$
- Memoryless property: the time to the next arrival is independent of the past

Bernoulli/Poisson Relation

	Poisson	Bernoulli
Time of Arrival	Continuous	Discrete
Arrival rate	λ / unit time	p/ per trial
PMF of # of Arrivals	Poisson	Binomial
Inter arrival Time Distribution	Exponential	Geometric
Time to the kth arrival	Erlang	Passcal

Merging

- Sum of independent Poisson random variables is Poisson
- Merging of independent Poisson processes is Poisson

Example

- Assume Poisson fishing, $\lambda = 0.6/h$

  - fish for two hours

- if no catch, continue until first catch

a) P (fish for more than two hours)= P(0,2)=e_{-0.6*2}

b) P(fish for more than 2 and less than 5 hours) 0 in first 2, 1 in [2, 5]

c) P (catch at least two fish) catch 2+ in first 2 hours

d) E[number of fish]

e) E[future fishing time| fished for four hours]=

f) E[fishing time]Sum of independent Poisson random variables is Poisson

Example

- Light bulb example
- Each light bulb has independent exponential (λ) lifetime
- Install three light bulbs.
- Find expected time until last light bulb dies out
- E [max{ X, Y,Z}]

Splitting of Poisson processes

- Assume that traffic through an intersection is a Poisson process
- Destinations of different vehicles are independent
- Each output stream is Poisson

Random incidence for Poisson

- Poisson process that has been running forever
- Show up at some "random time"/ "arbitrary time"

- What is the distribution of the length of the chosen interarrival interval ?

Random incidence in “renewal processes”

- Series of successive arrivals

  - i.i.d. interarrival times (but not necessarily exponential)

- **Example**:

  Bus interarrival times are equally likely to be 5 or 10 minutes

- If you arrive at a “random time”

  -What is the probability that you selected a 5 minute interarrival interval

  - What is the expected time to next arrival?

Markov Processes

Outline

1. Checkout counter example
2. Markov process definition
3. n-step transition probabilities
4. Classification of states

Checkout counter model

- Discrete time n=0,1,…
- Customer arrivals: Bernoulli (p)
  - Geometric interarrival times
- Customer service times: geometric (q)
- " State" X_n: number of customers at time n

Finite state Markov chains

- X_n: state after n transitions
  - belongs to a finite set, e.g.. {1,…,m}
  - X_0 is either given or random
- Markov property/assumption:

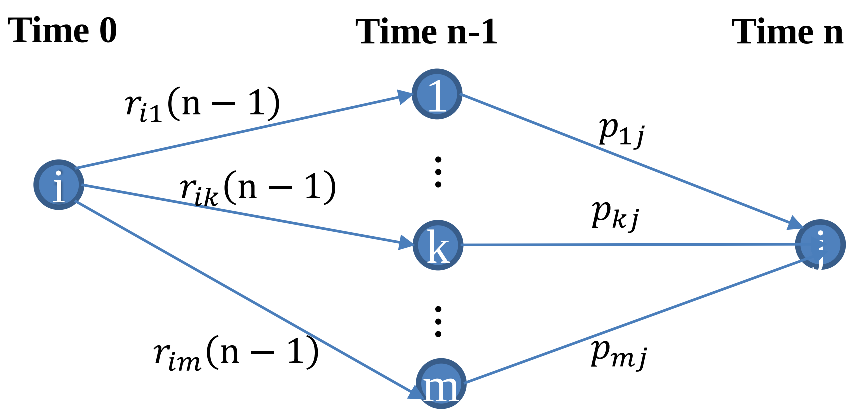
  (given current state, the past does not matter)

  $$p_{ij} = P(X_{n+1} = j | X_n = i) = P(X_{n+1} = j | X_n = i, X_{n-1}, \dots, X_0)$$
- Model specification:
  - Identify the possible states
  - Identify the possible transitions
  - Identify the transition probabilities

n-step transition probabilities

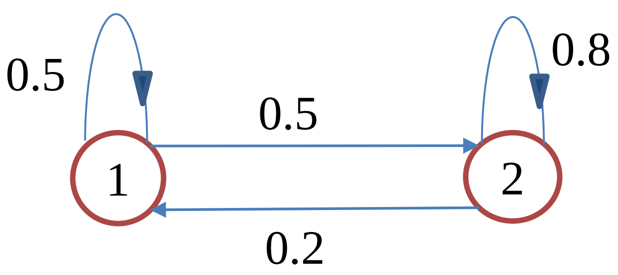
- State occupancy probabilities, given initial state i:

$$r_{ij}(\mathrm{n}) = P(X_n = j | X_0 = i)$$



- Key recursion: $r_{ij}(\mathrm{n}) = \sum_{k=1}^{m} r_{ik}(n-1) p_{kj}$
- With random initial state： $\mathrm{P}(X_n = j) = \sum_{i=1}^{m} \mathrm{P}(X_0 = i) r_{ij}(n)$

