Modelling Transportation Systems

Nonlinear Programming Models for Traffic Assignment

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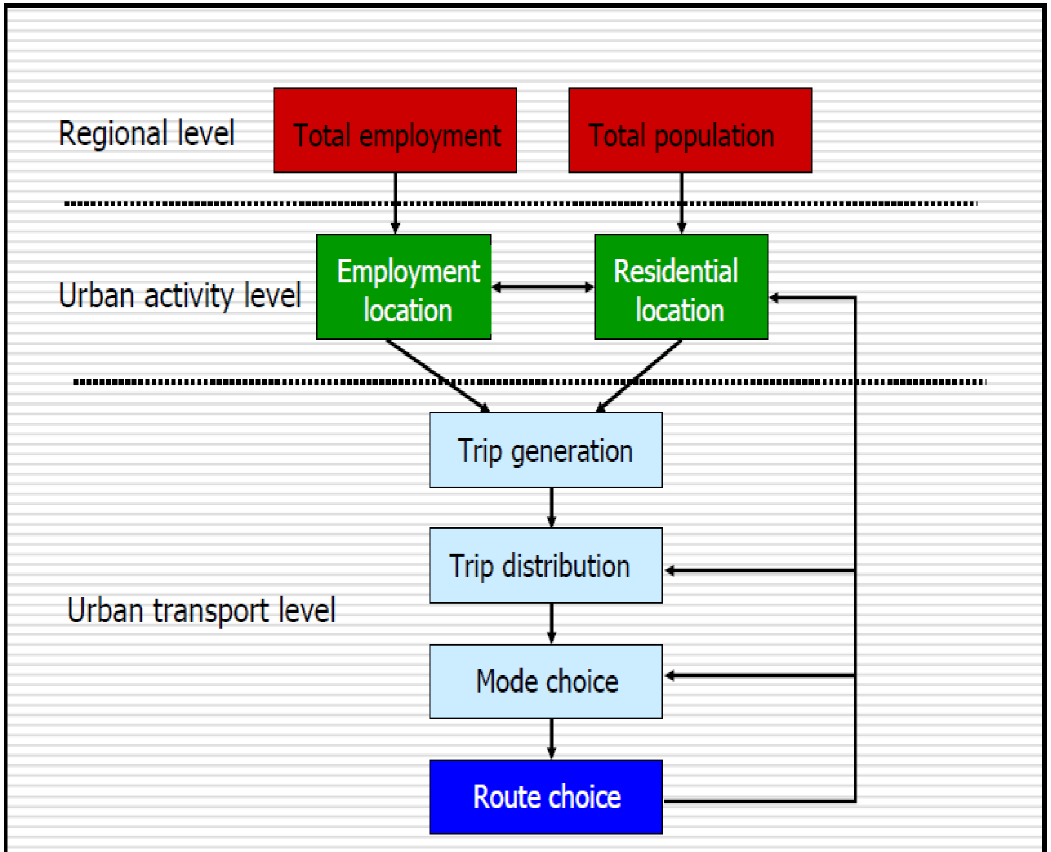
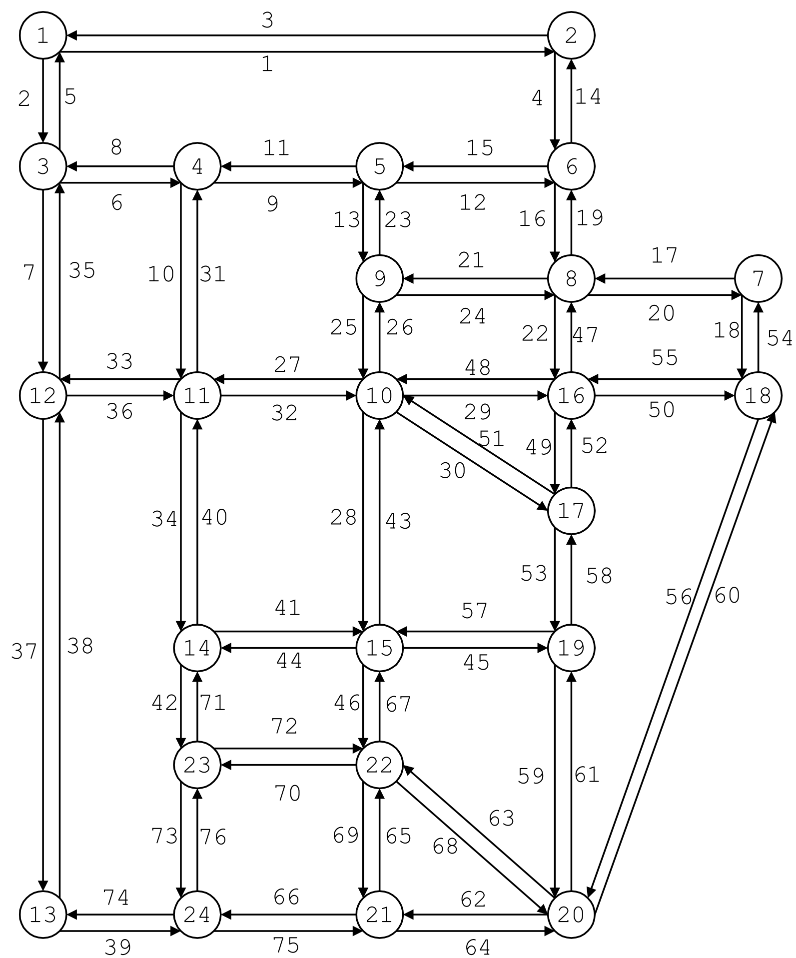
Preliminary Knowledge: Four-step method

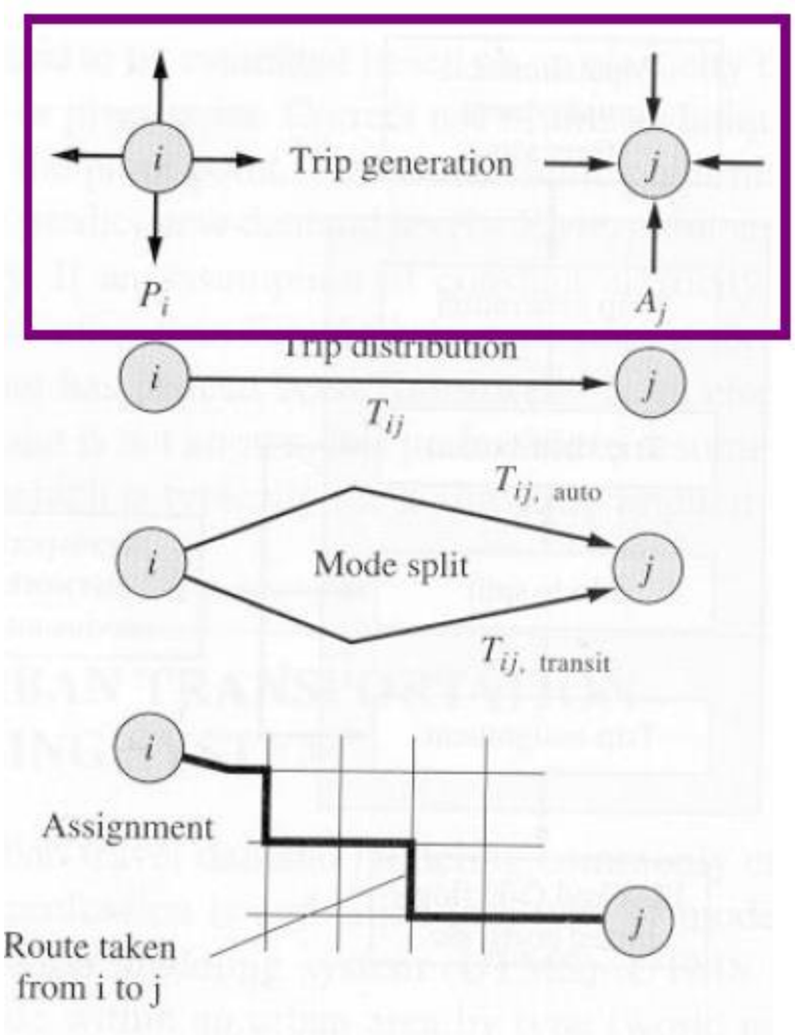
- Trip generation, Trip distribution, Mode choice, Route choice



Framework of Land-Use & Transportation Planning



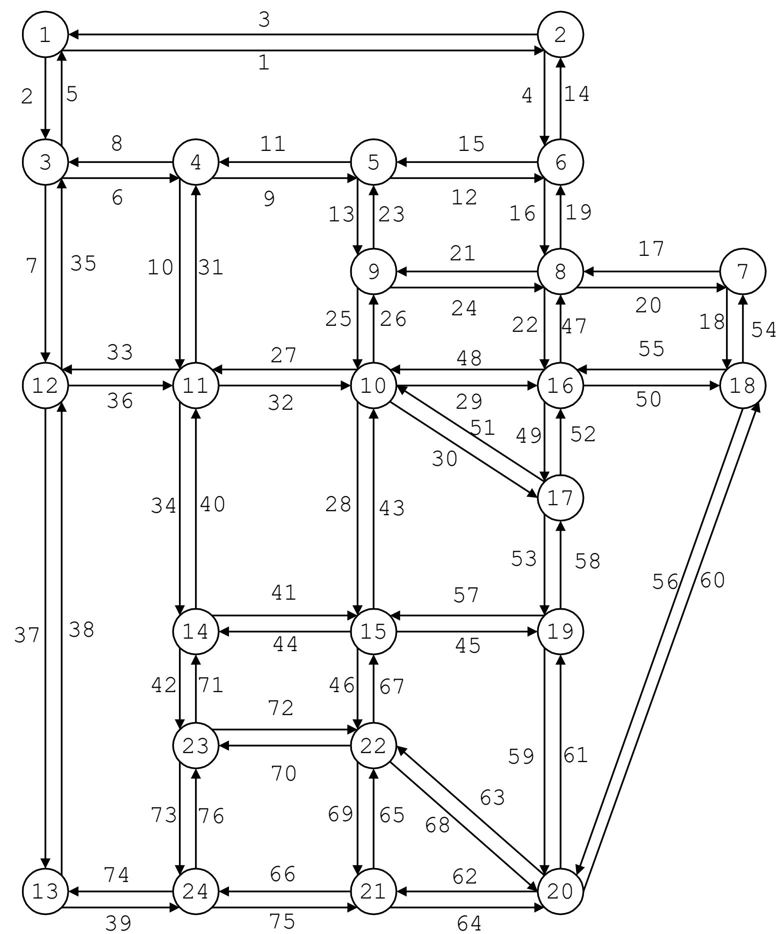
Preliminary Knowledge: Four-step method



1. Output: P/A table
2. Input: P/A table, distance matrix
   Output: OD table
3. Output: Multimodal OD table
4. Output: path/link traffic flow

Route Choice (Traffic Assignment)

