BDA - Assignment 3

Anonymous

```
library(ggplot2)
theme_set(theme_minimal())
library(tidyr)
library(aaltobda)
data("windshieldy1")
head(windshieldy1)
```

```
## [1] 13.357 14.928 14.896 15.297 14.820 12.067
```

Problem 1: Inference for normal mean and deviation

We are assuming that the observations follow a normal distribution with an unknown standard deviation σ and unknown mean μ. So, the observational model:

$$p(y) = \mathcal{N}(\mu, \sigma)$$

A noninformative prior distribution, assuming prior independence of location and scale parameters, is uniform on $(\mu, \log \sigma)$ or, equivalently $p(\mu, \sigma) \propto (\sigma^2)^{-1}$

Under this conventional improper prior density, the joint posterior distribution is proportional to the likelihood function multiplied by the factor $1/\sigma^2$:

$$p(\mu, \sigma^2|y) = \sigma^{-n-2} exp(\frac{1}{2\sigma^2}[(n-1)s^2 + n(\bar{y} - \mu)^2])$$

where $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$

a) (Value of the unknown μ)

```
data <- windshieldy1
n <- length(data)
mu <- mean(data)
sigma <- sd(data)
num_samples <- 100000

x <- seq(10, 20, 0.01)
exact_posterior_mu <- dtnew(x, df=n-1, mean=mu, scale=sigma/sqrt(n))
emprical_posterior_mu <- dnorm(x, mu, sigma/sqrt(n))


mu_point_est <- function(data){
  n <- length(data)
  mu <- mean(data)
  sigma <- sd(data)
  rtg <- rt(num_samples, df=n-1)
  rr <- (rtg * sigma/sqrt(n) ) + mu
```

```r
  mu_post <- mean(rr)

  return(mu_post)
}

mu_interval <- function(data, prob){
  n <- length(data)
  mu <- mean(data)
  sigma <- sd(data)
  rtg <- rt(num_samples, df=n-1)
  rr <- (rtg * sigma/sqrt(n) ) + mu
  q <- quantile(rr, c((1-prob)/2, prob+(1-prob)/2), names = FALSE)
  return(q)
}


cat("The mean of the mu is: \n")
```

```
## The mean of the mu is:
```

```r
mu_point_est(data)
```

```
## [1] 14.61064
```

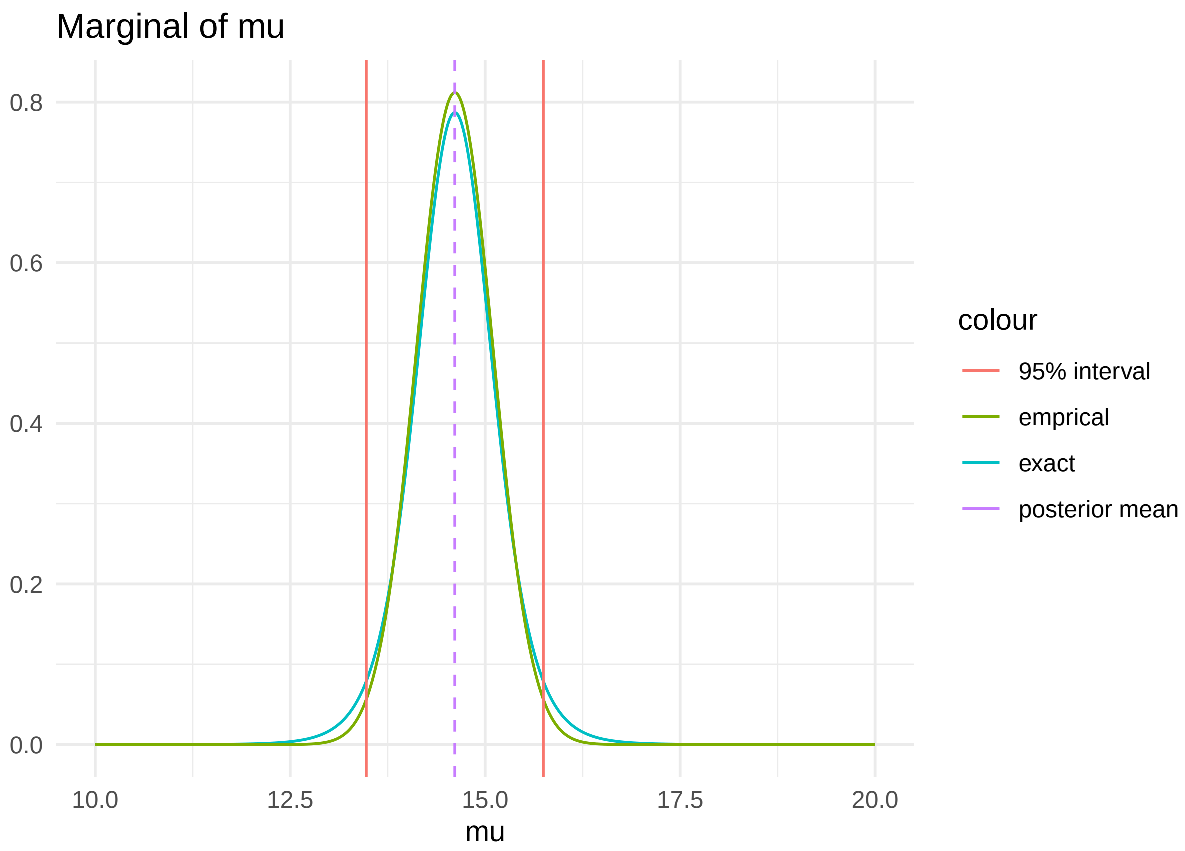
```r
cat("\n The interval estimates (95%) of the mu is: \n")
```

```
##
##  The interval estimates (95%) of the mu is:
```

```r
mu_interval(data, prob = 0.95)
```

```
## [1] 13.46443 15.75414
```

```r
ggplot() +
  geom_line(aes(x, exact_posterior_mu, color='exact')) +
  geom_line(aes(x, emprical_posterior_mu, color='emprical')) +
  geom_vline(aes(xintercept = mu_point_est(data), color = 'posterior mean'),
             linetype = 'dashed', show.legend = F) +
  geom_vline(aes(xintercept = c(mu_interval(data, prob = 0.95)), color = '95% interval'),
             linetype = 'solid', show.legend = F) +
  labs(title = 'Marginal of mu', x = 'mu', y = '')
```



