Modelling Transportation Systems

Linear Programming

Part 2： Sensitivity Analysis and Duality

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Learning Objectives

- Know how to perform sensitivity analysis of changes in the cost coefficients, right-hand sides
- Understand the duality theory in LP
- Use the duality theory to understand solution characteristics of both primal and dual problems
- Learn how to transform primal to dual and vice versa
- Understand the economic interpretation of dual solution

Sensitivity Analysis

Why Sensitivity Analysis?

$$\max \ z = CX$$

$$\begin{cases} AX_A \le b \\ X_A \ge 0 \end{cases}$$

"***What if***" analysis

- Coefficients in *A*, *C* and *b* are usually on estimates. There is a need to ascertain whether the solution remains optimal for the possible **range of variation** of the coefficients.
- If there is a need to add **new constraints or new activities**, there is no need to solve from scratch which can be quite expensive in real applications.
- Begins with the optimal solution of the original problem.
- Only one change at a time is studied in this module. Extensions can be made to more than one change.

Why Sensitivity Analysis?

- Six types of changes in an LP's parameters can change the optimal solution
  - Changing the objective function coefficient of a nonbasic variable (in terms of the optimal solution)
  - Changing the objective function coefficient of a basic variable (in terms of the optimal solution)
  - Changing the RHS of a constraint
  - Changing the column of a nonbasic variable
  - Adding a new variable or activity
  - Adding a new constraint

Giapetto’s Woodcarving

- Giapetto’s Woodcarving Inc., ***manufactures*** two types of wooden toys: soldiers and trains. The manufacture of each type of toys requires two types of skilled labor: finishing and carpentry. The amount of each resource needed to make each type of toy is given as following:

Resource	Soldiers	Trains
Finishing hours	2 hours	1 hours
Carpentry hours	1 hours	1 hours

- Currently, 100 finishing hours, and 80 carpentry hours are available. The profits of soldiers and trains are \$3 and \$2 each. Demand for trains is unlimited, but at most 40 soldiers are needed. Giapetto wants to maximize its profit.

Giapetto's Woodcarving (con't)

- Defining the decision variables as

$$x_1 = \text{number of soldiers produced}$$

$$x_2 = \text{number of trains produced}$$

- It is easy to see that Giapetto should solve the following LP:

$$\max\ z = 3x_1 + 2x_2$$

$$\begin{aligned}
\text{s.t.}\quad & 2x_1 + x_2 \le 100 && \text{(Constraint 1)}\\
& x_1 + x_2 \le 80 && \text{(Constraint 2)}\\
& x_1 \le 40 && \text{(Constraint 3)}\\
& x_1, x_2 \ge 0
\end{aligned}$$

$\Rightarrow$

$$\max\ z = 3x_1 + 2x_2 + 0x_3 + 0x_4 + 0x_5$$

$$\begin{aligned}
\text{s.t.}\quad & 2x_1 + x_2 + x_3 = 100\\
& x_1 + x_2 + x_4 = 80\\
& x_1 + x_5 = 40\\
& x_1, x_2, x_3, x_4, x_5 \ge 0
\end{aligned}$$

Giapetto's Woodcarving (con't)

- Use the Simplex Method introduced in 1a to find the optimal solution. The initial and final tableaus are presented as follows:

Initial Tableau

bv	x_1	x_2	x_3	x_4	x_5	b
x_3	2	1	1	0	0	100
x_4	1	1	0	1	0	80
x_5	1	0	0	0	1	40
$-z$	3	2	0	0	0	0

Final Tableau

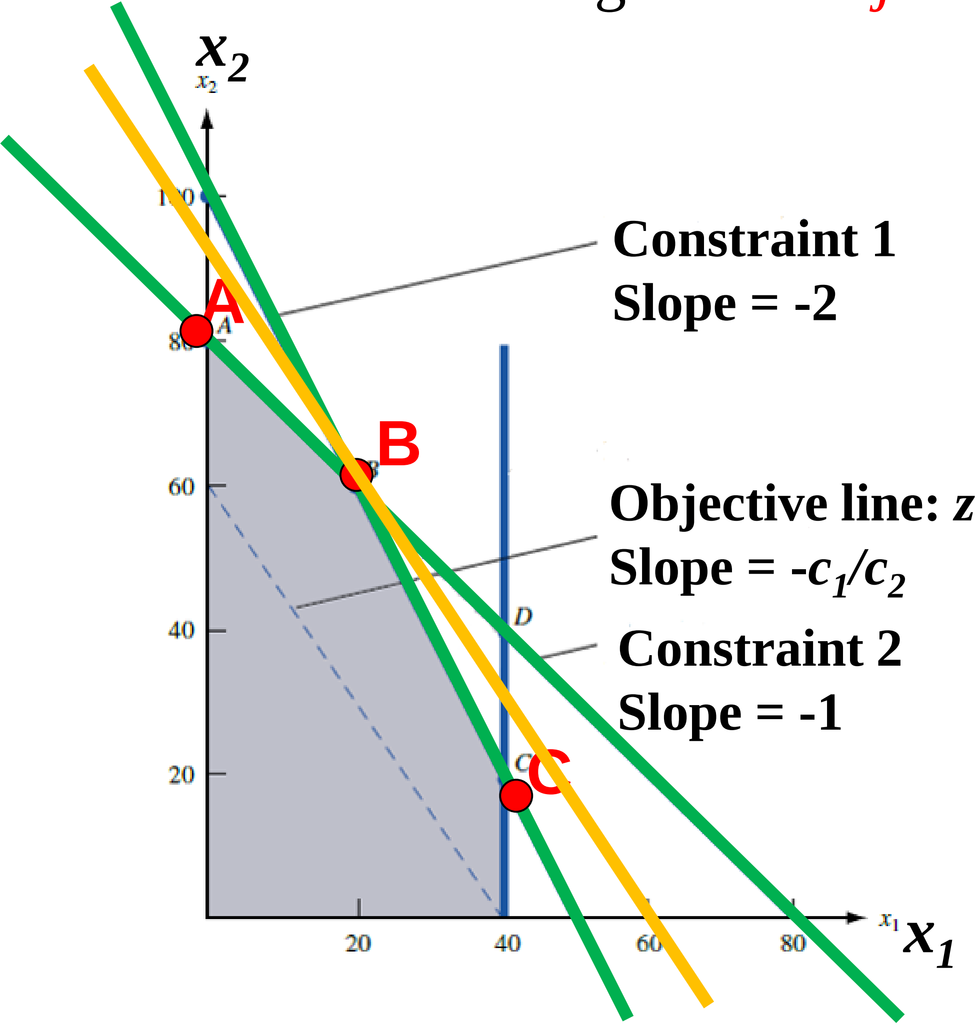
bv	x_1	x_2	x_3	x_4	x_5	b
x_2	0	1	-1	2	0	60
x_5	0	0	-1	1	1	20
x_1	1	0	1	-1	0	20
$-z$	0	0	-5	-4	0	-180

