

# Convex optimization problems (I)

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# Outline

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## □ Optimization Problems

- Basic Terminology
- Equivalent Problems
- Problem Descriptions

## □ Convex Optimization

- Standard Form
- Local and Global Optima
- An Optimality Criterion
- Equivalent Convex Problems
- Quasiconvex Optimization



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# Basic Terminology



Unconstrained  
when  $m = p = 0$

## □ Optimization Problems

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s. t.} \quad & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{aligned} \quad (1)$$

- Optimization variable:  $x \in \mathbf{R}^n$
- Objective function:  $f_0: \mathbf{R}^n \rightarrow \mathbf{R}$
- Inequality constraints:  $f_i(x) \leq 0$
- Inequality constraint functions:  $f_i: \mathbf{R}^n \rightarrow \mathbf{R}$
- Equality constraints:  $h_i(x) = 0$
- Equality constraint functions:  $h_i: \mathbf{R}^n \rightarrow \mathbf{R}$



# Basic Terminology

## □ Optimization Problems

$$\begin{array}{ll} \min & f_0(x) \\ \text{s. t.} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{array} \quad (1)$$

### ■ Domain

$$\mathcal{D} = \bigcap_{i=0}^m \text{dom } f_i \cap \bigcap_{i=1}^p \text{dom } h_i$$

- $x \in \mathcal{D}$  is **feasible** if it satisfies all the constraints
- The problem is **feasible** if there exists at least one feasible point



# Basic Terminology

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## □ Optimal Value $p^*$

$$p^* = \inf \{f_0(x) | f_i(x) \leq 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, p\}$$

- Infeasible problem:  $p^* = \infty$
- Unbounded below: if there exist  $x_k$  with  $f_0(x_k) \rightarrow -\infty$  as  $k \rightarrow \infty$ , then  $p^* = -\infty$

## □ Optimal Points

- $x^*$  is feasible and  $f_0(x^*) = p^*$

## □ Optimal Set

$$X_{\text{opt}} = \{x | f_i(x) \leq 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, p, f_0(x) = p^*\}$$

## □ $p^*$ is achieved if $X_{\text{opt}}$ is nonempty



# Basic Terminology

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## □ $\varepsilon$ -suboptimal Points

- a feasible  $x$  with  $f_0(x) \leq p^* + \varepsilon$

## □ $\varepsilon$ -suboptimal Set

- the set of all  $\varepsilon$ -suboptimal points

## □ Locally Optimal Points

$$\begin{array}{ll} \min & f_0(z) \\ \text{s. t.} & f_i(z) \leq 0, \quad i = 1, \dots, m \\ & h_i(z) = 0, \quad i = 1, \dots, p \\ & \|z - x\|_2 \leq R \end{array}$$

- $x$  is feasible and solves this above problem

## □ Globally Optimal Points



# Basic Terminology

## □ Types of Constraints

- If  $f_i(x) = 0$ ,  $f_i(x) \leq 0$  is **active** at  $x$
- If  $f_i(x) < 0$ ,  $f_i(x) \leq 0$  is **inactive** at  $x$
- $h_i(x) = 0$  is active at all feasible points
- **Redundant** constraint: deleting it does not change the feasible set

## □ Examples on $x \in \mathbf{R}$ and $\text{dom } f_0 = \mathbf{R}_{++}$

- $f_0(x) = 1/x$ :  $p^* = 0$ , the optimal value is not achieved
- $f_0(x) = -\log x$ :  $p^* = -\infty$ , unbounded below
- $f_0(x) = x \log x$ :  $p^* = -1/e$ ,  $x^* = 1/e$  is optimal





# Basic Terminology

## □ Feasibility Problems

$$\begin{array}{ll}\text{find} & x \\ \text{s. t.} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p\end{array}$$

- Determine whether constraints are consistent

## □ Maximization Problems

$$\begin{array}{ll}\max & f_0(x) \\ \text{s. t.} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p\end{array}$$

- It can be solved by minimizing  $-f_0$
- Optimal Value  $p^*$

$$p^* = \sup \{f_0(x) | f_i(x) \leq 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, p\}$$



# Basic Terminology

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## □ Standard Form

$$\begin{array}{ll}\min & f_0(x) \\ \text{s. t.} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p\end{array}$$

## □ Box constraints

$$\begin{array}{ll}\min & f_0(x) \\ \text{s. t.} & l_i \leq x_i \leq u_i, \quad i = 1, \dots, n\end{array}$$

