Assignment 2

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1

(1) 反证法：
假设 $f(x)$ 在 $c \in (a,b)$ 取得最大值，则 $f(c) > f(a)$ 且 $f(c) > f(b)$。则存在一点 $\xi \in (a,c), f^{'}(\xi) > 0$, 存在一点 $\eta \in (c,b), f^{'}(\eta) < 0$, 则 $f^{'}(\eta) < f^{'}(\xi)$
由凸函数的性质可知：一元可微凸函数，一阶导数单调不减，则有 $f^{'}(\eta) \geq f^{'}(\xi)$，所以假设不成立，$f(x)$ 的最大值只能在边界取得。
(2) 反证法：
假设 $f(x)$ 在 $c \in (a,b)$ 取得最大值，则 $f(c) > f(a)$ 且 $f(c) > f(b)$。
则存在 $t \in (0,1)$，使得 $c = ta + (1-t)b$。
根据凸函数的定义可知 $f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$。
假设 $f(a) \leq f(b)$，则 $tf(a) + (1-t)f(b) \leq tf(b) + (1-t)f(b) = f(b)$。
所以 $f(c) = f(ta + (1-t)b) \leq f(b)$，假设不成立，命题得证。

2

由题意可知：

$$\frac{\partial f(x)}{\partial x_1} = 3x_1^2 - x_2 - 2 = 0$$
$$\frac{\partial f(x)}{\partial x_2} = -x_1 + 2x_2 + 3 = 0$$

解得：$x_1 = \frac{1}{2}, x_2 = -\frac{5}{4}$，$\quad x_1 = -\frac{1}{3}, x_2 = -\frac{5}{3}$。

$$\frac{\partial f^2(x)}{\partial x_1^2} = 6x_1 \quad \frac{\partial f^2(x)}{\partial x_1 x_2} = -1$$

$$\frac{\partial f^2(x)}{\partial x_2 x_1} = -1 \quad \frac{\partial f^2(x)}{\partial x_2^2} = 2$$

所以黑塞矩阵为：

$$\boldsymbol{H} = \begin{bmatrix} 6x_1 & -1 \\ -1 & 2 \end{bmatrix}$$

当 $x_1 = \frac{1}{2}, x_2 = -\frac{5}{4}$ 时，黑塞矩阵正定，所以取极小值。
当 $x_1 = -\frac{1}{3}, x_2 = -\frac{5}{3}$ 时，黑塞矩阵负定，所以取极大值。

3

$$\frac{\partial f(x)}{\partial x_1} = 2x_2x_3 - 4x_3 + 2x_1 - 2$$
$$\frac{\partial f(x)}{\partial x_2} = 2x_1x_3 - 2x_3 + 2x_2 - 4$$
$$\frac{\partial f(x)}{\partial x_3} = 2x_1x_2 - 4x_1 - 2x_2 + 2x_3 + 4$$

于是有：

$$\frac{\partial f^2(x)}{\partial x_1^2} = 2 \quad \frac{\partial f^2(x)}{\partial x_1 x_2} = 2x_3 \quad \frac{\partial f^2(x)}{\partial x_1 x_3} = 2x_2 - 4$$
$$\frac{\partial f^2(x)}{\partial x_2 x_1} = 2x_3 \quad \frac{\partial f^2(x)}{\partial x_2^2} = 2 \quad \frac{\partial f^2(x)}{\partial x_1 x_3} = 2x_1 - 2$$
$$\frac{\partial f^2(x)}{\partial x_3 x_1} = 2x_2 - 4 \quad \frac{\partial f^2(x)}{\partial x_3 x_2} = 2x_1 - 2 \quad \frac{\partial f^2(x)}{\partial x_3^2} = 2$$

黑塞矩阵如下：

$$\boldsymbol{H} = \begin{bmatrix} 2 & 2x_3 & 2x_2 - 4 \\ 2x_3 & 2 & 2x_1 - 2 \\ 2x_2 - 4 & 2x_1 - 2 & 2 \end{bmatrix}$$