Optimal solution:

$$\begin{cases} x_1 = 20 \\ x_2 = 60 \\ z = 180 \end{cases}$$

Graphical Analysis of SA

- Effects of a change in an *objective function coefficient* c_1



$$z = c_1 x_1 + c_2 x_2$$

$$x_2 = \frac{z}{c_2} - \frac{c_1}{c_2} x_1$$

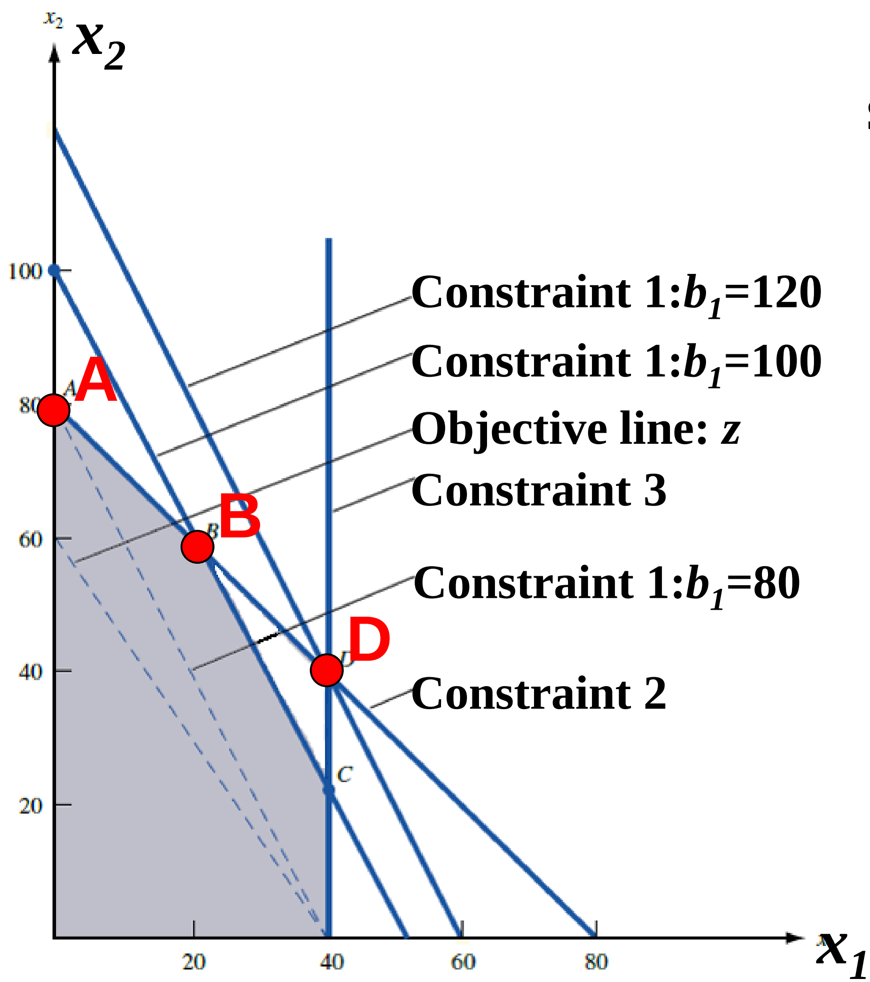
$$slope = -\frac{c_1}{c_2}$$

$$-2 \le -\frac{c_1}{c_2} \le -1$$

$$c_2 = 2, \text{ so } 2 \le c_1 \le 4$$

Graphical Analysis of SA (con't)

- Effects of a change in the *RHS*



$$\max \ z = 3x_1 + 2x_2$$

s.t.

$$2x_1 + x_2 \le 100 \qquad \text{(Constraint 1)}$$

$$x_1 + x_2 \le 80 \qquad \text{(Constraint 2)}$$

$$x_1 \le 40 \qquad \text{(Constraint 3)}$$

$$x_1, x_2 \ge 0$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$x_2 = \frac{b_1}{a_{12}} - \frac{a_{11}}{a_{12}} x_1$$

$$\text{For Constraint 1: } 80 \le \frac{b_1}{a_{12}} \le 120$$

$$\Rightarrow 80 \le b_1 \le 120$$

Shadow Price

- It is often important for managers to determine how a change in a constraint's RHS affects the LP's optimal *z*-value. With this in mind, we define the ***shadow price*** *for the change of optimal z-value with regard to a marginal change of b*.

$$\frac{\partial z}{\partial b} = \frac{\partial\left(C_B B^{-1} b\right)}{\partial b} = C_B B^{-1}$$

- This definition requires the current optimal solution to be still optimal. So, the definition of shadow price is related to the study of Sensitivity Analysis.

Shadow Price (con't)

- Effects of a change in the *RHS*

$$
\begin{aligned}
&\max\ z = 3x_1 + 2x_2 \\
\text{s.t.}\quad & 2x_1 + x_2 \le 100 \qquad \text{(Constraint 1)} \\
& x_1 + x_2 \le 80 \qquad \text{(Constraint 2)} \\
& x_1 \le 40 \qquad \text{(Constraint 3)} \\
& x_1, x_2 \ge 0
\end{aligned}
$$

- **For constraint 1:** suppose $b_1 = 100 + \Delta$, thus

At the optimal solution:

$$
\begin{cases} 2x_1 + x_2 = 100 + \Delta \\ x_1 + x_2 = 80 \end{cases} \Rightarrow \begin{cases} x_1 = 20 + \Delta \\ x_2 = 60 - \Delta \end{cases}
$$

$$
z = 3x_1 + 2x_2 = 3(20 + \Delta) + 2(60 - \Delta) = 180 + \Delta
$$

- As long as the current basis (vertex) remains optimal, a one unit increase in the number of available finishing hours will increase the optimal *z*-value by \$1.
- So the ***shadow price*** of the first constraint is \$1.

Shadow Price (con't)

- Effects of a change in the *RHS*
- **For constraint 2:** suppose $b_2 = 80 + \Delta$, thus

$$\begin{cases} 2x_1 + x_2 = 100 \\ x_1 + x_2 = 80 + \Delta \end{cases} \Rightarrow \begin{cases} x_1 = 20 - \Delta \\ x_2 = 60 + 2\Delta \end{cases}$$

$$z = 3x_1 + 2x_2 = 3(20 - \Delta) + 2(60 + 2\Delta) = 180 + \Delta$$

- The shadow price for Constraint 2 is $1.

- **For unbinding constraint 3 :** suppose $b_3 = 40 + \Delta$, thus

$$\begin{cases} 2x_1 + x_2 = 100 \\ x_1 + x_2 = 80 \end{cases} \Rightarrow \begin{cases} x_1 = 20 \\ x_2 = 60 \end{cases} \Rightarrow z = 3x_1 + 2x_2 = 180$$

- The shadow price for Constraint 3 is 0.