## □ Reformulation

$$\begin{array}{ll}\min & f_0(x) \\ \text{s. t.} & l_i - x_i \leq 0, \quad i = 1, \dots, n \\ & x_i - u_i \leq 0, \quad i = 1, \dots, n\end{array}$$



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# Equivalent Problems

## □ Two Equivalent Problems

- If from a solution of one, a solution of the other is readily found, and vice versa

## □ A Simple Example

$$\begin{aligned} \min \quad & \tilde{f}(x) = \alpha_0 f_0(x) \\ \text{s. t.} \quad & \tilde{f}_i(x) = \alpha_i f_i(x) \leq 0, \quad i = 1, \dots, m \\ & \tilde{h}_i(x) = \beta_i h_i(x) = 0, \quad i = 1, \dots, p \end{aligned}$$

- $\alpha_i > 0, i = 0, \dots, m$
- $\beta_i \neq 0, i = 1, \dots, p$
- Equivalent to the problem (1)



# Change of Variables

- $\phi: \mathbf{R}^n \rightarrow \mathbf{R}^n$  is one-to-one and  $\phi(\text{dom } \phi) \supseteq \mathcal{D}$ , and define

$$\begin{aligned}\tilde{f}_i(z) &= f_i(\phi(z)), & i &= 0, \dots, m \\ \tilde{h}_i(z) &= h_i(\phi(z)), & i &= 1, \dots, p\end{aligned}$$

- An Equivalent Problem

$$\begin{aligned}\min \quad & \tilde{f}_0(z) \\ \text{s.t.} \quad & \tilde{f}_i(z) \leq 0, & i &= 1, \dots, m \\ & \tilde{h}_i(z) = 0, & i &= 1, \dots, p\end{aligned}$$

- If  $z$  solves it,  $x = \phi(z)$  solves the problem (1)
- If  $x$  solves (1),  $z = \phi^{-1}(x)$  solves it



# Transformation of Functions

- $\psi_0: \mathbf{R} \rightarrow \mathbf{R}$  is monotone increasing
- $\psi_1, \dots, \psi_m: \mathbf{R} \rightarrow \mathbf{R}$  satisfy  $\psi_i(u) \leq 0$  if and only if  $u \leq 0$
- $\psi_{m+1}, \dots, \psi_{m+p}: \mathbf{R} \rightarrow \mathbf{R}$  satisfy  $\psi_i(u) = 0$  if and only if  $u = 0$
- Define 
$$\begin{aligned} \tilde{f}_i(x) &= \psi_i(f_i(x)), & i &= 0, \dots, m \\ \tilde{h}_i(x) &= \psi_{m+i}(h_i(x)), & i &= 1, \dots, p \end{aligned}$$
- An Equivalent Problem

$$\begin{aligned} \min \quad & \tilde{f}_0(x) \\ \text{s. t.} \quad & \tilde{f}_i(x) \leq 0, & i &= 1, \dots, m \\ & \tilde{h}_i(x) = 0, & i &= 1, \dots, p \end{aligned}$$



# Example

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## □ Least-norm Problems

$$\min \|Ax - b\|_2$$

- Not differentiable at any  $x$  with  $Ax - b = 0$

## □ Least-norm-squared Problems

$$\min \|Ax - b\|_2^2 = (Ax - b)^T (Ax - b)$$

- Differentiable for all  $x$



# Slack Variables

□  $f_i(x) \leq 0$  if and only if there is an  $s_i \geq 0$  that satisfies  $f_i(x) + s_i = 0$

□ An Equivalent Problem

$$\begin{array}{ll} \min & f_0(x) \\ \text{s. t.} & s_i \geq 0, \quad i = 1, \dots, m \\ & f_i(x) + s_i = 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{array}$$

- $s_i$  is the slack variable associated with the inequality constraint  $f_i(x) \leq 0$
- $x$  is optimal for the problem (1) if and only if  $(x, s)$  is optimal for the above problem, where  $s_i = -f_i(x)$





# Eliminating Equality Constraints

□ Assume  $\phi: \mathbf{R}^k \rightarrow \mathbf{R}^n$  is such that  $x$  satisfies

$$h_i(x) = 0, \quad i = 1, \dots, p$$

if and only if there is some  $z \in \mathbf{R}^k$  such that

$$x = \phi(z)$$

□ An Equivalent Problem

$$\min \quad \tilde{f}_0(z) = f_0(\phi(z))$$

$$\text{s. t.} \quad \tilde{f}_i(z) = f_i(\phi(z)) \leq 0, \quad i = 1, \dots, m$$

