Modelling Transportation Systems

Linear Programming

Part 1： The Simplex Method

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Linear Programming (LP)

- Learning Objectives
  - Able to transform any LP to the standard form.
  - Familiar with the basic terms of LP and its solution.
  - Understand the geometric basis for the Simplex Method.
  - Understand the procedures of the Simplex Method.
  - Identify optimality, unboundedness and infeasibility conditions of an LP.

Some Questions?

- 对于学习过线性规划、甚至多次复习过的同学，是否已抓住了问题的本质？可通过回答下面几个问题来检验自己、培养正确的思维方式。
  - 为什么需要把一个模型转换成标准型？
  - 为什么需要求“基解”？
  - 单纯形法确定“选入”与“选出”变量的目的是什么？
  - 满足单纯形法的停止规则为什么能得到最优解？

Formulating an LP

- Reddy Mikks produces both interior and exterior paints from two raw materials, M1 and M2. The following table provides the basic data of the problem:

	Tons of row material per ton of		Maximum daily availability (tons)
	Exterior paint	Interior paint	
M1	6	4	24
M2	1	2	6
Profit per ton ($1000)	5	4	

A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also, the maximum daily demand for interior paint is 2 tons.

Reddy Mikks wants to determine the ***optimum (best)*** product mix of interior and exterior paints that maximizes the total daily profit.

Formulating an LP

- Basic components:
- Activities: tons of interior paints
  tons of exterior paints
- Constraints: material limit of M1
  material limit of M2
  market limit
  demand limit
- Objective: maximize the total daily profit

Formulating an LP

- Model formulation

Define decision variables:

x_1 = Tons produced daily of exterior paint *(Activity 1)*

x_2 = Tons produced daily of interior paint *(Activity 2)*

Establish objective function:

Total profit from exterior paint $= 5x_1$ (thousand) dollars

Total profit from interior paint $= 4x_2$ (thousand) dollars

$$Maximize\ \ z = 5x_1 + 4x_2$$

Formulating an LP

- Model formulation

Build constraints:

$$6x_1 + 4x_2 \leq 24 \qquad \text{(Raw material M1)}$$

$$x_1 + 2x_2 \leq 6 \qquad \text{(Raw material M2)}$$

$$x_2 - x_1 \leq 1 \qquad \text{(Market limit)}$$

$$x_2 \leq 2 \qquad \text{(Demand limit)}$$

Formulating an LP

- Complete model

$$Max\ z = 5x_1 + 4x_2$$

subject to

$$6x_1 + 4x_2 \leq 24$$

$$x_1 + 2x_2 \leq 6$$

$$x_2 - x_1 \leq 1$$

$$x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

Definition of the LP

- **Three basic components**
  - Decision ***variables*** that we seek to determine;
  - ***Objective*** (goal) to optimize (maximize or minimize);
  - ***Constraints*** that the solution must satisfy.
- **Characteristics:**
  - Decision ***variables*** are nonnegative and continuous;
  - The ***objective*** function is linear;
  - ***Constraints*** are either linear equalities or linear inequalities.

Canonical form （规范形式）

- **Structure of the LP formulation:**

Maximization: $\max\ z = C_A X_A$

Subject to: $AX_A \le b$

and $X_A \ge 0$

$$C_A = (c_1, c_2, \ldots, c_k)$$

$$X_A = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_k \end{bmatrix};\ P_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \ldots \\ a_{mj} \end{bmatrix};\ b = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{bmatrix};\ A = \begin{pmatrix} a_{11} & a_{12} & \ldots a_{1k} \\ \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots a_{mk} \end{pmatrix}$$

Standard form （标准形式）

- **Transforming inequality constraints to equality:**

A general case with slack and surplus variables

$$\max\ z = 500x_1 + 450x_2$$

Subject to

$$
\begin{aligned}
6x_1 + 5x_2 &\geq 60 \\
10x_1 + 20x_2 &\leq 150 \\
x_1 &\leq 8 \\
x_1,\ x_2 &\geq 0
\end{aligned}
$$

$c_i x_i$

$$\max\ z = 500x_1 + 450x_2 + 0x_3 + 0x_4 + 0x_5$$

Subject to

surplus

a_{ij}

$$
\begin{aligned}
6x_1 + 5x_2 - x_3 \quad &= 60 \\
10x_1 + 20x_2 \quad + x_4 \quad &= 150 \\
x_1 \quad + x_5 &= 8 \\
x_1,\ x_2, x_3, x_4, x_5 &\geq 0
\end{aligned}
$$

b_j

slack

For the ease of presentation, we all focus on the "slack" case. For the "surplus" case, artificial variables are needed, which is discussed in the end of this lecture at the "two-phase simplex method".

Standard form （标准形式）

❑**Transforming inequality constraints to equality:**

LP: $\max\ z = C_A X_A$

s.t.

$$AX_A \le b$$

$$X_A \ge 0$$

Introduce the slack variables:

$$X_S = (x_{s1}, x_{s2}, \ldots, x_{sm})^T$$

The parameter matrix of X_S in the constraints is an identity matrix $I_{m\times m}$

Then LP can be rewritten as follows:

$$\max\ z = C_A X_A + OX_S$$

$$\text{s.t.}\quad AX_A + IX_S = b$$

$$X_A, X_S \ge 0$$

$$\boxed{\begin{aligned} \bar{A} &= (A, I) \\ X &= (X_A, X_S) \\ C &= (C_A, O) \end{aligned}}$$

Standard form （标准形式）

- **Structure of the LP formulation:** 一般形式: $n = m + k$

矩阵形式:

Maximization: $z = CX$

Subject to: $\bar{A}X = b$

and $X \geq 0, b > 0$

where $C = (c_1, c_2, \ldots, c_n)$

$$\max\ z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

$$\boxed{\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array}}$$

$$x_1, x_2, \ldots, x_n \geq 0$$

向量形式:

$$\max\ z = \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} P_j x_j = b$$

$$x_j \geq 0,\ j = 1, 2, \ldots, n$$

$$\bar{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \ldots & \ldots & & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} = (P_1, P_2, \ldots, P_n)$$

$$\bar{A} = (A, I)$$

Standard form (con't)

- **The case above is a very special form with the following characteristics:**
- ✓All decision variables are constrained to be non-negative.
- ✓All constraints (except non-negativity) are stated as equalities.
- ✓Right-hand-side (RHS) of constraints are non-negative.

Standard form (con't)

- **Special cases: <u>lower bound constraints</u> and <u>unconstrained variables</u>**

$$\max \ z = CX$$

s.t.

$$2x_1 + 3x_2 + 4x_3 = 10$$

$$x_1 - x_2 + 2x_3 = 8$$

$$x_1 \geq 0, x_2 \geq 3, x_3 \ \text{is unrestricted}$$

✓ Lower bound constraints for x_2

Introduce new variable $x_{2l} \geq 0$

Thus, $x_2 = x_{2l} + 3$

Substitute in all occurrences of x_2

✓ Unconstrained variable x_3

Introduce 2 new variables $x_{3l} \geq 0, \ x_{3u} \geq 0$

Thus, $x_3 = x_{3u} - x_{3l}$

Substitute in all occurrences of x_3

Standard form (con't)

$$\max\ z = CX$$

s.t.

$$2x_1 + 3x_2 + 4x_3 = 10$$
$$x_1 - x_2 + 2x_3 = 8$$
$$x_1 \geq 0, x_2 \geq 3, x_3 \ \ is\ \ unrestricted$$

- After substitutions:

$$2x_1 + 3(x_{2l} + 3) + 4(x_{3u} - x_{3l}) = 10$$
$$x_1 + (x_{2l} + 3) - 2(x_{3u} - x_{3l}) = 8$$

- Rearrange:

$$2x_1 + 3x_{2l} + 4x_{3u} - 4x_{3l} = 1$$
$$x_1 + x_{2l} - 2x_{3u} - 2x_{3l} = 11$$
$$x_1, x_{2l}, x_{3u}, x_{3l} \geq 0$$

Basic Geometrical Concepts

❑Convex Sets （凸集）

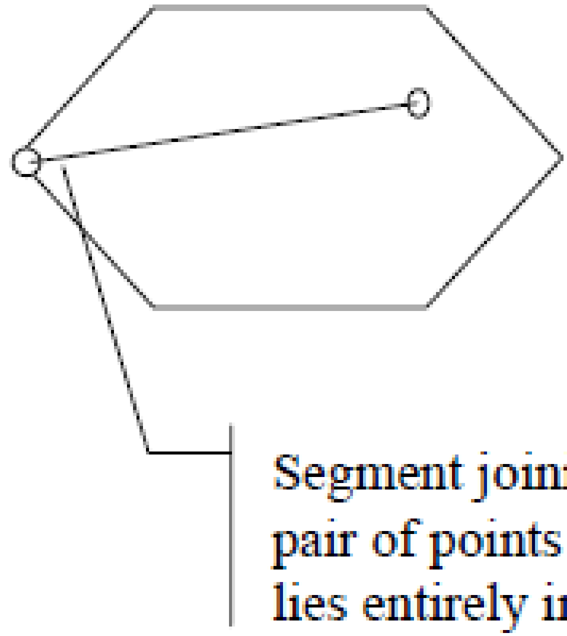
A subset $K \subseteq \Re^n$ is convex if every convex combination of any pair of points in **K** is also in

i.e.

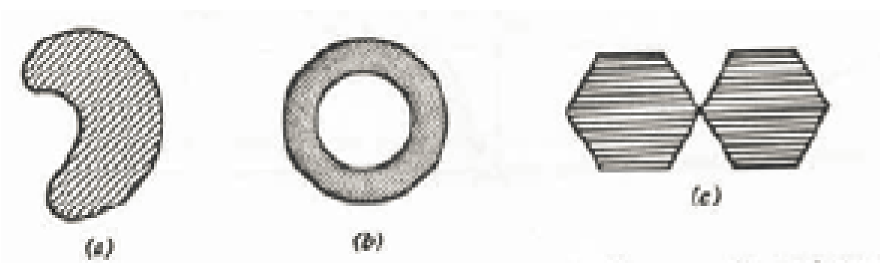
If $x_1 \in K \; and \; x_2 \in K$

then $\alpha x_1 + (1-\alpha)\, x_2 \in K$

where $0 \le \alpha \le 1$



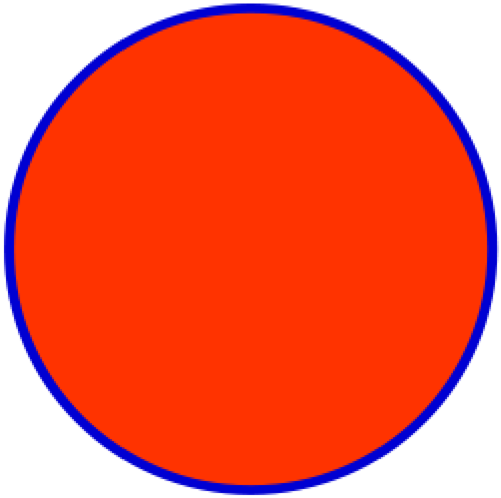
Segment joining any pair of points in K lies entirely in K



Basic Geometrical Concepts(con't)

❑Open and Closed set （开集、闭集）

Open sets can be defined as those sets which contain an **open ball** around each of their points (or, equivalently, a set is open if it doesn't contain any of its boundary points)



The points (x, y) satisfying $x^2 + y^2 = r^2$ are colored blue. The points (x, y) satisfying $x^2 + y^2 < r^2$ are colored red. **The red points form an open set. The blue points form a boundary set.**

The union of the red and blue points is a **closed set**.

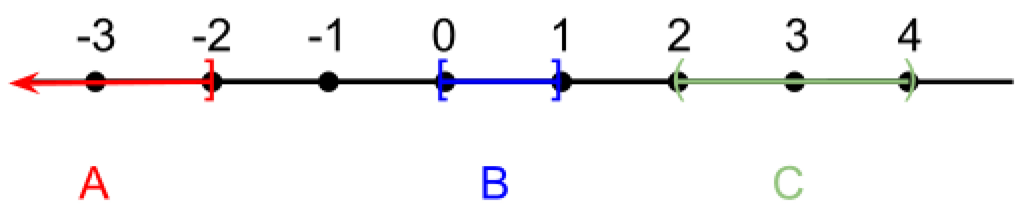
A **closed set** is a set whose complement is an **open set**.

Basic Geometrical Concepts(con't)

❑Compact set （紧集）

Compactness is a property that generalizes the notion of a subset of Euclidean space being **closed** (that is, containing all its limit points) and **bounded** (that is, having all its points lie within some fixed distance of each other).