Matrix Analysis of SA

- Six types of changes in an LP's parameters can change the optimal solution
  - Changing the objective function coefficient of a nonbasic variable (in terms of the optimal solution)
  - Changing the objective function coefficient of a basic variable (in terms of the optimal solution)
  - Changing the RHS of a constraint
  - Changing the column of a nonbasic variable
  - Adding a new variable or activity
  - Adding a new constraint

Matrix Analysis of SA (For Max LP)

- Changing the objective function coefficient of a **nonbasic variable** x_j

Assume that the variation of c_j is Δc_j. The **optimal basis** is unchanged if :

$$\sigma_j' = (c_j + \Delta c_j) - C_B B^{-1} P_j$$
$$= c_j - C_B B^{-1} P_j + \Delta c_j = \sigma_j + \Delta c_j \le 0$$

$$\Rightarrow \quad \Delta c_j \le -\sigma_j$$

Δc_j is upper bounded.

$$\boxed{z = C_B B^{-1} b + (C_N - C_B B^{-1} N) x_N}$$

Matrix Analysis of SA (For Max LP)

- Changing the objective function coefficient of a **nonbasic variable**
  - As the increase of c_j, the contribution of the variable to the objective is increasing. When the increment is relatively large, the ***test number*** will be large than zero, resulting in the change of optimal basis;
  - When c_j is decreasing, the ***test number*** become smaller, the optimum basis will be unchanged.
  - Thus, the **range of the change** of c_j is

$$\boxed{-\infty \le \Delta c_j \le -\sigma_j}$$

Matrix Analysis of SA (For Max LP)

- Changing the objective function coefficient of a **nonbasic variable**
- **Economic interpretation**
  - c_j represents the price/profit of a product in the objective function (recall the example of Giapetto's Woodcarving).
  - The increase of c_j indicates that the contribution of the product increases.
  - If the increment of c_j is large enough, the company has to change their production plan because the current plan is not optimal any more.

Matrix Analysis of SA (For Max LP)

- Changing the objective function coefficient of a **basic variable** x_j

$$\boxed{z = C_B B^{-1} b + (C_N - C_B B^{-1} N) x_N}$$

The change of c_j will result in the change of $C_B B^{-1}$, which leads to the change of ***all test numbers***. The ***optimal basis*** is unchanged if

$$\sigma'_k = c_k - (C_B + \Delta C_B) B^{-1} P_k \le 0, k \in \Omega_N$$

Assume that c_j is the r *th* ***basic variable*** of the present basis,

$$\Delta c_j = (\Delta C_B)_r$$

$$\Delta C_B = (0, \ldots, (\Delta C_B)_r, \ldots, 0)$$
$$= (0, \ldots, \Delta c_j, \ldots, 0)$$

Matrix Analysis of SA (For Max LP)

- Changing the objective function coefficient of a **basic variable** x_j

$$\sigma_k^{'} = c_k - (C_B + \Delta C_B)B^{-1}P_k$$
$$= c_k - C_B B^{-1}P_k - (0,\ldots,\Delta c_j,\ldots,0)B^{-1}P_k$$
$$= \sigma_k - \Delta c_j (B^{-1}P_k)_r \le 0$$

Let $\alpha_{rk} = (B^{-1}P_k)_r$

$$\sigma_k^{'} = \sigma_k - \Delta c_j \alpha_{rk} \le 0, k \in \Omega_N$$

$$\boxed{z = C_B B^{-1} b + (C_N - C_B B^{-1} N) x_N}$$

Matrix Analysis of SA (For Max LP)

- Changing the objective function coefficient of a **basic variable**

$$\sigma_k' = \sigma_k - \Delta c_j \alpha_{rk} \le 0, k \in \Omega_N$$

- **Case 1:**

If $\Delta c_j \le 0$, iff $\alpha_{rk} > 0$, the above inequality could be **false**, thus,

$$0 \ge \Delta c_j \ge \max_k \left\{ -\infty \middle| \alpha_{rk} \le 0, \sigma_k / \alpha_{rk} \middle| \alpha_{rk} > 0 \right\}$$

- **Case 2:**

If $\Delta c_j > 0$, iff $\alpha_{rk} < 0$, the above inequality could be **false**, thus,

$$0 < \Delta c_j \le \min_k \left\{ +\infty \middle| \alpha_{rk} \ge 0, \sigma_k / \alpha_{rk} \middle| \alpha_{rk} < 0 \right\}$$

Matrix Analysis of SA (For Max LP)

- Changing the objective function coefficient of a **basic variable**

$$\sigma_k' = \sigma_k - \Delta c_j \alpha_{rk} \le 0$$

- Thus, the **range of the change** of c_j is

$$\boxed{\begin{aligned}\max_k\left\{-\infty \middle| \alpha_{rk} \le 0, \sigma_k / \alpha_{rk} \middle| \alpha_{rk} > 0\right\} \le \Delta c_j \\ \le \min_k\left\{+\infty \middle| \alpha_{rk} \ge 0, \sigma_k / \alpha_{rk} \middle| \alpha_{rk} < 0\right\}\end{aligned}}$$

- **Notes:**
  - The variation of the objective function is $\Delta z = \Delta c_j \cdot x_j$ where x_j takes the optimal solution.
  - The optimal basis and **the value of the optimal solution** keep **unchanged.**

Matrix Analysis of SA (For Max LP)

- **Example**

$$\max\ z = 50x_1 + 30x_2$$

- subject to

$$4x_1 + 3x_2 \le 120$$

$$2x_1 + x_2 \le 50$$

$$x_1 \ge 0, x_2 \ge 0$$

Final Tableau

c_B	bv	x_1	x_2	x_3	x_4	$B^{-1}b$
30	x_2	0	1	1	-2	20
50	x_1	1	0	-1/2	3/2	15
	$-z$	0	0	-5	-15	-1350

Matrix Analysis of SA (For Max LP)

$$\max_k\left\{-\infty\middle|\alpha_{rk}\le 0,\sigma_k/\alpha_{rk}\middle|\alpha_{rk}>0\right\}\le\Delta c_j$$
$$\le\min_k\left\{+\infty\middle|\alpha_{rk}\ge 0,\sigma_k/\alpha_{rk}\middle|\alpha_{rk}<0\right\}$$

- **Example**

Final Tableau

c_B	bv	x_1	x_2	x_3	x_4	$B^{-1}b$
30	x_2	0	1	1	-2	20
50	x_1	1	0	-1/2	3/2	15
	$-z$	0	0	-5	-15	-1350

- SA for c_1

x_1 locates at the 2nd row of the basis the subscript of non-basic variables are $k=3\ and\ 4$, $\alpha_{23}=-1/2,\alpha_{24}=3/2$

$$\max\{-\infty,-15/1.5\}\le\Delta c_1\le\min\{+\infty,-5/(-0.5)\}$$
$$-10\le\Delta c_1\le 10$$

Matrix Analysis of SA (For Max LP)

