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PDF TITLE

On Real Analysis

PDF VERSION DATE

January 5, 2021

PDF DIRECTOR

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Real Analysis

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January 5, 2021

Contents

0.1 Preliminaries

This subject is a basis for much of calculus. It starts off with very little preliminary mathematics, such as basic set theory, notations and an understanding of logic and putting proofs together. This course is at Tier 2 level, and thus supposes concepts beneath Tier 2 are understood at least intuitively. However, we seek to honour the spirit of our guild by constructing the beautiful results in an intuitive and sensible manner.

0.2 Protractions

The intense subject of numerical analysis builds off of the results of real analysis and calculus, and so we direct the reader, as a sensible protraction, to Numerical Analysis.

0.3 Historical Works

- Abbott, 2015, Understanding Analysis

0.4 Set Theory and Notation

We direct the reader to any work that covers this. At Tier 2 this should be thoroughly known.

1 Real Numbers

1.1 A Discussion on Constructing $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$

By the end of this section, we will have gone from some preliminary knowledge and intuition to having constructed certain well-known sets of numbers. We will then prove that $\sqrt{2}$ is irrational, and end with two more theorems. We will make the following statement without much reasoning.

Statement 1 (The Naturals $\mathbb{N}$)**.** *Our construction begins with the natural numbers, $\mathbb{N}$, which are born from the process of counting.*

$$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$$

The counting process is achieved by incrementing the current 'count' to obtain the 'next'. That is to say, you add 1 to get the next natural number. We now state what 1 is.

Definition 2 (The Unit, 1.)**.** *The idea of 1 is whatever we want it to be, but it expresses a single unit of measure, be it metres, seconds, sheep, euros or electric charge.*

The ideas of 1 and of adding 1 by themselves produce both $\mathbb{N}$ and the operation of addition, where adding n is the same as incrementing n times.

Definition 3 (Adding $n \in \mathbb{N}$.)**.**

$$\textit{Addition} \quad \Rightarrow \quad m + n \equiv m + \overbrace{1 + 1 + \ldots + 1}^{n}$$

Now we have a set of numbers, $\mathbb{N}$, and an operation that can take place between any two numbers, called a binary operation $+$.

Definition 4 (Binary Operation)**.** *A binary operation $*$ is an operation defined on a set $\mathbb{A}$ such that*

$$(\forall a, b \in \mathbb{A}) \quad a * b \quad \textit{exists.}$$

The naturals and addition between naturals are now in our bank of knowledge. Now, we consider a property between a set of numbers and a binary operation on this set that will be central to our discussion.

Definition 5 (Couple)**.** *Consider a set of numbers $\mathbb{A}$ and a binary operation $*$ on $\mathbb{A}$. Then we will denote a **couple** as the pairing $(\mathbb{A}, *)$.*

Definition 6 (Closed Couple)**.** *A couple* $(\mathbb{A}, *)$ *will be called* ***closed*** *when*

$$(\forall a, b \in \mathbb{A}) \quad a * b \in \mathbb{A}$$

Statement 7 (Natural Addition is Closed)**.** *The couple* $(\mathbb{N}, +)$ *is closed. Stating this explicitly,*

$$(\forall n, m \in \mathbb{N}) \quad (n + m) \in \mathbb{N}$$

The significance of all of this is that if we only knew natural numbers, and addition between naturals, we also know the answer to any question that looks like $n + m$, $n, m \in \mathbb{N}$, since the answer is another natural number.

Except, along the line, we invent subtraction.

Definition 8 (Subtraction)**.** *If* $a + b = c$*, then we define the subtraction operation '*$-$*' such that* $c - b = a$*.*

Statement 9 (Natural Subtraction is not Closed)**.** *The couple* $(\mathbb{N}, -)$ *is not closed. Stating this explicitly,*

$$(\exists n, m \in \mathbb{N}) \quad (n - m) \notin \mathbb{N}$$

Proof. Note that $1, 2 \in \mathbb{N}$ but $(1 - 2) \notin \mathbb{N}$. This proves the statement. But in general,

$$(\forall n, m \in \mathbb{N}) \quad \begin{array}{ll} (n - m) \in \mathbb{N} & n > m \\ (n - m) \notin \mathbb{N} & n \leq m \end{array}$$

□

Hence, we cannot solve any subtraction question with natural numbers alone. We need new numerical ideas, new numbers to append to $\mathbb{N}$, that we can conceptually understand, specifically to answer the subtraction question. Particular notation is irrelevant, but we expect the reader to already know what history has decided the solution and its notation to be. We came up with zero and the negative numbers, and they solve this problem beautifully.