b) (the posterior predictive distribution

To draw from the posterior predictive distribution, we first draw (μ, σ^2) from the joint posterior distribution and then simulate $\tilde{y} \propto \mathcal{N}(\mu, \sigma^2)$. Posterior predictive distribution based on integrating (μ, σ^2):

$$p(\tilde{y}|\sigma^2, y) = \int p(\tilde{y}|\mu, \sigma^2, y)p(\mu|\sigma^2, y)d\mu = \mathcal{N}(\tilde{y}|\bar{y}, (1 + \frac{1}{n})\sigma^2)$$

Analytical form of posterior predictive distribution:

$$p(\tilde{y}|\sigma^2, y) = t_{n-1}(\bar{y}, (1 + \frac{1}{n})s^2)$$

```
x <- seq(0, 30, 0.01)
exact_posterior_pred <- dtnew(x, df=n-1, mean=mu, scale=sqrt(sigma*sqrt(1+1/n)))
emprical_posterior_pred <- dnorm(x, mu, sqrt(sigma*sqrt(1+1/n)))


mu_pred_point_est <- function(data){
  n <- length(data)
  mu <- mean(data)
  sigma <- sd(data)
  rtg <- rt(num_samples, df=n-1)
  rr <- (rtg * sigma*sqrt(1+(1/n)) ) + mu
  mu_post <- mean(rr)
```

```r
  return(mu_post)
}

mu_pred_interval <- function(data, prob = 0.95){
  n <- length(data)
  mu <- mean(data)
  sigma <- sd(data)
  rtg <- rt(num_samples, df=n-1)
  rr <- (rtg * sigma*sqrt(1+(1/n)) ) + mu
  q <- quantile(rr, c((1-prob)/2, prob+(1-prob)/2), names = FALSE)
  return(q)
}

cat("The mean of the posterior predictive is: \n")
```

```
## The mean of the posterior predictive is:
```

```r
mu_pred_point_est(data)
```

```
## [1] 14.61433
```

