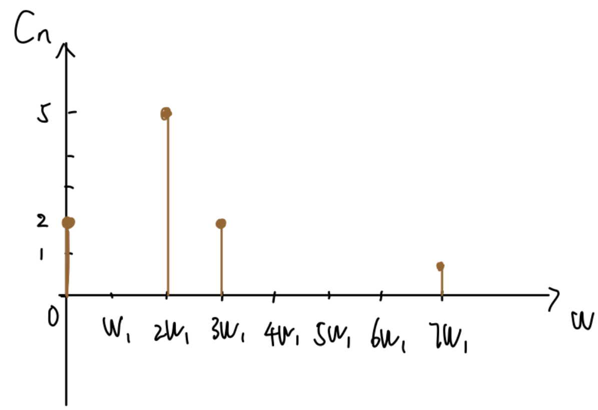

Figure 1: 1-幅度谱

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一. 已知周期信号 $x(t)$ 的傅里叶级数表示式为 $x(t) = 2 + 3\cos(2t) + 4\sin(2t) + 2\sin(3t + 30°) - \cos(7t + 150°)$：

(1) 求周期信号 $x(t)$ 的基波角频率；

(2) 画出周期信号 $x(t)$ 的幅度谱和相位谱。

解:

(1) $x(t) = 2+x_1(t)+x_2(t)+x_3(t)+x_4(t)$，$T_1 = T_2 = \pi, T_3 = \frac{2\pi}{3}, T_4 = \frac{2\pi}{7}$，因此 $T = 2T_1 = 2T_2 = 3T_3 = 7T_4 = 2\pi$，因此基波角频率为 $w_1 = \frac{2\pi}{T} = 1$。

(2) $x(t) = 2 + 5\cos(2t - 53°) + 2\cos(3t - 60°) + \cos(7t + 60°)$，所以其幅度谱和相位谱如图1图2：

二. 已知信号

$$x(t) = \begin{cases} 1 + \cos(t), & |t| \le \pi \\ 0, & |t| > \pi \end{cases}$$

求该信号的傅里叶变换。

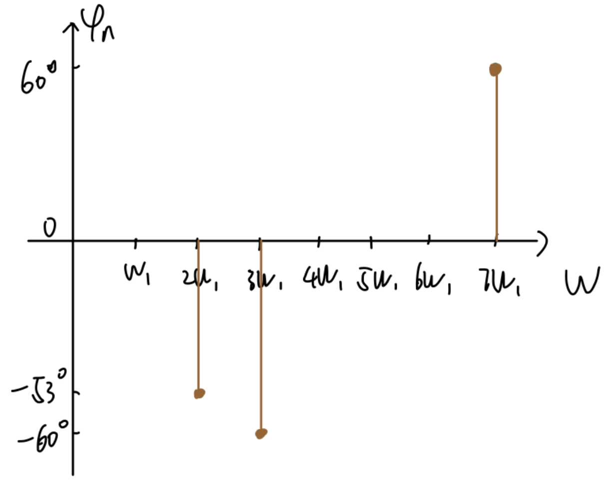
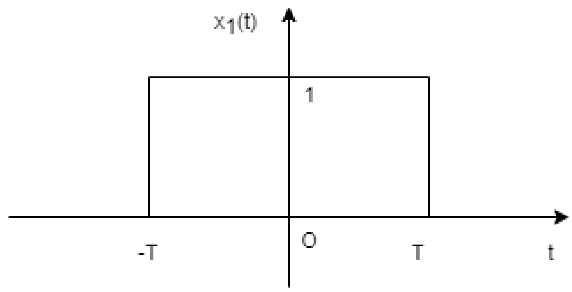


Figure 2: 1-相位谱

解：

$$\begin{aligned}
X(jw) &= \int_{-\infty}^{\infty} x(t)e^{-jwt}dt \\
&= \int_{-\pi}^{\pi}(1+\cos(t))(\cos(wt)-j\sin(wt))dt \\
&= \int_{-\pi}^{\pi}(1+\cos(t))\cos(wt)dt - \int_{-\pi}^{\pi} j(1+\cos(t))\sin(wt)dt \\
&= \int_{-\pi}^{\pi}(1+\cos(t))\cos(wt)dt - 0 \\
&= \int_{-\pi}^{\pi} \cos(wt) + \frac{1}{2}\cos((1-w)t) + \frac{1}{2}\cos((1+w)t)dt \\
&= \frac{2\sin(w\pi)}{w} + \frac{\sin((1-w)\pi)}{1-w} + \frac{\sin((1+w)\pi)}{1+w} \\
&= 2\pi Sa(w\pi) + \pi Sa((1-w)\pi) + \pi Sa((1+w)\pi)
\end{aligned}$$