Definition 10 (The Origin, 0.)**.** *We define the number* 0 *such that*

$$(\forall a) \quad a - a = 0$$

Definition 11 (Negation)**.** *When we ask a subtraction question* $a - b$*, we define the negative of this question as* $b - a$*, and we write*

$$(\forall a, b) \quad -(a - b) = b - a$$

and read this as 'the negative of $a - b$ *is* $b - a$*'. We then call the answer to this question the negative of the original answer. For example, we write*

$$3 - 5 = -(5 - 3) = -2$$

Historically, some people have had philosophical issues with seeing non-natural numbers as 'actual' numbers. At this point, we could either think of -2 as 'the negative of 2', where we understand 2 as the 'actual' number and the negative sign '$-$' as an operation on this number, or, alternatively, we can view -2 as just as much a number as 2 is. Take either view, but the consensus is that we should regard -2 as its own number. With that, we can then define the integers.

Definition 12 (The Nonnegative Integers $\mathbb{N}_0$)**.** *Simply append* 0 *to the beginning of* $\mathbb{N}$*, writing*

$$\mathbb{N}_0 = \{0\} \cup \mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$

Definition 13 (The Integers $\mathbb{Z}$)**.** *For each* $x \in \mathbb{N}$*, define its negative as* $-x$ *and write*

$$\mathbb{Z} = (-\mathbb{N}) \cup \{0\} \cup \mathbb{N} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

Statement 14. *Both* $(\mathbb{Z},+)$ *and* $(\mathbb{Z},-)$ *are closed. Explicitly,*

$$(\forall a,b\in\mathbb{Z}) \quad (a+b)\in\mathbb{Z} \quad \textit{and} \quad (a-b)\in\mathbb{Z}$$

We will then list the properties that should be known, without proof.

Statement 15. *Note the following.*

- $(\forall a,b,c\in\mathbb{Z}) \;\; a+b-c=a-c+b$
- $(\forall a,b\in\mathbb{Z}) \;\; a+b=a-(-b)$
- $(\forall a,b\in\mathbb{Z}) \;\; a-b=a+(-b)$

The next step is building the idea of multiplication. Similarly to how addition was repeatedly adding 1, or incrementing, multiplication can be firstly understood as repeated addition.

Definition 16. *We define multiplication of integers as*

$$(\forall n,m\in\mathbb{Z}) \quad nm=\overbrace{n+n+\ldots+n}^{m}$$

This will either be repeated addition or subtraction, depending on the signs of the factors.

Statement 17. *Multiplication is commutative. That is,*

$$(\forall n,m\in\mathbb{Z}) \quad nm=mn$$

Proof.

$$(\forall a,b\in\mathbb{Z})(a,b>0) \qquad ab=\left(\left.\overbrace{\begin{matrix}\bigcirc & \bigcirc & \dots & \bigcirc \\ \bigcirc & \bigcirc & \dots & \bigcirc \\ \vdots & \vdots & \ddots & \vdots \\ \bigcirc & \bigcirc & \dots & \bigcirc\end{matrix}}^{a}\right\} b\right)$$

Observe the rectangle representing ab, and that rotating it a right-angle will give ba, but will definitely have the same number of dots. Hence, $ab=ba$. $\square$

Statement 18. *Both* $(\mathbb{N},\times)$ *and* $(\mathbb{Z},\times)$ *are closed, where* $\times$ *denotes multiplication. Explicitly,*

$$(\forall a,b\in\mathbb{N},n,m\in\mathbb{Z}) \quad ab\in\mathbb{N} \quad \textit{and} \quad nm\in\mathbb{Z}$$

However, we then define division, similarly to subtraction.

Definition 19 (Division)**.** *If* $ab=c$*, then we define the division operation* $\div$ *such that* $c\div b=a$*. We also write*

$$\frac{c}{b}=a$$

Statement 20 (Division of 0)**.**

$$(\forall c) \quad \frac{c}{0} \quad \textit{has no answer}$$

Proof. Suppose the contrary, that we could answer the question of $\frac{c}{0}$ for some c with some a. That is,

$$(\exists a) \quad \frac{c}{0}=a$$

Yet, by the definition of division, this would imply

$$c=0a$$

which demands that $c = 0$. This means we definitely cannot answer the question for $c \neq 0$. This leaves the question then of

$$\frac{0}{0} = a$$

Again, this is equivalent to $0a = 0$, which is in fact true for any a. So, we have

$$(\forall a) \quad \frac{0}{0} = a$$

This is nonsense. We are uninterested in a question which has literally any number as an answer. Due to this, we disregard the case of $c = 0$ as well. Therefore, the statement holds for all c. $\square$

Definition 21. *We write $\mathbb{Z}_{\neq 0}$ as the integers without 0. That is,*

$$\mathbb{Z}_{\neq 0} = \mathbb{Z} \setminus \{0\}$$

This will be somewhat useful when writing statements about division, since it will remove the complications of dividing by zero being possible.