- **Example**

$$\max_k \left\{-\infty \middle| \alpha_{rk} \le 0, \sigma_k / \alpha_{rk} \middle| \alpha_{rk} > 0\right\} \le \Delta c_j \le \min_k \left\{+\infty \middle| \alpha_{rk} \ge 0, \sigma_k / \alpha_{rk} \middle| \alpha_{rk} < 0\right\}$$

Final Tableau

c_B	bv	x_1	x_2	x_3	x_4	$B^{-1}b$
30	x_2	0	1	1	-2	20
50	x_1	1	0	-1/2	3/2	15
	$-z$	0	0	-5	-15	-1350

- SA for c_2

x_2 locates at the 1st row of the basis the subscript of non-basic variables are $k = 3$ *and* 4, $\alpha_{13} = 1, \alpha_{14} = -2$

$$\max\{-\infty, -5/1\} \le \Delta c_2 \le \min\{+\infty, -15/(-2)\}$$
$$-5 \le \Delta c_2 \le 7.5$$

Matrix Analysis of SA (For Max LP)

- Changing the RHS of a constraint

Assume that only one term (i) in the RHS is changed

$$\Delta b = (0, \ldots, \Delta b_i, \ldots, 0)^T \qquad a_i x \le b_i$$

- **Case 1:** The i th constraint is ***non-binding:*** $a_i x < b_i$ or $a_i x > b_i$

Such a constraint would not affect the optimal solution iff it's always non-binding.

For $"\le"$: $a_i x \le b_i + \Delta b_i$ Still non-binding

$$\Rightarrow \quad a_i x - b_i \le \Delta b_i \le +\infty,$$

For $"\ge"$: $a_i x \ge b_i + \Delta b_i$

$$\Rightarrow \quad -\infty \le \Delta b_i \le a_i x - b_i$$

Matrix Analysis of SA (For Max LP)

- Changing the RHS of a constraint i
- **Case 2**: The i th constraint is ***binding (tight)***.

All the basis variable should satisfy:

$$x_B = B^{-1}(b + \Delta b)$$
$$= B^{-1}b + B^{-1}(0, \ldots, \Delta b_i, \ldots 0)^T \geq 0$$

We define a new notation here

$$\begin{bmatrix} \beta_{11} & \cdots & \beta_{1i} & \cdots & \beta_{1m} \\ \vdots & & \vdots & & \vdots \\ \beta_{k1} & \cdots & \beta_{ki} & \cdots & \beta_{km} \\ \vdots & & \vdots & & \vdots \\ \beta_{m1} & \cdots & \beta_{mi} & \cdots & \beta_{mm} \end{bmatrix} \begin{bmatrix} 0 \\ \ldots \\ \Delta b_i \\ \ldots \\ 0 \end{bmatrix} = \Delta b_i \begin{bmatrix} \beta_{1i} \\ \ldots \\ \beta_{ki} \\ \ldots \\ \beta_{mi} \end{bmatrix}$$

$$(x_B)_k = (B^{-1}b)_k + \boxed{\beta_{ki}} \Delta b_i \geq 0$$
$$k = 1, \ldots, m$$

Matrix Analysis of SA (For Max LP)

- Changing the RHS of a constraint
- **Case 2**: The i th constraint is ***binding***.

$$(x_B)_k = (B^{-1}b)_k + \beta_{ki}\Delta b_i \geq 0 \qquad k = 1, \ldots, m$$

- **Case 2.1:**

If $\Delta b_i < 0$, iff $\beta_{ki} > 0$, the above inequality could be **false**, thus,

$$\Delta b_i \geq \max\left\{-\infty \middle| \beta_{ki} \leq 0, -(B^{-1}b)_k / \beta_{ki} \middle| \beta_{ki} > 0\right\}$$

- **Case 2.2:**

If $\Delta b_i > 0$, iff $\beta_{ki} < 0$, the above inequality could be **false**, thus,

$$\Delta b_i \leq \min\left\{+\infty \middle| \beta_{ki} \geq 0, -(B^{-1}b)_k / \beta_{ki} \middle| \beta_{ki} < 0\right\}$$

Matrix Analysis of SA (For Max LP)

- **Example**

Final Tableau

c_B	bv	x_1	x_2	x_3	x_4	$B^{-1}b$	
30	x_2	0	1	1	-2	20	b_1
50	x_1	1	0	-1/2	3/2	15	b_2
	$-z$	0	0	-5	-15	-1350	

$$B = \begin{array}{cc} x_2 & x_1 \\ \end{array} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

$$B^{-1} = \begin{array}{cc} x_2 & x_1 \\ \end{array} \begin{bmatrix} 1 & -2 \\ -1/2 & 3/2 \end{bmatrix}$$

$$B^{-1}B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- **SA for** b_1

b_1 locates at the 1st column of B^{-1} : $\beta_{11} = 1, \beta_{21} = -0.5$

$$\max\{-\infty, -20/1\} \le \Delta b_1 \le \min\{+\infty, -15/(-0.5)\}$$

$$-20 \le \Delta b_1 \le 30$$

Matrix Analysis of SA (For Max LP)

- **Example**

Final Tableau

c_B	bv	x_1	x_2	x_3	x_4	$B^{-1}b$	
30	x_2	0	1	1	-2	20	b_1
50	x_1	1	0	-1/2	3/2	15	b_2
	$-z$	0	0	-5	-15	-1350	

$$B = \begin{array}{cc} x_2 & x_1 \\ \end{array} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

$$B^{-1} = \begin{array}{cc} x_2 & x_1 \\ \end{array} \begin{bmatrix} 1 & -2 \\ -1/2 & 3/2 \end{bmatrix}$$

$$B^{-1}B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- **SA for** b_2

b_2 locates at the 2nd column of B^{-1} : $\beta_{12} = -2, \beta_{22} = 1.5$

$$\max\{-\infty, -15/1.5\} \le \Delta b_2 \le \min\{+\infty, -20/(-2)\}$$

$$-10 \le \Delta b_2 \le 10$$

Duality Theory

Duality Theory

- What is the ***Dual*** ?
- Associated with any LP is another LP, called the ***dual***. Knowing the relation between an LP and its dual is vital to understanding advanced topics in linear and nonlinear programming. This relation is important because it gives us interesting economic insights.
- Knowledge of duality will also provide additional insights into sensitivity analysis.

Primal-Dual Relationships

Primal problem:

$$\max\ z = CX$$

$$\begin{cases} AX \le b \\ X \ge 0 \end{cases}$$

$$\max\ z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \le \begin{bmatrix} b_1 \\ \cdots \\ b_m \end{bmatrix}$$

$$x_1, x_2, \ldots, x_n \ge 0$$

Dual problem:

$$\min\ \omega = Yb$$

$$\begin{cases} YA \ge C \\ Y \ge 0 \end{cases}$$

$$\min\ \omega = y_1b_1 + y_2b_2 + \ldots + y_mb_n$$

$$(y_1, y_2, \ldots, y_m) \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix} \ge (c_1, c_2, \ldots, c_m)$$

$$y_1, y_2, \ldots, y_m \ge 0$$

<u>with a little abuse of notation…</u>

A Vitamin Example

- A family is trying to make a minimal cost diet from six available primary foods so that the diet contains at least 9 units of vitamin A and 19 units of vitamin C.