Example

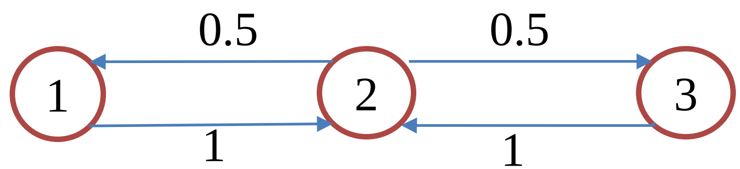


	$n = 0$	$n = \mathbf{1}$	$n = \mathbf{2}$	$n = \mathbf{100}$	$n = \mathbf{101}$
$r_{11}(n)$	1	0.5	0.35	2/7	2/7
$r_{12}(n)$	0	0.5	0.65	5/7	5/7
$r_{21}(n)$	0	0.2			
$r_{22}(n)$	1	0.8			

Steady state

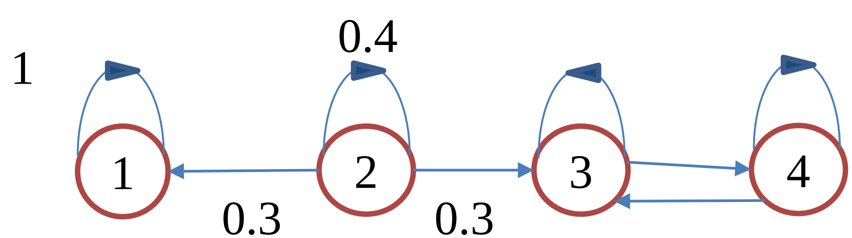
Generic convergence questions

- Does $r_{ij}(\mathrm{n})$ converge to something?



$n\ odd$: $r_{22}(\mathrm{n}) =$ $n\ even$: $r_{22}(\mathrm{n}) =$

- Does the limit depend on initial state



- $r_{11}(\mathrm{n}) = 1$ $r_{31}(\mathrm{n}) = 0$ $r_{21}(\mathrm{n}) = 0.5,\ \ large\ n$

Recurrent and transient states

- State i is recurrent if:
  - Starting from i, and from wherever you go,
  - there is a way of returning to i
- If not recurrent, called transient
  - i transient:
  - $P(X_n = i) \to 0$. i visited finite number of times
  - Recurrent

Lecture outline

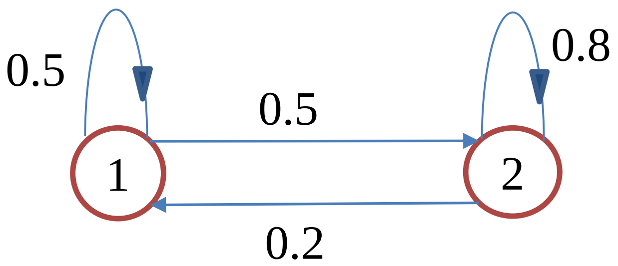
- Steady state behaviour
  - steady state convergence theorem
  - balance equations
- Birth-death processes

Visit frequency interpretation

$$\pi_j = \sum_k \pi_k p_{kj}$$

- (long run) frequency of being in j : π_j
- Frequency of transitions $k \to j$: $\pi_k p_{kj}$
- Frequency of transitions into j: $\sum_k \pi_k p_{kj}$