For example：

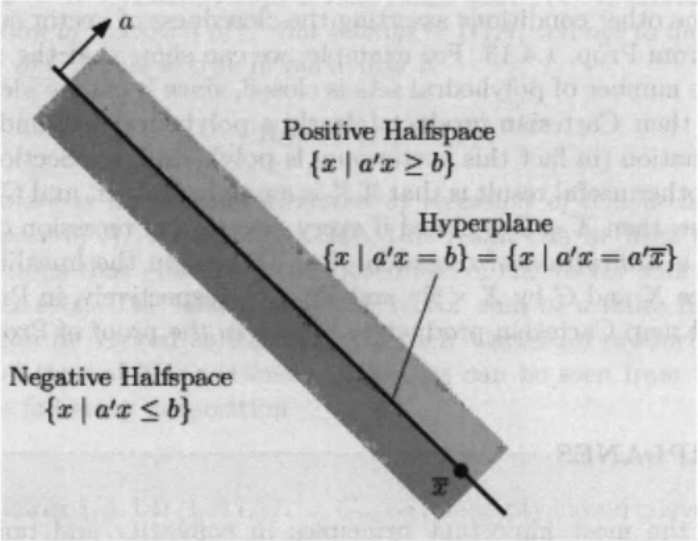


The interval $A = (-\infty, -2]$ is not compact because it is not bounded. The interval $C = (2, 4)$ is not compact because it is not closed. The interval $B = [0, 1]$ is compact because it is both closed and bounded.

Basic Geometrical Concepts(con't)

❑Hyperplane（超平面）

- In geometry a hyperplane is a subspace of one dimension less than its ambient space (外围空间). If a pace is 3-dimensional then its hyperplanes are the 2-dimensional planes.
- Definition：超平面是n 维欧式空间 $\mathfrak{R}^n$中形如 $\{x|a^T x = b\}$ 的集合，其中 a 是 $\mathfrak{R}^n$ 中的非零列向量而 b 是标量。



- 超平面 $H = \{x|a^T x = b\}$ 示意图
- 取H中的任意向量 $\overline{x}$，则超平面H可被等价的描述为

$$H = \{x|a^T x = a^T \overline{x}\} = \overline{x} + \{a^T x = 0\}$$

- 如图所示，该超平面把整个空间分割成了两个半空间

Basic Geometrical Concepts(con't)

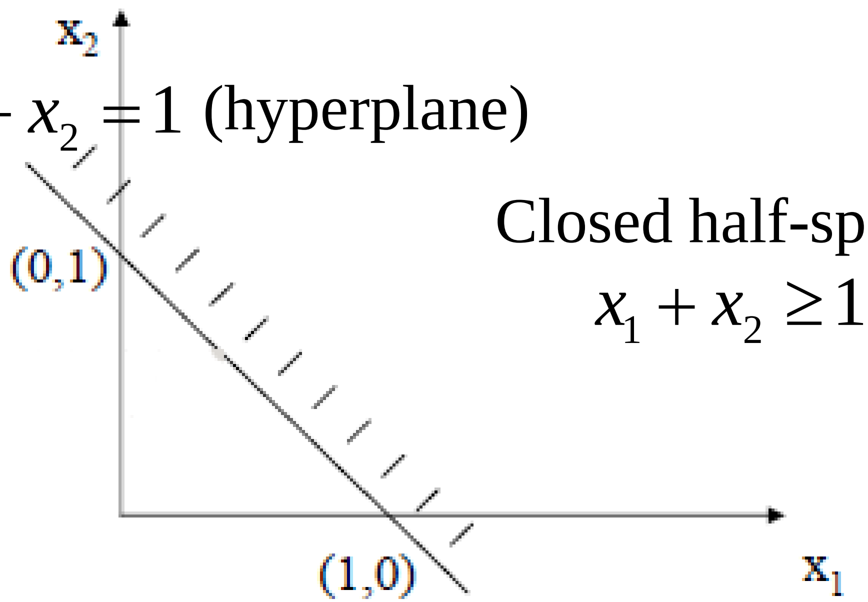
❑Half-space （半空间）

Consider an inequality constraint $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1$

The set of all $x \in \mathbf{R}^n$ satisfying this inequality is the set of all points lying on one side of the hyperplane.

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$x_1 + x_2 = 1$ (hyperplane)



Closed half-space

$$x_1 + x_2 \geq 1$$

The complement of a half-space is a half-space.

Basic Geometrical Concepts(con't)

Half-space and LPs

In the canonical form, all LP constraints may be expressed as linear inequality constraints.

Question: What about equality constraints?

Each equality constraint is equivalent to a pair of inequality constraints.

$$x_1 + x_2 = 1 \quad \Longleftrightarrow \quad \begin{array}{l} x_1 + x_2 \geq 1 \\ -x_1 - x_2 \geq -1 \end{array}$$

Thus, in LP problems, the set of feasible solution is the **intersection** of a finite number of closed half-spaces.

Basic Geometrical Concepts(con't)

Polyhedron （多面体）

A ***polyhedron*** is defined as the solution set of a finite number of linear equalities and inequalities:

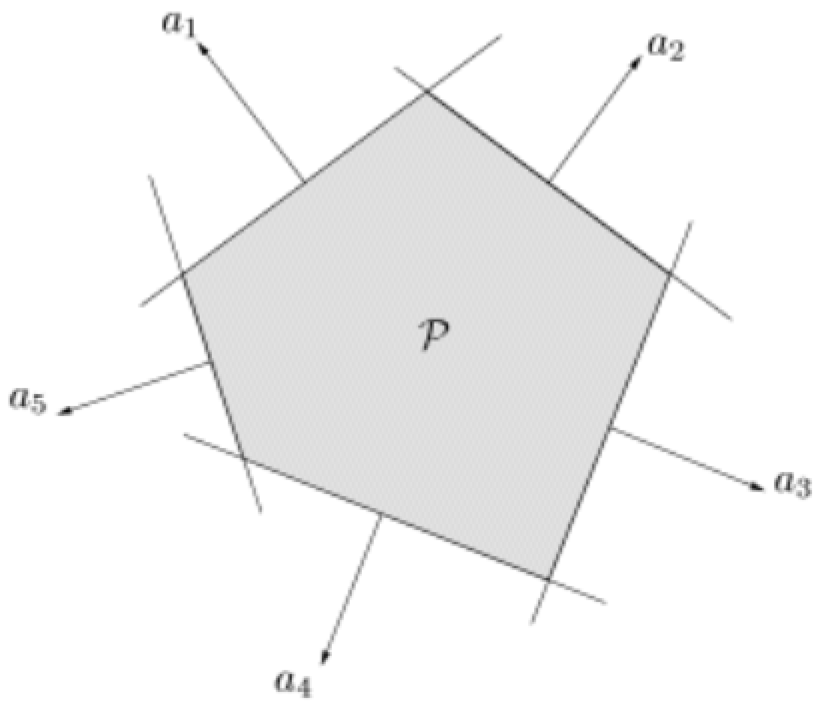
$$P = \left\{ x \middle| a_j^T x \le b_j,\ j = 1, \ldots, m \qquad c_j^T x = d_j,\ j = 1, \ldots, p \right\}$$

A polyhedron is thus the intersection of a finite number of half-spaces and hyperplanes. Affine sets* (e.g. subspaces, hyperplanes, lines), rays, line segments, and half-spaces are all polyhedral. It is easily shown that polyhedral are convex sets. A <u>bounded polyhedron</u> is called a ***polytope*** (多胞形) .

*<u>Affine set（仿射集） 中的任意两点连成的直线都在该set中</u>

Basic Geometrical Concepts(con't)

- **Polyhedron （多面体）**



The polyhedron *P* (shown shaded) is the intersection of five half-spaces, with outward normal vectors $a_1,\ldots a_5$

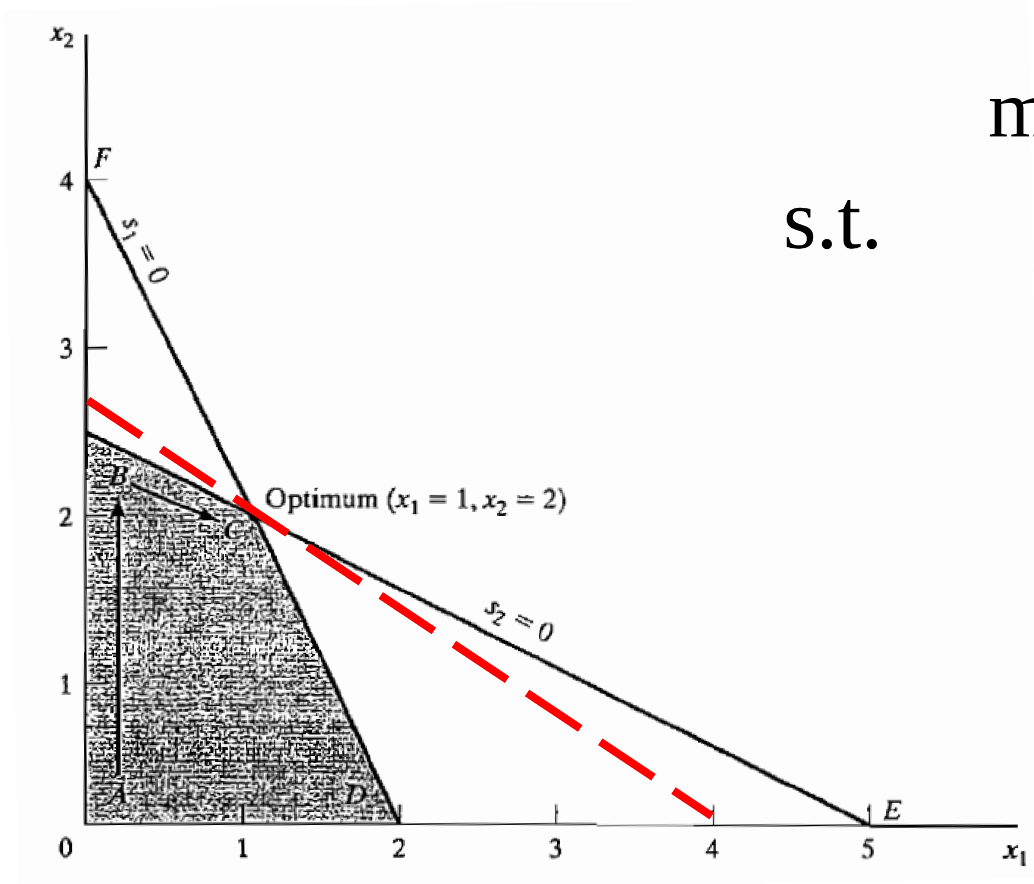
Basic Geometrical Concepts(con't)

Convex Sets and LP （think about proof?）

- Every half-space is a convex set. Closed? Bounded? Compact?
- Intersection of a family of convex sets is a convex set.
- The set of feasible set of an LP is therefore a convex polyhedron but may not be a convex polytope. May not be bounded, but closed!

How to solve a LP?

- **A simple 2-dimension example.**



$$\max \ z = 2x_1 + 3x_2$$

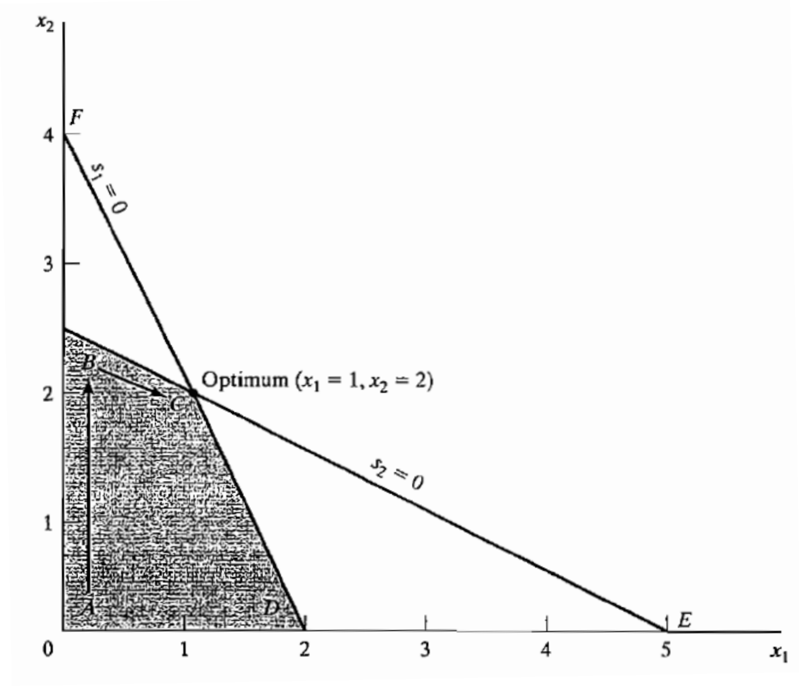
$$\text{s.t.} \quad \begin{aligned} 2x_1 + \ \ x_2 &\le 4 \\ x_1 + 2x_2 &\le 5 \\ x_1, x_2 &\ge 0 \end{aligned}$$

$$x_2 = -\frac{2}{3}x_1 + \frac{z}{3}$$

How to solve a LP?