Statement 22. $(\mathbb{Z}_{\neq 0}, \div)$ *is not closed. Explicitly,*

$$(\exists a, b \in \mathbb{Z}_{\neq 0}) \quad \frac{a}{b} \notin \mathbb{Z}_{\neq 0}$$

Proof. We only need to show that $1, 2 \in \mathbb{Z}_{\neq 0}$ but $\frac{1}{2} \notin \mathbb{Z}_{\neq 0}$. $\square$

So again we find ourselves unequipped to answer questions of division. We solve this problem similarly to the subtraction problem. We simply define, using the operation itself, numbers that satisfy the operational question. That is, as we defined negatives, we now define ratios.

Definition 23 (The Rationals, $\mathbb{Q}$)**.** *Let*

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z},\ q \neq 0 \right\} = \left\{ \frac{p}{q} \mid p \in \mathbb{Z},\ q \in \mathbb{N} \right\} = \left\{ \pm\frac{p}{q} \mid p, q \in \mathbb{N} \right\}$$

Definition 24. *Let*

$$\mathbb{Q}_{\neq 0} = \{x \in \mathbb{Q} \mid x \neq 0\} = \mathbb{Q} \setminus \{0\}$$

Statement 25. *All of* $(\mathbb{Q}, +)$, $(\mathbb{Q}, -)$, $(\mathbb{Q}, \times)$, *and* $(\mathbb{Q}_{\neq 0}, \div)$ *are closed. Explicitly,*

$$(\forall a, b \in \mathbb{Q}) \quad (a + b), (a - b), ab \in \mathbb{Q}$$

$$(\forall a, b \in \mathbb{Q}_{\neq 0}) \quad \frac{a}{b} \in \mathbb{Q}_{\neq 0}$$

Proof. Let $a, b, c, d \in \mathbb{Z}$, with $b, d \neq 0$. Then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Note that $(ad + bc) \in \mathbb{Z}$ since $(\mathbb{Z}, +)$ and $(\mathbb{Z}, \times)$ are closed, and that $bd \in \mathbb{Z}$ with $bd \neq 0$. So $(\mathbb{Q}, +)$ indeed closed. Similarly, we find $(\mathbb{Q}, -)$ is closed, as

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

Then, by commutativity,

$$\frac{a}{b}\frac{c}{d} = \frac{ac}{bd}$$

implies $(\mathbb{Q}, \times)$ is closed. Similarly, we find $(\mathbb{Q}_{\neq 0}, \div)$ is closed, since, for $c \neq 0$,

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$$

$\square$

Things become interesting as we define exponentiation and rooting (also known as radicalisation). A motivating reason for exploring exponents is that they neaten up repeated multiplication, although it must be said at the onset that this is not the defining feature of why exponents are crucial mathematical constructs. But it is fair to say that exponentiation is to multiplication what multiplication is to addition. Although, we lose commutativity for exponentiation. We now state the exponent laws without proof.

Statement 26 (Exponent Laws)**.** *Note the following.*

- $(\forall a) \quad a^1 = a$
- $(\forall a, b, c)(a^2 + c^2 \neq 0)(a^2 + b^2 \neq 0) \quad a^b a^c = a^{b+c}, \qquad (\forall a \neq 0) \;\; \frac{a^b}{a^c} = a^{b-c}$
- $(\forall a, n)(a^2 + n^2 \neq 0) \quad a^{-n} = \frac{1}{a^n}$
- $(\forall a \neq 0) \quad a^0 = 1$
- $(\forall a, b, c)(a^2 + b^2 + c^2 \neq 0) \quad (a^b)^c = a^{bc}, \qquad (\forall c \neq 0) \;\; \sqrt[c]{a^b} = a^{\frac{b}{c}}$

Note the following similarity between exponentiation and multiplication.

Statement 27. $(\mathbb{N}, \wedge)$ *is closed while* $(\mathbb{Z}_{\neq 0}, \wedge)$ *is not closed, where* $\wedge$ *denotes exponentiation. Explicitly,*

$$(\forall n, m \in \mathbb{N}) \quad nm \in \mathbb{N}$$

$$(\exists a, b \in \mathbb{Z}) \quad a^b \notin \mathbb{Z}$$

Proof. The fact that $(\mathbb{N}, \wedge)$ is closed is trivial. For the other,

$$(2, -1 \in \mathbb{Z}) \quad 2^{-1} = \frac{1}{2} \notin \mathbb{Z}$$

□

Next, we want to establish a few theorems.