Example



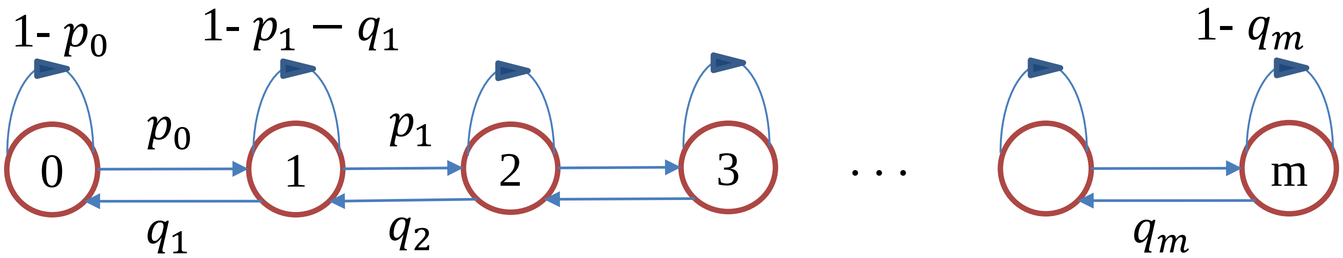
$$\pi_1 = \pi_1 * 0.5 + \pi_2 * 0.2$$

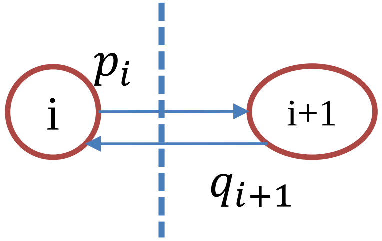
$$\pi_2 = \pi_1 * 0.5 + \pi_2 * 0.8$$

$$0.5\pi_1 = 0.2\pi_2$$

$$\pi_1 + \pi_2 = 1$$

Birth-death processes





$$\pi_i p_i = \pi_{i+1} q_{i+1}$$

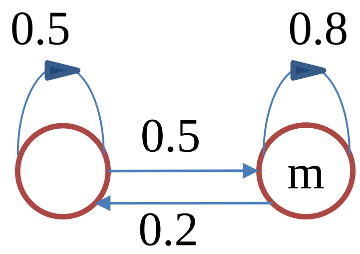
- Special case: $p_i = p \; and \; q_i = q \; for \; all \; i, \quad \rho = \frac{p}{q} = load \; factor$

$$\pi_{i+1} = \pi_i \frac{p}{q} = \pi_i \rho$$

$\pi_i = \pi_0 \rho^{\,i}$, i=0, 1, …, m

- Assume $p < q$ and $m \approx \infty$
- $\pi_0 = 1 - \rho \quad E[X_n] = \frac{\rho}{1-\rho}$ (in steady state)

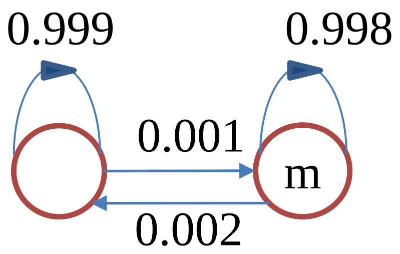
Example



$$\pi_1 = \frac{2}{7}, \pi_1 = \frac{5}{7}$$

- Assume process starts at state 1.
- Calculate: $P(X_1 = 1, and\ X_{100} = 1)$
- Calculate: $P(X_{100} = 1, and\ X_{101} = 2)$
- Calculate: $P(X_{100} = 1, and\ X_{200} = 1)$

Example



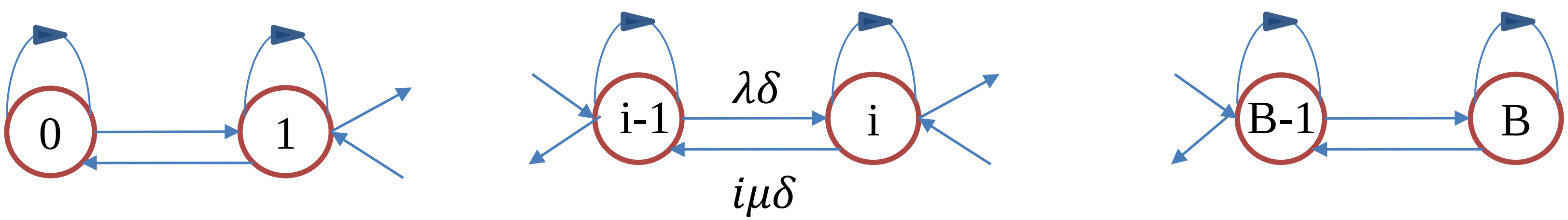
- Slow time scale, n>>1000 to apply steady state approximate

Lecture outline

- Probability of blocked phone calls
- Calculating absorption probabilities
- Calculating expected time to absorption

The phone company problem

- Calls originate as a Poisson process, rate λ
  - Each call duration is exponentially distributed (parameter μ)
  - B Lines available
- Discrete time intervals of (small) length δ