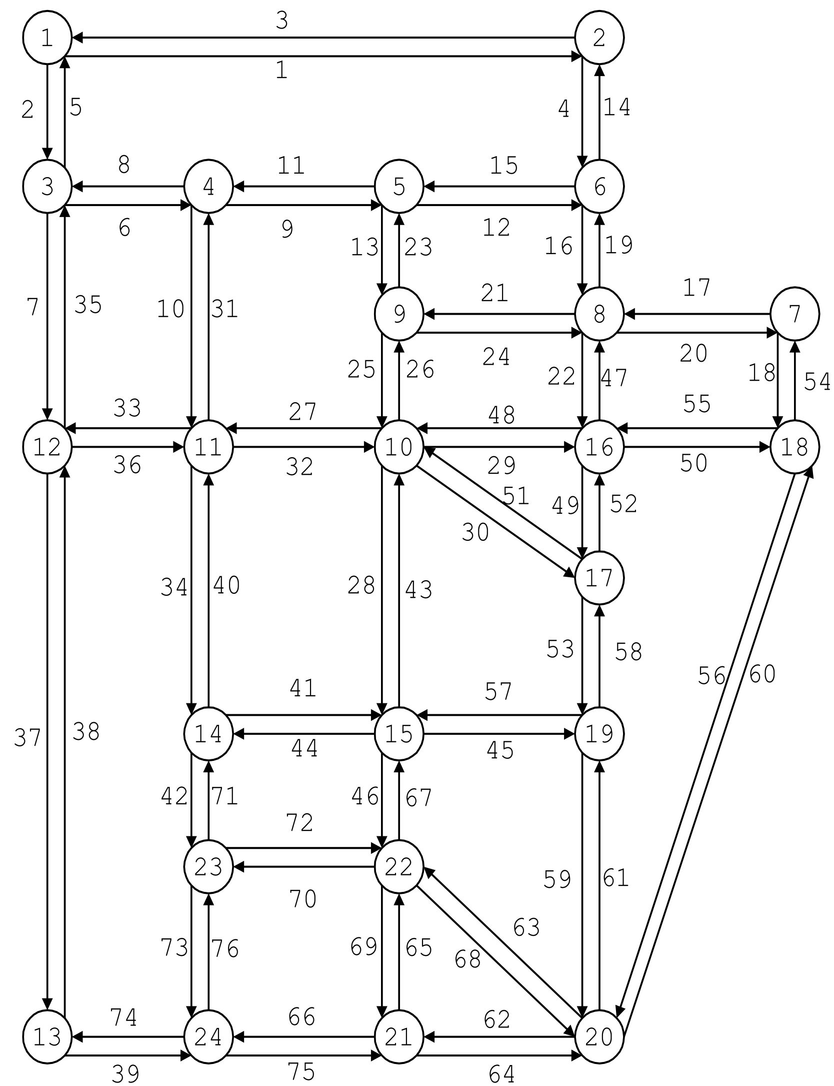
- Route Choice
  - Multiple OD demand
    - 1 → 18: 5000 Passenger-car-unit (pcu)/hour
    - 2 → 13: 4000 pcu/h
    - 10 → 20: 8000 pcu/h
  - For one OD, several routes could be chosen. How will the user choose?
  - For each link, the flow of different OD pairs will be loaded, to see the congestion level.



This is a Route Choice problem.

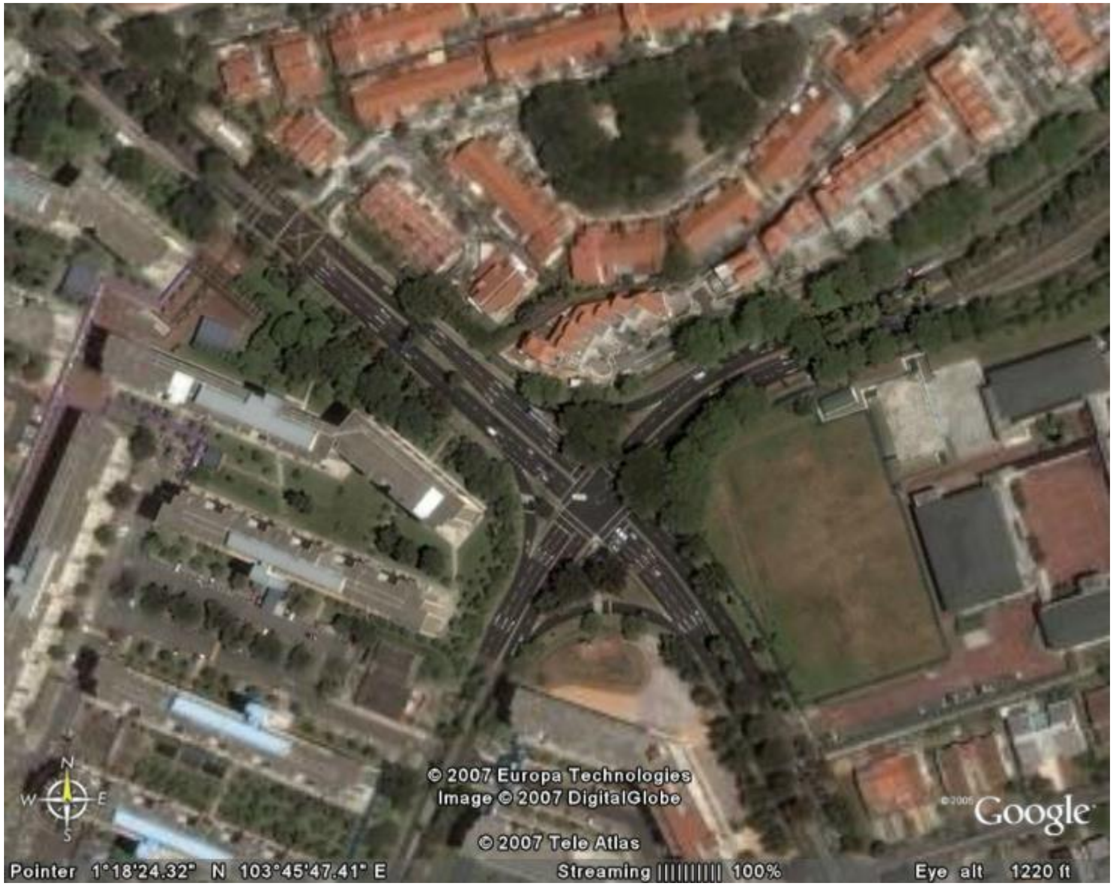
Notations of a transportation network

- Node (or vertices or points)
  - Road intersections
- Link (or arc or edge)
  - Connecting two nodes (head and tail)
  - Directed or undirected
- Path (or route)
  - Connecting a pair of OD nodes
  - Cyclic or acyclic



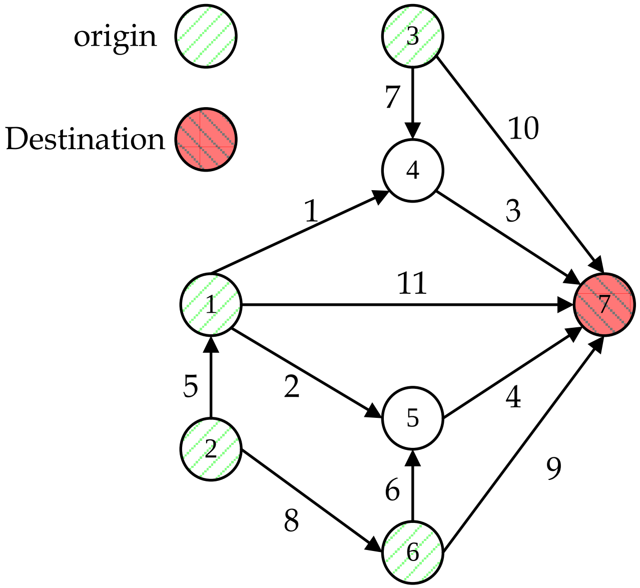
Network Representation

- Graph G=(N,A)
  - N: set of nodes
  - A: set of links
- Two subsets of set N
  - R: set of origins
  - S: set of destinations
- K_{rs}: set of paths from origin r to destination s



Assumptions of Route Choice

- Assumptions of the travel behaviours of Network users (drivers, commuters, travellers), for the "demand" side
  - Will choose the shortest path
  - Have perfect information about the link/path travel time
  - Not cooperative with other users, only concerns about their own travel time



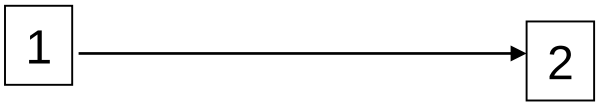
Assumptions of Route Choice

- Assumptions of the road network (supply side)
  - Link flows are continuous values
  - Link travel time is a strictly increasing function of its own link flow
    - BPR (Bureau of Public Roads) function

$$t(x) = t_0 \times \left(1 + \alpha \left(x/c \right)^{\beta} \right)$$

    - t: link travel time
    - t_0: free flow travel time
    - x: link flow (or use v to indicate link flows)
    - c: link capacity

α, β are parameters (usually $\alpha = 0.15, \beta = 4$)



Assumptions of Route Choice

❑Assumptions of the road network (supply side)

- Focus on the peak hour only (static network, not dynamic network), and the total OD demand q_{rs} is not changing.
- Flow conservation equation:
  - Path flow: $f_k^{rs} \geq 0$
  - Path-flow and OD demand incidence relationship

$$q_{rs} = \sum_{k \in K_{rs}} f_k^{rs}$$

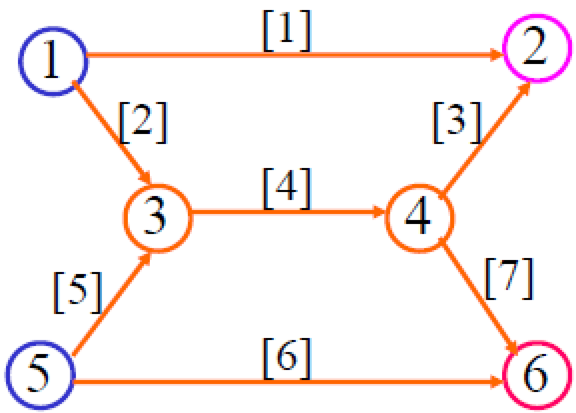
  - Path and link flow incidence relationship
  - $$x_a = \sum_{r \in R} \sum_{s \in S} \sum_{k \in K_{rs}} f_k^{rs} \delta_{ak}^{rs}$$