- **Findings and Inspirations from the simple example**
  - **Solution is always at a vertex of the feasible set, so just check the vertex values!**
  - **A vertex is the junction of two boundaries (constraints).**
  - **Simplex Method**

    1. Find a vertex point, called as <u>basic solution</u>
    2. A <u>criterion</u> to judge whether it's the optimum
    3. <u>Move</u> to a <u>better</u> vertex (basis) from the current one.



Simplex Method

- **Question 1: How to find a vertex (basis)?**

$$\max\ z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

$$\bar{A}X = b$$

$$\bar{A} = (A, I)$$

$$= (B, N)$$

$$\boxed{\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array}}$$

$$x_1, x_2, \ldots, x_n \geq 0$$

- n variables, m independent equations; n>m.
  - Question: what if n<m? n=m?
- So, we need a nonsingular square submatrix B.
  - **Note:** Nonsingular means invertible
- $BX_B = b$ so $X_B = B^{-1}b$, which gives a vertex point X_B

Basic LP Solution Characteristics

Basic concepts: Basis and Basic Variables

1. Any nonsingular square submatrix B of $\bar{A}$ is known as a ***basis*** for the system.

$$\bar{A} = (A, I) = (B, N)$$

2. B is reversible ⟺ $|B| \neq 0$
3. $\bar{A}$ can then be partitioned into basic and non-basic parts. Column vectors of $\bar{A}$ in B are the ***basic column vectors***, the other vectors are ***nonbasic column vectors*** which are grouped in N.
4. X_B : vector of variables associated with basic column vectors; they are the ***basic variables*** with respect to basis B.
5. X_N : the vector of ***nonbasic variables***.

Note that: We define set $\Omega = \{1, 2, ..., n\}$
Ω_B : set of the sequence of vectors in B
Ω_N : set of the sequence of vectors in N

Basic LP Solution Characteristics(con't)

Basic concepts: Basic Solutions

Let $X_N = 0$, thus

$$\bar{A}X = BX_B + NX_N = BX_B$$

$$\bar{A}X = b \Leftrightarrow BX_B = b$$

Because B is nonsingular, $BX_B = b$ has a unique solution.

$$X_B = B^{-1}b$$

Thus $X = (X_B, X_N) = (B^{-1}b, 0)$ is a ***basic solution*** of LP.

Pay attention that:

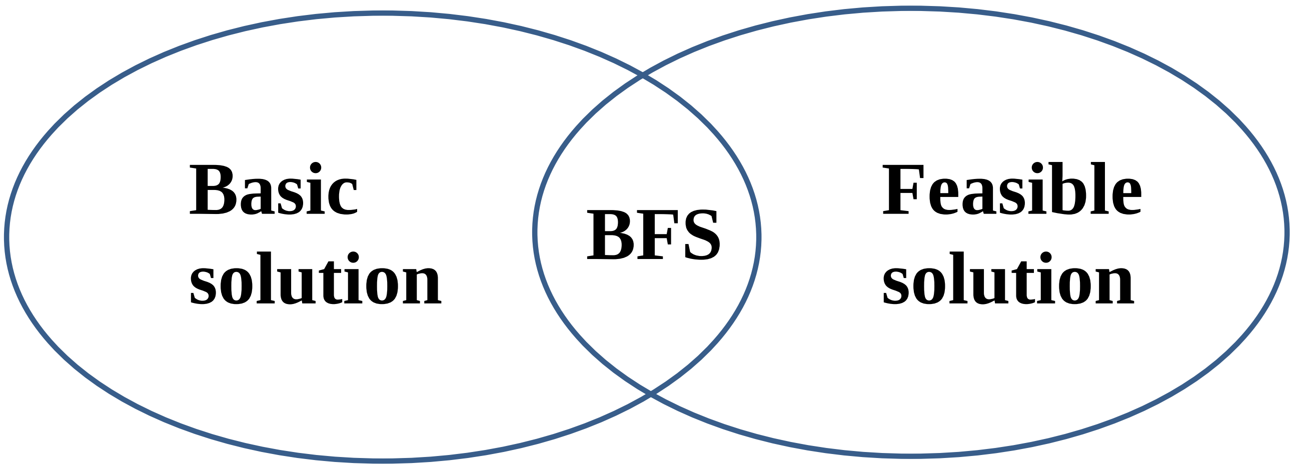
Such a basic solution may not be feasible, since the non-negativities are not considered.

Basic LP Solution Characteristics(con't)

❑**Basic concepts: Basic Feasible Solutions (BFS)**

- If the ***basic solution*** satisfies the non-negative constraints, it is the ***basic feasible solution***.

- Its corresponding ***basis*** is called the ***feasible basis***.



Basic LP Solution Characteristics(con't)

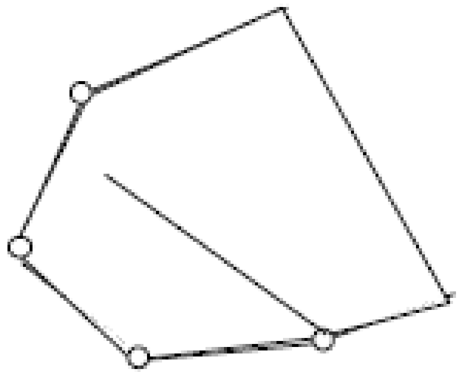
❑Basic concepts: Extreme Points

- A point x_E is an extreme point of convex set Γ if it is impossible to express it as a convex combination of any 2 other distinct points in Γ.

$$x_E = \alpha x_1 + (1-\alpha)\, x_2 \qquad 0 < \alpha < 1$$

- Another name for extreme points is corner points.
- **In the LP the extreme points are the**

basic feasible solutions (BFS).



Basic LP Solution Characteristics(con't)

- **Three important theorems**

$$\text{(LP.1)} \quad \max \; z = CX$$
$$\bar{A}X = b$$
$$X \geq 0$$

1. (Extreme point of the feasible set of LP.1) ≡ (BFS for the LP.1)
2. If LP.1 has a feasible solution, it has a BFS, and the total number of BFS is finite.
3. If LP.1 has an optimal feasible solution, then it has a BFS that is optimal.

Example

- **Question 1: How to find a vertex (basis)?**

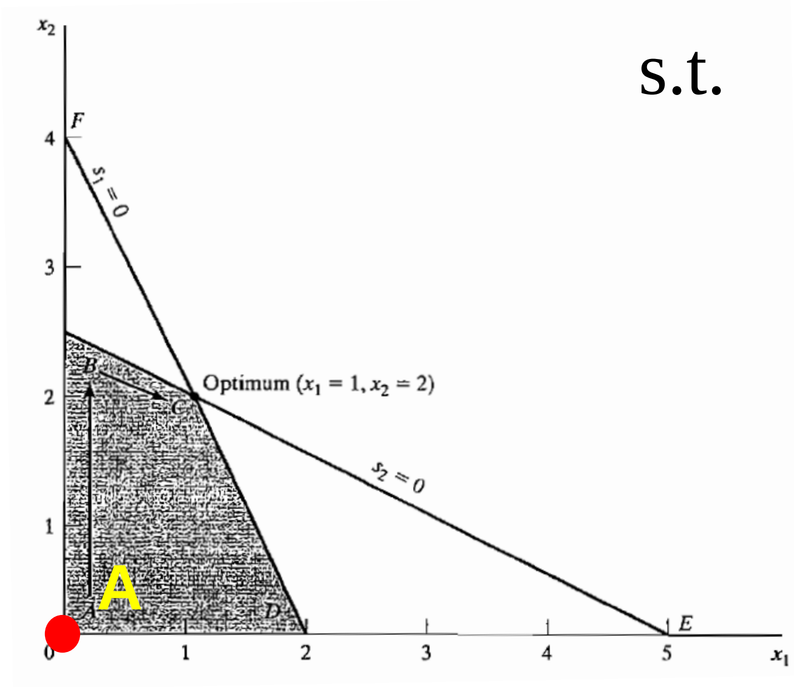
Standard form of the LP:

$$\max \; z = 2x_1 + 3x_2 + 0x_3 + 0x_4$$

$$\text{s.t.} \quad 2x_1 + x_2 + x_3 = 4$$

$$x_1 + 2x_2 + x_4 = 5$$

$$x_1, x_2, x_3, x_4 \geq 0$$



Start from Point A $(why?)$

Corner Point	Basic variables	Non-basic variables
A	x_3, x_4	x_1, x_2

BFS: $x_1 = x_2 = 0, x_3 = 4, x_4 = 5$

Simplex Method (con’t)

- **Question 2: A <u>criterion</u> to judge whether the vertex is optimum**
- Once we have obtained a basic feasible solution, we need to determine whether it is optimal; How? This is answered by question 2.
- Answer: To do this, we try to determine whether z can be increased by increasing ***one non-basic variable*** from its current value of zero while all the ***other non-basic variables*** are still null. So, we need to re-express z all by non-basic variables.
- If the BFS is not optimal, then we try to find another BFS. How to move? Get a BFS adjacent to the current BFS with a larger z-value. (this is answered by question 3).

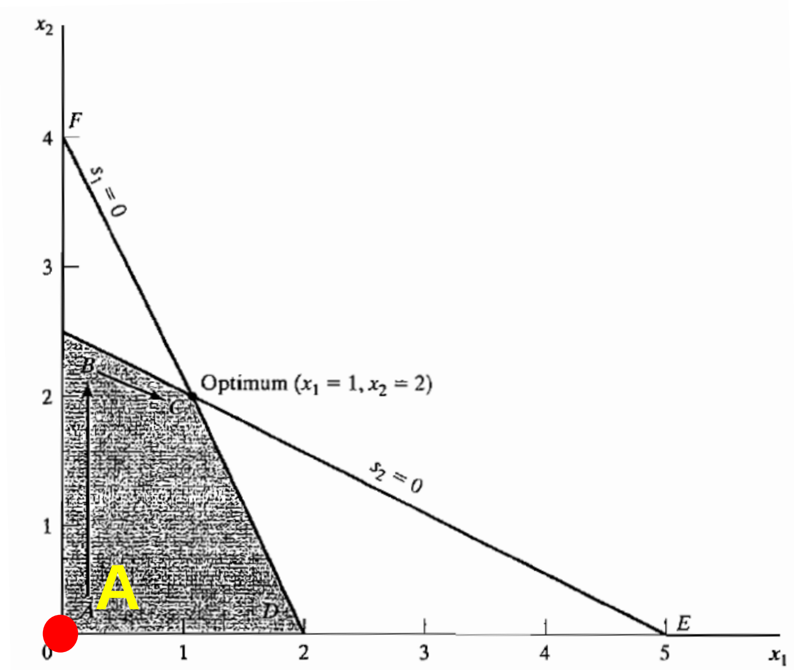
Simplex Method (con't)

- **Question 2: A <u>criterion</u> to judge whether the vertex is optimum**

Corner Point	Basic variables	Non-basic variables
A	x_3, x_4	x_1, x_2

$$z = 2x_1 + 3x_2 + 0x_3 + 0x_4$$

BFS: $x_1 = x_2 = 0, x_3 = 4, x_4 = 5$



If we increase x_1 by 1 (holding all other non-basic variables at zero). Then z will increase by 2.
Similarly, if we increase x_2 by 1, then z will increase by 3.
Thus, increasing any of the non-basic variables will increase z.

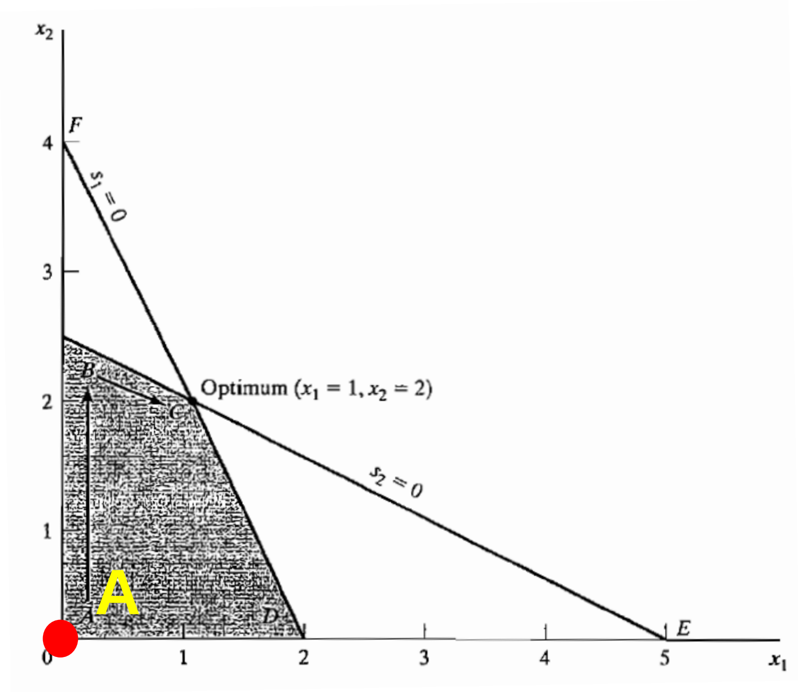
So, the current BFS is not optimal!