Definition 28 (Proper Subset)**.** *To be explicit, the proper subset symbol '$\subset$' is to mean, taking $A \subset B$, that*

$$(\forall x \in A) \;\; x \in B, \quad \textit{and} \quad (\exists x \in B) \;\; x \notin A$$

Lemma 29. $\mathbb{N} \subset \mathbb{Z}$*.*

Proof. This is trivial from how $\mathbb{Z}$ is defined. □

Statement 30. $\mathbb{Z} \subset \mathbb{Q}$*.*

Proof. Take some $x \in \mathbb{Q}$ and $p, q \in \mathbb{Z}$, with $q \neq 0$ so that $x = \frac{p}{q}$. By setting $q = 1$, we find that $\mathbb{Q} \ni x = p \in \mathbb{Z}$. So we see that any $p \in \mathbb{Z}$ can be associated with an equal $x \in \mathbb{Q}$. But $\frac{1}{2}$ is an obvious rational that is not an integer. □

Now we know that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$. To continue the chain and show that $\mathbb{Q} \subset \mathbb{R}$, we need to convince ourselves that all rationals are real but there is some $x \in \mathbb{R}$ that is not rational.

Assumption 31 (Real Numbers as Lengths)**.** *For now, we will make the explicit assumption that any point a on a number line can be associated with a real number describing its distance from the origin. We then write $a \in \mathbb{R}$ if this can be shown geometrically or otherwise. See figure (1).*

Corollary 32.

$$(\forall x \in \mathbb{Q}) \quad x \in \mathbb{R}$$

Corollary 33 (Pythagorean Realism)**.**

$$(\forall a, b \in \mathbb{R}) \quad \sqrt{a^2 + b^2} \in \mathbb{R}$$

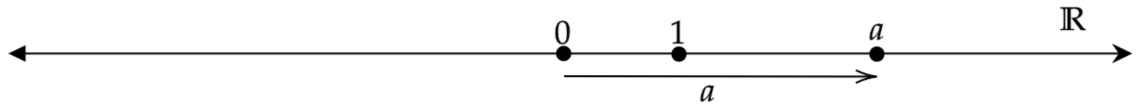


Figure 1: This is the real number line, showing how we can imagine any length being associated with a real number $a \in \mathbb{R}$.

Corollary 34.

$$\sqrt{2} \in \mathbb{R}$$

Theorem 35.

$$\sqrt{2} \notin \mathbb{Q}$$

Proof. Suppose, to the contrary, that $\sqrt{2} \in \mathbb{Q}$. This is equivalent to

$$(\exists p_0 \in \mathbb{N}, q_0 \in \mathbb{N}) \qquad \sqrt{2} = \frac{p_0}{q_0}$$

$$\Rightarrow \qquad p_0^2 = 2q_0^2 \tag{1}$$

This means p_0 is even, as p_0 cannot be odd. If it was, then p_0^2 would also be odd, but $2q_0^2 = p_0^2$ is an even number due to the factor of 2.

$$\therefore \quad (\exists p_1 \in \mathbb{N}) \qquad p_0 = 2p_1$$

$$\Rightarrow \qquad (2p_1)^2 = 2q_0^2 \qquad \Rightarrow \qquad q_0^2 = 2p_1^2$$

For the same reasons, q_0 is even and, following the same substitution, we get $q_1 \in \mathbb{N}$ such that

$$(2q_1)^2 = 2p_1^2 \qquad \Rightarrow \qquad p_1^2 = 2q_1^2$$

But (p_1, q_1) obeys the same condition, equation (1), as (p_0, q_0). Hence, from the same line of reasoning, we will find p_1, q_1 to both be even as well. If we keep writing $p_k = 2p_{k+1}$, then we will find arbitrarily large n such that p_n, q_n are even. In other, more informal, words, we get that p_0, q_0 are infinitely halve-able, while still being finite integers. But this is nonsense. No integers have an infinite number of factors of 2. This absurdity that confronts us leaves us with no choice but to conclude, as difficult or unbelievable as it may be, that there exist no suitable p_0, q_0 for $\sqrt{2}$ to be rational. And therefore, $\sqrt{2}$ is irrational. □

Corollary 36.

$$\mathbb{Q} \subset \mathbb{R}$$

Corollary 37.

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

1.2 Axioms of Bound

In this section, we introduce concepts of boundedness on sets, and once that is done provide two important axioms that, once we accept, can use to prove fundamental results. These are axioms are almost trivial when we think of the way the relevant numbers behave.

1.2.1 Definitions

Definition 38 (Bounds). *An upper bound of a set* $\mathbb{A} \subseteq \mathbb{R}$ *is defined as a number* $A \in \mathbb{R}$ *such that*

$$(\forall a \in \mathbb{A}) \quad a \leq A$$

We then define the set $\operatorname{upp}(\mathbb{A})$ *as the set of all upper bounds of* $\mathbb{A}$.

$$\operatorname{upp}(\mathbb{A}) = \{A \in \mathbb{R} \mid (\forall a \in \mathbb{A})\ a \leq A\}$$

We say that $\mathbb{A}$ *is bounded above if* $\operatorname{upp}(\mathbb{A}) \neq \emptyset$, *and that* $\mathbb{A}$ *is bounded above by all* $A \in \operatorname{upp}(\mathbb{A})$.