	NO.of Units of Nutrients per kg of food						Min Req.(in Units)
Nutrient	1	2	3	4	5	6	
vitA	1	0	2	2	1	2	9
vitB	0	1	3	1	3	2	19
cost of food (cts/kg)	35	30	60	50	27	22	

- LP formulation:

$$
\begin{array}{cccccccc}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & & \\
1 & 0 & 2 & 2 & 1 & 2 & \geq & 9 \\
0 & 1 & 3 & 1 & 3 & 2 & \geq & 19 \\
35 & 30 & 60 & 50 & 27 & 22 & = & z(x) \quad \text{minimize}
\end{array}
$$

A Vitamin Example (con't)

- Consider now a **manufacturer** venturing to make synthetic pills of each nutrient (Vitamins A&B) and to sell them to the family. It has to meet all the nutrient requirements by using the pills instead of the primary foods. The prices have to be competitive in order to induce the family to take on the pills.
- **There will be constraints on the prices that can be set.**
- Let π_1 and π_2 are the prices for vit A and B, respectively (cents/unit). Food 3 which has an intrinsic cost worth of $2\pi_1 + 3\pi_2$ while it costs 60 cents/kg. Unless $2\pi_1 + 3\pi_2 \leq 60$, <u>the family will conclude that the prices are not competitive enough</u>.
- **Moreover, they will need to maximize revenue from the sales for an equivalent minimum requirements the family would buy.**

A Vitamin Example (con't)

	NO.of Units of Nutrients per kg of food						Min Req.(in Units)
Nutrient	1	2	3	4	5	6	
vitA	1	0	2	2	1	2	9
vitB	0	1	3	1	3	2	19
cost of food (cts/kg)	35	30	60	50	27	22	

- Dual problem LP:

$$
\begin{array}{ccccc}
\pi_1 & \pi_2 & & & \\
1 & 0 & \leq & 35 & \\
0 & 1 & \leq & 30 & \\
2 & 3 & \leq & 60 & \\
2 & 1 & \leq & 50 & \\
1 & 3 & \leq & 27 & \\
2 & 2 & \leq & 22 & \\
9 & 19 & = & v(\pi) & \max
\end{array}
$$

A Vitamin Example (con't)

Primal

x_1	x_2	x_3	x_4	x_5	x_6			
1	0	2	2	1	2	$\geq$	9	
0	1	3	1	3	2	$\geq$	19	
35	30	60	50	27	22	$=$	$z(x)$	minimize

Dual

π_1	π_2			
1	0	$\leq$	35	
0	1	$\leq$	30	
2	3	$\leq$	60	
2	1	$\leq$	50	
1	3	$\leq$	27	
2	2	$\leq$	22	
9	19	$=$	$v(\pi)$	max

❑Characteristics：

- 1. Input-output coefficient tableau is transpose of the other；
- 2. Right-hand side constants are the coefficients of the objective function of the other；
- 3. Each column vector leads to a constraint of the other；
- 4. There is a variable corresponding to each constraint, and a constraint corresponding to each variable of the other；

A Vitamin Example (con't)

Primal

x_1	x_2	x_3	x_4	x_5	x_6			
1	0	2	2	1	2	$\geq$	9	
0	1	3	1	3	2	$\geq$	19	
35	30	60	50	27	22	=	$z(x)$	minimize

Dual

π_1	π_2			
1	0	$\leq$	35	
0	1	$\leq$	30	
2	3	$\leq$	60	
2	1	$\leq$	50	
1	3	$\leq$	27	
2	2	$\leq$	22	
9	19	=	$v(\pi)$	max

❑**Characteristics：**

- 5. Minimization problem with $\geq$ constraints, and maximization problem with $\leq$ constraints；
- 6. From the arguments of the family, they will not buy pills unless they provide the required nutrients just as cheap as the foods.

❑**Definition of Duality:** $\max\ v(\pi) \leq \min\ z(x)$

✓对同一事物（问题）从不同的角度（立场）观察，有两种**相对的**表述，例如买卖双方。**Same question, different stakeholders.**

Example

- Recall the example of Giapetto's Woodcarving
- Primal:

$$\max\ z = 3x_1 + 2x_2$$

$$\begin{aligned}
\text{s.t.}\quad & 2x_1 + x_2 \le 100 && \text{(Constraint 1:Finishing hour)}\\
& x_1 + x_2 \le 80 && \text{(Constraint 2:Carpentry hour)}\\
& x_1 \le 40 && \text{(Constraint 3:Demand)}\\
& x_1, x_2 \ge 0 &&
\end{aligned}$$

- Dual:

$$\min\ w = 100y_1 + 80y_2 + 40y_3$$

$$\begin{aligned}
\text{s.t.}\quad & 2y_1 + y_2 + y_3 \ge 3 && \text{(Constraint 1:Soldiers)}\\
& y_1 + y_2 \ge 2 && \text{(Constraint 2:Trains)}\\
& y_1, y_2, y_3 \ge 0 &&
\end{aligned}$$

Economic Interpretation of the Dual Problem

- Dual problem of Giapetto's Woodcarving

$$\min\ w = 100y_1 + 80y_2 + 40y_3$$

s.t.

$$2y_1 + y_2 + y_3 \geq 3 \quad \text{(Constraint 1:Soldiers)}$$

$$y_1 + y_2 \geq 2 \quad \text{(Constraint 2:Trains)}$$

$$y_1, y_2, y_3 \geq 0$$

- Now, an ***entrepreneur*** wants to purchase all of Giapetto's resources. Then the entrepreneur must determine the price he or she is willing to pay for a unit of Giapetto's resources. With this in mind, we define y_1 = price paid for 1finishing hour

  y_2 = price paid for 1 carpentry hour

  y_3 = price paid for additional demand

Economic Interpretation of the Dual Problem

- **Objective function**

The resource y_1, y_2, y_3 should be determined by solving the dual problem. The total price that should be paid for these resources is $100y_1 + 80y_2 + 40y_3$. Since the cost of purchasing the resources is to be minimized,

$$\min\ w = 100y_1 + 80y_2 + 40y_3$$

is the objective function for the dual.

- **Constraints**

The entrepreneur must offer Giapetto at least \$3 for a combination of resources that includes 2 finishing hours, 1 carpentry hours and 1 additional demand, because Giapetto could, if it desires, *use there resources to produce a soldier that can be sold for \$3*.