三. 已知 $x_1(t)$ 和 $x(t)$ 的波形图如图3所示，$x_1(t)$ 的傅里叶变换为 $X_1(j\omega) = 2T \cdot Sa(\omega T)$，试利用傅里叶变换的尺度变换、位移和线性性质求 $x(t)$ 的傅里叶变换。



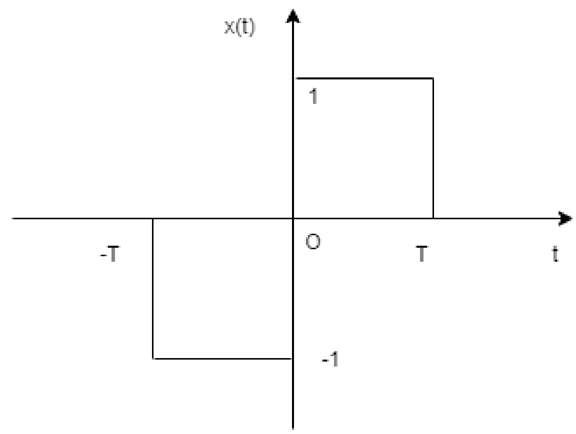


Figure 3: 题目三图

解：

易知 $x(t) = x_1(2t-T) - x_1(2t+T)$，所以

$$\begin{aligned}\mathcal{F}[x(t)] &= \mathcal{F}[x_1(2t-T)] - \mathcal{F}[x_1(2t+T)] \\ &= \frac{1}{2}X_1(\frac{jw}{2})e^{-j\frac{wT}{2}} - \frac{1}{2}X_1(\frac{jw}{2})e^{j\frac{wT}{2}} \\ &= \frac{1}{2}\times 2T\cdot Sa(\frac{wT}{2})e^{-j\frac{wT}{2}} - \frac{1}{2}\times 2T\cdot Sa(\frac{wT}{2})e^{j\frac{wT}{2}} \\ &= T\cdot Sa(\frac{wT}{2})(e^{-j\frac{wT}{2}} - e^{j\frac{wT}{2}})\end{aligned}$$