Similarly, a lower bound of a set $\mathbb{B} \subseteq \mathbb{R}$ *is defined as a number* $B \in \mathbb{R}$ *such that*

$$(\forall b \in \mathbb{B}) \quad b \geq B$$

We then define the set $\operatorname{low}(\mathbb{B})$ *as the set of all lower bounds of* $\mathbb{B}$.

$$\operatorname{low}(\mathbb{B}) = \{B \in \mathbb{R} \mid (\forall b \in \mathbb{B})\ b \geq B\}$$

We say that $\mathbb{B}$ *is bounded below if* $\operatorname{low}(\mathbb{B}) \neq \emptyset$, *and that* $\mathbb{B}$ *is bounded below by all* $B \in \operatorname{low}(\mathbb{B})$.

Definition 39 (Sup and Inf). *We define the supremum of the set* $\mathbb{A} \subseteq \mathbb{R}$ *as the least upper bound of* A, *and, denoting it as* $\sup(\mathbb{A})$, *we write*

$$\sup(\mathbb{A}) = A_0 \qquad \Leftrightarrow \qquad (\exists A_0 \in \mathbb{R})(\exists \mathbb{A} \subseteq \mathbb{R})(\operatorname{upp}(\mathbb{A}) \neq \emptyset)(\forall A \in \operatorname{upp}(\mathbb{A})) \quad A_0 \leq A$$

Similarly, we define the infimum of the set $\mathbb{B} \subseteq \mathbb{R}$ *as the greatest lower bound of* B, *and, denoting it as* $\inf(\mathbb{B})$, *we write*

$$\inf(\mathbb{B}) = B_0 \qquad \Leftrightarrow \qquad (\exists B_0 \in \mathbb{R})(\exists \mathbb{B} \subseteq \mathbb{R})(\operatorname{low}(\mathbb{B}) \neq \emptyset)(\forall B \in \operatorname{low}(\mathbb{B})) \quad B_0 \geq B$$

Definition 40 (Max and Min). *We define the maximum of a set* $\mathbb{A} \subset \mathbb{R}$ *as the greatest element of* $\mathbb{A}$. *Denoting it as* $\max(\mathbb{A})$, *we write*

$$\max(\mathbb{A}) = A \qquad \Leftrightarrow \qquad (\exists A \in \mathbb{A})(\exists \mathbb{A} \subseteq \mathbb{R})(\mathbb{A} \neq \emptyset)(\forall a \in \mathbb{A}) \quad A \geq a$$

Similarly, we define the minimum of a set $\mathbb{B} \subset \mathbb{R}$ *as the least element of* $\mathbb{A}$. *Denoting it as* $\min(\mathbb{B})$, *we write*

$$\min(\mathbb{B}) = B \qquad \Leftrightarrow \qquad (\exists B \in \mathbb{B})(\exists \mathbb{B} \subseteq \mathbb{R})(\mathbb{B} \neq \emptyset)(\forall b \in \mathbb{B}) \quad B \leq b$$

Example 41. *The simplest example of the difference between the supremum and maximum (and, likewise, the infimum and minimum), can be explained shortly the following way. Take the following set*

$$X = \{x \in \mathbb{R} \mid x < 2\}$$

Then $\sup(X) = 2$ *but* X *has no maximum. Why? However, take the set*

$$Y = \{y \in \mathbb{R} \mid y \leq 2\}$$

and then $\sup(Y) = \max(Y) = 2$.

Statement 42 (Also Min and Max). *From the stated definitions, we can also say*

$$\max(A) = a_0 \qquad \Leftrightarrow \qquad a_0 = \sup(A) \quad \textit{and} \quad a_0 \in A$$

which is that the maximum is nothing but the supremum that is also in the set. In general, the supremum does not have to be in the set. Similarly,

$$\min(B) = b_0 \qquad \Leftrightarrow \qquad b_0 = \inf(A) \quad \textit{and} \quad b_0 \in B$$

which is that the minimum is nothing but the infimum that is also in the set.

Proof. Exercise. $\square$

1.2.2 Supremum Properties

Statement 43 (Some Supremum Properties)**.** *Take* $K \subseteq \mathbb{R}$*, and let* $a + bK = \{a + bk \mid k \in K\}$ *for the sake of notation. Then the following makes sense.*

$$\sup(-K) = -\inf(K)$$

$$(\forall c \in \mathbb{R}) \quad \sup(c + K) = c + \sup(K)$$

$$(\forall a \in \mathbb{R})(a > 0) \quad \sup(aK) = a \sup(K)$$

Proof. Verifying these properties comes down to the behaviour of inequalities. From the definition, saying $\sup(K) = k_0$ means

$$(\exists k_0 \in \mathbb{R})(\exists K \subseteq \mathbb{R})(\operatorname{upp}(K) \neq \emptyset)(\forall u \in \operatorname{upp}(K)) \quad k_0 \leq u$$

by definition. We leave it to you to fill in the details of how the definition reveals the two properties. □

The first of the above properties lets us know that any result about the supremum can be connected a result about the infimum.