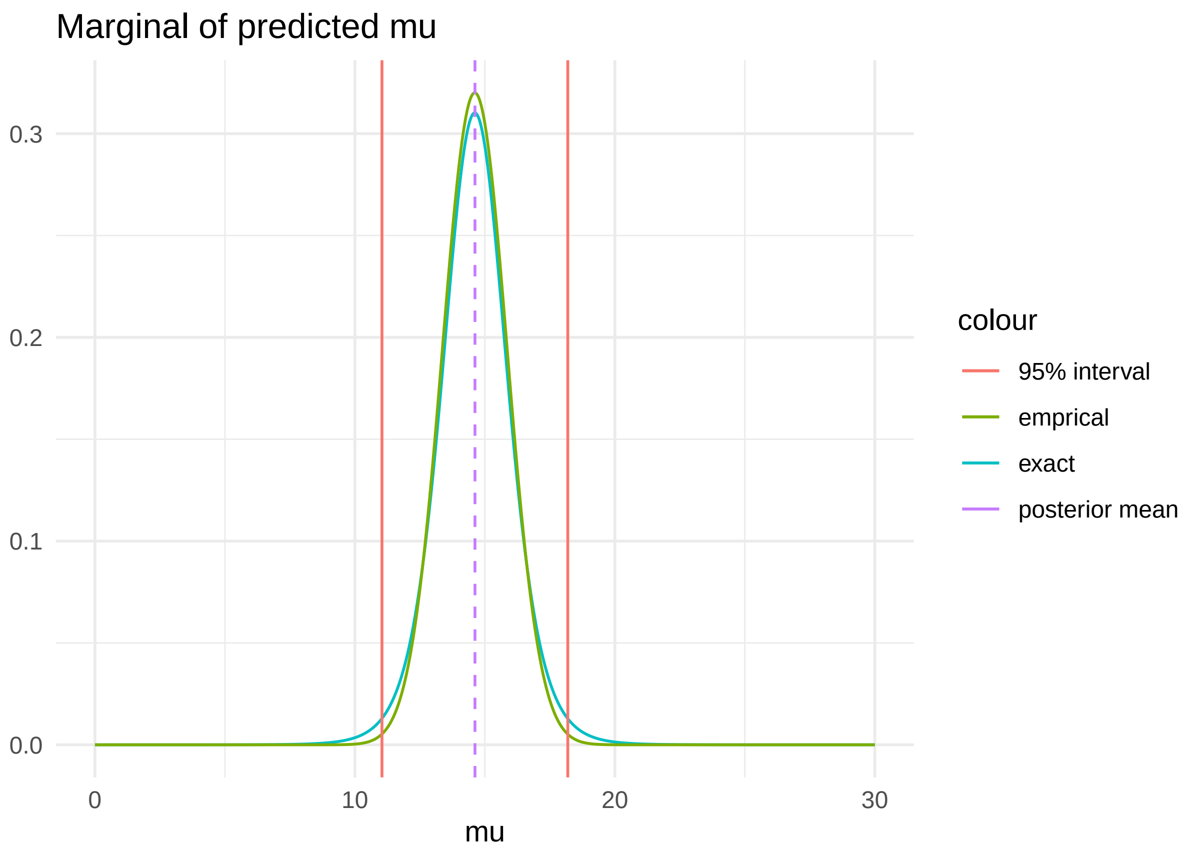
```r
cat("\n The interval estimates (95%) of the posterior predictive is: \n")
```

```
##
##  The interval estimates (95%) of the posterior predictive is:
```

```r
mu_pred_interval(data, prob = 0.95)
```

```
## [1] 11.02353 18.18185
```

```r
ggplot() +
  geom_line(aes(x, exact_posterior_pred, color='exact')) +
  geom_line(aes(x, emprical_posterior_pred, color='emprical')) +
  geom_vline(aes(xintercept = mu_pred_point_est(data), color = 'posterior mean'),
             linetype = 'dashed', show.legend = F) +
  geom_vline(aes(xintercept = c(mu_pred_interval(data, prob = 0.95)), color = '95% interval'),
             linetype = 'solid', show.legend = F) +
labs(title = 'Marginal of predicted mu', x = 'mu', y = '')
```



Problem 2: Inference for the difference between proportions

The observational model:

$$p(y_0, y_1) \propto p_0^{y_0}(1 - p_0)^{n_0 - y_0} p_1^{y_1}(1 - p_1)^{n_1 - y_1}$$