- If  $z$  is optimal for this problem,  $x = \phi(z)$  is optimal for the problem (1)
- If  $x$  is optimal for (1), there is at least one  $z$  which is optimal for this problem



# Eliminating linear equality constraints

□ Assume the equality constraints are all linear  $Ax = b$ , and  $x_0$  is one solution

□ Let  $F \in \mathbf{R}^{n \times k}$  be any matrix with  $\mathcal{R}(F) = \mathcal{N}(A)$ , then

$$\{x | Ax = b\} = \{Fz + x_0 | z \in \mathbf{R}^k\}$$

□ An Equivalent Problem ( $x = Fz + x_0$ )

$$\begin{array}{ll} \min & f_0(Fz + x_0) \\ \text{s. t.} & f_i(Fz + x_0) \leq 0, \quad i = 1, \dots, m \end{array}$$

■  $k = n - \text{rank}(A)$



# Introducing Equality Constraints

## □ Consider the problem

$$\begin{array}{ll}\min & f_0(A_0x + b_0) \\ \text{s.t.} & f_i(A_ix + b_i) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p\end{array}$$

■  $x \in \mathbf{R}^n$ ,  $A_i \in \mathbf{R}^{k_i \times n}$  and  $f_i: \mathbf{R}^{k_i} \rightarrow \mathbf{R}$

## □ An Equivalent Problem

$$\begin{array}{ll}\min & f_0(y_0) \\ \text{s.t.} & f_i(y_i) \leq 0, \quad i = 1, \dots, m \\ & y_i = A_ix + b_i, \quad i = 0, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p\end{array}$$

■ Introduce  $y_i \in \mathbf{R}^{k_i}$  and  $y_i = A_ix + b_i$



# Optimizing over Some Variables

□ Suppose  $x \in \mathbf{R}^n$  is partitioned as  $x = (x_1, x_2)$ , with  $x_1 \in \mathbf{R}^{n_1}, x_2 \in \mathbf{R}^{n_2}$  and  $n_1 + n_2 = n$

□ Consider the problem

$$\begin{array}{ll} \min & f_0(x_1, x_2) \\ \text{s.t.} & f_i(x_1) \leq 0, \quad i = 1, \dots, m_1 \\ & \tilde{f}_i(x_2) \leq 0, \quad i = 1, \dots, m_2 \end{array}$$

□ An Equivalent Problem

$$\begin{array}{ll} \min & \tilde{f}_0(x_1) \\ \text{s.t.} & f_i(x_1) \leq 0, \quad i = 1, \dots, m_1 \end{array}$$

■ where

$$\tilde{f}_0(x_1) = \inf \{f_0(x_1, z) \mid \tilde{f}_i(z) \leq 0, i = 1, \dots, m_2\}$$



# Example

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## □ Minimize a Quadratic Function

$$\begin{aligned} \min \quad & x_1^\top P_{11} x_1 + 2x_1^\top P_{12} x_2 + x_2^\top P_{22} x_2 \\ \text{s. t.} \quad & f_i(x_1) \leq 0, \quad i = 1, \dots, m \end{aligned}$$

## □ Minimize over $x_2$

$$\begin{aligned} \inf_{x_2} \quad & (x_1^\top P_{11} x_1 + 2x_1^\top P_{12} x_2 + x_2^\top P_{22} x_2) \\ & = x_1^\top (P_{11} - P_{12} P_{22}^{-1} P_{12}^\top) x_1 \end{aligned}$$

## □ An Equivalent Problem

$$\begin{aligned} \min \quad & x_1^\top (P_{11} - P_{12} P_{22}^{-1} P_{12}^\top) x_1 \\ \text{s. t.} \quad & f_i(x_1) \leq 0, \quad i = 1, \dots, m \end{aligned}$$