将各点带入判断黑塞矩阵的正定性得：
点 $(0, 3, 1)$ 的黑塞矩阵为负定，取极大值。点 $(0, 1, -1)$ 的黑塞矩阵为负定，取极大值。
点 $(1, 2, 0)$ 的黑塞矩阵为正定，取极小值。点 $(2, 1, 1)$ 的黑塞矩阵为负定，取极大值。
点 $(2, 3, -1)$ 的黑塞矩阵为负定，取极大值。

4

$$
\begin{gathered}
f(x) = -x_1^2 - 2x_2^2 + 2x_1x_2 + 2x_2 \\
\frac{\partial f(x)}{\partial x_1} = -2x_1 + 2x_2 \\
\frac{\partial f(x)}{\partial x_2} = -4x_2 + 2x_1 + 2
\end{gathered}
$$

$$
\begin{aligned}
g(\alpha) &= f(x - \alpha \nabla f(x)) \\
&= -[x_1 - \alpha(-2x_1 + 2x_2)]^2 - 2[x_2 - \alpha(-4x_2 + 2x_1 + 2)]^2 + \\
&\quad 2[x_1 - \alpha(-2x_1 + 2x_2)][x_2 - \alpha(-4x_2 + 2x_1 + 2)] + 2[x_2 - \alpha(-4x_2 + 2x_1 + 2)]
\end{aligned}
$$

得:

$$
\begin{aligned}
g^{'}(\alpha) =& 2(-2x_1 + 2x_2)[x_1 - \alpha(-2x_1 + 2x_2)] + 4(-4x_2 + 2x_1 + 2)[x_2 - \alpha(-4x_2 + 2x_1 + 2)] \\
& - 2(-2x_1 + 2x_2)[x_2 - \alpha(-4x_2 + 2x_1 + 2)] - 2(-4x_2 + 2x_1 + 2)[x_1 - \alpha(-2x_1 + 2x_2)] \\
& - 2(-4x_2 + 2x_1 + 2)
\end{aligned}
$$

第一次迭代：
$\frac{\partial f(x)}{\partial x_1} = 0 \quad \frac{\partial f(x)}{\partial x_2} = 2 \quad g(\alpha) = -8\alpha^2 - 4\alpha \quad g^{'}(\alpha) = -16\alpha - 4 = 0 \quad \alpha = -\frac{1}{4}$
得：$x_1 = 0 \quad x_2 = \frac{1}{2}$
第二次迭代：
$\frac{\partial f(x)}{\partial x_1} = 1 \quad \frac{\partial f(x)}{\partial x_2} = 0 \quad g(\alpha) = -\alpha^2 - \frac{1}{2} - \alpha + 1 \quad g^{'}(\alpha) = -2\alpha - 1 = 0 \quad \alpha = -\frac{1}{2}$
得：$x_1 = \frac{1}{2} \quad x_2 = \frac{1}{2}$
第三次迭代：
$\frac{\partial f(x)}{\partial x_1} = 0 \quad \frac{\partial f(x)}{\partial x_2} = 1 \quad g(\alpha) = -\frac{1}{4} - 2(\frac{1}{2} - \alpha)^2 + (\frac{1}{2} - \alpha) + 2(\frac{1}{2} - \alpha)$
$g^{'}(\alpha) = 4(\frac{1}{2} - \alpha) - 3 = 0 \quad \alpha = -\frac{1}{4}$
得：$x_1 = \frac{1}{2} \quad x_2 = \frac{3}{4}$
第四次迭代：
$\frac{\partial f(x)}{\partial x_1} = \frac{1}{2} \quad \frac{\partial f(x)}{\partial x_2} = 0 \quad g(\alpha) = -(\frac{1}{2} - \frac{1}{2}\alpha)^2 - 2(\frac{3}{4})^2 + \frac{3}{2}(\frac{1}{2} - \frac{1}{2}\alpha) + \frac{3}{2}$
$g^{'}(\alpha) = \frac{1}{2}(1 - \alpha) - \frac{3}{4} = 0 \quad \alpha = -\frac{1}{2}$
得：$x_1 = \frac{3}{4} \quad x_2 = \frac{3}{4}$