四. 求图4所示对称周期矩形信号的傅里叶级数，三角形式和指数形式。

解：

首先写出三角形式傅里叶级数：

$$x(t) = a_0 + \sum_{n=1}^{\infty}[a_n\cos(nw_1t) + b_n\sin(nw_1t)]$$

其中 $w_1 = \frac{2\pi}{T}$。由于 $x(t)$ 为奇信号且无直流分量，所以 $a_0 = 0, a_n = 0$。

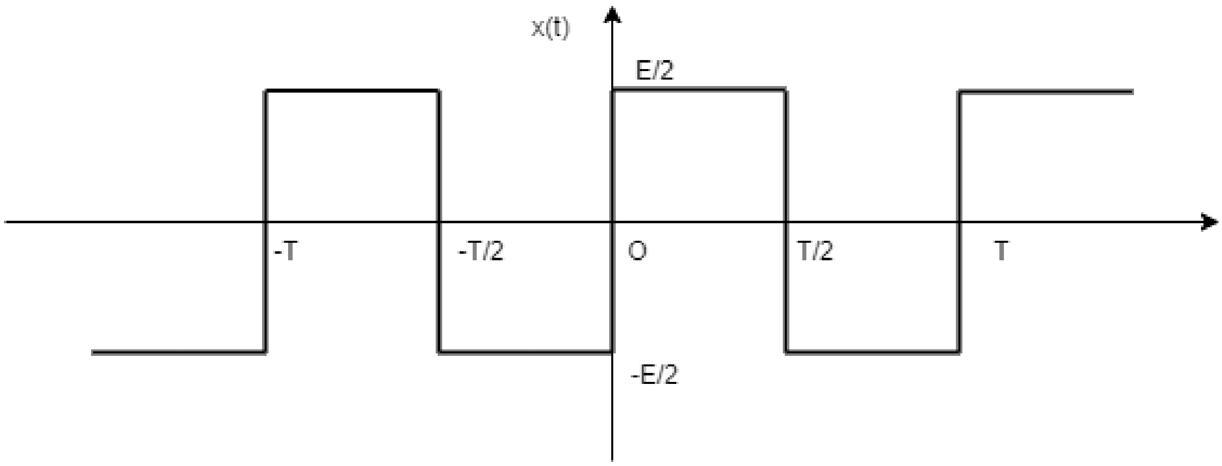


Figure 4: 题目四图

而

$$\begin{aligned} b_n &= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\sin(nw_1 t)dt \\ &= \frac{4}{T}\int_{0}^{\frac{T}{2}} x(t)\sin(nw_1 t)dt \\ &= \frac{4}{T}\int_{0}^{\frac{T}{2}} \frac{E}{2}\sin(nw_1 t)dt \\ &= \frac{E}{n\pi}(1-\cos(n\pi)) \end{aligned}$$

所以三角形式

$$\begin{aligned} x(t) &= \sum_{n=1}^{\infty}\frac{E}{n\pi}(1-\cos(n\pi))\sin(nw_1 t) \\ &= \sum_{n=1,3,\cdots}\frac{2E}{n\pi}\sin(\frac{2n\pi t}{T}) \end{aligned}$$

指数形式为

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jnw_1 t}$$

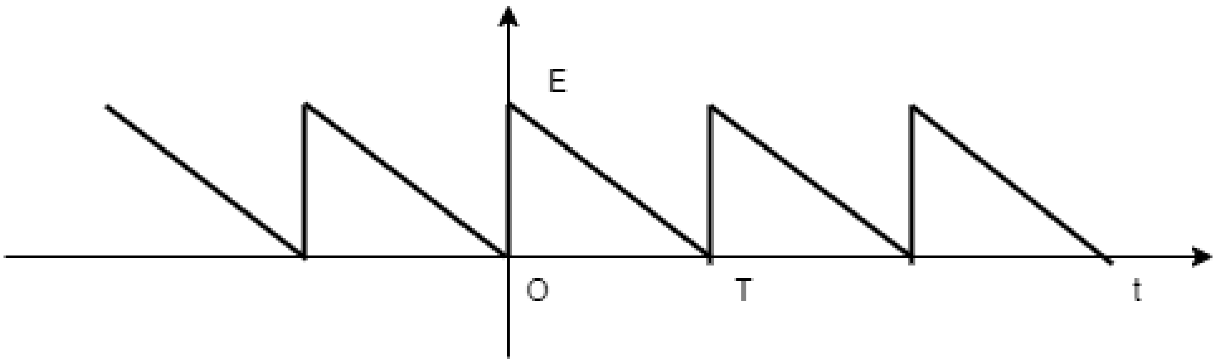


Figure 5: 题目五图

$$
\begin{aligned}
X_n &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jnw_1t}dt \\