# Epigraph Problem Form

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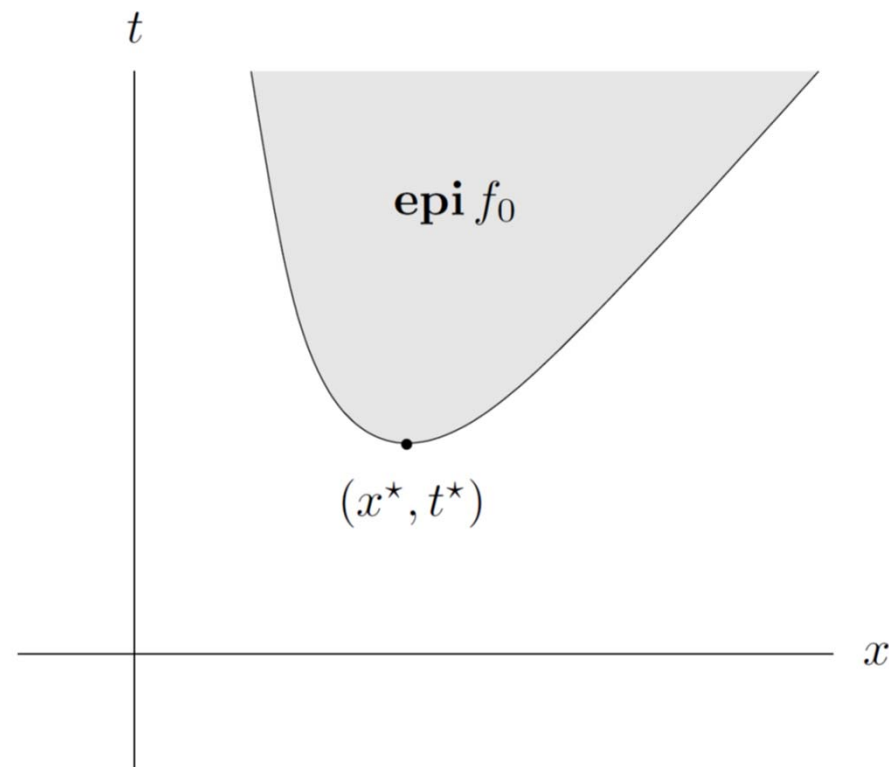
## □ Epigraph Form

$$\begin{array}{ll}\min & t \\ \text{s. t.} & f_0(x) - t \leq 0 \\ & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p\end{array}$$

- Introduce a variable  $t \in \mathbf{R}$
- $(x, t)$  is optimal for this problem if and only if  $x$  is optimal for (1) and  $t = f_0(x)$
- The objective function of the epigraph form problem is a **linear function** of  $x, t$

# Epigraph Problem Form

## □ Geometric Interpretation



- Find the point in the epigraph that minimizes  $t$



# Making Constraints Implicit

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## □ Unconstrained problem

$$\min F(x)$$

- $\text{dom } F = \{x \in \text{dom } f_0 \mid f_i(x) \leq 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, p\}$
- $F(x) = f_0(x)$  for  $x \in \text{dom } F$
- It has not make the problem any easier
- It could make the problem more difficult, because  $F$  is probably not differentiable





# Making Constraints Explicit

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## □ A Unconstrained Problem

$$\min f(x)$$

■ where

$$f(x) = \begin{cases} x^T x & Ax = b \\ \infty & \text{otherwise} \end{cases}$$

■ An implicit equality constraint  $Ax = b$

## □ An Equivalent Problem

$$\begin{array}{ll} \min & x^T x \\ \text{s.t.} & Ax = b \end{array}$$

■ Objective and constraint functions are differentiable



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# Problem Descriptions

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## □ Parameter Problem Description

- Functions have some analytical or closed form
- Example:  $f_0(x) = x^T P x + q^T x + r$ , where  $P \in \mathbf{S}^n, q \in \mathbf{R}^n$  and  $r \in \mathbf{R}$
- Give the values of the parameters

## □ Oracle Model (Black-box Model)

- Can only query the objective and constraint functions by oracle
- Evaluate  $f(x)$  and its gradient  $\nabla f(x)$
- Know some prior information



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# Convex Optimization Problems

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## □ Standard Form

$$\begin{array}{ll}\min & f_0(x) \\ \text{s. t.} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & a_i^\top x = b_i, \quad i = 1, \dots, p\end{array}$$

- The objective function must be **convex**
- The inequality constraint functions must be **convex**
- The equality constraint functions  $h_i(x) = a_i^\top x - b_i$  must be **affine**



# Convex Optimization Problems

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## □ Properties

- Feasible set of a convex optimization problem is convex

$$\bigcap_{i=0}^m \text{dom } f_i \cap \bigcap_{i=1}^m \{x | f_i(x) \leq 0\} \cap \bigcap_{i=1}^p \{x | a_i^T x = b_i\}$$

- ✓ Minimize a convex function over a convex set
- $\varepsilon$ -suboptimal set is convex
- The optimal set is convex
- If the objective is strictly convex, then the optimal set contains at most one point



# Concave Maximization Problems

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## □ Standard Form

$$\begin{array}{ll}\max & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & a_i^\top x = b_i, \quad i = 1, \dots, p\end{array}$$