Simplex Method (con't)

- **Question 3: Move to a better vertex (basis) from the current one**

$$z = 2x_1 + 3x_2 + 0x_3 + 0x_4$$

BFS: $x_1 = x_2 = 0, x_3 = 4, x_4 = 5$



Because a unit increase in x_2 causes the ***largest rate of increase*** in z, we choose to increase x_2 from its current value of zero. If x_2 is to increase from its current value of zero, then it will become a ***basic variable***. For this reason, we call x_2 the ***entering variable***.

Simplex Method (con't)

- **Question 3: Move to a better vertex (basis) from the current one**

$$z = 2x_1 + 3x_2 + 0x_3 + 0x_4$$

- ✓ We choose the ***entering variable*** to be the new ***basic variable*** with the ***largest positive coefficient*** in the objective function.
- ✓ Because each one-unit increase of x_2 increases z by 3, we take x_2 as large as possible. What ***limits*** the increase of x_2?
- ✓ Then, there must be a ***leaving variable***, because we need a square matrix: as x_2 increases, the values of the current basic variables (x_3, x_4) will decrease towards zero. The one that ***restricts*** the largest increase of x_2 is ***leaving variable.***

$$2x_1 + x_2 + x_3 = 4$$

$$x_1 + 2x_2 + x_4 = 5$$

Simplex Method (con't)

- **Question 3: Move to a better vertex (basis) from the current one**
- Recall the standard form of the LP

$$z = 2x_1 + 3x_2 + 0x_3 + 0x_4$$

$$\text{s.t.} \quad 2x_1 + x_2 + x_3 = 4$$

$$x_1 + 2x_2 + x_4 = 5$$

$$x_1, x_2, x_3, x_4 \geq 0$$

From row 1 of the constraints, we see that $x_3 = 4 - x_2$ (remember that $x_1 = 0$). Because the non-negative constraint $x_3 \geq 0$ must be satisfied, we can only increase x_2 as long as $x_3 \geq 0$, or $x_2 \leq 4$

For row 2

$$x_4 = 5 - 2x_2 \geq 0 \Rightarrow x_2 \leq \frac{5}{2}$$

Simplex Method (con't)

- **The Ratio Test**

$$\frac{\text{RHS of the constraint}}{\text{coefficient of the entering variable}}$$

- Summarizing,

$$x_3 \geq 0 \quad \text{for} \quad x_2 \leq 4$$

$$x_4 \geq 0 \quad \text{for} \quad x_2 \leq 5/2$$

- This means that to keep all the basic variables nonnegative, the largest that we can make x_2 is $\min\{4, 5/2\} = 5/2$.
- If we make $x_2 > 5/2$,then x_4 will become negative, and we will no longer have a BFS.
- When entering a variable into the basis, compute the ratio for every constraint in which the entering variable has a positive coefficient. **The constraint** with the smallest ratio is called the **winner of the ratio test**.

Simplex Method (con't)

❑The Ratio Test

If the entering variable has a zero coefficient in a constraint, what will happen to the entering variable?

For example, if there exists a constraint $x_1 \leq 5$

Standard form: $x_1 + x_5 = 5$, x_5 is the slack variable.

✓ The basic variable x_5 will remain positive for any value of the entering variable x_2 .

For such case, what will happen in the table of Simplex Method?

Simplex Method (con't)

$$\begin{aligned} 2x_1 + \ x_2 + x_3 &= 4 \\ x_1 + 2x_2 + x_4 &= 5 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

- **Find a new BFS**
- Returning to the example, constraint 2 is the winner of the ratio test.
- To make x_2 a basic variable in constraint 2, we use elementary row operations to convert x_2 to have a coefficient of 1 in constraint 2 and a coefficient of 0 in other rows. (*Why?*)
- This procedure is called **pivoting** on constraint 2; and constraint 2 is the **pivot row**.

$$\left(\begin{array}{cccc|c} 2 & 1 & 1 & 0 & 4 \\ 1 & 2 & 0 & 1 & 5 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 3/2 & 0 & 1 & -1/2 & 3/2 \\ 1/2 & 1 & 0 & 1/2 & 5/2 \end{array}\right)$$

Pivot term

- The outcome is that x_2 replaces x_4 as a basic variable.

Simplex Method (con't)

$$\left(\begin{array}{cccc|c} 3/2 & 0 & 1 & -1/2 & 3/2 \\ 1/2 & 1 & 0 & 1/2 & 5/2 \end{array}\right)$$

- **Find a new BFS**
- Consequently,

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 4 \\ x_1 + 2x_2 + x_4 &= 5 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \frac{3}{2}x_1 + x_3 - \frac{1}{2}x_4 &= \frac{3}{2} \\ \frac{1}{2}x_1 + x_2 + \frac{1}{2}x_4 &= \frac{5}{2} \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

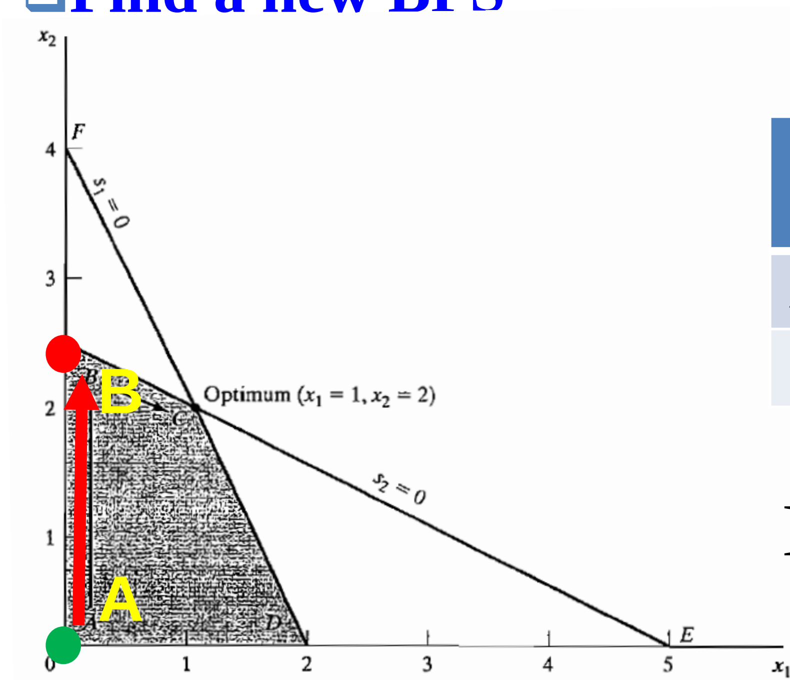
- Then, the objective function of LP is changed as follows

$$z = 2x_1 + 3x_2 + 0x_3 + 0x_4 \quad \Rightarrow \quad z = \frac{1}{2}x_1 + 0x_2 + 0x_3 - \frac{3}{2}x_4 + \frac{15}{2}$$

- The coefficient of the basic variables in the standard form's objective function **should be zero**. Why?

Simplex Method (con't)

- **Find a new BFS**



Corner Point	Basic variables	Non-basic variables
A	x_3, x_4	x_1, x_2
B	x_2, x_3	x_1, x_4

$$\text{BFS: } x_1 = 0, x_2 = \frac{5}{2}, x_3 = \frac{3}{2}, x_4 = 0$$

Is this BFS optimal? $z = \frac{1}{2}x_1 - \frac{3}{2}x_4 + \frac{15}{2}$

$$\frac{3}{2}x_1 + x_3 - \frac{1}{2}x_4 = \frac{3}{2}$$

$$\frac{1}{2}x_1 + x_2 + \frac{1}{2}x_4 = \frac{5}{2}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$x_3 \geq 0$ for $x_1 \leq 1$

$x_2 \geq 0$ for $x_1 \leq 5$

Simplex Method (con't)

- **Find a new BFS**

✓Thus, x_1 is the new ***entering variable***.

✓ Constraint 1 is the ***winner of the ratio test.***

Pivot term

$$\left(\begin{array}{cccc|c} 3/2 & 0 & 1 & -1/2 & 3/2 \\ 1/2 & 1 & 0 & 1/2 & 5/2 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 2/3 & -1/3 & 1 \\ 0 & 1 & -1/3 & 2/3 & 2 \end{array}\right)$$

$$\frac{3}{2}x_1 + x_3 - \frac{1}{2}x_4 = \frac{3}{2}$$
$$\frac{1}{2}x_1 + x_2 + \frac{1}{2}x_4 = \frac{5}{2}$$
$$x_1, x_2, x_3, x_4 \geq 0$$

$\Rightarrow$

$$x_1 + \frac{2}{3}x_3 - \frac{1}{3}x_4 = 1$$
$$x_2 - \frac{1}{3}x_2 + \frac{2}{3}x_4 = 2$$
$$x_1, x_2, x_3, x_4 \geq 0$$

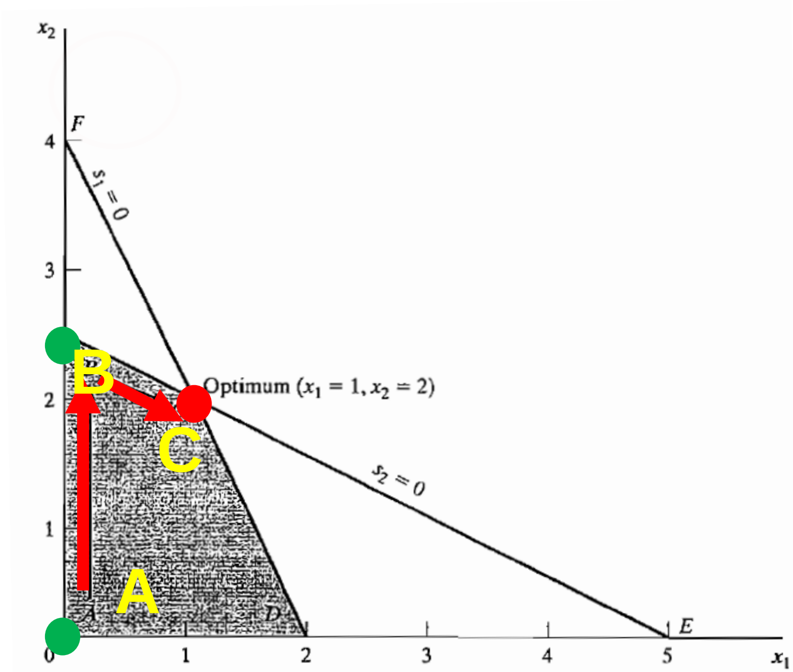
$$z = \frac{1}{2}x_1 - \frac{3}{2}x_4 + \frac{15}{2}$$

$\Downarrow$

$$z = -\frac{1}{3}x_3 - \frac{4}{3}x_4 + 8$$

Simplex Method (con't)

- **Find a new BFS**



Corner Point	Basic variables	Non-basic variables
A	x_3, x_4	x_1, x_2
B	x_2, x_3	x_1, x_4
C	x_1, x_2	x_3, x_4

$$z = -\frac{1}{3}x_3 - \frac{4}{3}x_4 + 8$$

A BFS is optimal (for a max problem) if each **nonbasic variable** has a **nonpositive coefficient** in the standard form's objective function. Why?

BFS: $x_1 = 1, x_2 = 2, x_3 = 0, x_4 = 0$

Optimum

$$z = 8$$

Simplex Method (con’t)

- Now, the three key questions are answered, and based on the same logic, it is quite straightforward to further develop an iterative matrix expression of the processes of Simplex Method.
- Such an iterative matrix expression is mainly used for computer programming which is important for solving the large scale problems.
- Make sure you understand the logic behind the key questions. Don’t be drown in the iterative matrix expressions.

Dimension of the used matrixes

$\overline{A}: m \times n \qquad \overline{A} = (A, I) = (B, N)$

$A: m \times k \qquad n = m + k$

$I: m \times m$

$B: m \times m$

$N: m \times k$

$b: m \times 1$

$P_j: m \times 1,\ j \in \Omega, \Omega = \left(\Omega_B, \Omega_N\right)$

$\left|\Omega\right| = n$

$\left|\Omega_B\right| = m$

$\left|\Omega_N\right| = n - m = k$

$X: n \times 1$

$X_A: k \times 1$

$X_S: m \times 1$

$X_B: m \times 1$

$X_N: k \times 1$

$C: 1 \times n$

$C_A: 1 \times k$

$C_B: 1 \times m$

$C_N: 1 \times k$

How to understand the matrix representation of the Simplex Method properly and accurately?

- **Recall the original structure of the LP formulation**

$$\max \ z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

s.t.

$$\boxed{\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array}}$$

$$x_1, x_2, \ldots, x_n \geq 0$$

$\Rightarrow$

$$\max \ z = CX$$

s.t.

$$\bar{A}X = b$$

$$X \geq 0$$

**The X in the standard form contains original variables and slack variables.*

How to understand the matrix representation of the Simplex Method properly and accurately?(con't)

- **Introducing slack variables**

LP: $\max\ z = C_A X_A$

s.t.