- Path travel time

$$c_k^{rs} = \sum_{a \in A} t_a \left(x_a \right) \delta_{ak}^{rs}$$

An Example for Flow Conservation

❑ **Example**

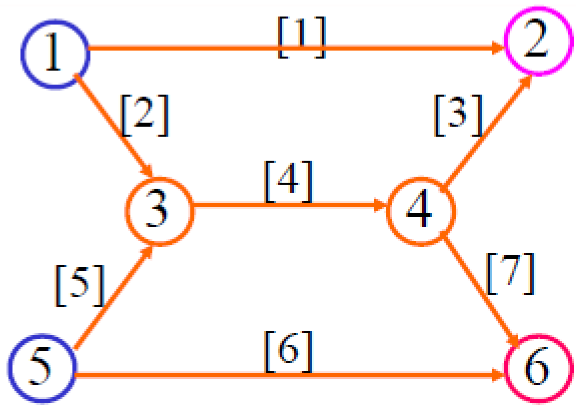


- Set of nodes: N={1,2,3,4,5,6}
- Set of links: A={1,2,3,4,5,6,7}

Link No	Start Node	End Node
1	1	2
2	1	3
3	4	2
4	3	4
5	5	3
6	5	6
7	4	6

- Sets of origins: R={1,5}
- Sets of destinations: S={2,6}

An Example for Flow Conservation



- Set of path between O-D pair 1-2

$K_{12}=\{1,2\}$:

Path 1: $1 \xrightarrow{[1]} 2$

Path 2: $1 \xrightarrow{[2]} 3 \xrightarrow{[4]} 4 \xrightarrow{[3]} 2$

- Set of path between O-D pair 1-6

$K_{16}=\{1\}$:

Path 1: $1 \xrightarrow{[2]} 3 \xrightarrow{[4]} 4 \xrightarrow{[7]} 6$

- Set of path between O-D pair 5-2

$K_{52}=\{1\}$:

Path 1: $5 \xrightarrow{[5]} 3 \xrightarrow{[4]} 4 \xrightarrow{[3]} 2$

- Set of path between O-D pair 5-6

$K_{56}=\{1,2\}$:

Path 1: $5 \xrightarrow{[6]} 6$

Path 2: $5 \xrightarrow{[5]} 3 \xrightarrow{[4]} 4 \xrightarrow{[7]} 6$

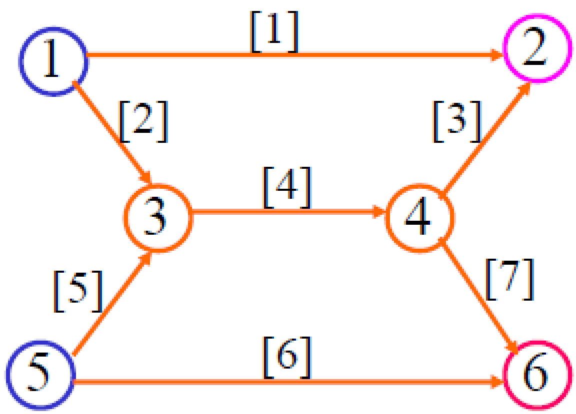
An Example for Flow Conservation

- **How to Represent Link Using Feasible Path Flow?**

x_a : Traffic flow on link $a \in A$

$$\begin{cases} x_1 = f_1^{12} \\ x_2 = f_2^{12} + f_1^{16} \\ x_3 = f_2^{12} + f_1^{52} \\ x_4 = f_2^{12} + f_2^{56} + f_1^{16} + f_1^{52} \\ x_5 = f_2^{56} + f_1^{52} \\ x_6 = f_1^{56} \\ x_7 = f_2^{56} + f_1^{16} \end{cases}$$

$$x_a = \sum_{r \in R} \sum_{s \in S} \sum_{k \in K_{rs}} f_k^{rs} \delta_{a,k}^{rs}$$



$$f_1^{12} + f_2^{12} = q_{12} = 125,\ f_1^{52} = q_{52} = 530,$$

$$f_1^{16} = q_{16} = 501,\ f_1^{56} + f_2^{56} = q_{56} = 489,$$

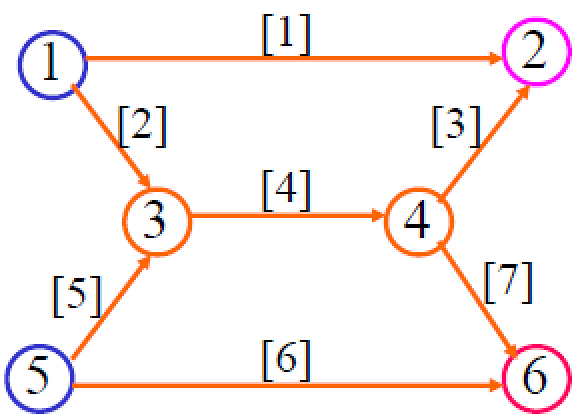
$$f_1^{12}, f_2^{12}, f_1^{16}, f_1^{52}, f_1^{56}, f_2^{56} \geq 0$$

An Example for Flow Conservation

- Path-link incidence matrix $\Delta = \left(\delta_{ak}^{rs}\right)$

$$\delta_{ak}^{rs} = \begin{cases} 1 & \text{if link } a \text{ is on path } k \text{ between O-D pair } r\text{-}s \\ 0 & \text{otherwise} \end{cases}$$

- **Example**

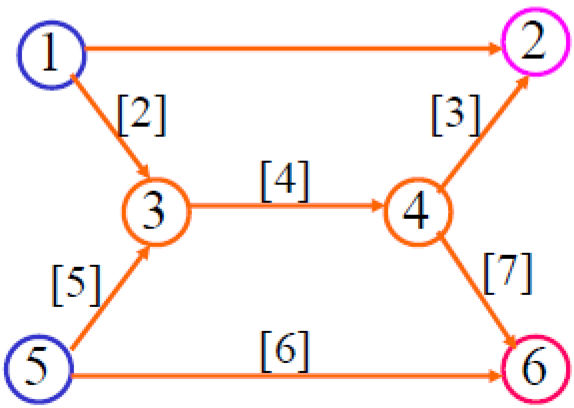


$$\Delta = \left(\delta_{ak}^{rs}\right)_{7\times 6} = \begin{array}{c} \begin{array}{cccccc} \multicolumn{2}{c}{\overbrace{\hspace{3em}}^{1-2}} & \multicolumn{2}{c}{\overbrace{\hspace{3em}}^{5-6}} & \overbrace{\hspace{0.5em}}^{1-6} & \overbrace{\hspace{0.5em}}^{5-2} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

An Example for Flow Conservation

- **Given an O-D Matrix (or Table), What Conditions Should A Feasible Path Flow Solution Satisfy ?**

f_k^{rs} : Traffic flow on path k from origin r to destination s



$\{q_{rs}, r \in R = \{1,5\}, s \in S = \{2,6\}\}$

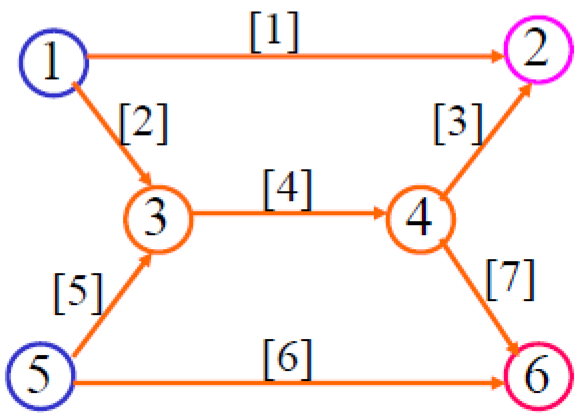
o \ d	2	6
1	125	501
5	530	489

$$f_1^{12} + f_2^{12} = q_{12} = 125 \text{ trips/hour} \qquad f_1^{52} = q_{52} = 530$$

$$f_1^{16} = q_{16} = 501 \text{ trips/hour} \qquad f_1^{56} + f_2^{56} = q_{56} = 489$$

$$\sum_{k \in K_{rs}} f_k^{rs} = q_{rs}, r \in R, s \in S$$

An Example for Path Travel Time



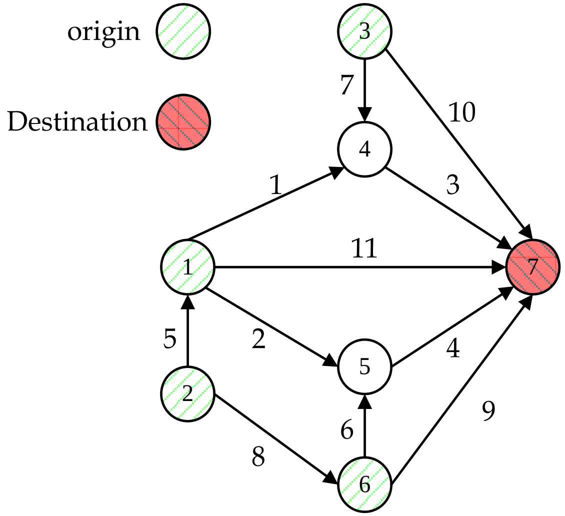
$$
\begin{cases}
c_1^{12} = t_1(x_1) \\
c_2^{12} = t_2(x_2) + t_4(x_4) + t_3(x_3) \\
c_1^{56} = t_6(x_6) \\
c_2^{56} = t_5(x_5) + t_4(x_4) + t_7(x_7) \\
c_1^{16} = t_2(x_2) + t_4(x_4) + t_7(x_7) \\
c_1^{52} = t_5(x_5) + t_4(x_4) + t_3(x_3)
\end{cases}
$$

$$c_k^{rs} = \sum_{a \in A} \delta_{a,k}^{rs} t_a(x_a), k \in K_{rs}, r \in R, s \in S$$

Travel time of a path is defined as sum of travel times of all links defining the path

Assumptions of Route Choice

- Based on day-by-day adjustments, the link traffic flows will come to an equilibrium.
- All the users try to choose the shortest path, so it is obvious that at the equilibrium:
  - Users between the same OD pair will use different paths
  - Travel time on all the used paths should be the same, otherwise…
  - Travel time on one path could be larger, only if no one want to use this path!



User Equilibrium

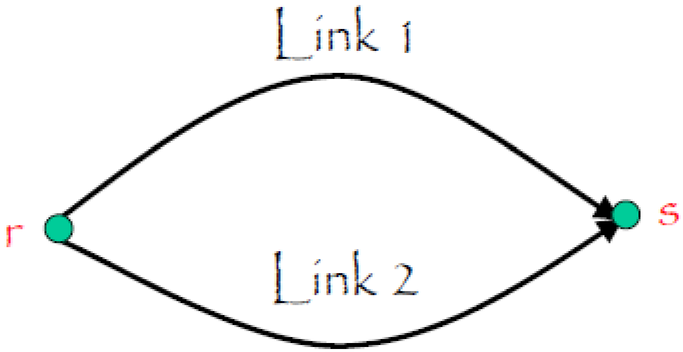
- **Wardrop's First Principle (UE conditions)**
  - "The journey time on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route."

- Reference:
  - Wardrop, J. G. (1952) Some Theoretical Aspects of Road Traffic Research. *Proceedings of the Institute of Civil Engineers*, Part II, pp. 325-378.