&= \frac{1}{T}[\int_{-\frac{T}{2}}^{0} x(t)e^{-jnw_1t}dt + \int_{0}^{\frac{T}{2}} x(t)e^{-jnw_1t}dt] \\
&= \frac{1}{T}[\int_{-\frac{T}{2}}^{0} \frac{-E}{2}e^{-jnw_1t}dt + \int_{0}^{\frac{T}{2}} \frac{E}{2}e^{-jnw_1t}dt] \\
&= \frac{1}{T}[\frac{E}{2jnw_1}(1-e^{jn\pi}) - \frac{E}{2jnw_1}(e^{-jn\pi}-1)] \\
&= \frac{E}{4jn\pi}(2-e^{jn\pi}-e^{-jn\pi}) \\
&= \frac{E}{2jn\pi}(1-\frac{e^{jn\pi}+e^{-jn\pi}}{2}) \\
&= \frac{E}{2jn\pi}(1-\cos(n\pi)) = \begin{cases} \frac{E}{jn\pi} & \text{n is odd} \\ 0 & \text{n is even} \end{cases}
\end{aligned}
$$

所以指数形式为

$$
x(t) = \sum_{n=\pm1,\pm3,\cdots} \frac{E}{jn\pi}e^{\frac{2n\pi t}{T}}
$$

五. 求图5所示周期锯齿信号的指数形式傅里叶级数，并大致画出频谱图。

解：

令 $w_1 = \frac{2\pi}{T}$，指数形式为

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jnw_1 t}$$

$$\begin{aligned} X_n &= \frac{1}{T}\int_0^T x(t)e^{-jnw_1 t}dt \\ &= \frac{1}{T}\int_0^T E(1-\frac{t}{T})e^{-jnw_1 t}dt \\ &= \frac{E}{T}[\int_0^T e^{-jnw_1 t}dt - \frac{1}{T}\int_0^T te^{-jnw_1 t}dt] \end{aligned}$$