Economic Interpretation of the Dual Problem

- **Constraints**

Since the entrepreneur is offering $2y_1 + y_2 + y_3$ for the resources used to produce a soldier, he or she may choose y_1, y_2, y_3 to satisfy

$$2y_1 + y_2 + y_3 \geq 3 \quad \text{(Constraint 1:Soldiers)}$$

Similarly, the second dual constraint,

$$y_1 + y_2 \geq 2 \qquad \text{(Constraint 2:Trains)}$$

The sign restrictions must also hold,

$$y_1, y_2, y_3 \geq 0$$

- When the primal is a normal max problem, the dual variables are related to the value of the resources available to the decision maker. For this reason, the dual variables are often referred to as resource ***shadow prices***.

Economic Interpretation of the Dual Problem

- Dual problem of Giapetto's Woodcarving

$$\min \ w = 100y_1 + 80y_2 + 40y_3$$

s.t.

$$2y_1 + y_2 + y_3 \geq 3 \quad \text{(Constraint 1:Soldiers)}$$

$$y_1 + y_2 \geq 2 \quad \text{(Constraint 2:Trains)}$$

$$y_1, y_2, y_3 \geq 0$$

- Optimal solution:

$$\boxed{\begin{aligned} y_1 &= 1 \\ y_2 &= 1 \\ y_3 &= 0 \end{aligned}}$$

$$\boxed{w = 180}$$

- It is clear that the shadow prices can be obtained from the ***sensitivity analysis of RHS*** and the ***optimal solution of dual problem***.
- The objective values of primal and dual problem are the same.

Characteristics of duality

	x_1 x_2 $\cdots$ x_n	Primal	$\min\ \omega$
y_1 $\cdots$ y_m	a_{11} a_{12} $\cdots$ a_{1n} $\cdots$ $\cdots$ $\cdots$ a_{m1} a_{m2} $\cdots$ a_{mm}	$\leq$ $\cdots$ $\leq$	b_1 $\cdots$ b_m
Dual	$\geq$ $\geq$ $\cdots$ $\geq$	$\max\ z = \min\ \omega$	
$\max\ z$	c_1 c_2 $\cdots$ c_n		

What about the equalities? See below

Characteristics of duality -- a special case with equations

$$\max\ z = \sum_{j=1}^{n} c_j x_j$$

$$\begin{cases} \sum_{j=1}^{n} a_{ij} x_j = b_i, \ i = 1, 2, \ldots, m \\ x_j \geq 0, \qquad j = 1, 2, \ldots, n \end{cases}$$

- ***Step 1:*** Convert the equality constraints into inequality constraints

$$\begin{cases} \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \ i = 1, 2, \ldots, m \\ -\sum_{j=1}^{n} a_{ij} x_j \leq -b_i, \ i = 1, 2, \ldots, m \\ x_j \geq 0, \qquad j = 1, 2, \ldots, n \end{cases}$$

Characteristics of duality-a special case with equations

- ***Step 2:*** Obtain the dual problem

$$\min\ \omega = \sum_{i=1}^{m} b_j y_j^{'} + \sum_{i=1}^{m} (-b_j y_j^{''})$$

$$\begin{cases} \sum_{i=1}^{m} a_{ij} y_i^{'} + \sum_{i=1}^{m} (-a_{ij} y_i^{''}) \geq c_j, & j = 1, 2, \ldots, n \\ y_i^{'}, y_i^{''} \geq 0, & i = 1, 2, \ldots, m \end{cases}$$

$y_i^{'}$ *and* $y_i^{''}$ are the dual variables for the inequality constraints, respectively.

$$\min\ \omega = \sum_{i=1}^{m} b_i (y_i^{'} - y_i^{''})$$

$$\sum_{i=1}^{m} a_{ij} (y_i^{'} - y_i^{''}) \geq c_j, \quad j = 1, 2, \ldots, n$$

Characteristics of duality-a special case with equations

- ***Step 2:*** Obtain the dual problem

$$\max\ \omega = \sum_{i=1}^{m} b_i (y_i^{'} - y_i^{''})$$

$$\sum_{i=1}^{m} a_{ij} (y_i^{'} - y_i^{''}) \geq c_j,\ \ j = 1, 2, \ldots, n$$

Let $y_i = y_i^{'} - y_i^{''}$, $y_i^{'}, y_i^{''} \geq 0$. It is obvious that y_i is unrestricted.

Thus, the dual problem can be rewritten as:

$$\max\ \omega = \sum_{i=1}^{m} b_i y_i$$

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j,\ \ j = 1, 2, \ldots, n$$

$$y_i \ \ is\ unrestricted.$$

Fundamental Theorems of duality

- **Equivalence Theorem**
- **Weak Duality Theorem**
- **Corollaries of the Weak Duality Theorem**
  - Pay attention to the lower- and upper-bound of P and D.
- **Strong Duality Theorem**
- **Complementary Slackness Theorem**

Fundamental properties (con't)

- **Duals of equivalent problems are equivalent**

Let P be the LP and D the dual.

If P' is the equivalent to P, and D' is its dual,

then D' is equivalent to D.

$$\max\ z = CX \quad \begin{cases} AX \le b \\ X \ge 0 \end{cases}$$

(P)

$$\min\ \omega = Yb \quad \begin{cases} YA \ge C \\ Y \ge 0 \end{cases}$$

(D)

equivalent

$$\max\ (-\omega) = -Yb \quad \begin{cases} -YA \le -C \\ Y \ge 0 \end{cases}$$

(D')

equivalent

(P)

$$\min\ (-\omega') = -CX \quad \begin{cases} -AX \ge -b \\ X \ge 0 \end{cases}$$

(D)

Fundamental properties (con't)

Weak Duality Theorem

- In a primal-dual pair of LPs $z(\bar{X}) \le \omega(\bar{Y})$

where $\bar{X}$ and $\bar{Y}$ being the **primal and dual** ***feasible solutions***.

Proof.

$$A\bar{X} \le b \Rightarrow \bar{Y}A\bar{X} \le \bar{Y}b$$

(D) $\min\ \omega = Yb$

$$\begin{cases} YA \ge C \\ Y \ge 0 \end{cases} \Rightarrow \bar{Y}A \ge C \Rightarrow \bar{Y}A\bar{X} \ge C\bar{X}$$