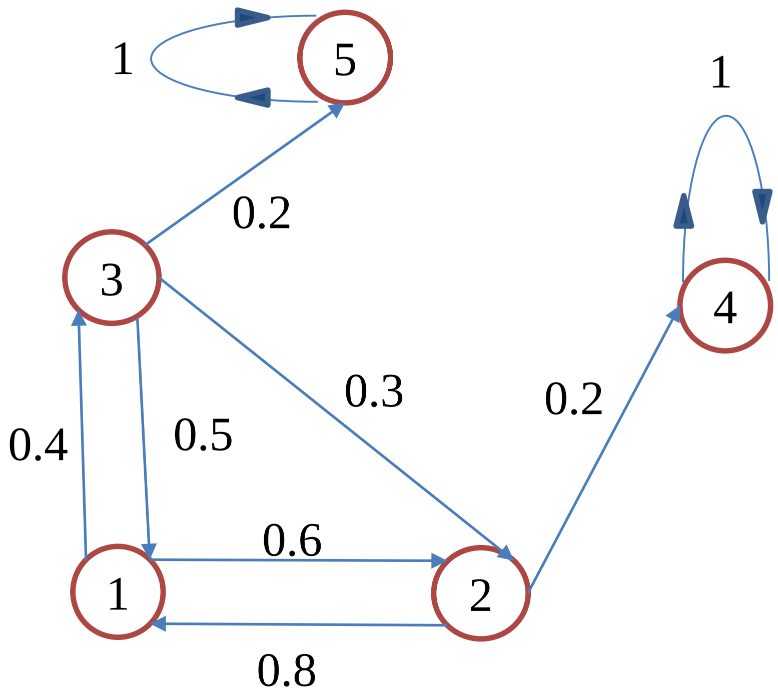
- Balance equations: $\lambda\pi_{i-1} = i\mu\pi_i$

$$\pi_i = \pi_0 \frac{\lambda^i}{\mu^i i!} \qquad \pi_0 = 1/\sum_{i=0}^{B} \frac{\lambda^i}{\mu^i i!}$$

$\pi_B = \mathrm{P(busy)} \approx 1\%\ get\ B = ?$

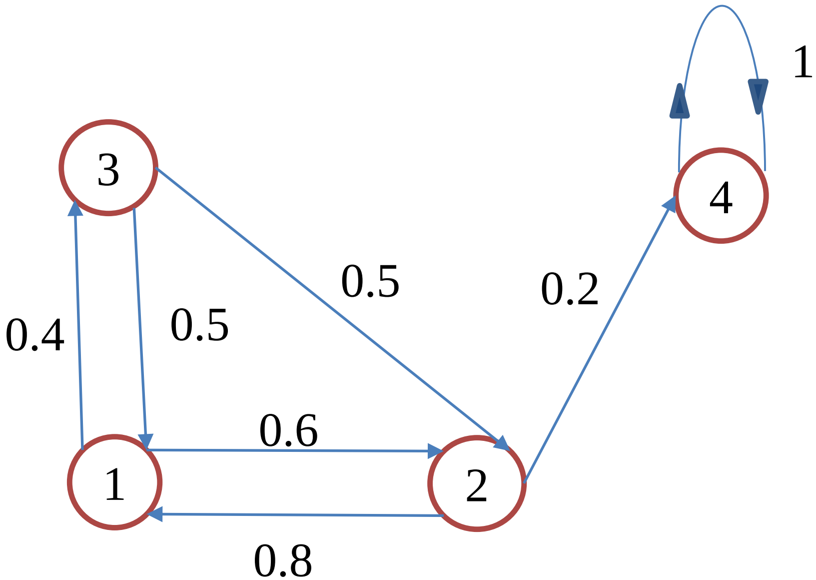
Calculating absorption probabilities

- What is the probability a_i that: process eventually settles in state 4, given that the initial state is i ?



For $i = 4$, $a_i =$
For $i = 5$, $a_i =$
$a_i = \sum_j p_{ij}\, a_j$, for all i
--unique solution

Expected time to absorption



- Find expected number of transitions μ_i , until reaching the absorbing state, given that the initial state is i ?
- $\mu_i = 0$ for $i =$
- For all other i: $\mu_i = 1 + \sum_j p_{ij}\, a_j$
- Unique solution

Mean first passage and recurrence times

- Chain with one recurrent class; fix s recurrent
- Mean first passage time from i to s:

$$t_i = \boldsymbol{E}[min\{n \geq 0 \; such \; that \; X_n = s\}|X_0 = \boldsymbol{i}]$$

- $t_1, t_2, \dots, t_m$ are the unique solution to
- $t_s = 0,\, t_i = 1 + \sum_j p_{ij}\, t_j$, for all $i \neq s$
- Mean recurrence time of s :

$$t_s^* = \boldsymbol{E}[min\{n \geq 1 \; such \; that \; X_n = s\}|X_0 = \boldsymbol{s}]$$

$$t_s^* = 1 + \sum_j p_{sj}\, t_j$$