- It is referred as a convex optimization problem if  $f_0$  is concave and  $f_1, \dots, f_m$  are convex
- It is readily solved by minimizing the convex objective function  $-f_0$

# Abstract From Convex Optimization Problem



## □ Consider the Problem

$$\begin{array}{ll}\min & f_0(x) = x_1^2 + x_2^2 \\ \text{s. t.} & f_1(x) = x_1/(1 + x_2^2) \leq 0 \\ & h_1(x) = (x_1 + x_2)^2 = 0\end{array}$$

- Not a convex optimization problem
  - ✓  $f_1$  is not convex and  $h_1$  is not affine
- But the **feasible set** is indeed convex
- Abstract convex optimization problem

## □ An Equivalent Convex Problem

$$\begin{array}{ll}\min & f_0(x) = x_1^2 + x_2^2 \\ \text{s. t.} & f_1(x) = x_1 \leq 0 \\ & h_1(x) = x_1 + x_2 = 0\end{array}$$





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# Local and Global Optima

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□ Any locally optimal point of a convex problem is also (globally) optimal

□ Proof by Contradiction

■  $x$  is locally optimal implies

$$f_0(x) = \inf\{f_0(z) \mid z \text{ feasible}, \|z - x\|_2 \leq R\}$$

for some  $R$

■ Suppose  $x$  is not globally optimal, i.e., there exists  $f_0(y) < f_0(x)$  and  $\|y - x\|_2 > R$

■ Define

$$z = (1 - \theta)x + \theta y, \theta = \frac{R}{2\|y - x\|_2} \in (0,1)$$



# Local and Global Optima

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- By convexity of the feasible set  
 $z$  is feasible

- It is easy to check

$$\|z - x\|_2 = \|\theta(y - x)\|_2 = \left\| \frac{R(y - x)}{2\|y - x\|_2} \right\|_2 = \frac{R}{2} < R$$

- By convexity of  $f_0$

$$f_0(z) \leq (1 - \theta)f_0(x) + \theta f_0(y) < f_0(x)$$

which contradicts

$$f_0(x) = \inf\{f_0(z) \mid z \text{ feasible}, \|z - x\|_2 \leq R\}$$



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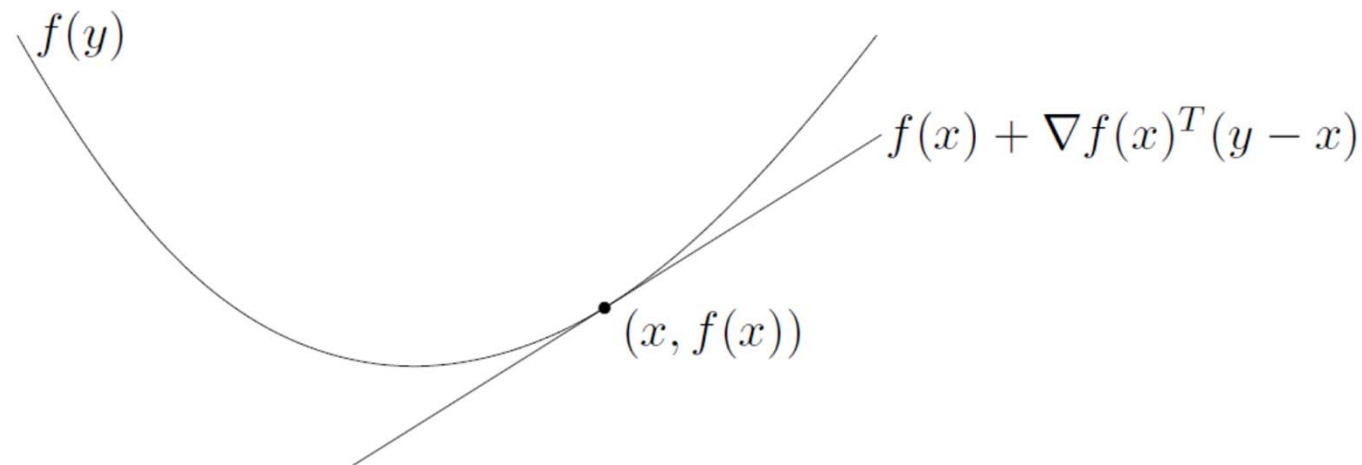
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# An Optimality Criterion for Differentiable $f_0$



□ Suppose  $f_0$  is differentiable

$$f_0(y) \geq f_0(x) + \nabla f_0(x)^T (y - x), \forall x, y \in \text{dom } f_0$$