We use independent Beta distribution as priors:

$$p(p_i) = Beta(\alpha_i, \beta_i)$$

So posterior distributions are independent:

$$p(p_i|y_i) = Beta(y_i + \alpha_i, n_i - y_i + \beta_i)$$

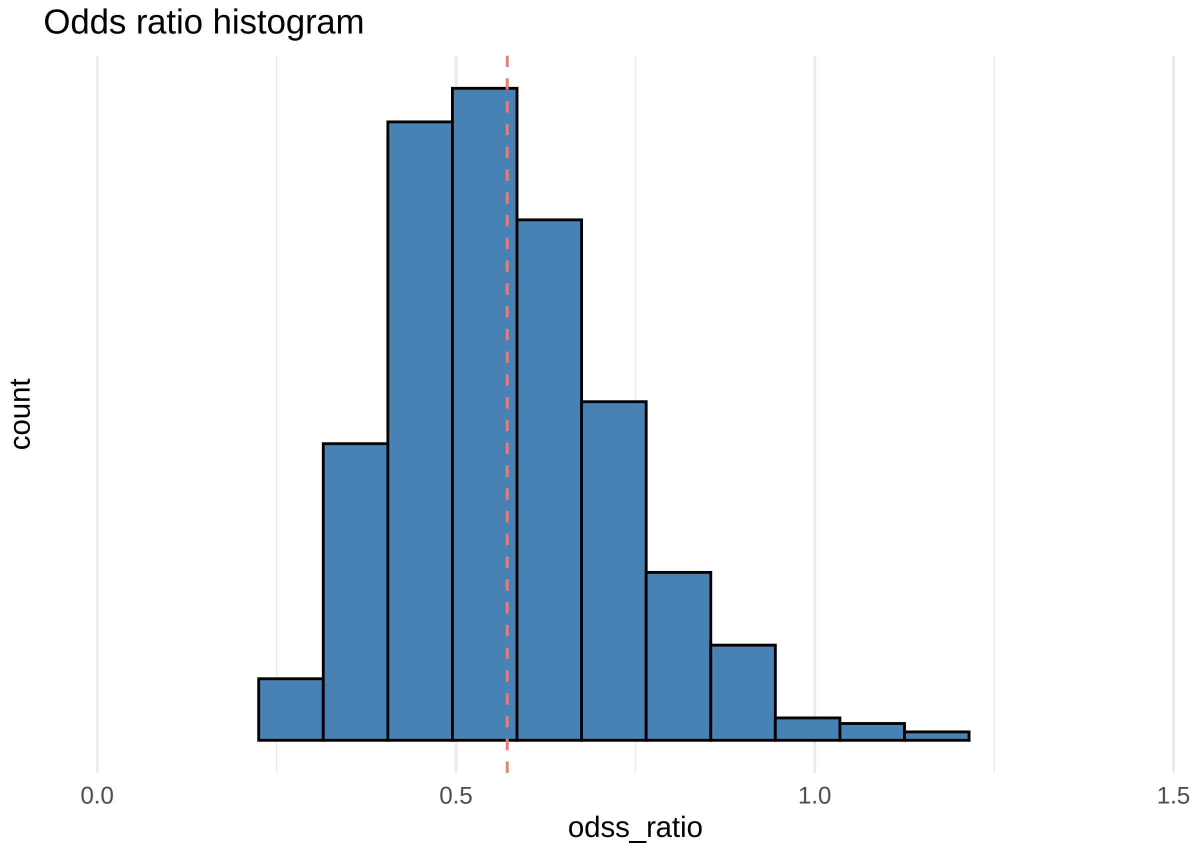
where $i \in 0, 1, n_0 = 674, n_1 = 680, y_0 = 39, y_1 = 22$

$$p(p_0, p_1|y_0, y_1) \propto p(p_0|y_0)p(p_1|y_1) \propto Beta(\alpha_0, \beta_0)Beta(\alpha_1, \beta_1) \propto Beta(\alpha_0 + \alpha_1, \beta_0 + \beta_1)$$

a)