5

根据题意由拉格朗日乘子法得：

$$
L(x, \lambda) = 4(x_1 - 2)^2 + 3(x_2 - 4)^2 + \lambda_1(x_1 + x_2 - 5) + \lambda_2(1 - x_1) + \lambda_3(2 - x_2)
$$

得：

$$\frac{\partial L}{\partial x_1} = 8(x_1 - 2) + \lambda_1 - \lambda_2$$

$$\frac{\partial L}{\partial x_2} = 6(x_2 - 4) + \lambda_1 - \lambda_2$$

$$\frac{\partial L}{\partial \lambda_1} = x_1 + x_2 - 5$$

$$\frac{\partial L}{\partial \lambda_2} = 1 - x_1$$

$$\frac{\partial L}{\partial \lambda_3} = 2 - x_2$$

由 KKT 条件可得：

(1)

$$\frac{\partial L}{\partial x_1} = 8(x_1 - 2) + \lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 6(x_2 - 4) + \lambda_1 - \lambda_3 = 0$$

(2)

$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

(3)

$$\lambda_1(x_1 + x_2 - 5) = 0 \quad \lambda_2(1 - x_1) = 0 \quad \lambda_3(2 - x_2) = 0$$

(4)

$$x_1 + x_2 - 5 \leq 0 \quad 1 - x_1 \leq 0 \quad 2 - x_2 \leq 0$$

若 $\lambda_1 = 0 \quad \lambda_2 = 0 \quad \lambda_3 = 0$，则条件 (4) 不成立。
若 $\lambda_1 \neq 0 \quad \lambda_2 = 0 \quad \lambda_3 = 0$，则 $x_1 = \frac{11}{7} \quad x_2 = \frac{24}{7}$。
若 $\lambda_1 = 0 \quad \lambda_2 \neq 0 \quad \lambda_3 = 0$，则条件 (2) 不成立。
若 $\lambda_1 \neq 0 \quad \lambda_2 \neq 0 \quad \lambda_3 = 0$，则条件 (1) 不成立。
若 $\lambda_1 = 0 \quad \lambda_2 = 0 \quad \lambda_3 \neq 0$，则条件 (2) 不成立。
若 $\lambda_1 \neq 0 \quad \lambda_2 = 0 \quad \lambda_3 \neq 0$，则条件 (2) 不成立。
若 $\lambda_1 = 0 \quad \lambda_2 \neq 0 \quad \lambda_3 \neq 0$，则条件 (2) 不成立。
若 $\lambda_1 \neq 0 \quad \lambda_2 \neq 0 \quad \lambda_3 \neq 0$，则条件 (3) 不成立。
综上所述，$f(x)$ 的最小值在 $\boldsymbol{X} = (\frac{11}{7}, \frac{24}{7})^T$ 处取得，$\min f(x) = \frac{12}{7}$。

6

7

(1)

$$\frac{\partial z(x)}{\partial x_1} = 2x_1 - 3 \quad \frac{\partial z(x)}{\partial x_2} = 2x_2 - 4$$

第一次迭代：$\nabla z(x^1) = (-\frac{5}{2}, -\frac{7}{2})$

$$\min g(y) = -\frac{5}{2}y_1 - \frac{7}{2}y_2$$

$$s.t. \begin{cases} y_1 + y_2 \leq 1 \\ y_1, y_2 \geq 0 \end{cases}$$

得 $y^1 = (0,1)$。$x^1 + \alpha(y^1 - x^1) = (\frac{1}{4}, \frac{1}{4}) + \alpha[(0,1) - (\frac{1}{4}, \frac{1}{4})] = (\frac{1}{4} - \alpha\frac{1}{4}, \frac{1}{4} + \alpha\frac{3}{4})$
$\min f(x^1 - \alpha(y^1 - x^1)) = (\frac{1}{4} - \alpha\frac{1}{4})^2 + (\frac{1}{4} + \alpha\frac{3}{4})^2 - 3(\frac{1}{4} - \alpha\frac{1}{4}) - 4(\frac{1}{4} + \alpha\frac{3}{4}) \quad \alpha \in [0,1]$
得 $\alpha = 1$，$x^2 = (0,1)$。
第二次迭代：$\nabla z(x^2) = (-3, -2)$