# An Optimality Criterion for Differentiable $f_0$

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□ Suppose  $f_0$  is differentiable

$$f_0(y) \geq f_0(x) + \nabla f_0(x)^\top (y - x), \forall x, y \in \text{dom } f_0$$

□ Let  $X$  denote the feasible set

$$X = \{x | f_i(x) \leq 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, p\}$$

□  $x$  is optimal if and only if  $x \in X$  and

$$\nabla f_0(x)^\top (y - x) \geq 0 \text{ for all } y \in X$$

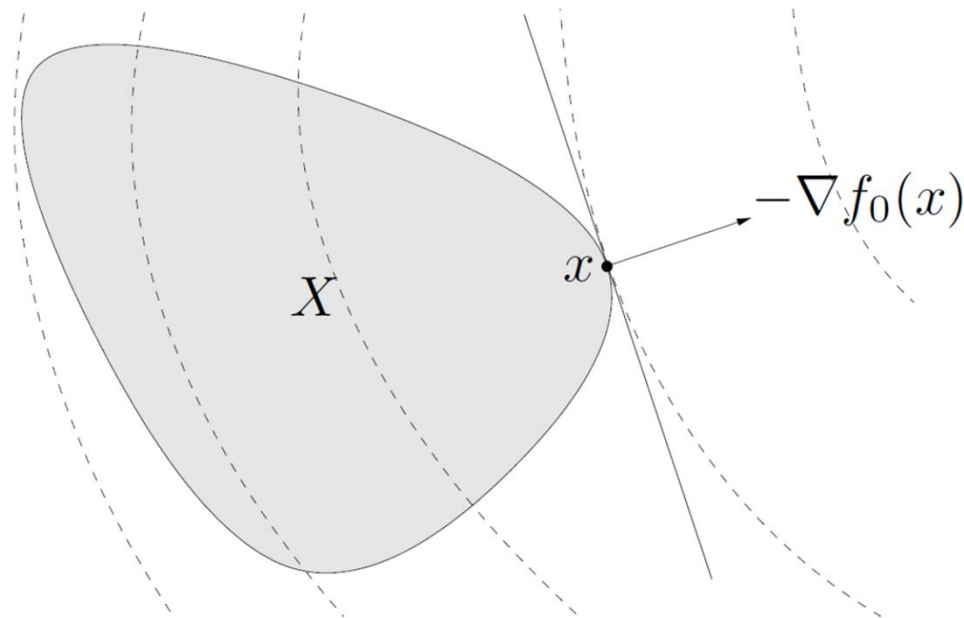
# An Optimality Criterion for Differentiable $f_0$



□  $x$  is optimal if and only if  $x \in X$  and

$$\nabla f_0(x)^\top (y - x) \geq 0 \text{ for all } y \in X$$

□  $-\nabla f_0(x)$  defines a supporting hyperplane to the feasible set at  $x$





# Proof of Optimality Condition

## □ Sufficient Condition

$$\left. \begin{array}{l} \nabla f_0(x)^\top (y - x) \geq 0 \\ f_0(y) \geq f_0(x) + \nabla f_0(x)^\top (y - x) \end{array} \right\} \Rightarrow f_0(y) \geq f_0(x)$$

## □ Necessary Condition

- Suppose  $x$  is optimal but

$$\exists y \in X, \nabla f_0(x)^\top (y - x) < 0$$

- Define  $z(t) = ty + (1 - t)x, t \in [0, 1]$

$$f_0(z(0)) = f_0(x), \quad \left. \frac{d}{dt} f_0(z(t)) \right|_{t=0} = \nabla f_0(x)^\top (y - x) < 0$$

- So, for small positive  $t$ ,  $f_0(z(t)) < f_0(x)$





# Unconstrained Problems

□  $x$  is optimal if and only if  $\nabla f_0(x) = 0$

■ Consider  $y = x - t\nabla f_0(x)$  and  $t > 0$

■ When  $t$  is small,  $y$  is feasible

$$\nabla f_0(x)^\top (y - x) = -t \|\nabla f_0(x)\|_2^2 \geq 0 \Leftrightarrow \nabla f_0(x) = 0$$

□ Unconstrained Quadratic Optimization

$$\min f_0(x) = (1/2)x^\top Px + q^\top x + r, \quad \text{where } P \in \mathbf{S}_+^n$$