We have odds ratio as $\psi = \frac{p_1/(1-p_1)}{p_0/(1-p_0)}$. For computing the posterior of odds ratio we use sampling from $p(p_i|yi_)$ and then simulate ψ based on $\psi = \frac{p_1/(1-p_1)}{p_0/(1-p_0)}$.

```r
n0 <- 674
y0 <- 39
n1 <- 680
y1 <- 22
a0 <- 1
b0 <- 1
a1 <- 1
b1 <- 1

post_alpha0 <- a0 + y0
post_beta0 <- b0 + n0 - y0
post_dist0 <- rbeta(1000, post_alpha0, post_beta0)

post_alpha1 <- a1 + y1
post_beta1 <- b1 + n1 - y1
post_dist1 <- rbeta(1000, post_alpha1, post_beta1)



posterior_odds_ratio_point_est <- function(p0, p1){
  psi <- (p1/(1-p1))/(p0/(1-p0))
  return(mean(psi))
}

posterior_odds_ratio_interval <- function(p0, p1, prob = 0.9){
  psi <- (p1/(1-p1))/(p0/(1-p0))
  q <- c(quantile(psi, (1-prob)/2), quantile(psi, prob+(1-prob)/2))
  return(q)
}

odds_ratio <- (post_dist1/(1-post_dist1))/(post_dist0/(1-post_dist0))

ggplot() +
  geom_histogram(aes(odds_ratio), binwidth = 0.09, fill = 'steelblue', color = 'black') +
  coord_cartesian(xlim = c(0, 1.5)) +
  scale_y_continuous(breaks = NULL) +
  labs(title = 'Odds ratio histogram', x = 'odss_ratio')+
  geom_vline(aes(xintercept = mean(odds_ratio), color = 'q'),
             linetype = 'dashed', show.legend = F)
```



```
posterior_odds_ratio_point_est(post_dist0, post_dist1)
```

```
## [1] 0.571362
```

```
posterior_odds_ratio_interval(post_dist0, post_dist1, prob = 0.95)
```

```
##      2.5%     97.5% 
## 0.3269635 0.9080281
```

The point estimate is 0.5718645 and 95% interval [0.3232104, 0.9281028].

b) (Prior sensitivity analysis)

You can see the sensitivity to the prior in Table. With different parameters for priors, posterior have not been pulled towards prior, so posterior is not sensitive to the prior.

```
A0 <- c(1, 2, 0.5, 5)
B0 <- c(1, 10, 10 ,100)
A1 <- c(1, 2, 0.4, 4)
B1 <- c(1, 10, 10, 100)
post_mean = c()
post_int = matrix(rep(0,2*length(A0)), ncol=2)
for(i in 1:length(A0)){
  a0 <- A0[i]
  b0 <- B0[i]
  a1 <- A1[i]
  b1 <- B1[i]

  post_alpha0 <- a0 + y0
  post_beta0 <- b0 + n0 - y0
  prior_dist0 <- rbeta(1000, a0, b0)
  post_dist0 <- rbeta(1000, post_alpha0, post_beta0)

  post_alpha1 <- a1 + y1
  post_beta1 <- b1 + n1 - y1
  prior_dist1 <- rbeta(1000, a1, b1)
  post_dist1 <- rbeta(1000, post_alpha1, post_beta1)

  prior_dist <- rbeta(1000, a0+a1, b0+b1)
  post_mean[i] <- posterior_odds_ratio_point_est(post_dist0, post_dist1)
  post_int[i, ] <- posterior_odds_ratio_interval(post_dist0, post_dist1, prob = 0.95)


}

post_mean
```

```
## [1] 0.5723527 0.5780003 0.5598457 0.5942766
```

```
post_int
```

```
##           [,1]      [,2]
## [1,] 0.3307545 0.9414562
## [2,] 0.3146429 0.9296768
## [3,] 0.3123140 0.9098332
## [4,] 0.3458481 0.9274029
```

Parameters of the prior distribution		Summaries of the posterior distribution	
$\frac{\alpha_0+\alpha_1}{\alpha_0+\beta_0+\alpha_1+\beta_1}$	$\alpha_0+\beta_0+\alpha_1+\beta_1$	mean of ψ	95% posterior interval for π
0.5	4	0.5706	[0.3137, 0.9642]
0.1667	24	0.5849	[0.3311, 0.9221]
0.0431	20.9	0.5664	[0.3155, 0.9026]
0.0431	209	0.5956	[0.3369, 0.9474]

Problem 3: Inference for the difference between normal means

a)

Uninformative joint prior: $p(\mu, \sigma_2) \propto \frac{1}{\sigma_2^2}$ likelihood: $p(y_2|\mu, \sigma_2) = \mathcal{N}(\mu, \sigma_2)$ Marginal posterior for μ: $p(\mu|y_2) = t_{n-1}(y_2, \frac{s^2}{n})$