$AX_A \le b$

$X_A \ge 0$

$\Rightarrow$

$\max\ z = C_A X_A + OX_S$

s.t. $AX_A + IX_S = b$

$X_A, X_S \ge 0$

$\bar{A}X = b \quad \bar{A} = (A, I) = (B, N)$

- **An New Example**

$\max\ z = 2x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5$

s.t.

$$2x_1 + x_2 + x_3 = 4$$

$$x_1 + 2x_2 + x_4 = 5$$

$$\boxed{x_1 - x_2 + x_5 = -1}$$

$$x_1, x_2, x_3, x_4, x_5 \ge 0$$

$$\bar{A} = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix}_{3\times 5}$$

$A_{3\times 2}$ $\quad$ $I_{3\times 3}$

Matrix representation of the simplex method (con't)

❑**Division of basic and nonbasic matrix and coefficients**

$$\max\ z = C_A X_A + O X_S$$

$$\text{s.t.}\quad A X_A + I X_S = b$$

$$X_A, X_S \geq 0$$

B is non-singular

Let B denote the initial feasible basis matrix.

Thus, the coefficient matrix $\overline{A} = (A, I)$ can be divided into two parts:

$$(B, N)$$

where B is the basis matrix.

N is the coefficient matrix of nonbasic variables.

Matrix representation of the simplex method (con't)

- **Division of basic and nonbasic matrix and coefficients**

Accordingly, the decision variables can be divided into

$$\begin{bmatrix} X_B \\ X_N \end{bmatrix}_{n\times 1}$$

The coefficient matrix in the objective function can be divided into C_B *and* C_N

Matrix representation of the simplex method (con't)

❑Division of basic and nonbasic variables

$$\max\ z = C_A X_A + OX_S \quad \Longleftrightarrow \quad \max\ z = C_B X_B + C_N X_N$$

$$\text{s.t.}\quad AX_A + IX_S = b \qquad\qquad \text{s.t.}\quad BX_B + NX_N = b$$

$$X_A, X_S \geq 0 \qquad\qquad X_B, X_N \geq 0$$

❑Then $BX_B = b - NX_N$

$$\Downarrow B^{-1}$$

$$X_B = B^{-1}(b - NX_N) = B^{-1}b - B^{-1}NX_N$$

Remember that X_N is always zero?

❑Substitute X_B in the objective function, and it gives

$$z = C_B X_B + C_N X_N$$

$$= C_B(B^{-1}b - B^{-1}NX_N) + C_N X_N$$

$$= C_B B^{-1}b + (C_N - C_B B^{-1}N)X_N$$

Matrix representation of the simplex method (con't)

- **Recall the example**

$$z = 2x_1 + 3x_2 + 0x_3 + 0x_4$$
$$\text{s.t.}\quad 2x_1 + x_2 + x_3 = 4$$
$$x_1 + 2x_2 + x_4 = 5$$
$$x_1, x_2, x_3, x_4 \geq 0$$

$$C_N = \begin{pmatrix} 2 & 3 \end{pmatrix}, C_B = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccc} N & B & b \\ \left(\begin{array}{cc|cc|c} 2 & 1 & 1 & 0 & 4 \\ 1 & 2 & 0 & 1 & 5 \end{array}\right) \end{array}$$

$$X_B = B^{-1}b - B^{-1}NX_N$$

Zero!

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

BFS: $x_1 = x_2 = 0, x_3 = 4, x_4 = 5$

$$z = C_B B^{-1} b + (C_N - C_B B^{-1} N) X_N \qquad \boxed{X_N = 0}$$

Matrix representation of the simplex method (con't)

$$-z+(C_N-C_BB^{-1}N)X_N=-C_BB^{-1}b$$

- **Complete formulation**

Think about the three questions based on this matrix expression?

$$\max\ z=C_BB^{-1}b+(C_N-C_BB^{-1}N)X_N$$

s.t. $X_B=B^{-1}b-B^{-1}NX_N$

$X_B, X_N\geq 0$

- Let $X_N=0$, then $X_B=B^{-1}b,\ z=C_BB^{-1}b$
- If $B^{-1}b\geq 0$, $X=(B^{-1}b,0)$ is a BFS.

or current value

	Basic Var.	Nonbasic Var.	RHS	
Coef. Matrix	$I_{m\times m}$	$B^{-1}N$	$B^{-1}b$	x_B
$-z$	0	$C_N-C_BB^{-1}N$	$-C_BB^{-1}b$	$-z$

Matrix representation of the simplex method (con't)

- **Example**

$$\max\ z = 2x_1 + 3x_2 + 0x_3 + 0x_4$$

$$\text{s.t.}\quad 2x_1 + x_2 + x_3 = 4$$

$$x_1 + 2x_2 + x_4 = 5$$

$$x_1, x_2, x_3, x_4 \geq 0$$

- **Matrix representation**

$$\max\ z = C_N X_N + C_B X_B = \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

- **The augmented matrix of the coef. of constraints**

$$\begin{array}{ccc} \left(\begin{array}{cc|cc|c} 2 & 1 & 1 & 0 & 4 \\ 1 & 2 & 0 & 1 & 5 \end{array}\right) \\ N \quad\quad\quad B \quad\quad b \end{array}$$

Matrix representation of the simplex method (con't)

❑The simplex tableau

RHS

Initial Tableau

bv	x_1	x_2	x_3	x_4	b
x_3	2	1	1	0	4
x_4	1	2	0	1	5
$-z$	2	3	0	0	0

Basic variables

$$\begin{array}{cc|c} N & B & b \\ \left(\begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix}\right. & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \left.\begin{matrix} 4 \\ 5 \end{matrix}\right) \end{array}$$

$$Bx_B + Nx_N = b$$

Objective row

- **Question 1:**

$$x_B = B^{-1}b - B^{-1}NX_N$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$z = C_B B^{-1} b + (C_N - C_B B^{-1} N) X_N$$

$$= C_B B^{-1} b = 0$$

BFS: $x_1 = x_2 = 0, x_3 = 4, x_4 = 5$

Matrix representation of the simplex method (con't)

❑The simplex tableau

RHS

Initial Tableau

bv	x_1	x_2	x_3	x_4	b
x_3	2	1	1	0	4
x_4	1	2	0	1	5
$-z$	2	3	0	0	0

Basic variables

$$z = C_B B^{-1} b + (C_N - C_B B^{-1} N) x_N$$

- **Question 2:**

This BFS is not optimal because the **nonbasic variables** have **positive coefficients** in the objective row.

$$C_N - C_B B^{-1} N$$

$$= \begin{pmatrix} 2 & 3 \end{pmatrix} - \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \end{pmatrix}$$

Matrix representation of the simplex method (con't)

- **The simplex tableau**

Initial Tableau

bv	x_1	x_2	x_3	x_4	b (RHS)	Ratio
x_3	2	1	1	0	4	$4/1$
x_4	1	2	0	1	5	$5/2$
$-z$	2	3	0	0	0	

Pivot term

Leaving variable Entering variable

$$C_N - C_B B^{-1} N$$
$$= \begin{pmatrix} 2 & 3 \end{pmatrix} - \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 3 \end{pmatrix}$$

- **Question 3**

- Step 1: Decide the entering variable; $i = \arg\max_{j \in \Omega_N} \left\{ C_j - C_B B^{-1} P_j \right\}$
- Step 2: Ratio test, decide the leaving variable.

$$B^{-1} b = \left(\bar{b}_j, j = 1, 2, \ldots, m \right) \qquad B^{-1} P_i = \left(\pi_j, j = 1, 2, \ldots, m \right) \qquad k = \arg\min_{j=1,2,\ldots,m} \left\{ \frac{\bar{b}_j}{\pi_j} \,\middle|\, \frac{\bar{b}_j}{\pi_j} > 0 \right\}$$

Matrix representation of the simplex method (con't)

❑ The simplex tableau

Initial Tableau

bv	x_1	x_2	x_3	x_4	b
x_3	2	1	1	0	4
x_4	1	2	0	1	5
$-z$	2	3	0	0	0

Pivot term

Iteration 1

bv	x_1	x_2	x_3	x_4	b
x_3	3/2	0	1	-1/2	3/2
x_2	1/2	1	0	1/2	5/2
$-z$	1/2	0	0	-3/2	-15/2

$$z = C_B B^{-1} b + (C_N - C_B B^{-1} N) x_N$$

New BFS:

$$X_B = B^{-1} b - B^{-1} N X_N$$

$$= \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 3/2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3/2 \\ 5/2 \end{pmatrix}$$

Why?

BFS:

$$x_1 = 0, x_2 = \frac{5}{2}, x_3 = \frac{3}{2}, x_4 = 0$$

Still not optimal.

Matrix representation of the simplex method (con't)

❑The simplex tableau

Initial Tableau

bv	x_1	x_2	x_3	x_4	b
x_3	2	1	1	0	4
x_4	1	2	0	1	5
$-z$	2	3	0	0	0

Pivot term

Iteration 1

bv	x_1	x_2	x_3	x_4	b	Ratio
x_3	3/2	0	1	-1/2	3/2	1
x_2	1/2	1	0	1/2	5/2	5
$-z$	1/2	0	0	-3/2	-15/2	

Pivot term

Leaving variable

Entering variable

Matrix representation of the simplex method (con't)

The simplex tableau

Iteration 1

bv	x_1	x_2	x_3	x_4	b
x_3	3/2	0	1	-1/2	3/2
x_2	1/2	1	0	1/2	5/2
$-z$	1/2	0	0	-3/2	-15/2

Pivot term

Final Tableau

bv	x_1	x_2	x_3	x_4	b
x_1	1	0	2/3	-1/3	1
x_2	0	1	1/3	2/3	2
$-z$	0	0	-7/3	-4/3	-8

New BFS:

$$X_B = B^{-1}b - B^{-1}NX_N$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

BFS:

$$x_1 = 1, x_2 = 2, x_3 = 0, x_4 = 0$$

Optimal!

$$z = C_B B^{-1} b$$

$$= (2 \quad 3) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 8$$

Summary

❑The structure of Simplex tableau

✓Divided by basic and nonbasic variables: $\bar{A} = (B, N)$

$B^{-1}B = I$

	Basic Var.	Nonbasic Var.	RHS
Coef. Matrix	$I_{m\times m}$	$B^{-1}N$	$B^{-1}b$
σ_j	0	$C_N - C_B B^{-1}N$	$-C_B B^{-1}b$

$C_B - C_B B^{-1}B$

✓ The coefficient matrix of basic variable is always a identity matrix
✓ The test number of the basic variable is always zero.

Summary

$$B^{-1}I = B^{-1}$$

❑**The structure of Simplex tableau**

✓Divided by structure and slack variables: $\bar{A} = (A, I)$

	X_A	X_S	RHS
Coef. Matrix	$B^{-1}A$	B^{-1}	$B^{-1}b$
σ_j	$C_A - C_B B^{-1}A$	$-C_B B^{-1}$	$-C_B B^{-1}b$

$$O - C_B B^{-1}I$$

$$C = (C_B, C_N)$$

$$\bar{A} = (B, N)$$

Remember this term!

- Dual solution
- Simplex Multiplier
- Shadow Price
- ……

Summary of the Simplex Algorithm for a Max Problem

- **Step 1:** Convert the LP to standard form.
- **Step 2:** Find a BFS **(Question 1).**
- **Step 3**: Determine whether the current BFS is optimal **(Question 2).**
- **Step 4:** If the current BFS is not optimal, then determine which nonbasic variable ***(entering variable)*** should become a basic variable and which basic variable ***(leaving variable)*** should become a nonbasic variable to find a new BFS with a better objective function value.
- **Step 5:** Use elementary row operations to find the new BFS with the better objective function value **(Question 3).** Go back to **Step 3**.

Discussion about the solution

- **Determining the optimality**

Question: How many possible conditions of the solution in LP?

Four　　Unique optimal solution、Alternative optimal solutions、Unbounded Solution、No feasible solution

- Remind that

$$z = C_B B^{-1} b + (C_N - C_B B^{-1} N) X_N$$

Because X_N is always equal to zero during iterations, its coefficient in the objective function has **no direct impact** on the value of z.