- For one OD pair, the travel times on all the used paths are identical, which is termed as the OD travel time (OD travel impedance), denoted by u^{rs}

Example 1 for UE

❑ Example



- Link travel functions:

$$t_1(x_1) = 40 + 0.5x_1$$

$$t_2(x_2) = 10 + 0.25x_2^2$$

- Case 1. O-D demand: $q_{rs} = 10$ units

Assuming that both paths are used by travelers, we have

$$\begin{cases} 40 + 0.5x_1 = 10 + 0.25x_2^2 \\ x_1 + x_2 = 10 \end{cases}$$

$$x_1^* = -0.8743 \Rightarrow t_1(x_1^*) = 39.563$$

$$x_2^* = 10.8743 \Rightarrow t_2(x_2^*) = 39.563$$

➢ **UE solution**

$$x_1^{UE} = 0 \Rightarrow t_1(x_1^{UE}) = 40$$

$$x_2^{UE} = 10 \Rightarrow t_2(x_2^{UE}) = 35$$

Example 1 for UE

- Case 2. O-D demand: $q_{rs} = q \geq \sqrt{120}$ units

  Assuming that both paths are used by travelers, we have

$$\begin{cases} 40 + 0.5x_1 = 10 + 0.25x_2^2 \\ x_1 + x_2 = q \end{cases}$$

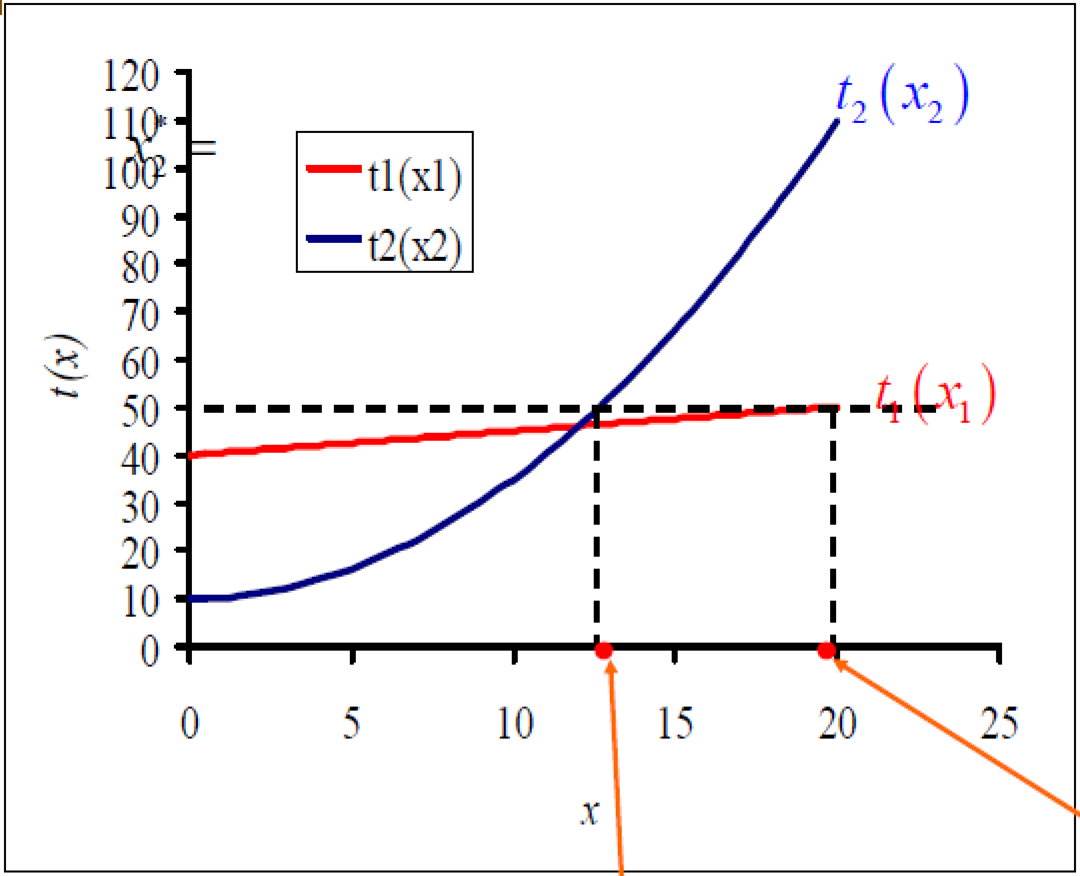
$$\Downarrow$$

➢ **UE solution**

$$x_1^* = (q+1) - \sqrt{2q+121} \geq 0 \Rightarrow t_1(x_1^*) = 40.5 + 0.5q - 0.5\sqrt{2q+121}$$

$$x_2^* = \sqrt{2q+121} - 1 \geq 0 \Rightarrow t_2(x_2^*) = 40.5 + 0.5q - 0.5\sqrt{2q+121}$$

Example 1 for UE



$$t_1(x_1) = 40 + 0.5x_1$$

$$t_2(x_2) = 10 + 0.25x_2^2$$

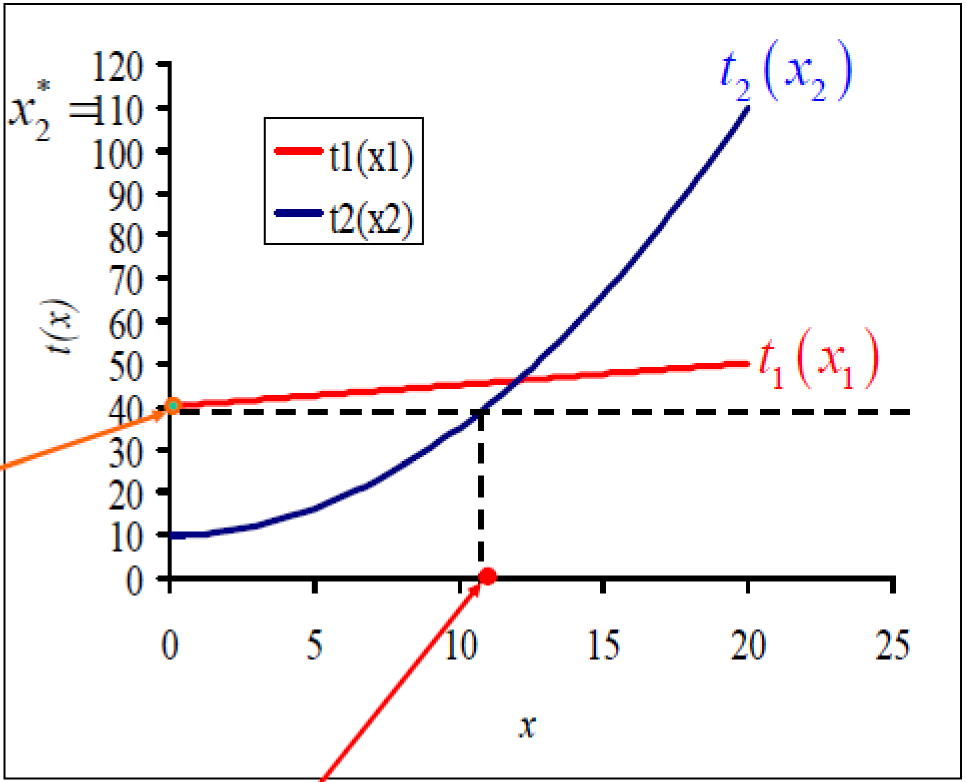
$$t_2(x_2) = 50 \Rightarrow x_2^* = \sqrt{160}$$

$$t_1(x_1) = 50 \Rightarrow x_1^* = 20$$

Example 1 for UE

$$t_1(x_1) = 40 + 0.5x_1$$

$$t_2(x_2) = 10 + 0.25x_2^2$$



Free flow travel time of path *1*

$$10 + 0.25x_2^2 = 40 \Rightarrow x_2^* = \sqrt{120}$$

Example 2 for UE

- **Example 2**
  - Link travel time functions:

$$t_1(x_1) = 3 + \frac{1}{2}x_1$$

$$t_2(x_2) = 1 + 2x_2$$

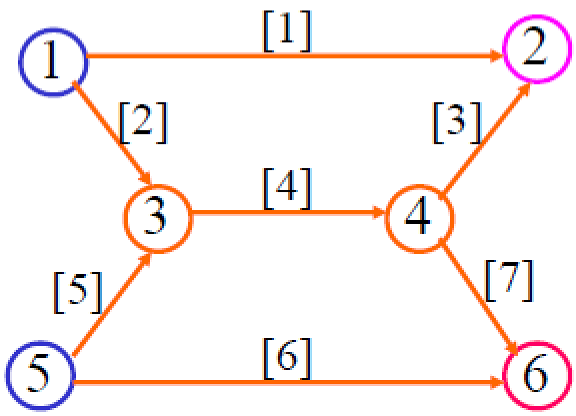
$$t_3(x_3) = \frac{1}{2} + x_3$$

$$t_4(x_4) = 1 + \frac{1}{2}x_4$$

$$t_5(x_5) = 2 + \frac{1}{2}x_5$$

$$t_6(x_6) = 4 + x_6$$

$$t_7(x_7) = 1 + \frac{1}{2}x_7$$

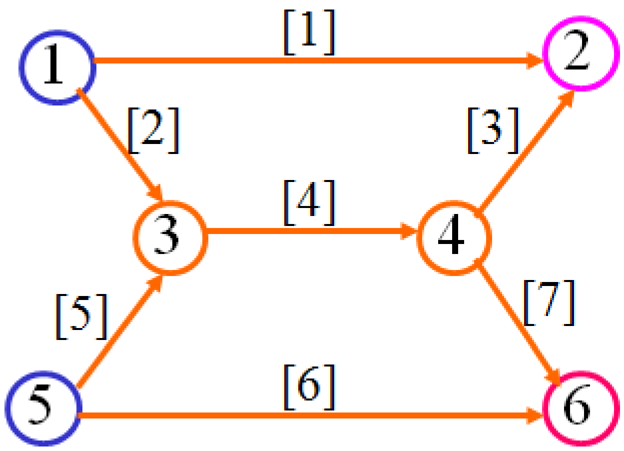


- O-D matrix

$$q_{12} = 100, \qquad q_{16} = 0$$

$$q_{52} = 0, \qquad q_{56} = 50$$

Example 2 for UE



- **O-D pair <1, 2>**

$$3+\frac{1}{2}x_1 = t_2(x_2)+t_4(x_4)+t_3(x_3)$$

$$\Updownarrow$$

$$3+\frac{1}{2}f_1^{12} = 1+2\left(f_2^{12}\right)+1+\frac{1}{2}\left(f_2^{12}+f_2^{56}\right)+0.5+f_2^{12}$$