μ_d will be calculated by sampling from μ_1 and μ_2 and then calculating $\mu_1 - \mu_2$.

```
data("windshieldy1")
data("windshieldy2")


post_mean <- function(data){
  n <- length(data)
  mu <- mean(data)
  sigma <- sd(data)
  rtg <- rt(num_samples, df=n-1)
  rr <- (rtg * sigma/sqrt(n) ) + mu
  return(rr)
}



data1 <- windshieldy1
data2 <- windshieldy2

n2 <- length(data2)
mu_difference <- post_mean(data1) - post_mean(data2)
posterior_mean <- mean(mu_difference)

cat("mean \n")
```

```
## mean
```

```
posterior_mean
```

```
## [1] -1.207597
```

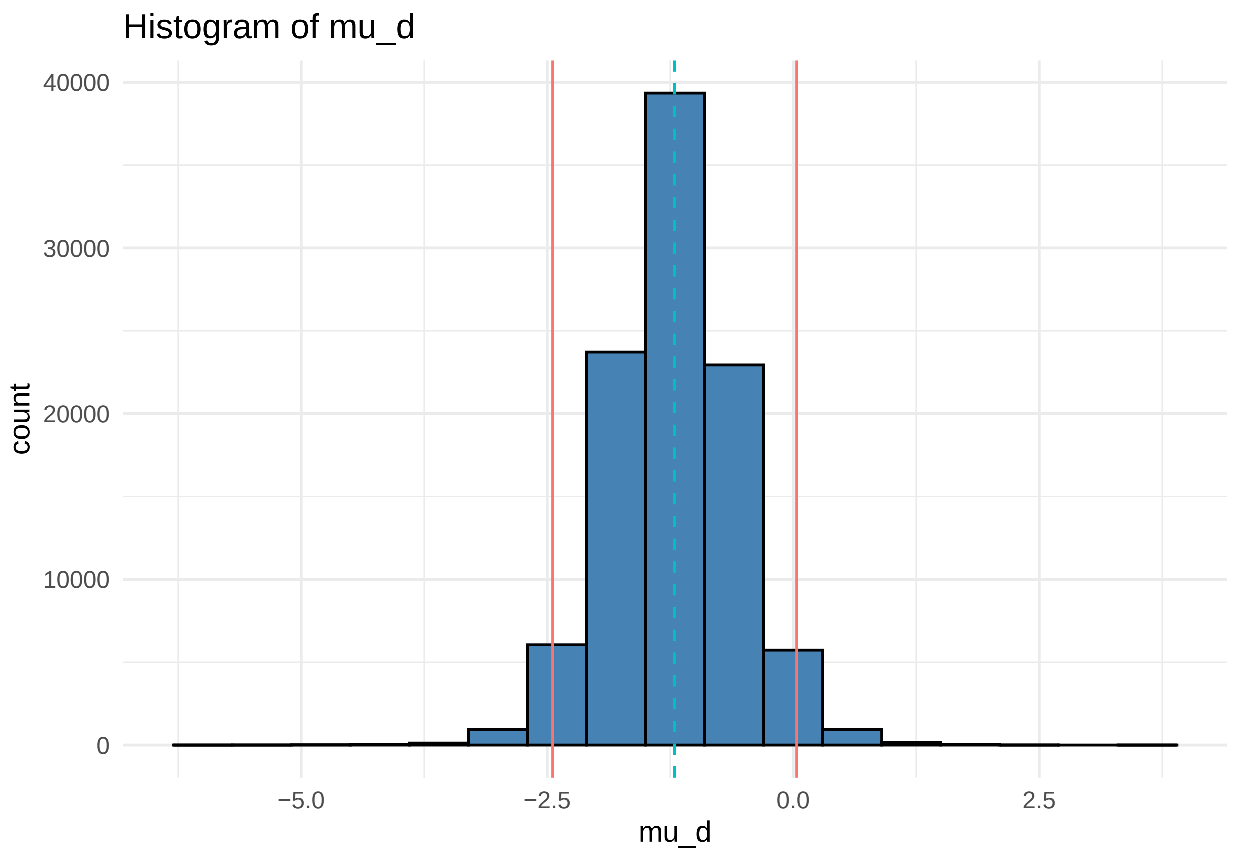
```
prob <- 0.95
posterior_interval<- quantile(mu_difference, c((1-prob)/2, prob+(1-prob)/2), names = FALSE)

cat("\n interval estimates (95%) \n")
```

```
##
##  interval estimates (95%)
```

```
posterior_interval
```

```
## [1] -2.44349942  0.03631815
```

```
labs <- c('posterior mean')
ggplot() +
  geom_histogram(aes(mu_difference), binwidth = 0.6, fill = 'steelblue', color = 'black') +
  labs(title = 'Histogram of mu_d ', x = 'mu_d')+
  geom_vline(aes(xintercept = posterior_mean, color = 'q'),
             linetype = 'dashed', show.legend = F) +
  geom_vline(aes(xintercept = c(posterior_interval), color = '95% interval'),
             linetype = 'solid', show.legend = F)
```



```
p_mu2 = sum(mu_difference<0)/num_samples
p_mu2
```

```
## [1] 0.97196
```