$$\min g(y) = -3y_1 - 2y_2$$

$$s.t. \begin{cases} y_1 + y_2 \leq 1 \\ y_1, y_2 \geq 0 \end{cases}$$

得 $y^2 = (1,0)$。$x_2 + \alpha(y^2 - x^2) = (0,1) + \alpha[(1,0) - (0,1)] = (\alpha, 1-\alpha)$
$\min f(x_2 + \alpha(y^2 - x^2)) = \alpha^2 + (1-\alpha)^2 - 3\alpha - 4(1-\alpha) \quad \alpha \in (0,1)$ 得 $\alpha = \frac{1}{4}$，$x^3 = (\frac{1}{4}, \frac{3}{4})$
第三次迭代：$\nabla z(x^3) = (-\frac{5}{2}, -\frac{5}{2})$

$$\min g(y) = -\frac{5}{2}y_1 - \frac{5}{2}y_2$$

$$s.t. \begin{cases} y_1 + y_2 \leq 1 \\ y_1, y_2 \geq 0 \end{cases}$$

已经迭代至最优解。

(2)

$$L(x, \lambda) = x_1^2 + x_2^2 - 3x_1 - 4x_2 + \lambda_1(x_1 + x_2 - 1) + \lambda_2(-x_1) + \lambda_3(-x_2)$$

得：

$$\frac{\partial L}{\partial x_1} = 2x_1 - 3 + \lambda_1 - \lambda_2$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - 4 + \lambda_1 - \lambda_3$$

<1>

$$2x_1 - 3 + \lambda_1 - \lambda_2 = 0 \quad 2x_2 - 4 + \lambda_1 - \lambda_3 = 0$$

<2>

$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

<3>

$$\lambda_1(x_1 + x_2 - 1) = 0 \quad \lambda_2(-x_1) = 0 \quad \lambda_3(-x_2) = 0$$

<4>

$$x_1 + x_2 - 1 \leq 0 \quad -x_1 \leq 0 \quad -x_2 \leq 0$$

带入点 $(\frac{1}{4}, \frac{3}{4})$，得 $\lambda_1 = \frac{5}{2} \quad \lambda_2 = 0 \quad \lambda_3 = 0$，满足 KKT 的所有条件。

8

根据题意可得：

$$\begin{cases} 2 + x_1^2 = 1 + 3x_2 \\ x_1 + x_2 = x_3 = q \end{cases}$$

解得:$x_1 = \frac{-3+\sqrt{12q+5}}{2} \quad x_2 = q - \frac{-3+\sqrt{12q+5}}{2} \quad x_3 = q$，
$t(x_1) = t(x_2) = \frac{6q+11-3\sqrt{12q+5}}{2} \quad t(x_3) = 3 + q$。

9

(1) 由图可知 A 到 D 的有三条不同的路线，
即 $A \to B \to D \quad A \to C \to D \quad A \to C \to B \to D$。
$\hat{t_1}(x_1) = t(x_1) + x_1 t^{'}(x_1) = 21 + 0.02x_1 \quad \hat{t_2}(x_2) = 8 + 0.2x_2$
$\hat{t_3}(x_3) = 4 + 0.04x_3 \quad \hat{t_4}(x_4) = 19 + 0.02x_4 \quad \hat{t_5}(x_5) = 6 + 0.2x_5$

$$\begin{cases} 21 + 0.02f_1 + 6 + 0.2f_1 = 8 + 0.2f_2 + 19 + 0.02f_2 = 8 + 0.2f_3 + 4 + 0.04f_3 + 6 + 0.2f_3 \\ f_1 + f_2 + f_3 = 4 \end{cases}$$

$f_1 = f_2 = 0 \quad f_3 = 4$
(2) 去掉 link C—B 后变为两条路线。

$$\begin{cases} 21 + 0.02f_1 + 6 + 0.2f_1 = 8 + 0.2f_2 + 19 + 0.02f_2 \\ f_1 + f_2 = 4 \end{cases}$$

$f_1 = f_2 = 2$

$27.44 * 2 * 2 > (18 + 0.44 * 4) * 4$，出行总成本增加。