Statement 44 (Supremum uniqueness)**.** *Take some set* $K \subseteq \mathbb{R}$*, bounded above. Then the number* $\sup(K)$ *is unique. That is, there is only one real number* k_0 *such that* $k_0 = \sup(K)$*.*

Proof. This should be trivial to prove by contradiction. Assume the contrary,

$$(\exists k_1, k_2 \in \mathbb{R})(\exists K \subseteq \mathbb{R})(k_1 \neq k_2) \qquad k_1 = \sup(K) \quad \text{and} \quad k_2 = \sup(K)$$

Since k_1, k_2 are supremums of K, by definition we have $k_1, k_2 \in \operatorname{upp}(K)$. Also by definition, $(\forall k \in \operatorname{upp}(K))$ $k_1 \leq k$ and $k_2 \leq k$, as they are the least upper bounds. That gives us

$$k_1 \leq k_2 \qquad \text{and} \qquad k_2 \leq k_1$$

Hence,

$$k_1 = k_2$$

□

Statement 45. *Any real number* $x \in \mathbb{R}$ *can, in effect, be defined by*

$$x = \sup\{a \in \mathbb{R} \mid a < x\}$$

or

$$x = \inf\{a \in \mathbb{R} \mid a > x\}$$

1.2.3 The Two Fundamental Axioms

Axiom 46 (The Well-Ordering Principle, of $\mathbb{N}$)**.** *Every non-empty subset of the natural numbers has a least element. That is, any set of natural numbers contains a number that is less than every other number in the set. Writing this in gobbledygook,*

$$(\forall \mathbb{A} \subseteq \mathbb{N})(\mathbb{A} \neq \emptyset)(\exists n_0 \in \mathbb{A})(\forall n \in \mathbb{A} \setminus \{n_0\}) \qquad n_0 < n$$

This axiom particularly distinguishes the nature of $\mathbb{N}$ from $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$. The natural numbers 'start somewhere'. The following axiom distinguishes $\mathbb{R}$ from $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$, by providing a rigorous way of putting that 'the real numbers have no holes'.

Axiom 47 (The Completeness Axiom, on $\mathbb{R}$)**.** *Every non-empty subset of* $\mathbb{R}$ *that is bounded above has a least upper bound. In gobbledygook,*

$$(\forall A \subseteq \mathbb{R})(A \neq \emptyset) \qquad \operatorname{upp}(A) \neq \emptyset \quad \Rightarrow \quad (\exists a_0 \in \mathbb{R}) \quad a_0 = \sup(A)$$

1.2.4 The Fruit: Theorems

Theorem 48. *Given some $n \in \mathbb{N}$, if you can find some $k \in \mathbb{N}$ such that*

$$k < \sqrt{n} < k+1,$$

that is, n is not a perfect square, then $\sqrt{n} \notin \mathbb{Q}$.

Proof. We are given that

$$(\exists n, k \in \mathbb{N}) \quad k < \sqrt{n} < k+1$$

Next, suppose, to the contrary, that $\sqrt{n} \in \mathbb{Q}$. Then, by definition, we must have

$$(\exists p, q \in \mathbb{N}) \quad \sqrt{n} = \frac{p}{q} \qquad \Rightarrow \qquad q\sqrt{n} \in \mathbb{N}$$

Put $\mathbb{N} \supseteq S = \{q \in \mathbb{N} \mid q\sqrt{n} \in \mathbb{N}\}$. We know that S is non-empty since, supposedly, $\frac{p}{\sqrt{n}} \in S$. So, by the Well-Ordering Principle, S has a least element, $q_0 \in \mathbb{N}$.

By assumption, we have

$$0 < \sqrt{n} - k < 1,$$

so putting $q_1 = (\sqrt{n} - k)q_0$ reveals $0 < q_1 < q_0$, and since $q_1 = q_0\sqrt{n} - kq_0$, which is a difference of two integers, we get $q_1 \in S$. This is a contradiction, as q_0 was meant to be the least element in S. The supposition is thus charged false, deeming the theorem proven. □

Corollary 49. *There are infinitely many irrational numbers.*

Example 50. *We know that $\sqrt{2}$ is irrational and only have to state*

$$1 < \sqrt{2} < 2$$

that is, $k = 1$, to prove it, in light of the above theorem. We also know $\sqrt{3} \notin \mathbb{Q}$ since $1 < \sqrt{3} < 2$ as well. The theorem can be continued to be used to trivially prove the irrationality of $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}, \ldots$

Note that if it is a perfect square being square rooted, it is obviously rational, meaning that the converse of the above theorem is true too. This is an interesting result, but we are yet to investigate the 'density' of $\mathbb{Q}$ in $\mathbb{R}$. This next result proves the Pythagorean Realism we assumed earlier, built on the Completeness Axiom.