$$C\bar{X} \le \bar{Y}A\bar{X} \le \bar{Y}b$$

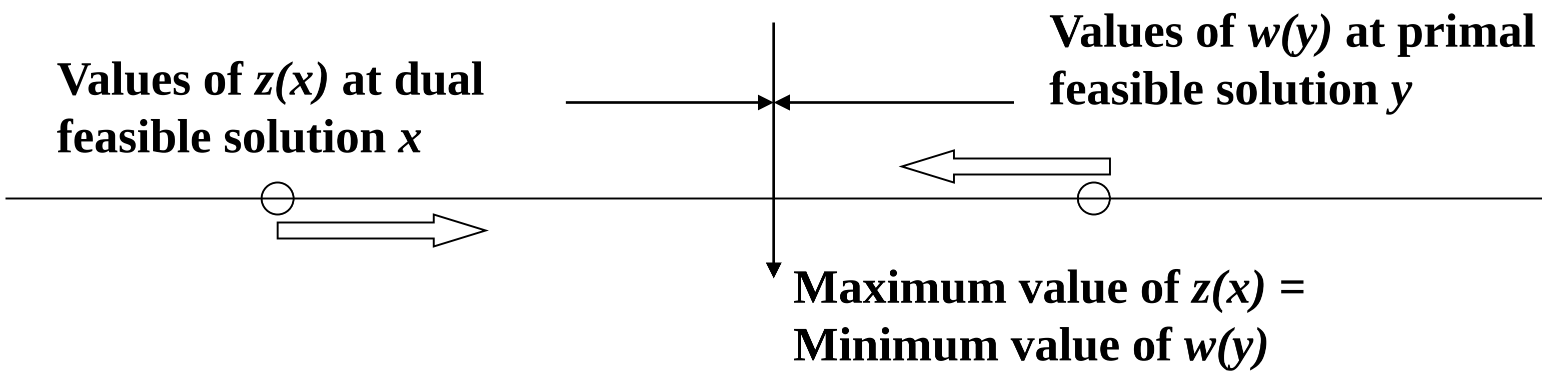
$$C\bar{X} \le \bar{Y}b$$

Fundamental properties (con't)

- **Corollaries of the Weak Duality Theorem**

1.The objective value of any primal feasible solution (max) is the **lower bound** to the optimal objective value of the dual problem.

2.Conversely, the objective value of any dual solution (min) is the **upper bound** to the optimal objective value of primal problem.



Fundamental properties (con't)

- **Corollaries of the Weak Duality Theorem**

If the primal problem is feasible and its objective is unbounded, then the dual problem does not have any feasible solution;

- Note that the above condition is not "vice versa", namely, if the primal problem has no feasible solution, then the dual problem may have unbounded solution or has no feasible solution.

If **P** is unbounded, **D** has no feasible solution;
If **D** is unbounded, **P** has no feasible solution;
If **P** has no feasible solution, **D** is unbounded **or** has no feasible solution.

Fundamental properties (con't)

- **Strong Duality Theorem**

If either the primal or the dual problem has an optimal solution, then the other one also has optimal solution; and the two optimal objective values are equal.

$$\max\ z = CX \qquad\qquad \min\ \omega = Yb$$

$$\begin{cases} AX \le b \\ X \ge 0 \end{cases} \Longrightarrow \begin{cases} YA \ge C \\ Y \ge 0 \end{cases}$$

$$CX \le YAX \le Yb$$

$$z^* = \omega^*$$

$$z^* = C_B B^{-1} b$$

$$\omega^* = Y^* b$$

$$\Longrightarrow \quad Y^* = C_B B^{-1}$$

<u>Shadow Price!</u>

Fundamental properties (con't)

- **Complementary Slackness Theorem/Condition**

X and Y are optimal solutions to the prime and dual problem iff:

Constraints of D Constraints of P

$$\begin{cases} (C-YA)X = 0 \\ Y(b-AX) = 0 \end{cases} \Longleftrightarrow \begin{cases} (c_j - y_j a_j)x_j = 0, & 1 \le j \le n \\ y_i(b_i - a_i x_i) = 0, & 1 \le i \le m \end{cases}$$

Solution of D Solution of P

$$\Updownarrow$$

$$\begin{cases} v_j x_j = 0 \\ u_i y_i = 0 \end{cases} \quad v_j \text{ and } u_i \text{ are slacks}$$

- Both the slacks and variables can be 0 at the same time
- The unrestricted variables are not included (equality constraints in its dual problem)

Corollary:

Variable > 0, then the associated constraint is tight.

Fundamental properties (con’t)

- **Use the Complementary Slackness Theorem to solve a LP**

Return to the Vitamin example

x_1	x_2	x_3	x_4	x_5	x_6			
1	0	2	2	1	2	$\geq$	9	
0	1	3	1	3	2	$\geq$	19	
35	30	60	50	27	22	$=$	$z(x)$	minimize

Optimal solution:

$$x^* = (0,0,0,0,5,2)^T$$

Applying complementary slackness property:

$$\begin{cases} (\pi_1 + 3\pi_2 - 27)\cdot x_5 = 0 \\ (2\pi_1 + 2\pi_2 - 22)\cdot x_6 = 0 \end{cases}$$

Since x_5 and x_6 are positive, the constraints 5 and 6 of the dual are tight (no slack)

$$\begin{cases} \pi_1 + 3\pi_2 - 27 = 0 \\ 2\pi_1 + 2\pi_2 - 22 = 0 \end{cases} \Rightarrow \begin{cases} \pi_1 + 3\pi_2 = 27 \\ 2\pi_1 + 2\pi_2 = 22 \end{cases} \Rightarrow \begin{cases} \pi_1 = 3 \\ \pi_1 = 8 \end{cases}$$

Is this dual feasible?

The Dual Simplex Method (Reading materials)

The Dual Simplex Method

- **Recall the Strong Duality Theorem and**
- **the structure of Simplex tableau**

P:

$$\max \ z = CX$$

$$\begin{cases} AX \le b \\ X \ge 0 \end{cases}$$

$\Rightarrow$

D:

$$\min \ \omega = Yb$$

$$\begin{cases} YA \ge C \\ Y \ge 0 \end{cases}$$

$$z^* = \omega^*$$

$$z^* = C_B B^{-1} b$$

$$\omega^* = Y^* b$$

$$\Downarrow$$

$$Y^* = C_B B^{-1}$$

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Summary

$B^{-1}I = B^{-1}$

The structure of Simplex tableau

✓Divided by structure and slack variables: $\bar{A} = (A, I)$

	X_A	X_S	RHS
Coef. Matrix	$B^{-1}A$	B^{-1}	$B^{-1}b$
σ_j	$C_A - C_B B^{-1} A$	$-C_B B^{-1}$	$-C_B B^{-1} b$

$0 - C_B B^{-1} I$

$C = (C_B, C_N)$

$\bar{A} = (B, N)$

Remember this term!

- Dual solution
- Simplex Multiplier
- Shadow Price
- ……

The Dual Simplex Method

- **The structure of Simplex tableau**

	X_A	X_S	RHS
Coef. Matrix	$B^{-1}A$	B^{-1}	$B^{-1}b$
σ_j	$C_A - C_B B^{-1}A$	$-C_B B^{-1}$	$-C_B B^{-1}b$

P solution

D solution

- The Primal Simplex Method:
  - Keep **P feasible** solution, try to find the **D feasible** solution
- The Dual Simplex Method:
  - Keep **D feasible** solution, try to find the **P feasible** solution
- **P feasible:** $B^{-1}b \geq 0$
- **D feasible:** $-C_B B^{-1} \leq 0$

The Dual Simplex Method

- **Primal Simplex Method**
- When we use the simplex method to solve a max problem
  - We begin with a **P feasible** solution (each constraint in the initial tableau has a nonnegative RHS).
  - At least one of the test number of the initial tableau has a positive coefficient, so our initial **P solution** is not **D feasible**.
  - Through a sequence of simplex pivots, we maintain **P feasibility** and obtain an optimal solution when **D feasibility** (non-positive test numbers) is attained.