$$\Updownarrow$$

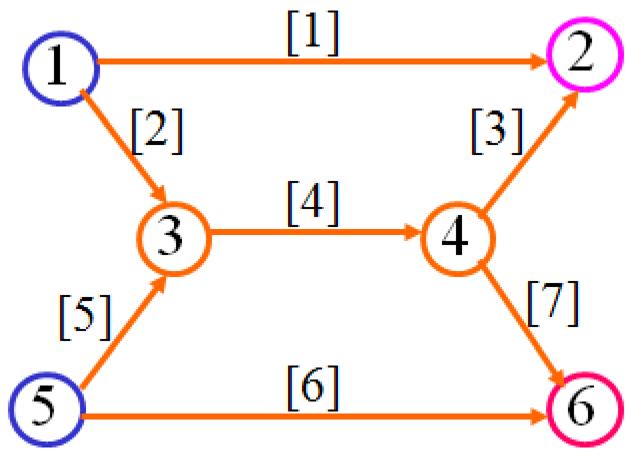
$$3+\frac{1}{2}f_1^{12} = 2.5+3.5f_2^{12}+0.5f_2^{56}$$

$$\Updownarrow$$

$$-0.5f_1^{12}+3.5f_2^{12}+0.5f_2^{56}=0.5 \quad ①$$

$$f_1^{12}+f_2^{12}=100 \quad ②$$

Example 2 for UE



$$4+f_1^{56}=t_5(x_5)+t_4(x_4)+t_7(x_7)$$

$$\Updownarrow$$

$$4+f_1^{56}=2+\frac{1}{2}\left(f_2^{56}\right)+1+\frac{1}{2}\left(f_2^{12}+f_2^{56}\right)+1+\frac{1}{2}\left(f_2^{56}\right)$$

$$\Updownarrow$$

$$-f_1^{56}+1.5f_2^{56}+0.5f_2^{12}=0 \qquad ③$$

$$f_1^{56}+f_2^{56}=50$$

Solving equations ①—④ yields that

UE path flow solution: $f_1^{12}=89.615,\ f_2^{12}=10.385,\ f_1^{56}=32.977,\ f_2^{56}=17.923$

UE link flow solution: $x_1=89.615, x_2=10.385, x_3=10.385, x_4=28.308$

$x_5=17.923, x_6=32.977, x_7=17.923$

1. Equivalence Condition

❑Mathematical Representation/Definition of UE

$$(1)\ f_k^{rs} \times \left[c_k^{rs} - u^{rs} \right] = 0, \forall k \in K_{rs}, r \in R, s \in S$$

$$(2)\ c_k^{rs} \geq u^{rs} \geq 0, \forall k \in K_{rs}, r \in R, s \in S$$

❑Beckmann's Transformation (an optimization model for UE)

$$\min z(x) = \sum_{a \in A} \int_0^{x_a} t_a(\omega) d\omega$$

subject to:

$$x_a = \sum_{r \in R} \sum_{s \in S} \sum_{k \in K_{rs}} f_k^{rs} \delta_{ak}^{rs}, \forall a \in A$$

$$\sum_{k \in K_{rs}} f_k^{rs} = q_{rs}, \forall r \in R, \forall s \in S$$

$$f_k^{rs} \geq 0, \forall k \in K_{rs}, \forall r \in R, \forall s \in S$$

1. Equivalence Condition

❑KKT Conditions of Beckmann's Transformation

- KKT Equations

$$\frac{\partial z(x)}{\partial f_k^{rs}} + \sum_{r\in R}\sum_{s\in S} \frac{\partial\left(q_{rs} - \sum_{k\in K_{rs}} f_k^{rs}\right)}{\partial f_k^{rs}} u_{rs} + \frac{-\partial f_k^{rs}\lambda_k^{rs}}{\partial f_k^{rs}} = 0 \Longrightarrow \sum_{a\in A} t_a \delta_{ak}^{rs} - u_{rs} = \lambda_k^{rs}$$

- Complementary slackness conditions

$$\lambda_k^{rs} \times f_k^{rs} = 0$$

- Non-negativity $\lambda_k^{rs} \geq 0$
- Feasibility

$$\boxed{\begin{gathered} f_k^{rs} \times \left[c_k^{rs} - u^{rs}\right] = 0 \\ c_k^{rs} \geq u^{rs} \geq 0 \end{gathered}}$$

1. Equivalence Condition

- The KKT Conditions of Beckmann's Transformation shows that its optimal solution can fulfill the UE condition.
- This property is called as "Equivalence", which is an important process for every proposed transport models.

2. Convexity of Beckmann's Transformation

- Hessian Matrix of Objective Function of Beckmann's Transformation

$$\nabla^2 z(x) = \begin{pmatrix} t'(v_1) & 0 & 0 & \cdots \\ 0 & \ddots & 0 & \cdots \\ 0 & 0 & t'(v_a) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- Since the link travel time function is a strictly increasing function of link flows, so its first order derivative is always larger than zero.
  - BPR function

$$t_a(x_a) = t_a^0 \times \left[1.0 + \alpha_a (x_a / C_a)^{\beta_a} \right]$$

2. Convexity of Beckmann's Transformation

- The Hessian Matrix is a diagonal matrix.
- So the values of its Principal Minor Determinants are all positive, indicating that it is positive definite. So, the objective function of Beckmann's Transformation is strictly convex.

2. Convexity

- Feasible sets of the UE only contains linear constraints, so it is obviously a convex set.

$$x_a = \sum_{r\in R}\sum_{s\in S}\sum_{k\in K_{rs}} f_k^{rs}\delta_{ak}^{rs}, \forall a \in A$$

$$\sum_{k\in K_{rs}} f_k^{rs} = q_{rs}, \forall r \in R, \forall s \in S$$

$$f_k^{rs} \geq 0, \forall k \in K_{rs}, \forall r \in R, \forall s \in S$$

- Thus, we know that the UE model have <u>strictly convex objective functions</u> and also are defined over <u>a convex set</u>.

2. Convexity

- The convexity of the model, guarantees another two important properties
  - Existence of the solution.
  - Uniqueness of the solution.

Why the existence and uniqueness properties are important?
If no, what are the impacts on the solution algorithm?
How to get all the optimal solutions, if more than one?

"All-or-nothing" Transit Assignment

- How to find the flow pattern that minimizes the total travel over the network, given the (fixed and known) values of the link travel times and the OD matrix?

- **Solution:** "All-or-nothing" Transit Assignment
  - All the flow for a given O-D pair *r-s*, q_{rs} , is assigned to the minimum-travel-time path connecting this pair.
  - All other paths connecting this O-D pair do not carry flow.

  - The resulting flow pattern is both an equilibrium situation (since no user will be better off by switching paths) and an optimal assignment (since the total travel time in the system is obviously minimized).

3. Solution Algorithm for UE

- The Frank-Wolfe Method is an efficient algorithm for solving a convex programming problem

$$\min z(x) = \sum_{a \in A} \int_0^{x_a} t_a(\omega) d\omega$$

subject to:

$$x_a = \sum_{r \in R} \sum_{s \in S} \sum_{k \in K_{rs}} f_k^{rs} \delta_{ak}^{rs}, \forall a \in A$$

$$\sum_{k \in K_{rs}} f_k^{rs} = q_{rs}, \forall r \in R, \forall s \in S$$

$$f_k^{rs} \geq 0, \forall k \in K_{rs}, \forall r \in R, \forall s \in S$$

3. Solution Algorithm for UE

- Step 0: Choose an initial feasible point $x^{(0)}$ and set k=0
  - Usually use all-or-nothing assignment based on the free flow travel time to get the initial feasible point
  - Here x is a vector including all the link flows.
- Step 1: Find the descent direction $d^{(k)}=y^{(k)}-x^{(k)}$ where $y^{(k)}$ is the optimal solution of the LP

$$\min_y z(y) = \sum_{i=1}^{n}\left[\left(\partial f\left(x^{(k)}\right)/\partial x_i\right)y_i\right]$$

  - For the UE model, this objective function becomes:

$$\min_y z(y) = \sum_{a\in A}\left[t_a\left(x_a^{(k)}\right)\times y_a\right]$$

Subject to

Flow conservation equations

3. Solution Algorithm for UE

- $t_a\left(x_a^{(k)}\right)$ is fixed, rather than a function. We use $t_a^{(k)}$ to replace it, thus the sub-problem of Frank-Wolfe Method becomes:

$$\min_y z(y) = \sum_{a \in A} t_a^{(k)} y_a$$

Subject to

$$y_a = \sum_{r \in R} \sum_{s \in S} \sum_{k \in K_{rs}} f_k^{rs} \delta_{ak}^{rs}, \forall a \in A$$

$$\sum_{k \in K_{rs}} f_k^{rs} = q_{rs}, \forall r \in R, \forall s \in S$$

$$f_k^{rs} \geq 0, \forall k \in K_{rs}, \forall r \in R, \forall s \in S$$

<u>This is a all-or-nothing assignment problem, where the demand can be assigned to the shortest paths!</u>

3. Solution Algorithm for UE

- Step 2: If the stop criterion is fulfilled, then stop. Otherwise, go to Step 3
- Step 3: Find an α_k, which is the optimal solution of the following one-dimension minimization problem, by using a line search method

$$\min_{0\le\alpha\le1} f\left(x^{(k)}+\alpha\left(y^{(k)}-x^{(k)}\right)\right)$$

$$\min_{0\le\alpha\le1}\sum_{a\in A}\int_0^{x^{(k)}+\alpha\left(y^{(k)}-x^{(k)}\right)} t_a(\omega)d\omega$$

Subject to

Flow conservation equations

3. Solution Algorithm for UE

- Then, generate the new link flow pattern, and repeat the procedure by going back to Step 1.

$$x_a^{(n+1)} = x_a^{(n)} + \alpha_n \left(y_a^{(n)} - x_a^{(n)} \right), \forall a \in A$$

Summary of Frank-Wolfe Method

- **Step 0**: (Initialization) Perform all-or-nothing traffic assignment based on $\left\{t_a = t_a(0), a \in A\right\}$.This yields $\left\{x_a^{(1)}, a \in A\right\}$. Set counter $n := 1$
- **Step 1:** (update) Set $t_a^{(n)} = t_a\left(x_a^{(n)}\right), \forall a \in A$
- **Step 2**: (Direction finding) Perform all-or-nothing traffic assignment based on $\left\{t_a^{(n)}, a \in A\right\}$. This yields a set of (auxiliary) flows $\left\{y_a^{(n)}, a \in A\right\}$.
- **Step 3:** (Line search) Find α_n that solves $\min_{0 \le \alpha \le 1} \sum_{a \in A} \int_0^{x_a^{(n)} + \alpha\left[y_a^{(n)} - x_a^{(n)}\right]} t_a(\omega) d\omega$
- **Step 4:** (Move) $x_a^{(n+1)} = x_a^{(n)} + \alpha_n\left(y_a^{(n)} - x_a^{(n)}\right), \forall a \in A$
- **Step 5:** (Convergence test) If a convergence criterion is met, then stop (the current solution, $\left\{x_a^{(n+1)}, a \in A\right\}$, is the set of equilibrium flows); otherwise set n:=n+1 and go to Step 1