■  $x$  is optimal if and only if  $\nabla f_0(x) = Px + q = 0$

■ If  $q \notin \mathcal{R}(P)$ , no solution,  $f_0$  is unbound below

■ If  $P \succ 0$ , unique minimizer  $x^* = -P^{-1}q$

■ If  $P$  is singular, but  $q \in \mathcal{R}(P)$ ,  $X_{\text{opt}} = -P^\dagger q + \mathcal{N}(P)$



# Problems with Equality Constraints Only

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□ Consider the Problem

$$\begin{array}{ll}\min & f_0(x) \\ \text{s. t.} & Ax = b\end{array}$$

□  $x$  is optimal if and only if

$$\nabla f_0(x)^\top (y - x) \geq 0, \forall Ay = b$$

# Problems with Equality Constraints Only



## □ Consider the Problem

$$\begin{array}{ll}\min & f_0(x) \\ \text{s. t.} & Ax = b\end{array}$$

Lagrange Multiplier  
Optimality Condition

$$\begin{array}{l}Ax = b \\ \nabla f_0(x) + A^\top v = 0\end{array}$$

## □ $x$ is optimal if and only if

$$\left. \begin{array}{l} \nabla f_0(x)^\top (y - x) \geq 0, \forall Ay = b \\ \{y | Ay = b\} = \{x + v | v \in \mathcal{N}(A)\} \end{array} \right\}$$

$$\Leftrightarrow \nabla f_0(x)^\top v \geq 0, \forall v \in \mathcal{N}(A)$$

$$\Leftrightarrow \nabla f_0(x)^\top v = 0, \forall v \in \mathcal{N}(A)$$

$$\Leftrightarrow \nabla f_0(x) \perp \mathcal{N}(A) \Leftrightarrow \nabla f_0(x) \in \mathcal{N}(A)^\perp = \mathcal{R}(A^\top)$$

$$\Leftrightarrow \exists v \in \mathbf{R}^p, \nabla f_0(x) + A^\top v = 0$$



# Minimization over the Nonnegative Orthant

## □ Consider the Problem

$$\begin{array}{ll}\min & f_0(x) \\ \text{s. t.} & x \geq 0\end{array}$$

## □ $x$ is optimal if and only if

$$\begin{aligned} & \nabla f_0(x)^\top (y - x) \geq 0, \forall y \geq 0 \\ \Leftrightarrow & \begin{cases} \nabla f_0(x) \geq 0 \\ -\nabla f_0(x)^\top x \geq 0 \end{cases} \Leftrightarrow \begin{cases} \nabla f_0(x) \geq 0 \\ \nabla f_0(x)^\top x = 0 \end{cases} \end{aligned}$$

## □ The Optimality Condition

$$x \geq 0, \quad \nabla f_0(x) \geq 0, \quad x_i (\nabla f_0(x))_i = 0, i = 1, \dots, n$$

■ The last condition is called **complementarity**



# Outline

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## □ Optimization Problems

- Basic Terminology
- Equivalent Problems
- Problem Descriptions

## □ Convex Optimization

- Standard Form
- Local and Global Optima
- An Optimality Criterion
- Equivalent Convex Problems
- Quasiconvex Optimization



# Equivalent Convex Problems

## □ Standard Form

$$\begin{array}{ll}\min & f_0(x) \\ \text{s. t.} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & a_i^\top x = b_i, \quad i = 1, \dots, p\end{array}$$

## □ Eliminating Equality Constraints

$$\begin{array}{ll}\min & f_0(Fz + x_0) \\ \text{s. t.} & f_i(Fz + x_0) \leq 0, \quad i = 1, \dots, m\end{array}$$

- $A = [a_1^\top; \dots; a_p^\top], b = (b_1; \dots; b_p)$
- $Ax_0 = b, \mathcal{R}(F) = \mathcal{N}(A)$
- The composition of a convex function with an affine function is convex



# Equivalent Convex Problems

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## □ Introducing Equality Constraints

- If an objective or constraint function has the form  $f_i(A_i x + b_i)$ , where  $A_i \in \mathbf{R}^{k_i \times n}$ , we can replace it with  $f_i(y_i)$  and add the constraint  $y_i = A_i x + b_i$ , where  $y_i \in \mathbf{R}^{k_i}$

## □ Slack Variables

- Introduce new constraint  $f_i(x) + s_i = 0$  and requiring that  $f_i$  is affine
- Introduce slack variables for linear inequalities preserves convexity of a problem

## □ Minimizing over Some Variables

- It preserves convexity.  $f_0(x_1, x_2)$  needs to be jointly convex in  $x_1$  and  $x_2$



# Equivalent Convex Problems

## □ Epigraph Problem Form

$$\begin{array}{ll}\min & t \\ \text{s. t.} & f_0(x) - t \leq 0 \\ & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & a_i^\top x = b_i, \quad i = 1, \dots, p\end{array}$$

- The objective is linear (hence convex)
- The new constraint function  $f_0(x) - t$  is also convex in  $(x, t)$
- This problem is convex
- Any convex optimization problem is readily transformed to one with linear objective.