The Dual Simplex Method

Dual Simplex Method?

- However, in many situations, it is easier to solve an LP by beginning with a tableau in which each test number is non-positive (so the tableau is **D feasible**) and at least one constraint has a negative right-hand side (so the tableau is **P infeasible**).
- The **Dual Simplex Method** maintains the non-positive test number (**D feasible**) and eventually obtains a tableau in which each RHS is non-negative (**P feasible**).
- At this point, an optimal tableau has been obtained. Because this technique maintains dual feasibility, it is called the **Dual Simplex Method**.

The Dual Simplex Method

- **Algorithmic Framework**
- Step 1: Check the **P** and **D feasibility**.
  - Is the test number of each variable non-positive? And is the RHS of each constraint non-negative? If so, an optimal solution has been found; if not, at least one constraint has a negative RHS, and we go to Step 2.
- Step 2: Find the **leaving variable**.
  - Choose the most negative basic variable as the **leaving variable**.

$$r = \arg\min_i \{(B^{-1}b)_i \,\big|\, (B^{-1}b)_i < 0\}$$

  - We choose x_r as the leaving variable.

The Dual Simplex Method

- **Algorithmic Framework**
- Step 3: Find the **entering variable**.
  - If all the elements in the basis of **row r** is non-negative, the LP has no feasible solution.
  - Otherwise, perform the following **Ratio test**

$$\theta = \min_j \left\{ \frac{\sigma_j}{a_{rj}} \middle| a_{rj} < 0 \right\} = \frac{\sigma_j}{a_{rs}},\ j \in \Omega_N$$

Why Min?

  - Choose the corresponding variable x_s as the entering variable, and the a_{rs} as the pivot term.
- Step 4: Perform the elementary row operations; Return to Step 1.

The Dual Simplex Method

- **Algorithmic Framework**
- **Remarks 1**
- Assume that the entering variable is x_s
- The new test numbers are

$$\sigma_j' = \sigma_j - \frac{a_{rj}}{a_{rs}}\sigma_s$$

In order to maintain **D feasible**, we have $\sigma_j' = \sigma_j - \frac{a_{rj}}{a_{rs}}\sigma_s \le 0$

$$\sigma_j' = \sigma_j - \frac{a_{rj}}{a_{rs}}\sigma_s = a_{rj} \cdot \left(\frac{\sigma_j}{a_{rj}} - \frac{\sigma_s}{a_{rs}} \right) \le 0$$

The Dual Simplex Method

- **Algorithmic Framework**
- **Remarks 1**

$$\sigma_j' = \sigma_j - \frac{a_{rj}}{a_{rs}}\sigma_s = a_{rj} \cdot \left(\frac{\sigma_j}{a_{rj}} - \frac{\sigma_s}{a_{rs}} \right) \le 0$$

- If $a_{rj} \ge 0$, the above inequality is always true.

- If $a_{rj} < 0$, the above inequality is true if and only if

$$\frac{\sigma_s}{a_{rs}} \le \frac{\sigma_j}{a_{rj}}$$

- Thus,

$$s = \arg\min_j \left\{ \frac{\sigma_j}{a_{rj}} \middle| a_{rj} < 0 \right\}, j \in \Omega_N$$

The Dual Simplex Method

- **Algorithmic Framework**
- **Remarks 2**

Why we said **"If all the elements in the basis of row r is non-negative, the LP has no feasible solution."**

- For **row r**

$$x_r = (B^{-1}b)_r - \sum_{j \in \Omega_N}^{n-m} a_{rj} x_j$$

- For the current solution, all the $x_j (j \in \Omega_N)$ are non-basic variable, and the values of x_j are zero, then $x_r = (B^{-1}b)_r < 0$.
- If all the $a_{rj} \geq 0$, the x_r cannot become non-negative no matter how x_j change from zero. Thus, this LP has no feasible solution.

The Dual Simplex Method

- **Example**

$$\max\ z = 2x_1 + x_2$$

s.t.

$$x_1 + x_2 + x_3 = 5$$
$$2x_2 + x_3 \le 5$$
$$4x_2 + 6x_3 \ge 9$$
$$x_1, x_2, x_3 \ge 0$$

$-z$	2	1	0	0	0	0

$-z$	0	-1	-2	0	0	10

bv	x_1	x_2	x_3	x_4	x_5	b
x_1	1	1	1	0	0	5
x_4	0	2	1	1	0	5
x_5	0	-4	-6	0	1	-9
σ	0	-1	-2	0	0	10

D feasible

P infeasible

The Dual Simplex Method

- **Example**

bv	x_1	x_2	x_3	x_4	x_5	b
x_1	1	1	1	0	0	5
x_4	0	2	1	1	0	5
x_5	0	-4	-6	0	1	-9
σ	0	-1	-2	0	0	10
θ		1/4	1/3			

Pivot

$$r = \arg\min_i \{(B^{-1}b)_i \mid (B^{-1}b)_i < 0\}$$

$$s = \arg\min_j \left\{ \frac{\sigma_j}{a_{rj}} \middle| a_{rj} < 0 \right\}$$

The Dual Simplex Method

- **Example**

bv	x_1	x_2	x_3	x_4	x_5	b
x_1	1	0	-1/2	0	1/4	11/4
x_4	0	0	-2	1	1/2	1/2
x_2	0	1	3/2	0	1/2	9/4
σ	0	0	-1/2	0	-1/4	31/4

P feasible

D feasible

The Dual Simplex Method

- **When we use the Dual Simplex Method?**
- **Reason 1:**

Recall that **What should we do if the LP has ≥ constraints?**

Change the ≥ to ≤ and introduce the artificial variable.

Example:

$$\max\ z = 2x_1 + x_2$$

$$\text{s.t.}\quad x_1 + x_2 + x_3 = 5$$

$$2x_2 + x_3 \le 5$$

$$4x_2 + 6x_3 \ge 9$$

$$x_1, x_2, x_3 \ge 0$$

bv	x_1	x_2	x_3	x_4	x_5	b
x_1	1	1	1	0	0	5
x_4	0	2	1	1	0	5
x_5	0	-4	-6	0	1	-9
σ	0	-1	-2	0	0	0

Then, use the **Modified Simplex Method** or **Dual Simplex Method**

The Dual Simplex Method

Reason 2: Calculation Simplication

- For the **LP with larger number of variables than constraints** **(e.g. too many equation constraints)**, the Dual Simplex method can efficiently reduce the amount of calculation.
- For the **LP with larger number of constraints than variables**, we can **change the P into D**, and then use the Dual Simplex method.