Frank-Wolfe Method

❑ Stop Criteria

$$(1)\quad \left|\sum_{a\in A}\int_0^{x_a^{(n)}} t_a(\omega)\,d\omega - \sum_{a\in A}\int_0^{x_a^{(n+1)}} t_a(\omega)\,d\omega\right| \le \varepsilon$$

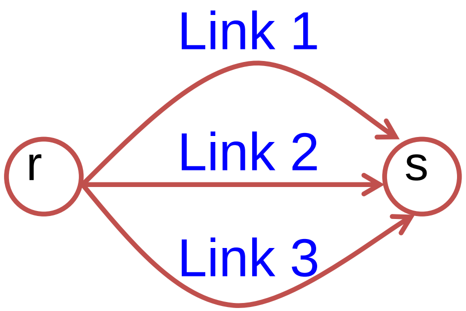
$$(2)\quad \sum_{r\in R}\sum_{s\in S}\frac{\left|u_{rs}^{(n+1)} - u_{rs}^{(n)}\right|}{u_{rs}^{(n)}} \le \varepsilon$$

where $u_{rs}^{(n)}$ denote the minimum path travel time between O-D pair <r,s> at the nth iteration

$$(3)\quad \frac{\sqrt{\sum_{a\in A}\left(x_a^{(n+1)} - x_a^{(n)}\right)^2}}{\sum_{a\in A} x_a^{(n)}} \le \varepsilon$$

Frank-Wolfe Method

❑ Example 1



- **Link travel time functions**

$$t_1(x_1) = 10\left(1 + 0.15(x_1/2)^4\right) \text{ time units}$$

$$t_2(x_2) = 20\left(1 + 0.15(x_2/4)^4\right) \text{ time units}$$

$$t_3(x_3) = 25\left(1 + 0.15(x_3/3)^4\right) \text{ time units}$$

- **O-D demand**

$$q_{rs} = 10 \text{ flow units}$$

- **Line search method**

Golden section algorithm

Frank-Wolfe Method

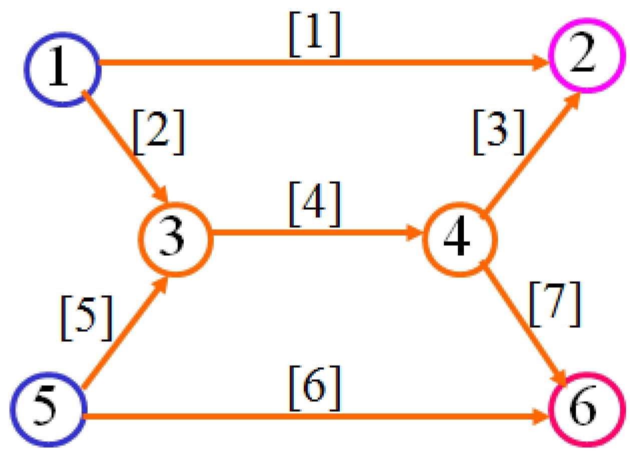
	Algorithmic step	Link 1	Link 2	Link 3	Objective function	Step size
0	Initialization	$t_1^{(0)} = 10.0$	$t_2^{(0)} = 20.0$	$t_3^{(1)} = 25.0$		
		$x_1^{(1)} = 10.00$	$x_2^{(1)} = 0.00$	$x_3^{(1)} = 0.00$	$z(x) = 1975.00$	
1	Update	$t_1^{(1)} = 947.0$	$t_2^{(1)} = 20.0$	$t_3^{(1)} = 25.0$		
	Direction	$y_1^{(1)} = 0$	$y_2^{(1)} = 10$	$y_3^{(1)} = 0$		$\alpha_1 = 0.596$
	Move	$x_1^{(2)} = 4.04$	$x_2^{(2)} = 5.96$	$x_3^{(2)} = 0.00$	$z(x) = 197.00$	
2	Update	$t_1^{(2)} = 35.0$	$t_2^{(2)} = 35.0$	$t_3^{(2)} = 25.0$		
	Direction	$y_1^{(2)} = 0$	$y_2^{(2)} = 0$	$y_3^{(2)} = 10$		$\alpha_2 = 0.161$
	Move	$x_1^{(3)} = 3.39$	$x_2^{(3)} = 5.00$	$x_3^{(3)} = 1.61$	$z(x) = 189.98$	
3	Update	$t_1^{(3)} = 22.3$	$t_2^{(3)} = 27.3$	$t_3^{(3)} = 35.3$		
	Direction	$y_1^{(3)} = 10$	$y_2^{(3)} = 0$	$y_3^{(3)} = 0$		$\alpha_3 = 0.035$
	Move	$x_1^{(4)} = 3.62$	$x_2^{(4)} = 4.83$	$x_3^{(4)} = 1.55$	$z(x) = 189.44$	

Frank-Wolfe Method

	Algorithmic step	Link 1	Link 2	Link 3	Objective function	Step size
4	Update	$t_1^{(4)} = 26.1$	$t_2^{(4)} = 26.3$	$t_3^{(4)} = 25.3$		
	Direction	$y_1^{(4)} = 0$	$y_2^{(4)} = 0$	$y_3^{(4)} = 10$		$\alpha_4 = 0.020$
	Move	$x_1^{(5)} = 3.54$	$x_2^{(2)} = 4.73$	$x_3^{(2)} = 1.72$	$z(x) = 189.33$	
5	Update	$t_1^{(5)} = 24.8.0$	$t_2^{(5)} = 25.8$	$t_3^{(5)} = 25.4$		
	Direction	$y_1^{(5)} = 10$	$y_2^{(5)} = 0$	$y_3^{(5)} = 0$		$\alpha_5 = 0.007$
	Move	$x_1^{(6)} = 2.59$	$x_2^{(6)} = 4.70$	$x_3^{(3)} = 1.71$	$z(x) = 189.33$	
	Update	$t_1^{(6)} = 25.6$	$t_2^{(6)} = 25.7$	$t_3^{(6)} = 25.4$		

Frank-Wolfe Method

Example 2



- Link travel times

$$t_1(x_1) = 3 + \frac{1}{2}x_1$$

$$t_2(x_2) = 1 + 2x_2$$

$$t_3(x_3) = 0.5 + x_3$$

$$t_4(x_4) = 1 + \frac{1}{2}x_4$$

$$t_5(x_5) = 2 + \frac{1}{2}x_5$$

$$t_6(x_6) = 4 + x_6$$

$$t_7(x_7) = 1 + \frac{1}{2}x_7$$

- O-D matrix

$$q_{12} = 100, q_{16} = 0$$

$$q_{52} = 0, q_{56} = 50$$

Frank-Wolfe Method

Iteration 0: (All-or-Nothing)

$$x_1^{(1)} = 0, x_2^{(1)} = 100, x_3^{(1)} = 100, x_4^{(1)} = 100, \ x_5^{(1)} = 0, x_6^{(1)} = 50, x_7^{(1)} = 0$$

Iteration 1:

(1) Update

$$t_1(x_1^{(1)}) = 3, t_2(x_2^{(1)}) = 201, t_3(x_3^{(1)}) = 100.5, t_4(x_4^{(1)}) = 51$$

$$t_5(x_5^{(1)}) = 2, t_6(x_6^{(1)}) = 54, t_7(x_7^{(1)}) = 1$$

(2) Descent direction finding

All-or-nothing assignment

$$y_1^{(1)} = 100, y_2^{(1)} = 0, y_3^{(1)} = 0, y_4^{(1)} = 50, \quad y_5^{(1)} = 50, y_6^{(1)} = 0, y_7^{(1)} = 50$$

Descent direction

$$\left(y_a^{(1)} - x_a^{(1)}, a \in A\right)^T = \left(100, -100, -100, -50, 50, -50, 50\right)^T$$

Frank-Wolfe Method

(3) Line search

$$h(\alpha) = z(x^{(1)} + \alpha(y^{(1)} - x^{(1)})) = \sum_{a=1}^{7} \int_0^{x_a^{(1)} + \alpha(y_a^{(1)} - x_a^{(1)})} t_a(\omega) d\omega$$

Let $h'(\alpha) = 0$

$$h'(\alpha) = \sum_{a=1}^{7} [t_a(x_a^{(1)} + \alpha(y_a^{(1)} - x_a^{(1)})) \times (y_a^{(1)} - x_a^{(1)})] = 0$$

where

$$x_1^{(1)} + \alpha(y_1^{(1)} - x_1^{(1)}) = 100\alpha; \qquad x_2^{(1)} + \alpha(y_2^{(1)} - x_2^{(1)}) = 100 - 100\alpha$$

$$x_3^{(1)} + \alpha(y_3^{(1)} - x_3^{(1)}) = 100 - 100\alpha; \qquad x_4^{(1)} + \alpha(y_4^{(1)} - x_4^{(1)}) = 100 - 50\alpha$$

$$x_5^{(1)} + \alpha(y_5^{(1)} - x_5^{(1)}) = 50\alpha; \qquad x_6^{(1)} + \alpha(y_6^{(1)} - x_6^{(1)}) = 50 - 50\alpha$$

$$x_7^{(1)} + \alpha(y_7^{(1)} - x_7^{(1)}) = 50\alpha$$

Frank-Wolfe Method

$$h'(\alpha) = (3+50\alpha)\times 100 + (201-200\alpha)\times(-100) + (100.5-100\alpha)\times(-100)$$
$$+(51-25\alpha)\times(-50) + (2+25\alpha)\times 50 + (54-50\alpha)\times(-50)$$
$$+(1+25\alpha)\times 50 = 0 \quad \Rightarrow \quad \alpha_1 = 0.8737$$

(4) Move

$$x_a^{(2)} = x_a^{(1)} + \alpha_1 (y_a^{(1)} - x_a^{(1)}), a = 1, 2, \ldots 7$$

(5) Convergence test

Criterion 1: $\left| \sum_{a\in A} \int_0^{x_a^{(1)}} t_a(\omega)\, d\omega - \sum_{a\in A} \int_0^{x_a^{(2)}} t_a(\omega)\, d\omega \right| = 15268.78$

Criterion 2: $\sum_{r\in R} \sum_{s\in S} \frac{\left| u_{rs}^{(2)} - u_{rs}^{(1)} \right|}{u_{rs}^{(1)}} = 15.37$

Criterion 3: $\frac{\sqrt{\sum_{a\in A} \left(x_a^{(2)} - x_a^{(1)} \right)^2}}{\sum_{a\in A} x_a^{(1)}} = 0.499$

Frank-Wolfe Method

Criterion 3: $\dfrac{\sqrt{\sum_{a\in A}\left(x_a^{(n+1)}-x_a^{(n)}\right)^2}}{\sum_{a\in A} x_a^{(n)}}$