However, the sign of $C_N - C_B B^{-1} N$ decides the increase and decrease of z. (*Why?*)

* X_N 中的非基变量想要入基变为基变量，即由0变为正，其对目标函数值的影响由其系数决定

Discussion about the solution (con't)

- Recall that the Coef. matrix can be represented by column vectors, that is

$$P_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdots \\ a_{mj} \end{bmatrix}; \ \overline{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (P_1, P_2, \ldots, P_n)$$

- ***A*** can be divided into ***B*** and ***N*** based on the basic and nonbasic variables.

$$\text{Recall that } \Omega = \{1, 2, \ldots, n\}$$

$$\Omega_B : \text{set of the sequence of vectors in } \mathbf{B}$$

$$\Omega_N : \text{set of the sequence of vectors in } \mathbf{N}$$

Discussion about the solution (con't)

❑Test vector and test number

	Basic Var.	Nonbasic Var.	RHS
Coef. Matrix	$I_{m\times m}$	$B^{-1}N$	$B^{-1}b$
$-z$	0	$C_N - C_B B^{-1}N$	$-C_B B^{-1}b$

Test vector

The components of the test vector are defined as the test numbers:

$$\sigma_j = C_j - C_B B^{-1} P_j,\ \ j \in \Omega_N$$

P_j 为**N**中的一个列向量，维度是m*1（某个非基变量所对应的列向量）

Discussion about the solution (con't)

- **Case 1:Unique optimal solution**

Theorem 1:

Assume that B is a feasible basis, $X^* = (B^{-1}b, 0)$ is a BFS, if all the test number

$$\sigma_j = C_j - C_B B^{-1} P_j < 0,\ \ j \in \Omega_N$$

then X^* is the optimal solution.

✓Remember the **criterion** of **Question 2 ?**

- …, we want to see whether there is any way that z can be increased by increasing ***one non-basic variable*** from its current value of zero while holding all ***other non-basic variables*** at their current values of zero…

Discussion about the solution (con't)

❑Case 2: Alternative optimal solutions

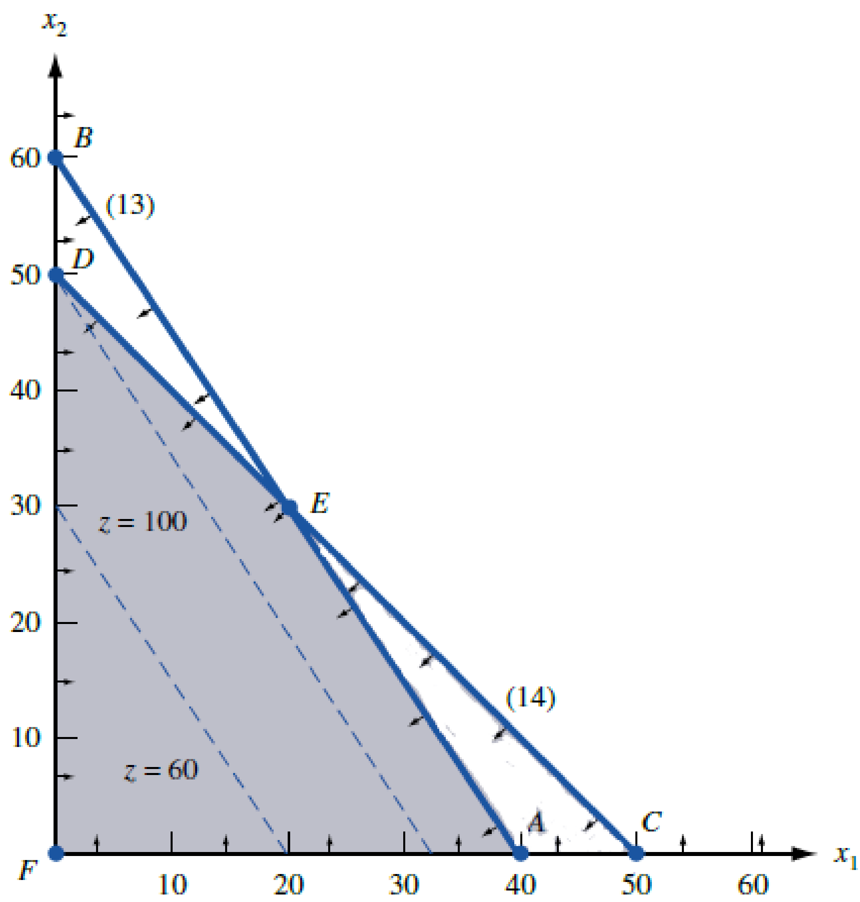
Example:

$$\max z = 3x_1 + 2x_2$$

$$\text{s.t.} \quad \frac{1}{40}x_1 + \frac{1}{60}x_2 \le 1$$

$$\frac{1}{50}x_1 + \frac{1}{50}x_2 \le 1$$

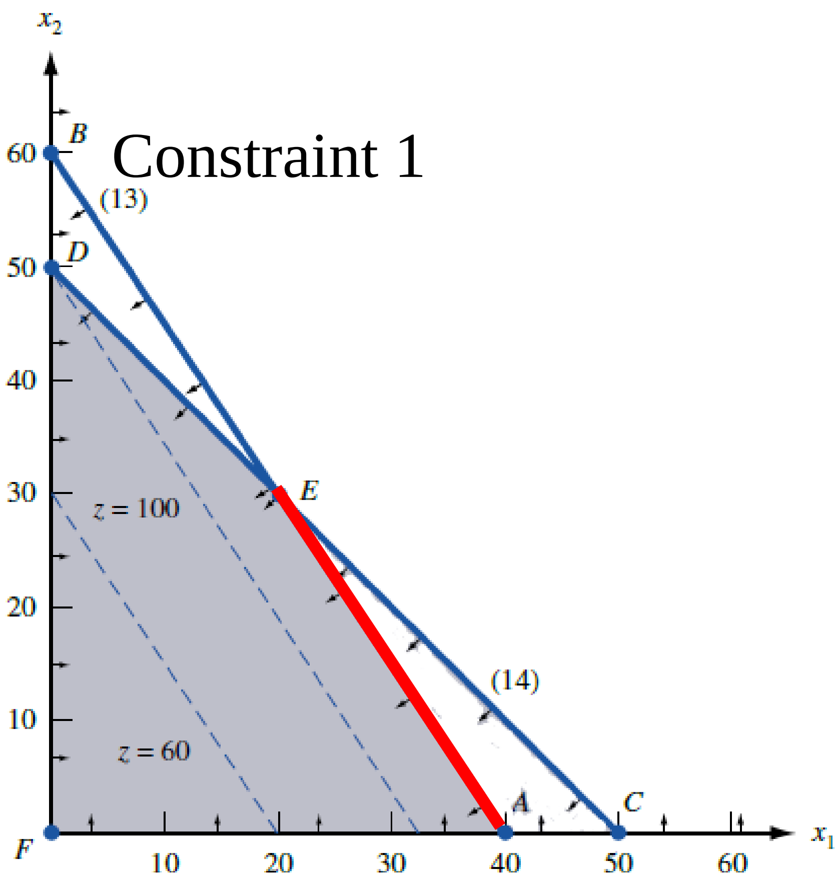
$$x_1, x_2 \ge 0$$



The feasible region is AEDF.

Discussion about the solution (con't)

- **Case 2: Alternative optimal solutions**



The examining line (the red one) parallel to the line of Constraint 1 in the direction of increasing *z*, we find that the last "point" in the feasible region to intersect the line of constraint 1 is the **entire** line segment *AE*. This means that any point on *AE* is optimal.

Discussion about the solution (con't)

❑ Case 2: Alternative optimal solutions

Theorem 2:

Assume that B is a feasible basis,

$X^{(0)} = (B^{-1}b, 0)$ is a BFS.

For all the $j \in \Omega_N, \sigma_j \leq 0$. The test number of one nonbasic variable equals zero, i.e. $\sigma_k = 0, k \in \Omega_N$, the LP has alternative optimal solutions.

（**think about proof?**）

Discussion about the solution (con't)

Case 2: Alternative optimal solutions

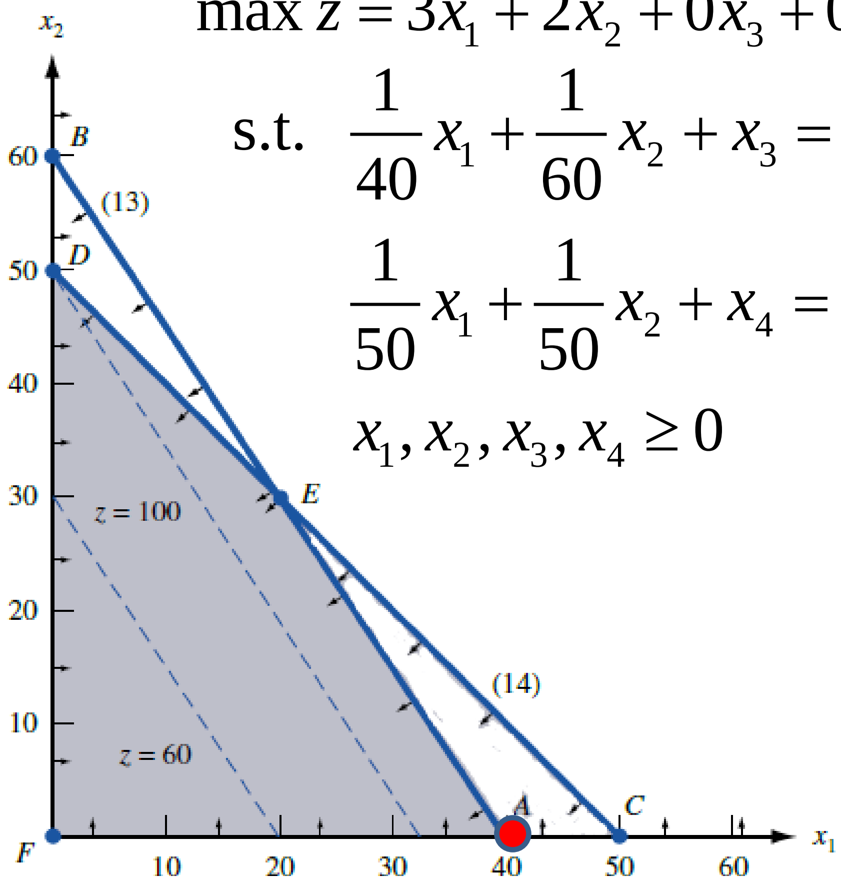
Example:

$$\max z = 3x_1 + 2x_2 + 0x_3 + 0x_4$$

$$\text{s.t.} \quad \frac{1}{40}x_1 + \frac{1}{60}x_2 + x_3 = 1$$

$$\frac{1}{50}x_1 + \frac{1}{50}x_2 + x_4 = 1$$

$$x_1, x_2, x_3, x_4 \geq 0$$



Initial Tableau

bv	x_1	x_2	x_3	x_4	b
x_3	1/40	1/60	1	0	1
x_4	1/50	1/50	0	1	1
$-z$	3	2	0	0	0

Pivot term

Iteration 1

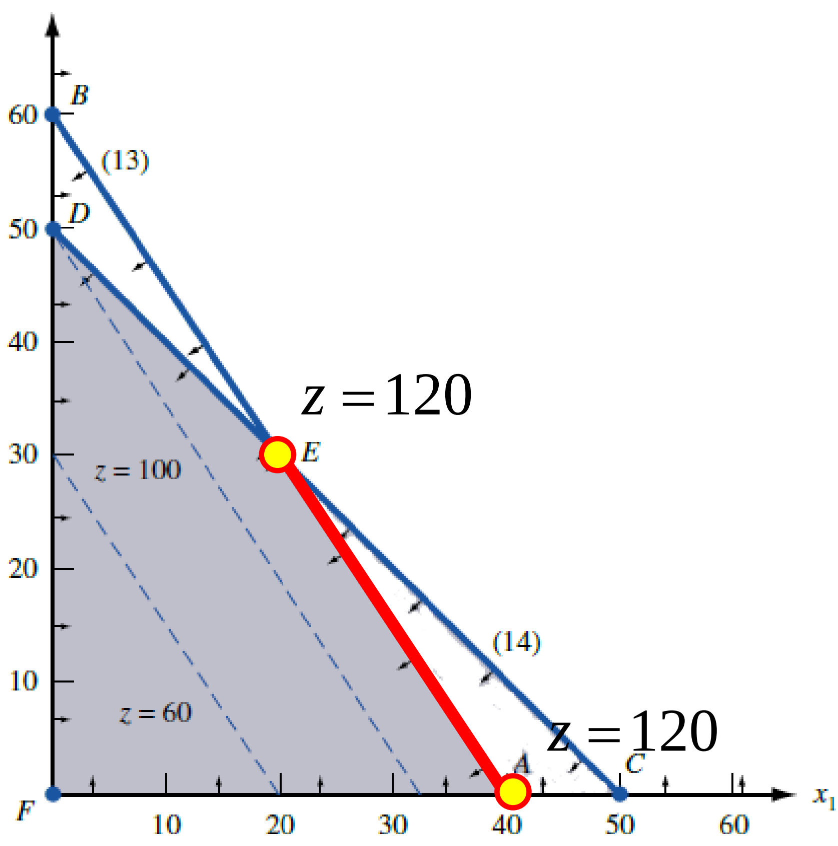
bv	x_1	x_2	x_3	x_4	b
x_1	1	2/3	40	0	40
x_4	0	1/150	-4/5	1	1/5
$-z$	0	0	-120	0	-120