Theorem 51. *For any positive $x \in \mathbb{R}$, we have $\sqrt{x} \in \mathbb{R}$ as well.*

Proof. Take the following for granted.

$$(\exists x \in \mathbb{R})(x > 0) \quad A = \{a \in \mathbb{R} \mid a^2 < x\}$$

and consider the set $A = \{a \in \mathbb{R} \mid a^2 < x\}$. The Completeness Axiom only tells us that there does exist some $M \in \mathbb{R}$, $M > 0$, which is the least upper bound for A. Using our intuition, we can be reasonably certain that this supremum of A is precisely $\sqrt{x}$. But this must be proven.

The proof for this is going to utilise the two stringent conditions placed on M, the first of which is that M is an upper bound for A, and secondly, it is the smallest number that is so. Thus, shift M is restricted from shifting by any distance to the left or right on the real number line, for any distance to the left and it can no longer be an upper bound, since it was already the smallest upper bound, and any distance to the right and it is now a larger upper bound, and hence no longer the smallest. This 'immovability' of M is exactly what we will exploit to show that our intuitions are correct, and $M = \sqrt{x}$. We begin our proof.

Suppose that $M^2 < x$. Tactically, we suspect that this supposition will contradict the condition that M is an upper bound for A. So, we suspect we may be able to arrive at $(\exists h \in \mathbb{R})(h > 0) \ \ (M + h) \in \mathbb{A}$. This motivates our exploration of the expression

$$(M+h)^2 = M^2 + 2Mh + h^2.$$

Constrict $h \leq 1$ to guarantee $h^2 \leq h$. Under this constriction,

$$(M+h)^2 \leq M^2 + (2M+1)h$$

Now, consider $(2M+1)h < x - M^2$. By the supposition, $x - M^2 > 0$, and, of course, $M > 0$, so all of the requirements on h are achievable. We can thus find a small enough $h > 0$ such that

$$(M+h)^2 < M^2 + (x - M^2) = x$$

This states that $(M+h) \in A$, which supplies the necessary contradiction as suspected. In conclusion, $M^2 < x$ is false, and so $M^2 \geq x$.

Suppose that $M^2 > x$. As before, we employ tactic, and suspect that this supposition will contradict that M is the least of all upper bounds for A. So, we suspect we may be able to arrive at the existence of an upper bound $(\exists h \in \mathbb{R}(h > 0) \;\; (M+h)$ is an upper bound for A. As before, we can find the necessary h to say

$$\begin{aligned}(M-h)^2 &= M^2 - 2Mh + h \\ &> M^2 - 2Mh \\ &> M^2 - (M^2 - x) \\ &= x\end{aligned} \tag{2}$$

which shows the necessary contradiction. Thus, we also conclude $M^2 > x$ is false. We are left with

$$x \leq M^2 \leq x, \qquad M^2 = x.$$

□

The next result encapsulates the intuition that real line has 'no gaps' by saying that 'zooming in infinitely' still provides a real number.

Theorem 52 (Nested Interval Property)**.** *Define*

$$(\forall n \in \mathbb{N})(\exists a_n, b_n \in \mathbb{R}) \qquad I_n = \{x \in \mathbb{R} \mid a_n \leq x \leq b_n\} \equiv [a_n, b_n]$$

such that $I_n \supseteq I_{n+1}$, that is, each I_n contains I_{n+1}. Then

$$\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$$

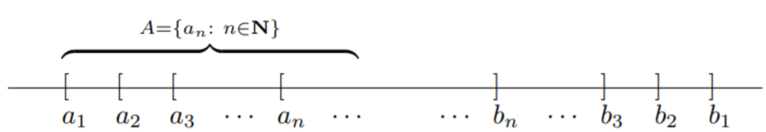
Proof. Consider the sets of all the lower bounds

$$A = \{a_n \mid n \in \mathbb{N}\}$$

and all the upper bounds

$$B = \{b_n \mid n \in \mathbb{N}\}$$

and observe



that $B \subseteq \text{upp}\,(A)$, which is to say that $(\forall n \in \mathbb{N})\; b_n$ is an upper bound for the set A. Now, since this makes A bounded, the Completeness Axiom states that

$$(\exists x_0 \in \mathbb{R}) \qquad x_0 = \sup(A)$$

which gives us $(\forall n \in \mathbb{N}) \quad a_n \leq x_0$ since $\sup(A) \in \text{upp}\,(A)$. Further, since $B \subseteq \text{up}(A)$, we have that $(\forall n \in \mathbb{N}) \quad b_n \in \text{upp}\,(A)$, and thereby, since x_0 is the least upper bound of A, that $(\forall n \in \mathbb{N}) \quad x_0 \leq b_n$. Thus, we conclude

$$(\forall n \in \mathbb{N}) \quad a_n \leq x_0 \leq b_n$$

which places $(\forall n \in \mathbb{N}) \quad x_0 \in I_n$. □

We are now ready to investigate how $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ sit between each other. The following results will establish some ideas of this.