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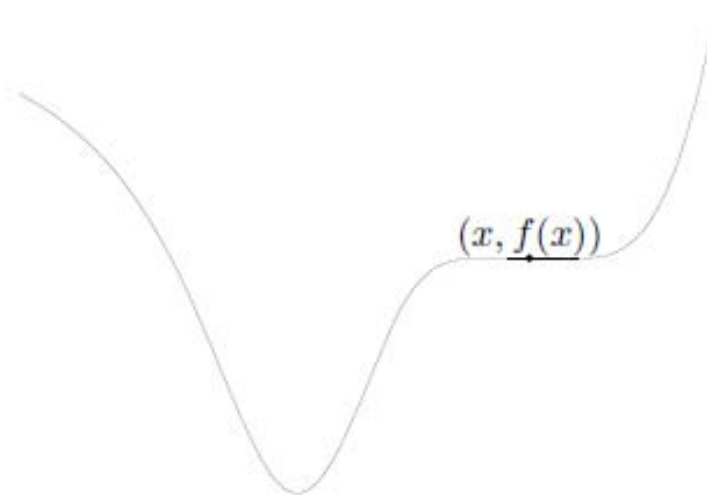
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# Quasiconvex Optimization

## □ Standard Form

$$\begin{array}{ll}\min & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & Ax = b\end{array}$$

- $f_0$  is quasiconvex and  $f_1, \dots, f_m$  are convex
- Have locally optimal solutions that are not (globally) optimal





# Quasiconvex Optimization

## □ Standard Form

$$\begin{array}{ll}\min & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & Ax = b\end{array}$$

- $f_0$  is quasiconvex and  $f_1, \dots, f_m$  are convex
- Have locally optimal solutions that are not (globally) optimal

## □ Optimality Conditions for Differentiable $f_0$

- Let  $X$  denote the feasible set,  $x$  is optimal if
$$x \in X, \quad \nabla f_0(x)^\top (y - x) > 0 \text{ for all } y \in X \setminus \{x\}$$
  1. Only a sufficient condition
  2. Requires  $\nabla f_0(x)$  to be nonzero

# Representation via family of convex functions



□ Represent the sublevel sets of a quasiconvex function  $f$  via inequalities of convex functions.

■  $\phi_t: \mathbf{R}^n \rightarrow \mathbf{R}$  is convex,  $t \in \mathbf{R}$

$$f(x) \leq t \Leftrightarrow \phi_t(x) \leq 0$$

■  $\phi_t$  is a nonincreasing function of  $t$

■ Examples

$$\phi_t(x) = \begin{cases} 0 & f(x) \leq t \\ \infty & \text{otherwise} \end{cases}$$

$$\phi_t(x) = \text{dist}(x, \{z | f(z) \leq t\})$$

# Quasiconvex Optimization via Convex Feasibility Problems



- Let  $\phi_t: \mathbf{R}^n \rightarrow \mathbf{R}, t \in \mathbf{R}$ , be a family of convex functions such that

$$f_0(x) \leq t \Leftrightarrow \phi_t(x) \leq 0$$

and for each  $x$ ,  $\phi_s(x) \leq \phi_t(x)$  whenever  $s \geq t$

- Let  $p^*$  be the optimal value of quasiconvex problem
- Consider the feasibility problem

$$\begin{array}{ll}\text{find} & x \\ \text{s. t.} & \phi_t(x) \leq 0 \\ & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & Ax = b\end{array}$$

- If it is feasible,  $p^* \leq t$ . Conversely,  $p^* \geq t$

# Bisection for Quasiconvex Optimization



## □ Algorithm

**given**  $l \leq p^*, u \geq p^*$ , tolerance  $\epsilon > 0$

**repeat**

1.  $t := (l + u)/2$

2. Solve the convex feasibility problem

3. **if** it is feasible,  $u := t$ ;    **else**  $l := t$

**until**  $u - l \leq \epsilon$

- The interval  $[l, u]$  is guaranteed to contain  $p^*$
- The length of the interval after  $k$  iterations is  $2^{-k}(u - l)$
- $\lceil \log_2((u - l)/\epsilon) \rceil$  iterations are required



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