BFS: $x_1 = 40, x_2 = 0, x_3 = 0, x_4 = 1/5, z = 120$

Discussion about the solution (con't)

❑Case 2: Alternative optimal solutions

Let x_2 be the entering variable.



bv	x_1	x_2	x_3	x_4	b
x_1	1	2/3	40	0	40
x_4	0	1/150	-4/5	1	1/5
$-z$	0	0	-120	0	120

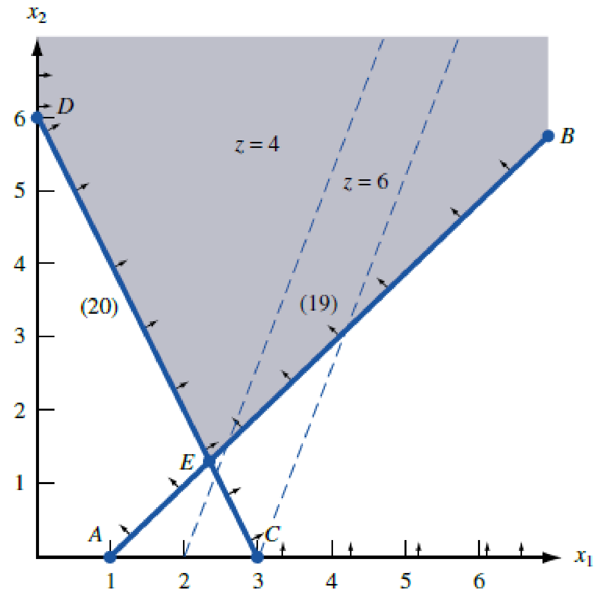
Pivot term

bv	x_1	x_2	x_3	x_4	b
x_1	1	0	120	-100	20
x_2	0	1	-120	150	30
$-z$	0	0	-120	0	120

BFS: $x_1 = 20, x_2 = 30, x_3 = 0, x_4 = 0, z = 120$

Discussion about the solution (con't)

- **Case 3: Unbounded LP**
- For a maximization problem, an unbounded LP occurs if it is possible to find points in the feasible region with arbitrarily large *z*-values.
- For a minimization problem, an LP is unbounded if there are points in the feasible region with arbitrarily small *z*-values.



Discussion about the solution (con't)

❑Unbounded LP in matrix representation

Theorem 3: **Not optimal**

If $\exists j \in \Omega_N,\ \sigma_j > 0$; and all the corresponding coefficients of this nonbasic variable x_j are negative, i.e.,$B^{-1}P_j \le 0$, then the LP has unbounded solution.

Proof:

Suppose that x_j is selected as the entering variable, thus,

$$IX_B + B^{-1}P_j x_j = B^{-1}b$$

Allow this nonbasic variable to be >0, while the other nonbasic variables are still zero

$$B^{-1}b = \left(\overline{b}_j,\ j = 1,2,\ldots m\right)$$

$$B^{-1}P_j = \left(\pi_j,\ j = 1,2,\ldots m\right)$$

$$k = \underset{j=1,2,\ldots m}{\arg\min}\left\{\frac{\overline{b}_j}{\pi_j} \mid \frac{\overline{b}_j}{\pi_j} > 0\right\}$$

Discussion about the solution (con't)

❑Unbounded LP in matrix representation

$$X_B = B^{-1}b - B^{-1}P_j x_j$$

$$\because B^{-1}b \geq 0, -B^{-1}p'_j \geq 0$$

$\therefore$ no matter x_j takes what value $X_B \geq 0$ is always true, for all the current basic variables.

So, x_j can take any large value, since $\sigma_j > 0$ the optimal solution is unbounded

Namely, when we select the leaving variable, there is no existing basic variable that can restrict the increase of the entering variable.

Discussion about the solution (con't)

- **Case 3: Unbounded LP**

Example:

$$\max z = 4x_1 + x_2$$

s.t.

$$-x_1 + x_2 \le 2$$

$$x_1 - 4x_2 \le 4$$

$$x_1 - 2x_2 \le 8$$

$$x_1, x_2 \ge 0$$

Initial Tableau

bv	x_1	x_2	x_3	x_4	x_5	b
x_3	-1	1	1	0	0	2
x_4	1	-4	0	1	0	4
x_5	1	-2	0	0	1	8
$-z$	4	1	0	0	0	0

Pivot term

Iteration 1

bv	x_1	x_2	x_3	x_4	x_5	b
x_3	0	-3	1	1	0	6
x_1	1	-4	0	1	0	4
x_5	0	2	0	-1	1	4
$-z$	0	17	0	-4	0	-16

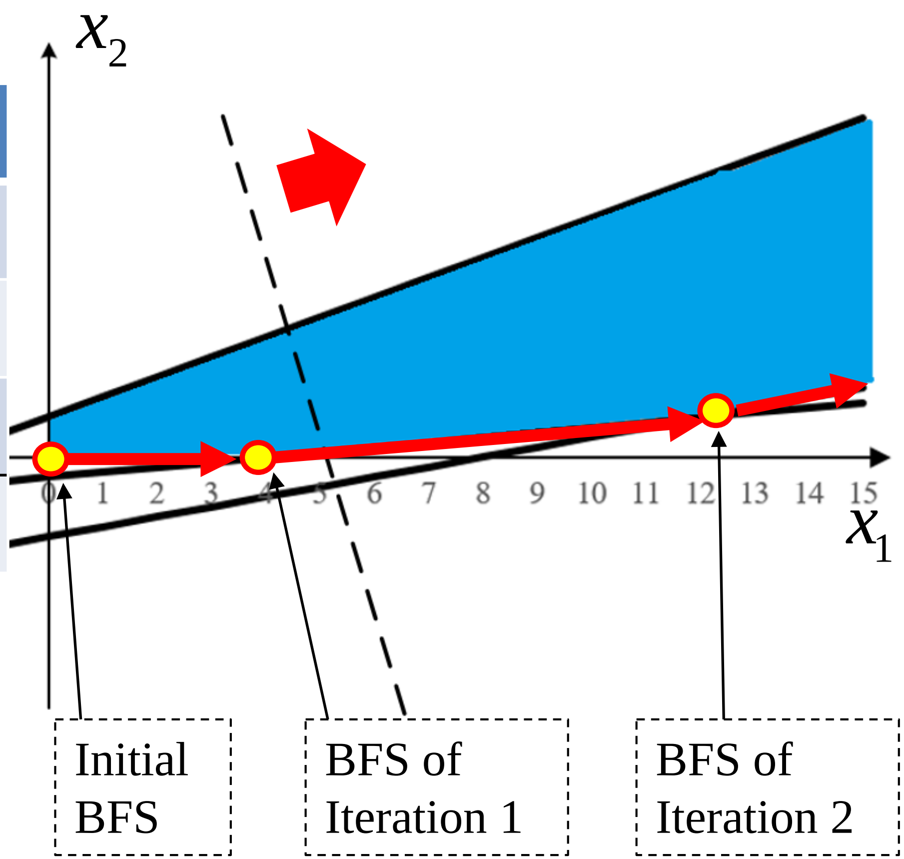
Pivot term

Discussion about the solution (con't)

❑Case 3: Unbounded LP

Iteration 2

bv	x_1	x_2	x_3	x_4	x_5	b
x_3	0	0	1	-1/2	3/2	12
x_1	1	0	0	-1	2	12
x_2	0	1	0	-1/2	1/2	2
$-z$	0	0	0	9/2	-17/2	-50



- 存在非基变量的检验数大于零；
- 且该变量对应的所有系数都小于零 （Ratio test 无法进行）
- 该LP有无界解（无有界解）。

Discussion about the solution (con't)

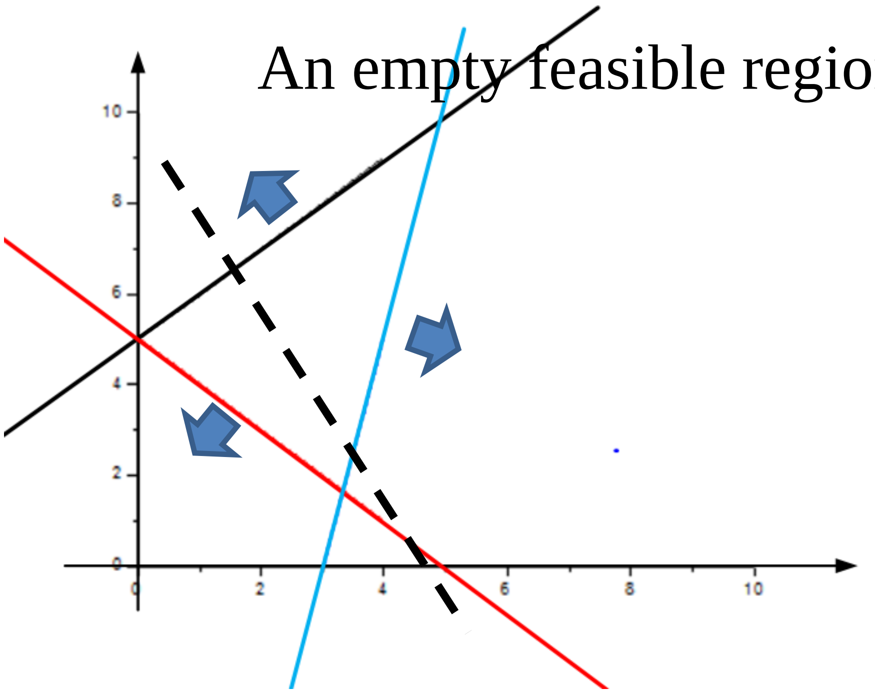
- **Case 4: Infeasible LP**

Example:

$$\max z = 3x_1 + 2x_2$$

$$\text{s.t.}\quad \begin{aligned} x_1 - x_2 &\le -5 \\ x_1 + x_2 &\le 5 \\ 5x_1 - x_2 &\ge 15 \\ x_1, x_2 &\ge 0 \end{aligned}$$

An empty feasible region



It is easy to check whether the feasible region is empty in a 2-dimentional space, but how to check the feasibility of the LP in multi-dimentional conditions?

Discussion about the solution (con't)

- **Case 4: Infeasible LP**

$$\max z = 3x_1 + 2x_2$$

$$\text{s.t.} \quad \begin{aligned} x_1 - x_2 &\le -5 \\ x_1 + x_2 &\le 5 \\ 5x_1 - x_2 &\ge 15 \\ x_1, x_2 &\ge 0 \end{aligned}$$

$$\Rightarrow$$

$$\max z = 3x_1 + 2x_2$$

$$\text{s.t.} \quad \begin{aligned} -x_1 + x_2 - x_3 &= 5 \\ x_1 + x_2 + x_4 &= 5 \\ 5x_1 - x_2 - x_5 &= 15 \\ x_1, x_2, x_3, x_4, x_5 &\ge 0 \end{aligned}$$

If the RHS is negative value, we need to covert it to a positive value.

Discussion about the solution (con't)

Case 4: Infeasible LP

$$\max z = 3x_1 + 2x_2$$

$$\text{s.t.} \quad -x_1 + x_2 - x_3 + x_6 = 5$$

$$x_1 + x_2 + x_4 = 5$$

$$5x_1 - x_2 - x_5 + x_7 = 15$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Standard form tableau

x_1	x_2	x_3	x_4	x_5	b
-1	1	-1	0	0	5
1	1	0	1	0	5
5	-1	0	0	-1	15
3	2	0	0	0	0

Where is the feasible basis?

x_6, x_7 ***are Artificial Variables***.

Modified Simplex Method

- **Why we need the modified Simplex Method?**
- Recall that the original simplex method requires an initial **BFS**. In all the problems we have discussed so far, we found an initial **BFS** by using the slack variables as our *basic variables*.
- If an LP has *any* $\geq$ *or* $=$ *constraints*, an initial BFS may not be available. (*Why?*)
- So we need modified Simplex Methods:
  - Big M Method
  - Two-phase Simplex Method
- **Question 1: What should we do if the LP has $\geq$ or $=$ constraints?**
- Introducing artificial variables!

Modified Simplex Method

- **Question 2: What is the necessary condition for a feasible solution for the original problem?**
- Any feasible solution to the augmented problem with all artificial variables equal to zero provides a feasible solution to the original problem.

- **Question 3: Why must the artificial variables be equal to zero?**
- They are introduced only to have an initial BFS. If they are not zero, the constraints are not fulfilled.

$$5x_1 - x_2 - x_5 = 15$$

$$5x_1 - x_2 - x_5 + x_7 = 15$$

The Big M Method

- **What dose the Big M refer to?**
- M represents a “very large” positive number.