Lemma 53. *Assume that, for some $A \subseteq \mathbb{R}$, some upper bound $s \in \mathbb{R}$ of A exists. Then s is not only an upper bound, but also the supremum of A, if and only if, for all $\varepsilon > 0$, we can find some $a \in A$ such that $a - \varepsilon < s$. In gobbledygook,*

$$(\exists A \subseteq \mathbb{R})(\exists s \in \operatorname{upp}(A)) \quad s = \sup(A) \Leftrightarrow [(\forall \varepsilon \in \mathbb{R})(\varepsilon > 0)(\exists a \in A) \quad s - \varepsilon < a]$$

Proof. If we think of the nature of the supremum, it acts as the precise boundary of a bounded set. The supremum is the very specific smallest-of-all upper bound. Intuitively, then, an arbitrarily small decrease of the supremum must necessarily lose its status as an upper bound, which is what $s - \varepsilon < a$ states, since s is the supremum, $-\varepsilon$ is an arbitrarily small decrease, with $(s - \varepsilon)$ being the result of that decrease, and finally $(s - \varepsilon) < a$ saying $(s - \varepsilon)$ is not an upper bound since it is no longer greater than all elements of A. $\square$

Theorem 54 (The Archimedean Property)**.** *The following are true.*

i) $(\forall x \in \mathbb{R})(\exists n \in \mathbb{N}) \quad n > x$.

ii) $(\forall y \in \mathbb{R})(\exists n \in \mathbb{N}) \quad 1/n < y$.

Proof. For i, suppose the contrary, that

$$(\exists x \in \mathbb{R})(\forall n \in \mathbb{N}) \quad x > n$$

Then $\mathbb{N}$ is a non-empty subset of $\mathbb{R}$ that is bounded above by x, and thus it has a least upper bound, by the Completeness Axiom. We write

$$(\exists s \in \mathbb{R})(s \leq x) \quad s = \sup(\mathbb{N})$$

By nature of s being the least upper bound of $\mathbb{N}$, we know that $s-1$ is not an upper bound, and therefore

$$(\exists m \in \mathbb{N}) \quad m > s - 1$$

But then we have that $m + 1 > s$ and $(m + 1) \in \mathbb{N}$. This contradicts that s is supposed to be an upper bound for $\mathbb{N}$. Hence, $\mathbb{N}$ is indeed unbounded from above.

Part ii follows immediately from part i by simply letting $x = 1/y$. See that then $n > 1/y$ becomes $1/n < y$ as required. $\square$

And we'll just note this small result as an exercise:

Statement 55 (Cut Property)**.** *If A and B are nonempty, disjoint sets, with $A \cup B = \mathbb{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists some $c \in \mathbb{R}$ such that $a \leq c$ for all $a \in A$ and $b \geq c$ for all $b \in B$. In gobbledygook,*

$$(\exists A, B \subseteq \mathbb{R})(A, B \neq \emptyset)(A \cap B = \emptyset)(A \cup B = \mathbb{R})(\forall a \in A)(\forall b \in B) \quad a < b$$

$$\Rightarrow \qquad (\exists c \in \mathbb{R})(\forall a \in A)(\forall b \in B) \quad a \leq c \leq b$$

Proof. This is a fairly obvious result. We leave it as exercise to use the Completeness Axiom to prove it. $\square$

1.3 Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem 56. *For any two real numbers a and b, we can find a rational number q such that q lies between a and b. In gobbledygook,*

$$(\forall a, b \in \mathbb{R})(\exists q \in \mathbb{Q}) \quad a < q < b$$

Proof. Assuming $q > 0$, which we can without loss of generality, we have, by definition,

$$(\exists n, m \in \mathbb{N}) \quad q = \frac{n}{m}$$

Then we want to show that

$$(\forall a, b \in \mathbb{R})(\exists n, m \in \mathbb{N}) \quad am < n < bm$$

Finding n is simple enough. Choose n to be the integer such that

$$n - 1 \leq am < n$$

which is to say that n is the nearest integer above am. Now it remains to choose m such that $n < bm$. Using the Archimedean Property, we can choose m large enough such that

$$\frac{1}{m} < b - a$$

meaning that an increment of $1/m$ cannot 'step over' the interval $[a, b]$ without entering it first. This would imply, at least intuitively, that n/m will be found within $[a, b]$. To complete the proof, we write

$$\begin{aligned} n &\leq am + 1 \\ &< \left(b - \frac{1}{m}\right) m + 1 \\ &= bm \end{aligned}$$

And thus we have $n/m < b$ as well as $n/m > a$, giving us the required $a < n/m < b$. $\square$