I counted the number of sample that are smaller than 0 and then divided them on total number of samples. The result means that with probability of 0.97 the μ_2 is bigger than μ_1.

b)

The means are not the same. As you can see in Figure mean of posterior is 1.2 and zero is not even in 95% posterior interval.

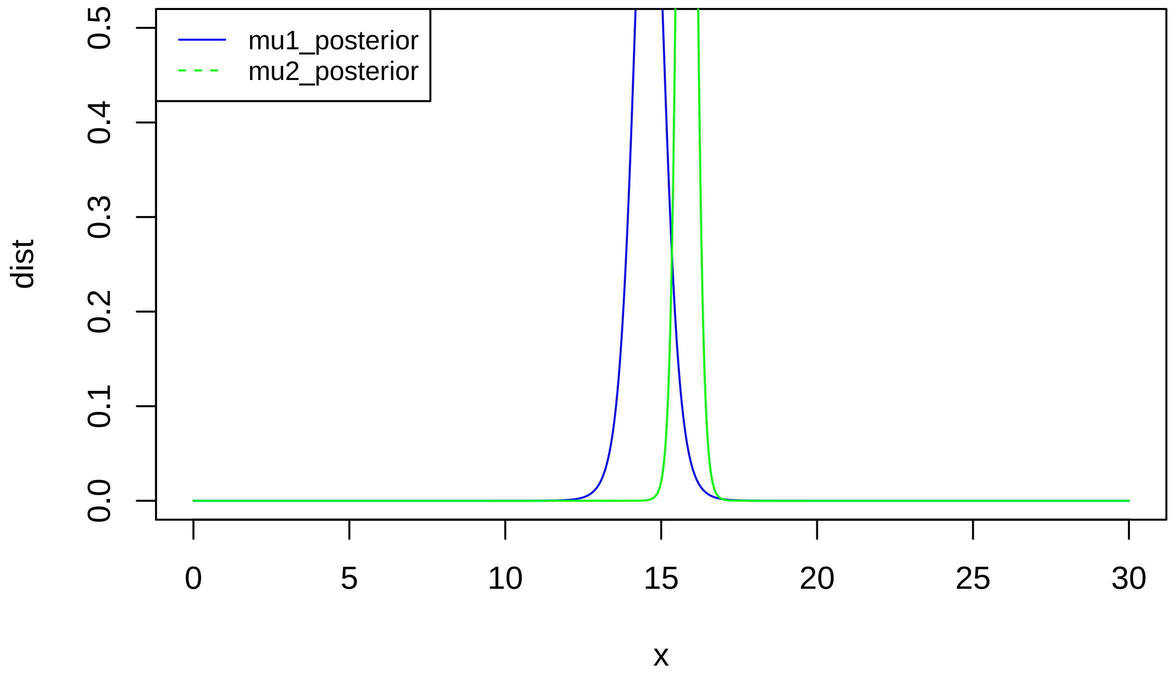
```
cat("mean of mu1 is: ",mu_point_est(data))
```

```
## mean of mu1 is:  14.61263
```

```
E2 <- mean(mu_point_est(data2))
cat("\n mean of mu2 is: ",E2)
```

```
##
##  mean of mu2 is:  15.82025
```

```
plot(x,dtnew(x,n-1,mean=mean(data1),scale=sd(data1)/sqrt(n)),type="l",col="blue",ylim=c(0,0.5),ylab="di
lines(x,dtnew(x,n2-1,mean=mean(data2),scale=sd(data2)/sqrt(n2)),type="l",col="green")
legend("topleft",legend=c("mu1_posterior", "mu2_posterior"),col=c("blue", "green"),lty=1:2, cex=0.8)
```



As you can see in plot, the means are not the same, because there is a noticable difference in the distributions.I also printed the means of both distribution which shows inequality.