- **Why we use the Big M?**

Example:

$$\min z = 2x_1 + 3x_2$$

s.t.

$$\frac{1}{2}x_1 + \frac{1}{4}x_2 \le 4$$

$$x_1 + 3x_2 \ge 20$$

$$x_1 + x_2 = 10$$

$$All\ \ x_i \ge 0$$

The conversion between Max and Min

❑ The relationship between Max and Min in LPs

- The conversion between Max and Min in LP can be simply performed by multiply the objective function by “-1” .
- In the Simplex Method, the difference between Max and Min lies on the sign of **test numbers**.

	Basic Var.	Nonbasic Var.	RHS
Coef. Matrix	$I_{m\times m}$	$B^{-1}N$	$B^{-1}b$
σ_j	0	$C_N - C_B B^{-1}N$	$-C_B B^{-1}b$

For Max, ≤ 0

For Min, ≥ 0

The Big M Method

- **Transform into the standard form**

LP1

$$\min z = 2x_1 + 3x_2$$

$$\text{s.t.} \quad \frac{1}{2}x_1 + \frac{1}{4}x_2 \le 4$$

$$x_1 + 3x_2 \ge 20$$

$$x_1 + x_2 = 10$$

$$All \;\; x_i \ge 0$$

LP2

$$\min z = 2x_1 + 3x_2$$

$$\text{s.t.} \quad \frac{1}{2}x_1 + \frac{1}{4}x_2 + s_1 = 4$$

$$x_1 + 3x_2 - e_2 + a_2 = 20$$

$$x_1 + x_2 + a_3 = 10$$

$$All \;\; x_i \ge 0$$

$$\text{BFS:} \quad x_1 = x_2 = 0, s_1 = 4, a_2 = 20, a_3 = 10$$

- **How to guarantee that the optimal solution to LP2 is the same as the optimal solution to LP1?**

The Big M Method

- LP2 $\min z = 2x_1 + 3x_2$

$$\Rightarrow \quad \min z = 2x_1 + 3x_2 + Ma_2 + Ma_3$$

- Notes:
  - In a Min problem, we can ensure that all the artificial variables will be zero by adding a term Ma_i to the objective function for each artificial variable a_i.
  - In a Max problem, add a term $-Ma_i$ to the objective function.
  - This implies that as long as the artificial variables are not zero, the objective function is highly affected. So, if we found an optimal solution, the artificial variables must be zero!
  - If the artificial variable is not zero in the optimal solution, then the original LP has no feasible solution!

The Big M Method

Initial Tableau

bv	x_1	x_2	s_1	e_2	a_2	a_3	b
s_1	1/2	1/4	1	0	0	0	4
a_2	1	3	0	-1	1	0	20
a_3	1	1	0	0	0	1	10
$-z'$	2	3	0	0	M	M	

Pivot term

Iteration 1

bv	x_1	x_2	s_1	e_2	a_2	a_3	b
s_1	5/12	0	1	1/12	-1/12	0	7/3
x_2	1/3	1	0	-1/3	1/3	0	20/3
a_3	2/3	0	0	1/3	-1/3	1	10/3
$-z'$	$\frac{2M-3}{3}$	0	0	$\frac{M-3}{3}$	$\frac{3-4M}{3}$	0	$-\frac{60+10M}{3}$

Pivot term

The Big M Method

Final Tableau

bv	x_1	x_2	s_1	e_2	a_2	a_3	b
s_1	0	0	1	-1/8	1/8	-5/8	1/4
x_2	0	1	0	-1/2	1/2	-1/2	5
x_1	1	0	0	1/2	-1/2	3/2	5
$-z'$	0	0	0	$-\frac{1}{2}$	$\frac{1-2M}{2}$	$\frac{3-2M}{2}$	25

Two-phase Simplex Method

- **Procedure of the Two-phase Simplex Method**
- **Step 1:** Modify the constraints so that the RHS of each constraint is nonnegative. This requires that each constraint with a negative RHS be multiplied by -1.
- **Step 2:** Convert each inequality constraint to the standard form. If constraint i is a $\leq$ constraint, then add a slack variable s_i. If constraint i is a $\geq$ constraint, subtract an excess (surplus) variable e_i.

Two-phase Simplex Method

- **Procedure of the Two-phase Simplex Method**
- **Step 3:** If constraint i (after **Step 1**) has an $\geq$ or $=$, add an artificial variable a_i. Also add the non-negativity $a_i \geq 0$.
- **Step 4:** Solve a new LP whose objective function is

min z' = sum of all the artificial variables

Why do we set such a objective function?

This is called the **Phase I LP**. The Phase I LP aims to nullify the artificial variables.

Two-phase Simplex Method

- **Solving the Phase I LP will result in one of following three cases:**
- **Case 1:** The optimal value of *z'* is equal to zero, and no artificial variables are in the optimal Phase I basis.

- **Case 2:** The optimal value of *z'* is equal to zero and at least one artificial variable is in the optimal Phase I basis.

- **Case 3:** The optimal value of *z'* is greater than zero.
  - **The original LP has no feasible solution. (Why?)**

We have provided three examples below to discuss each case.

Example: Case 1

$$\max z = 3x_1 - x_2 - x_3$$

s.t.

$$\begin{aligned} x_1 - 2x_2 + x_3 &\le 11 \\ -4x_1 + x_2 + 2x_3 &\ge 3 \\ -2x_1 + x_3 &= 1 \\ All\ \ x_i &\ge 0 \end{aligned}$$

- **Phase I LP** $\min z' = a_2 + a_3$

s.t.

$$\begin{aligned} x_1 - 2x_2 + x_3 + s_1 &= 11 \\ -4x_1 + x_2 + 2x_3 - e_2 + a_2 &= 3 \\ -2x_1 + x_3 + a_3 &= 1 \\ All\ \ x_i &\ge 0 \end{aligned}$$

Example: Case 1

❑ **Phase I Tableau**

Initial Tableau

bv	x_1	x_2	x_3	s_1	e_2	a_2	a_3	b
s_1	1	-2	1	1	0	0	0	11
a_2	-4	1	2	0	-1	1	0	3
a_3	-2	0	1	0	0	0	1	1
$-z'$	6	-1	-3	0	1	0	0	-4

Pivot term

Example: Case 1

❑Phase I Tableau

Iteration 1

bv	x_1	x_2	x_3	s_1	e_2	a_2	a_3	b
s_1	3	-2	0	1	0	0	0	10
a_2	0	1	0	0	-1	1	1	1
x_3	-2	0	1	0	0	0	1	1
$-z'$	0	-1	0	0	1	0	3	0

Pivot term

Example: Case 1

Phase I Tableau

Final Tableau

bv	x_1	x_2	x_3	s_1	e_2	a_2	a_3	b
s_1	3	0	0	1	-2	2	-5	12
x_2	0	1	0	0	-1	1	-2	1
x_3	-2	0	1	0	0	0	1	1
$-z'$	0	0	0	0	0	1	1	-1

- Now the Phase I Tableau is optimal!
- The current **BFS** is $x_1 = 0, x_2 = 1, x_3 = 1, s_4 = 12, e_2 = a_2 = a_3 = 0$
- In the Phase II Tableau, we should remove the artificial variables of the final tableau of Phase I.

Example: Case 1

❑Phase II Tableau

Initial Tableau

Pivot term

bv	x_1	x_2	x_3	s_1	e_2	b
s_1	3	0	0	1	-2	12
x_2	0	1	0	0	-1	1
x_3	-2	0	1	0	0	1
$-z$	1	0	0	0	-1	-2

$$\begin{cases} z = 3x_1 - x_2 - x_3 \\ x_2 - e_2 = 1 \\ -2x_1 + x_3 = 1 \end{cases}$$

$$\Downarrow$$

$$z = x_1 - e_2 - 2$$

Example: Case 1

$$\begin{cases} z = 3x_1 - x_2 - x_3 \\ x_2 - e_2 = 1 \\ -2x_1 + x_3 = 1 \end{cases} \Rightarrow z = x_1 - e_1 - 2$$

Phase II Tableau

$$z = C_B B^{-1} b + (C_N - C_B B^{-1} N) X_N$$

$$= (0,-1,-1)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 12 \\ 1 \\ 1 \end{pmatrix} +$$

$-z$	**3**	**-1**	**-1**	**0**	**0**	**0**

⇓

$-z$	**1**	**0**	**0**	**0**	**-1**	**-2**

$$\left[(3,0) - (0,-1,-1)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3 & -2 \\ 0 & -1 \\ -2 & 0 \end{pmatrix}\right] \begin{pmatrix} x_1 \\ e_2 \end{pmatrix}$$

$$= x_1 - e_2 - 2$$

Example: Case 1

Phase II Tableau

Final Tableau

bv	x_1	x_2	x_3	s_1	e_2	b
x_1	1	0	0	1/3	-2/3	4
x_2	0	1	0	0	-1	1
x_3	0	0	1	2/3	-4/3	9
$-z$	0	0	0	-1/3	-1/3	-2

Optimal!

- The optimal solution is

$$x_1 = 4, x_2 = 1, x_3 = 9, z = 2$$

Example: Case 2

$$\max z = -40x_1 - 10x_2 - 7x_5 - 14x_6$$

s.t.

$$\begin{aligned}
x_1 - x_2 \qquad\qquad + 2x_5 \qquad &= 0 \\
-2x_1 + x_2 \qquad\qquad - 2x_5 \qquad &= 0 \\
x_1 \qquad + x_3 \qquad + x_5 - x_6 &= 3 \\
2x_2 + x_3 + x_4 + 2x_5 + x_6 &= 4 \\
All\ \ x_i \geq 0 &
\end{aligned}$$

- We may use x_4 as a basic variable for the fourth constraint and use artificial variables a_1, a_2 and a_3 as basic variables for the first three constraints.

Example: Case 2

❑Phase I Tableau

Initial Tableau

bv	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	a_3	b
a_1	1	-1	1	0	2	0	1	0	0	0
a_2	-2	1	0	0	-2	0	0	1	0	0
a_3	1	0	0	0	1	-1	0	0	1	3
x_4	0	2	1	1	2	1	0	0	0	4
$-z'$	0	0	-1	0	-1	1	0	0	0	-3

Pivot term

Example: Case 2

❑Phase I Tableau

Final Tableau

bv	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	a_3	b
a_1	1	-1	0	0	2	0	1	0	0	0
a_2	-2	1	0	0	-2	0	0	1	0	0
x_3	1	0	1	0	1	-1	0	0	1	3
x_4	-1	2	0	1	1	2	0	0	-1	1
$-z'$	1	0	0	0	0	0	0	0	1	0

- ❑Now the Phase I Tableau is optimal!
- ❑And two artificial variables remain in the basis (a_1 and a_2).
- ❑We can drop the artificial variable a_3 in the Phase II tableau.

Example: Case 2

Phase II Tableau

Initial Tableau

bv	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2		b
a_1	1	-1	0	0	2	0	1	0		0
a_2	-2	1	0	0	-2	0	0	1		0
x_3	1	0	1	0	1	-1	0	0		3
x_4	-1	2	0	1	1	2	0	0		1
$-z$	-40	-10	0	0	-7	-14	0	0		0

- After several iterations, we can obtain the optimal solution:

$$x_1 = 0, x_2 = 0, x_3 = -7/2, x_4 = 0, x_5 = 0, x_6 = -1/2, z = 7$$

Example: Case 3

$$\max z = 2x_1 + 3x_2$$

s.t.

$$\frac{1}{2}x_1 + \frac{1}{4}x_2 \le 4$$
$$x_1 + 3x_2 \ge 36$$
$$x_1 + x_2 = 10$$
$$x_1, x_2 \ge 0$$

$$\max z = 2x_1 + 3x_2$$

s.t.

$$\frac{1}{2}x_1 + \frac{1}{4}x_2 + s_1 = 4$$
$$x_1 + 3x_2 - e_2 + a_2 = 36$$
$$x_1 + x_2 + a_3 = 10$$
$$x_1, x_2, s_1, e_2, a_2, a_3 \ge 0$$

Example: Case 3

Initial Tableau

bv	x_1	x_2	s_1	e_2	a_2	a_3	b
s_1	1/2	1/4	1	0	0	0	4
a_2	1	3	0	-1	1	0	36
a_3	1	1	0	0	0	1	10
$-z'$	-2	-4	0	0	0	0	-46

Pivot term

Final Tableau

bv	x_1	x_2	s_1	e_2	a_2	a_3	b
s_1	1/4	1/4	1	0	0	-1/4	3/2
a_2	-2	0	0	-1	1	-3	6
a_3	1	1	0	0	0	1	10
$-z'$	2	0	0	1	0	4	-6

Example: Case 3

Final Tableau

bv	x_1	x_2	s_1	e_2	a_2	a_3	b
s_1	1/4	1/4	1	0	0	-1/4	3/2
a_2	-2	0	0	-1	1	-3	6
a_3	1	1	0	0	0	1	10
$-z'$	2	0	0	1	0	4	-6

- No variable in the objective row has a negative coefficient, so this is an optimal tableau, and since the optimal value of z' is 6>0, the original LP must have no feasible solution. *(Why?)*