Iterative Results from Frank-Wolfe Method

$x_1^{(1)}=0$	$x_2^{(1)}=100$	$x_3^{(1)}=100$	$x_4^{(1)}=100$	$x_5^{(1)}=0$	$x_6^{(1)}=50$	$x_7^{(1)}=0$	α_n	$z\left(x^{(n)}\right)$	
$y_1^{(1)}=100$	$y_2^{(1)}=0$	$y_3^{(1)}=0$	$y_4^{(1)}=0$	$y_5^{(1)}=0$	$y_6^{(1)}=50$	$y_7^{(1)}=0.00$			
$x_1^{(2)}=87.38$	$x_2^{(2)}=12.63$	$x_3^{(2)}=12.63$	$x_4^{(2)}=12.63$	$x_5^{(2)}=0.00$	$x_6^{(2)}=50.00$	$x_7^{(2)}=0.00$	0.8737	3931	0.499
$y_1^{(2)}=100$	$y_2^{(2)}=0$	$y_3^{(2)}=0$	$y_4^{(2)}=50$	$y_5^{(2)}=50$	$y_6^{(2)}=0$	$y_7^{(2)}=50$			
$x_1^{(3)}=91.78$	$x_2^{(3)}=8.22$	$x_3^{(3)}=8.22$	$x_4^{(3)}=25.67$	$x_5^{(3)}=17.46$	$x_6^{(3)}=32.54$	$x_7^{(3)}=17.46$	0.3491	3550	0.193
$y_1^{(3)}=0$	$y_2^{(3)}=100$	$y_3^{(3)}=100$	$y_4^{(3)}=150$	$y_5^{(3)}=50$	$y_6^{(3)}=0$	$y_7^{(3)}=50$			
$x_1^{(4)}=89.71$	$x_2^{(4)}=10.29$	$x_3^{(4)}=10.29$	$x_4^{(4)}=28.49$	$x_5^{(4)}=18.19$	$x_6^{(4)}=31.81$	$x_7^{(4)}=18.19$	0.0226	3540	0.023
$y_1^{(4)}=0$	$y_2^{(3)}=100$	$y_3^{(3)}=100$	$y_4^{(3)}=100$	$y_5^{(3)}=0$	$y_6^{(3)}=50$	$y_7^{(3)}=0$			
$x_1^{(5)}=89.61$	$x_2^{(5)}=10.39$	$x_3^{(5)}=10.39$	$x_4^{(5)}=28.56$	$x_5^{(5)}=18.18$	$x_6^{(5)}=31.82$	$x_7^{(4)}=18.18$	0.0010	3540	0.00

System Optimum Conditions

❑Wardrop's Second Principle (i.e. SO Conditions)

- Wardrop, J. G. (1952) Some Theoretical Aspects of Road Traffic Research. Proceedings of the Institute of Civil Engineers, Part II, pp. 325-378.
- "The average journal travel time is minimal"
- Average journey travel time

$$\frac{\sum_{r \in R} \sum_{s \in S} \sum_{k \in K_{rs}} \left[c_k^{rs} \times f_k^{rs} \right]}{\sum_{r \in R} \sum_{s \in S} q_{rs}}$$

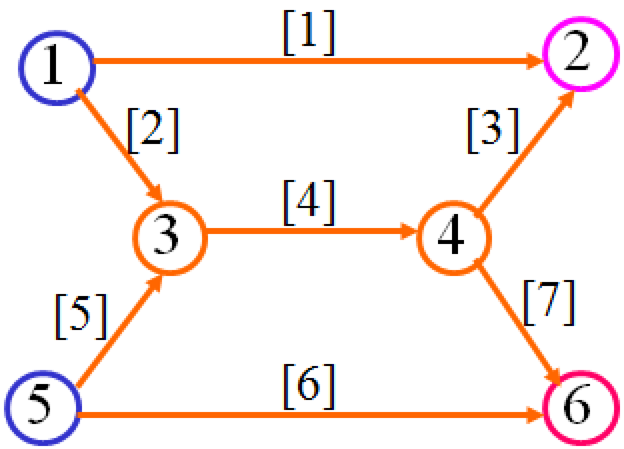
- Total system travel time

$$\tilde{z}(x) = \sum_{a \in A} \left[t_a(x_a) \times x_a \right] = \sum_{r \in R} \sum_{s \in S} \sum_{k \in K_{rs}} \left[c_k^{rs} \times f_k^{rs} \right]$$

System Optimum Conditions (cont'd)

- Minimizing the average journey travel time is equivalent to minimizing the total system travel time:

$$\frac{\sum_{r\in R}\sum_{s\in S}\sum_{k\in K_{rs}}\left[c_k^{rs}\times f_k^{rs}\right]}{\sum_{r\in R}\sum_{s\in S} q_{rs}} = \frac{\sum_{a\in A}\left[t_a(x_a)\times x_a\right]}{\sum_{r\in R}\sum_{s\in S} q_{rs}}$$



$$\sum_{r\in R}\sum_{s\in S}\sum_{k\in K_{rs}}\left[c_k^{rs}\times f_k^{rs}\right]=$$

$$\left(c_1^{12}\times f_1^{12}\right)+\left(c_2^{12}\times f_2^{12}\right)+\left(c_1^{56}\times f_1^{56}\right)+\left(c_2^{56}\times f_2^{56}\right)=$$

$$\left[t_1(x_1)\times f_1^{12}\right]+\left[\left(t_2(x_2)+t_4(x_4)+t_3(x_3)\right)\times f_2^{12}\right]+$$

$$\left(t_6(x_6)\times f_1^{56}\right)+\left[\left(t_5(x_5)+t_4(x_4)+t_7(x_7)\right)\times f_2^{56}\right]=$$

$$t_1(x_1)\, f_1^{12}+t_2(x_2)\, f_2^{12}+t_3(x_3)\, f_2^{12}+t_4(x_4)\left(f_2^{12}+f_2^{56}\right)$$

$$+t_5(x_5)\, f_2^{56}+t_6(x_6)\, f_1^{56}+t_7(x_7)\, f_2^{56}=\sum_{a=1}^{7} t_a(x_a)\, x_a$$

$$c_1^{12}=t_1(x_1)$$

$$c_2^{12}=t_2(x_2)+t_4(x_4)+t_3(x_3)$$

$$c_1^{56}=t_6(x_6)$$

$$c_2^{56}=t_5(x_5)+t_4(x_4)+t_7(x_7)$$

$$f_1^{12}+f_2^{12}=q_{12}$$

$$f_1^{56}+f_2^{56}=q_{56}$$

System Optimum Formulation

❑Convex Minimization Model

$$\min \tilde{z}(x) = \sum_{a \in A} x_a t_a(x_a)$$ ← total system travel time

subject to

$$x_a = \sum_{r \in R} \sum_{s \in S} \sum_{k \in K_{rs}} f_k^{rs} \delta_{ak}^{rs}, \forall a \in A$$

$$\sum_{k \in K_{rs}} f_k^{rs} = q_{rs}, \forall r \in R, \forall s \in S$$

$$f_k^{rs} \geq 0, \forall k \in K_{rs}, \forall r \in R, \forall s \in S$$

System-Optimal Formulation (cont'd)

- The link flow pattern that minimizes the previous minimization model does not generally represent an 'equilibrium' situation. Why?
- SO can result only from joint decisions by all travelers to act so as to minimize the total system travel time rather than their *own.*
- At the SO flow pattern, drivers may be able to increase their travel time by unilaterally changing routes.
- This situation is unlikely to sustain itself and consequently the SO flow pattern is not able and should not be used as a model for actual route choice behavior.

System-Optimal Formulation (cont'd)

- However, the value of SO objective function may serve as a yardstick by which different flow pattern can be measured.

- Total system travel time is a common measurement of effectiveness upon networks.

<u>Who is interested with the SO flow pattern?</u>

SO Solution vs. UE Solution

- **UE solution is identical to SO solution provided that travel time for each link is a <u>constant</u>**
  - Assumption

$$t_a\left(x_a\right) = t_a^0, \forall a \in A$$

where t_a^0 is the free-flow travel time on link $a \in A$

  - Why?

$$z\left(x\right) = \sum_{a\in A}\int_0^{x_a} t_a\left(\omega\right)d\omega = \sum_{a\in A}\int_0^{x_a} t_a^0 d\omega = \sum_{a\in A} x_a t_a^0 \quad \Rightarrow \quad z\left(x\right) = \tilde{z}\left(x\right)$$

$$\tilde{z}\left(x\right) = \sum_{a\in A} x_a t_a\left(x_a\right) = \sum_{a\in A} x_a t_a^0$$

- In what situation will link travel time be equal to link free-flow travel time?

SO Solution vs. UE Solution (cont'd)

- **Marginal Function of Link Travel Function**

$$t'(x_a) = \frac{dt_a(x_a)}{dx_a}, a \in A$$

- **Marginal Cost Function**

$$\tilde{t}_a(x_a) = t_a(x_a) + x_a t'(x_a), a \in A \qquad \tilde{t}_a(x_a) = \frac{d\left[x_a t_a(x_a)\right]}{dx_a}$$

- The generalized link travel time $\tilde{t}_a(x_a)$ can be interpreted as the marginal contribution of an additional traveler on the a[th] link to the total travel time on this link.
  - It is the sum of two components:

$t_a(x_a)$: the travel time experienced by that additional traveler when the total link flow is x_a

$t'(x_a)$: the additional travel-time burden that this traveler inflicts on each one of the travelers already using link a (there are x_a of them)

SO Solution vs. UE Solution (cont'd)

- **SO solution is the UE solution corresponding to network with the marginal cost functions.**

Note that

$$x_a t_a(x_a) = \int_0^{x_a} \left[t_a(\omega) + \omega t'(\omega) \right] d\omega = \int_0^{x_a} \tilde{t}_a(\omega) d\omega$$

Thus, we have

$$\min \tilde{z}(x) = \sum_{a \in A} x_a t_a(x_a)$$

s.t.

$$x_a = \sum_{r \in R} \sum_{s \in S} \sum_{k \in K_{rs}} f_k^{rs} \delta_{ak}^{rs}, \forall a \in A$$

$$\sum_{k \in K_{rs}} f_k^{rs} = q_{rs}, \forall r \in R, \forall s \in S$$

$$f_k^{rs} \geq 0, \forall k \in K_{rs}, \forall r \in R, \forall s \in S$$

$$\Leftrightarrow$$

$$\min \tilde{z}(x) = \sum_{a \in A} x_a t_a(x_a) = \sum_{a \in A} \int_0^{x_a} \tilde{t}_a(\omega) d\omega$$

s.t.

$$x_a = \sum_{r \in R} \sum_{s \in S} \sum_{k \in K_{rs}} f_k^{rs} \delta_{ak}^{rs}, \forall a \in A$$