当 $n=0$ 时，$X_0 = \frac{E}{T}(T-\frac{T^2}{2T}) = \frac{E}{2}$。当 $n \neq 0$ 时，上式的第一项积分为 $\frac{e^{-j2n\pi}-1}{-jnw_1} = \frac{\cos(2n\pi)-j\sin(2n\pi)-1}{-jnw_1} = 0$。而

$$\begin{aligned} \frac{1}{T}\int_0^T te^{-jnw_1 t}dt &= \int_0^T \frac{t}{-jnw_1 T}de^{-jnw_1 t} \\ &= \frac{1}{-jnw_1 T}(te^{-jnw_1 t}\Big|_0^T - \int_0^T e^{-jnw_1 t}dt) \\ &= \frac{e^{-j2n\pi}}{-jnw_1} \end{aligned}$$

因此

$$X_n = \frac{E}{T}(0 - \frac{e^{-j2n\pi}}{-jnw_1}) = \frac{E}{j2n\pi}$$

因此指数形式为

$$x(t) = \frac{E}{2} + \sum_{n\neq 0, n\in \mathbf{Z}} \frac{E}{j2n\pi}e^{jnw_1 t}$$

其幅度谱和相位谱如图6图7所示。

六. 求图8所示锯齿脉冲与单周正弦脉冲的傅里叶变换。

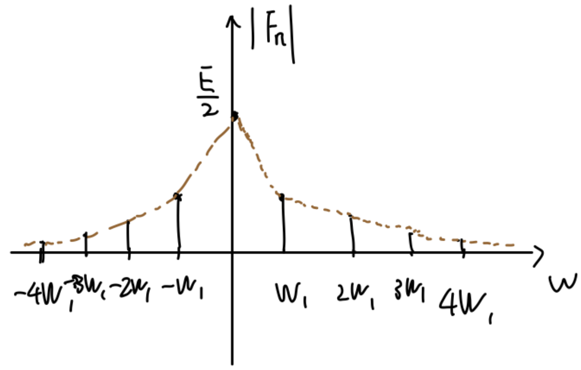


Figure 6: 5-幅度谱

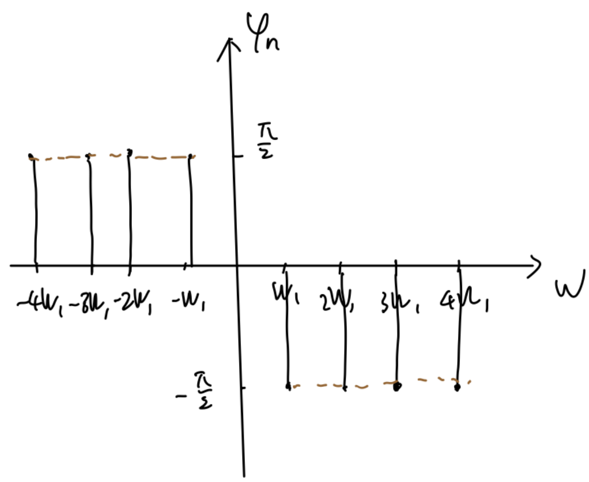


Figure 7: 5-相位谱

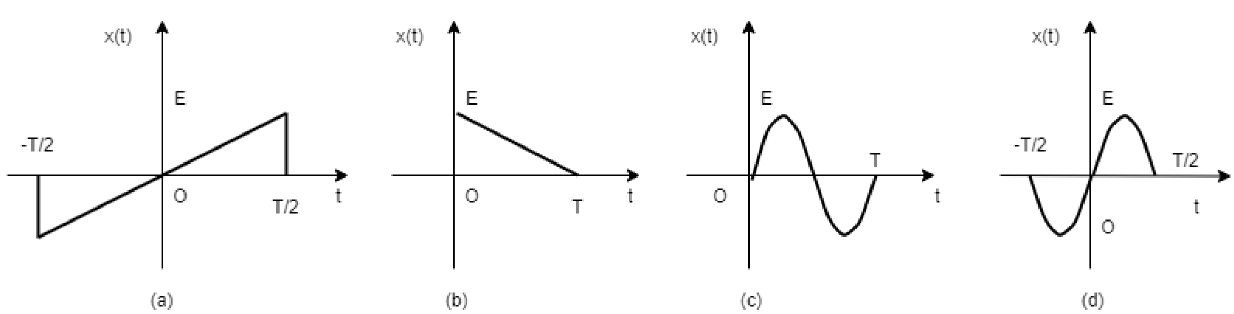


Figure 8: 题目六图

解：

(a) 当 $w \neq 0$ 时，

$$
\begin{aligned}
X(jw) &= \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{2Et}{T} e^{-jwt} dt \\
&= \frac{2E}{-jwT} \int_{-\frac{T}{2}}^{\frac{T}{2}} t d e^{-jwt} \\
&= \frac{2E}{-jwT} (T\cos(\frac{wT}{2}) - \frac{2}{w}\sin(\frac{wT}{2})) \\
&= \frac{2E}{-jw} (\cos(\frac{wT}{2}) - Sa(\frac{wT}{2}))
\end{aligned}
$$

而 $x(t)$ 无直流分量，所以 $X(0) = 0$。

(b) 当 $w \neq 0$ 时，

$$
\begin{aligned}
X(jw) &= \int_0^T (E - \frac{Et}{T}) e^{-jwt} dt \\
&= \int_0^T E e^{-jwt} dt - \frac{E}{T} \int_0^T t e^{-jwt} dt \\
&= \frac{E}{w^2 T} (1 - e^{-jwT} - jwT)
\end{aligned}
$$

当 $w = 0$ 时 $X(0) = \int_0^T (E - \frac{Et}{T}) dt = \frac{ET}{2}$。

(c) 令 $w_1 = \frac{2\pi}{T}$，则

$$
\begin{aligned}
X(jw) &= \int_0^T E\sin(w_1 t) e^{-jwt} dt \\
&= \frac{E}{2j} \int_0^T (e^{(w_1 - w)}t - e^{-j(w_1 + w)t}) dt \\
&= \frac{E w_1}{w_1^2 - w^2} (1 - e^{-jwT})
\end{aligned}
$$

当 $w = w_1$ 时，$X(jw_1) = \frac{E}{2j} \int_0^T (1 - e^{\frac{-j4\pi t}{T}}) dt = \frac{ET}{2j}$

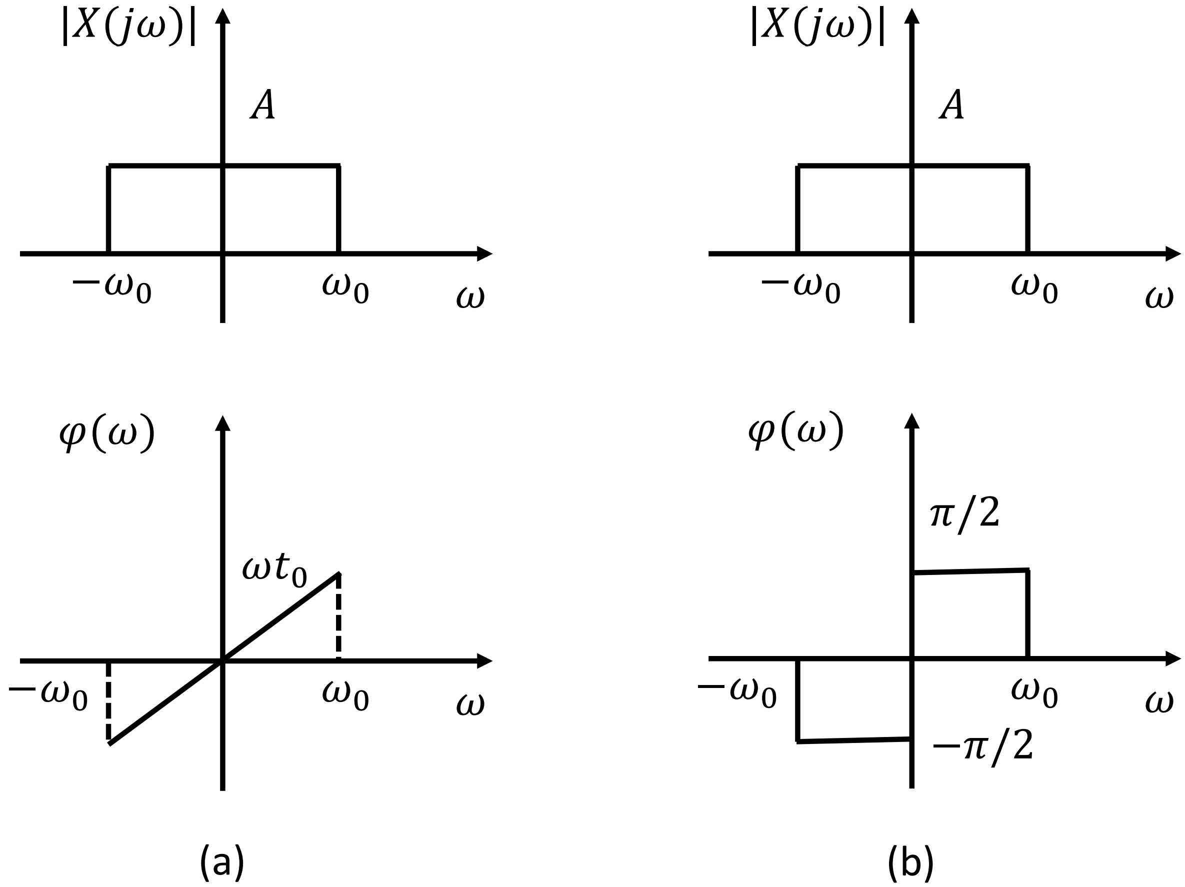


Figure 9: 题目七图

(d) 令 $w_1 = \frac{2\pi}{T}$，则

$$
\begin{aligned}
X(jw) &= \int_{-\frac{T}{2}}^{\frac{T}{2}} E\sin(w_1 t)e^{-jwt}dt \\
&= \frac{E}{2j}\int_{-\frac{T}{2}}^{\frac{T}{2}}(e^{(w_1-w)}t - e^{-j(w_1+w)t})dt \\
&= \frac{j2Ew_1}{w^2-w_1^2}\sin(\frac{wT}{2})
\end{aligned}
$$

当 $w = w_1$ 时，$X(jw_1) = \frac{E}{2j}\int_{-\frac{T}{2}}^{\frac{T}{2}}(1-e^{\frac{-j4\pi t}{T}})dt = \frac{ET}{2j}$

七. 分别求图9所示 $X(j\omega)$ 的傅里叶逆变换。

解:

(a) 易知 $X(jw)=Ae^{jwt_0}$，所以

$$\begin{aligned}x(t)&=\frac{1}{2\pi}\int_{-w_0}^{w_0}Ae^{jwt_0}e^{jwt}dw\\&=\frac{A}{2\pi}\int_{-w_0}^{w_0}e^{j(t_0+t)w}dw\\&=\frac{A}{\pi(t+t_0)}\sin(w_0(t+t_0))\\&=\frac{Aw_0}{\pi}Sa(w_0(t+t_0))\end{aligned}$$

(b) 易知 $X(jw)=Ae^{j\frac{\pi}{2}sgn(w)}$，其中 $sgn(x)$ 为符号函数，所以

$$\begin{aligned}x(t)&=\frac{1}{2\pi}\int_{-w_0}^{w_0}Ae^{j\frac{\pi}{2}sgn(w)}e^{jwt}dw\\&=\frac{1}{2\pi}[\int_{-w_0}^{0}-jAe^{jwt}dw+\int_{0}^{w_0}jAe^{jwt}dw]\\&=\frac{A}{\pi t}[\frac{e^{-jw_0t}+e^{jw_0t}}{2}-1]\\&=\frac{A}{\pi t}(\cos(w_0t)-1)\end{aligned}$$

八. 利用微分定理求图10所示梯形脉冲的傅里叶变换，并大致画出 $\tau=2\tau_1$ 情况下该脉冲的频谱图。

解:

容易画出 $x(t)$ 的一阶导数和二阶导数如图11。易得 $\mathcal{F}[\frac{d^2x(t)}{dt^2}]=(jw)^2X(jw)$，而

$$\frac{d^2x(t)}{dt^2}=\frac{2E}{\tau-\tau_1}[\delta(t+\frac{\tau}{2})-\delta(t+\frac{\tau_1}{2})-\delta(t-\frac{\tau_1}{2})+\delta(t-\frac{\tau}{2})]$$