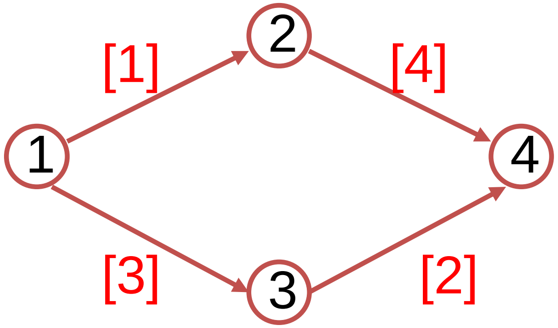
$$\sum_{k \in K_{rs}} f_k^{rs} = q_{rs}, \forall r \in R, \forall s \in S$$

$$f_k^{rs} \geq 0, \forall k \in K_{rs}, \forall r \in R, \forall s \in S$$

Solution Method Finding SO Solution

- **Any method solving UE is also available for solving SO**

  - Example: Find the SO link flow solution



➢O-D demand

$$q_{14} = 6$$

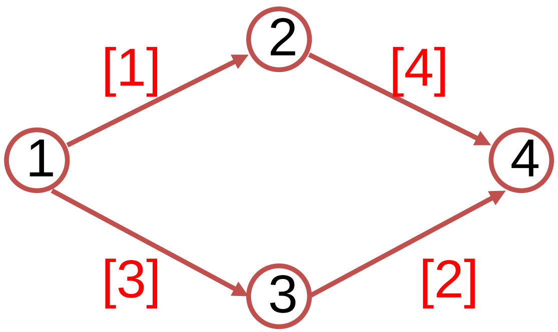
➢Link travel time functions

$$t_1(x_1) = 50 + x_1 \qquad t_2(x_2) = 50 + x_2$$

$$t_3(x_3) = 10x_3 \qquad t_4(x_4) = 10x_4$$

Solution Method Finding SO Solution (cont'd)

- **Solution:**



Step 1: Build the generalized link travel time functions.

$$\tilde{t}_1(x_1) = t(x_1) + x_1 t'(x_1) = 50 + 2x_1,$$

$$\tilde{t}_2(x_2) = 50 + 2x_2,\ \tilde{t}_3(x_3) = 20x_3,\ \tilde{t}_4(x_4) = 20x_4$$

Step 2: Solve the UE with generalized link travel time functions.

$$\begin{cases} 50 + 2f_1^{14} + 20f_1^{14} = 20f_2^{14} + 50 + 2f_2^{14} \\ f_1^{14} + f_2^{14} = 6 \end{cases} \Longrightarrow f_1^{(SO)14} = f_2^{(SO)14} = 3$$

$$\Downarrow$$

$$x_1^{SO} = x_2^{SO} = x_3^{SO} = x_4^{SO} = 3$$

Marginal-Cost Pricing Principle

- **How to achieve a SO solution by implementing a proper traffic management strategy**

  Let $\left\{x_a^{SO}, a \in A\right\}$ be a SO link flow solution

- Road pricing scheme
  - Charge the following toll in terms of time for each link in the network

$$Toll_a = x_a^{SO} t'\left(x_a^{SO}\right), a \in A$$

  - Link travel time functions with toll

$$\hat{t}_a\left(x_a\right) = t_a\left(x_a\right) + x_a^{SO} t'\left(x_a^{SO}\right), a \in A$$

Marginal-Cost Pricing Principle (cont'd)

- **Reasons:** Travelers follow UE Principle to choose their routes by taking into account the toll, namely,

$$\min z(x) = \sum_{a \in A} \int_0^{x_a} \left[t_a(\omega) + x_a^{SO} t'\left(x_a^{SO}\right) \right] d\omega$$

Its optimal solution is:

$$\left\{ x_a^{SO}, a \in A \right\}$$

subject to

$$x_a = \sum_{r \in R} \sum_{s \in S} \sum_{k \in K_{rs}} f_k^{rs} \delta_{ak}^{rs}, \forall a \in A$$

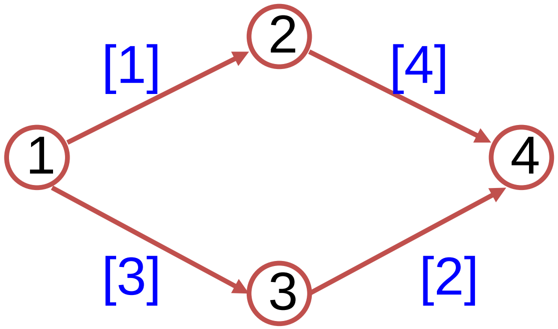
$$\sum_{k \in K_{rs}} f_k^{rs} = q_{rs}, \forall r \in R, \forall s \in S$$

$$f_k^{rs} \geq 0, \forall k \in K_{rs}, \forall r \in R, \forall s \in S$$

<u>How to convert the time toll into a realistic monetary toll</u>?

Braess's Paradox

- **Total system travel time is a key index to measure effectiveness of a transport management or control strategy**
  - Example
    - O-D demand: $q_{14} = 6$
    - Link travel time functions



$$t_1(x_1) = 50 + x_1 \qquad t_2(x_2) = 50 + x_2$$

$$t_3(x_3) = 10x_3 \qquad t_4(x_4) = 10x_4$$

- UE solution

$$x_1^{UE} = x_2^{UE} = x_3^{UE} = x_4^{UE} = 3$$

$$C_{14}^{UE} = 50 + x_1^{UE} + 10x_4^{UE} = 50 + x_2^{UE} + 10x_3^{UE} = 83$$

- Total system travel time

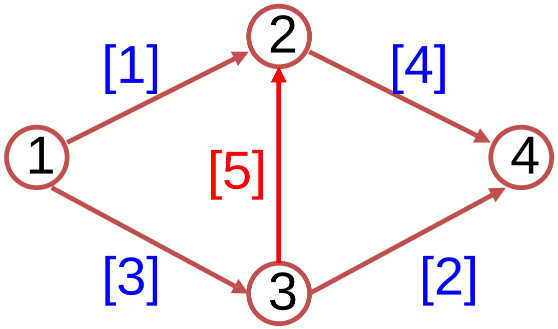
$$TSTT = \sum_{i=1}^{4}\left[t_i\left(x_i^{UE}\right)x_i^{UE}\right] = 498$$

$$TSTT = \sum_{r\in R}\sum_{s\in S} c_{rs}^{UE} q_{rs} = 83\times 6 = 498$$

Braess's Paradox

Build a new road (link 5) connecting nodes from 3 to 2 with link travel time function:

$$t_5\left(x_5\right)=10+x_5$$



UE solution

$$x_1^{UE}=2, x_2^{UE}=2, x_3^{UE}=4 \quad x_4^{UE}=4, x_5^{UE}=2$$

$$C_{14}^{UE}=50+x_1^{UE}+10x_4^{UE}=50+x_2^{UE}+10x_3^{UE}$$

$$=10x_3^{UE}+10+x_5^{UE}+10x_4^{UE}=92$$

After building a new road, the actual travel time is increased!

Total system travel time

$$TSTT=\sum_{r\in R}\sum_{s\in S}q_{rs}C_{rs}^{UE}=6\times 92=552=\sum_{i=1}^{5}\left[t_i\left(x_i^{UE}\right)x_i^{UE}\right]$$

After building a new road, the total system travel time is increased!

Braess's Paradox

- Braess's Paradox implies that road expansion (capacity increase) may even deteriorate the total travel time in the network
  - Some practical examples: a new arterial road or bridge in urban road network may intrigue new travel demand, thus cause high congestions in the annex building.
- This strongly supports the significance of a systematic planning to the entire transport network.
- A transport engineer/planner should always keep in mind about the Braess's Paradox.

Braess's Paradox

- At first glance, it is unconvincing that build more road would increase the total travel time in the network.

<u>Try to mathematically explain why it is so?</u>

Bi-level Modeling – Transport Planning and Mangement

- The bi-level model can be expressed mathematically as:

$$\min F(u, v(u))$$
$$s.t.\ G(u, v(u)) \leq 0$$

where *v(u)* is implicitly defined by:

$$\min\ f(u, v)$$
$$s.t.\ g(u, v) \leq 0$$

where *F* and *u* are the objective function and decision vector of upper-level decision-maker; *G* is the constraint set of upper-level decision vector; *f* and *v* are the objective function and decision vector of lower-level decision-maker; and *g* is the constraint set of lower-level decision vector.

Bi-level Modeling – Congestion Pricing

$$t_a(x_a) = t_a^0\left(1+\alpha\left(\frac{x_a}{C_a}\right)^{\beta}\right),\ a \in A$$
γ : value of time

- *Upper-level (SO):*

$$\min F = \sum_{a\in A} x_a t_a(x_a)$$
$$s.t.\quad \tau_a \geq 0,\ a \in \overline{A}$$
$$\text{flow conservation conditions}$$

The decision variable for upper-level is: $\boldsymbol{\tau} = (\tau_a, a \in \overline{A})$. $\mathbf{x} = (x_a, a \in A)$ is defined by lower-level problem.

- *Lower-level (UE):*

$$\min f = \sum_{a\in A\setminus\overline{A}} \int_0^{x_a} t_a(x)dx + \sum_{a\in\overline{A}} \int_0^{x_a} (t_a(x) + \tau_a/\gamma)dx$$
$$s.t.\quad \sum_k f_k^{rs} = q_{rs},\ \forall r,$$
$$f_k^{rs} \geq 0,\ \forall k, r, s$$
$$x_a = \sum_r \sum_s \sum_k f_k^{rs}\delta_{a,k}^{rs},\ a \in A$$

Bi-level Modeling – Network Design

Upper Level (SO):

$$\min_{z} F = \sum_{a\in A\setminus\bar{A}} x_a t_a(x_a) + \sum_{a\in\bar{A}}\sum_{i\in I_a} z_a^i x_a t_a^i(x_a)$$

$$s.t. \qquad \sum_{i\in I_a} z_a^i = 1, \forall a\in\bar{A}$$

$$\sum_{a\in\bar{A}}\sum_{i\in I_a} c_a^i z_a^i \le B$$

$$z_a^i \in \{0,1\}, \forall a\in\bar{A}, i\in I_a$$

and flow conservation conditions

Decision variable for upper level:

$\mathbf{z} := \left(z_a^i, a\in\bar{A}, i\in I_a\right)$

x_a is an implicit function of $\mathbf{z}$

$$z_a^i = \begin{cases} 1, \text{ only if } i \text{ lanes are added to link } a\in\bar{A} \\ 0, \qquad \text{otherwise} \end{cases}$$

c_a^i : Construction cost of adding i lanes to link $a\in\bar{A}, i\in I_a$.

I_a : Set of number of lanes to be added to link .

Lower Level (UE):

$$\min\ f = \sum_{a\in A\setminus\bar{A}} \int_0^{x_a} t_a(x)dx + \sum_{a\in\bar{A}}\sum_{i\in I_a} z_a^i \int_0^{x_a} t_a^i(x)dx$$

$$s.t.$$

Flow conservation conditions

Decision variable for lower level:

$\mathbf{x} := \left(x_a, a\in A\right)$

z_a^i is an implicit function of $\mathbf{x}$