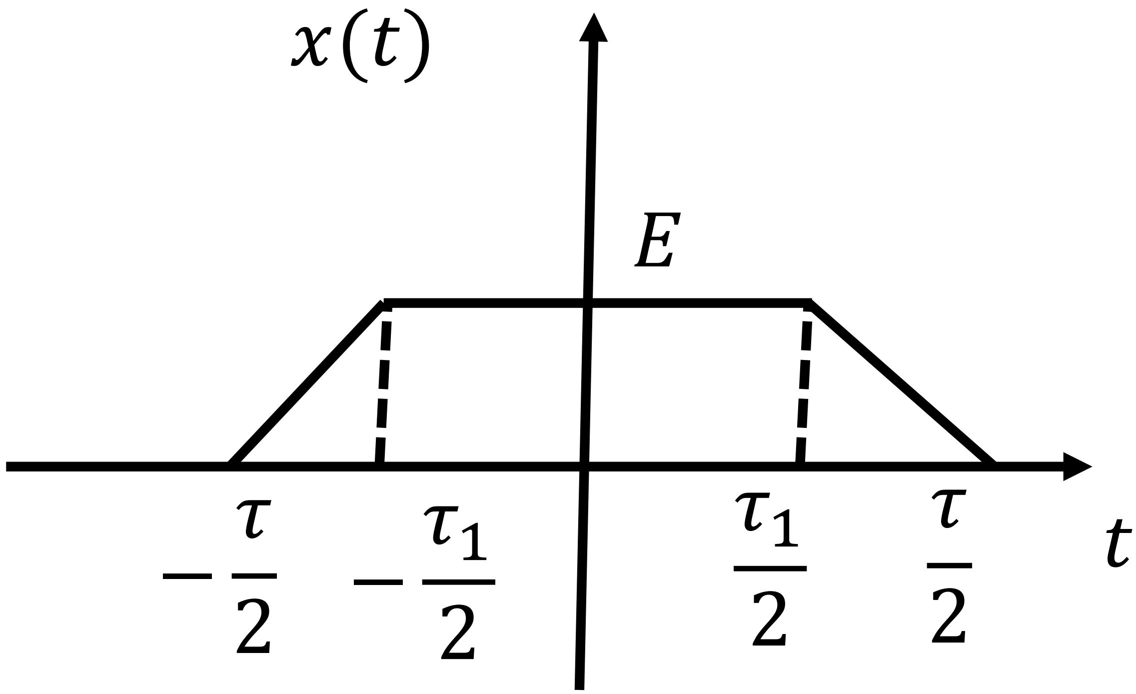


Figure 10: 题目八图

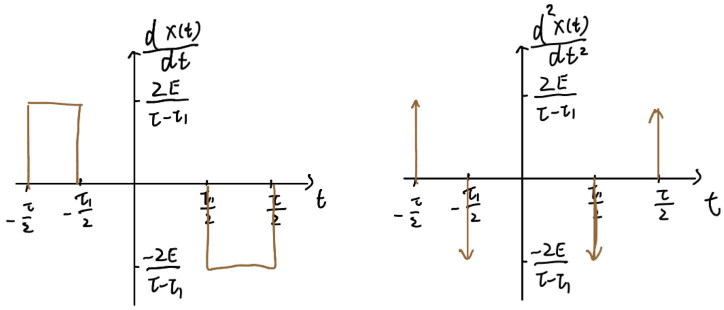


Figure 11: 8-导数

所以

$$
\begin{aligned}
-w^2X(jw) &= \frac{2E}{\tau-\tau_1}(e^{jw\frac{\tau}{2}} - e^{jw\frac{\tau_1}{2}} - e^{-jw\frac{\tau_1}{2}} + e^{-jw\frac{\tau}{2}}) \\
&= \frac{4E}{\tau-\tau_1}(\cos(\frac{w\tau}{2}) - \cos(\frac{w\tau_1}{2})) \\
&= \frac{8E}{\tau_1-\tau}\sin(\frac{w(\tau+\tau_1)}{4})\sin(\frac{w(\tau-\tau_1)}{4}) \\
&= -2Ew\sin(\frac{w(\tau+\tau_1)}{4})Sa(\frac{w(\tau-\tau_1)}{4})
\end{aligned}
$$

所以

$$
\begin{aligned}
X(jw) &= \frac{-2Ew}{-w^2}\sin(\frac{w(\tau+\tau_1)}{4})Sa(\frac{w(\tau-\tau_1)}{4}) \\
&= \frac{E(\tau+\tau_1)}{2}Sa(\frac{w(\tau+\tau_1)}{4})Sa(\frac{w(\tau-\tau_1)}{4})
\end{aligned}
$$

当 $\tau = 2\tau_1$ 时，$X(jw) = \frac{3E\tau}{4}Sa(\frac{3w\tau}{8})Sa(\frac{w\tau}{8})$。其频谱图如图12。

九. 若已知 $\mathcal{F}[x(t)] = X(j\omega)$，利用傅里叶变换的性质确定下列信号的傅里叶变换。

(1) $tx(2t)$

(2) $(t-2)x(t)$

(3) $(t-2)x(-2t)$

(4) $t\frac{dx(t)}{dt}$

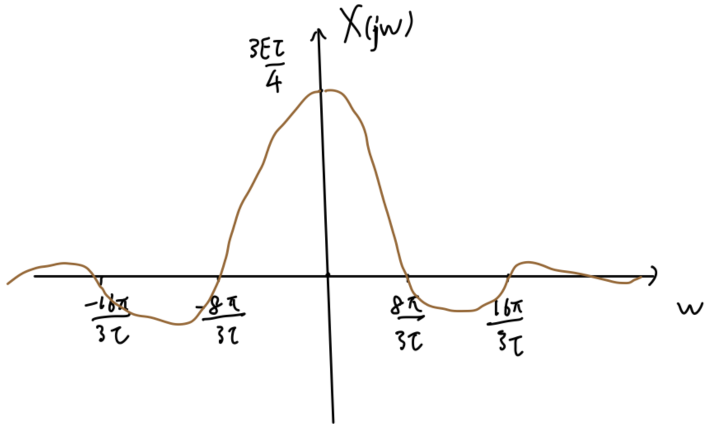


Figure 12: 8-频谱图

(5) $x(1-t)$

(6) $(1-t)x(1-t)$

(7) $x(2t-5)$

解：

(1) 由频域微分特性得 $\mathcal{F}[tx(t)] = j\frac{d}{dw}X(jw)$，再由尺度变换容易得 $\mathcal{F}[2tx(2t)] = \frac{j}{2}\frac{d}{d\frac{w}{2}}X(j\frac{w}{2})$，再由线性特性得 $\mathcal{F}[tx(2t)] = \frac{j}{2}\frac{d}{dw}X(\frac{jw}{2})$。

(2) $\mathcal{F}[(t-2)x(t)] = \mathcal{F}[(t-2)x(t)] = \mathcal{F}[tx(t)-2x(t)] = \mathcal{F}[tx(t)] - \mathcal{F}[2x(t)] = j\frac{d}{dw}X(jw) - 2X(jw)$。

(3) 令 $k(t) = x(-2t)$，则有 $\mathcal{F}[k(t)] = \frac{1}{2}X(\frac{jw}{-2})$。且 $\mathcal{F}[(t-2)k(t)] = j\frac{d}{dw}\frac{X(\frac{jw}{-2})}{2} - X(\frac{jw}{-2}) = \frac{j}{2}\frac{d}{dw}X(\frac{jw}{-2}) - X(\frac{jw}{-2})$。

(4) 令 $k(t) = \frac{d}{dt}x(t)$,则 $K(jw) = jwX(jw)$。而 $\mathcal{F}[t\frac{d}{dt}x(t)] = \mathcal{F}[tk(t)] = j\frac{d}{dw}K(jw) = -\frac{d}{dw}wX(jw) = -X(jw) - w\frac{d}{dw}X(jw)$。

(5) 因为 $\mathcal{F}[x(-t)] = X(-jw)$，所以 $\mathcal{F}[x(1-t)] = \mathcal{F}[x(-(t-1))] = X(-jw)e^{-jw}$。

(6) 令 $k(t) = x(1-t)$，则有 $K(jw) = \mathcal{F}[k(t)] = X(-jw)e^{-jw}$，所以 $\mathcal{F}[(1-t)k(t)] = K(jw) - j\frac{d}{dw}K(jw) = X(-jw)e^{-jw} - j[e^{-jw}\frac{d}{dw}X(-jw) - jX(-jw)e^{-jw}] = -je^{-jw}\frac{d}{dw}X(-jw)$。

(7) 因为 $\mathcal{F}[x(2t)] = \frac{1}{2}X(\frac{jw}{2})$，所以 $\mathcal{F}[2t-5] = \mathcal{F}[x(2(t-\frac{5}{2}))] = \frac{1}{2}X(\frac{jw}{2})e^{-jw\frac{5}{2}}$。