

Convex optimization problems (I)

Li jun Zhang

zlj@nju.edu.cn

<http://cs.nju.edu.cn/zlj>





Outline

□ Optimization Problems

- Basic Terminology
- Equivalent Problems
- Problem Descriptions

□ Convex Optimization

- Standard Form
- Local and Global Optima
- An Optimality Criterion
- Equivalent Convex Problems
- Quasiconvex Optimization



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Basic Terminology

□ Optimization Problems

$$\min f_0(x)$$

$$\text{s.t. } f_i(x) \leq 0, \quad i = 1, \dots, m \quad (1)$$

$$h_i(x) = 0, \quad i = 1, \dots, p$$

Unconstrained
when $m = p = 0$

- Optimization variable: $x \in \mathbf{R}^n$
- Objective function: $f_0: \mathbf{R}^n \rightarrow \mathbf{R}$
- Inequality constraints: $f_i(x) \leq 0$
- Inequality constraint functions: $f_i: \mathbf{R}^n \rightarrow \mathbf{R}$
- Equality constraints: $h_i(x) = 0$
- Equality constraint functions: $h_i: \mathbf{R}^n \rightarrow \mathbf{R}$



Basic Terminology

□ Optimization Problems

$$\min f_0(x)$$

$$\text{s.t. } f_i(x) \leq 0, \quad i = 1, \dots, m \quad (1)$$

$$h_i(x) = 0, \quad i = 1, \dots, p$$

■ Domain

$$\mathcal{D} = \bigcap_{i=0}^m \text{dom } f_i \cap \bigcap_{i=1}^p \text{dom } h_i$$

- $x \in \mathcal{D}$ is **feasible** if it satisfies all the constraints
- The problem is **feasible** if there exists at least one feasible point



Basic Terminology

□ Optimal Value p^*

$$p^* = \inf \{f_0(x) | f_i(x) \leq 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, p\}$$

- Infeasible problem: $p^* = \infty$
- Unbounded below: if there exist x_k with $f_0(x_k) \rightarrow -\infty$ as $k \rightarrow \infty$, then $p^* = -\infty$

□ Optimal Points

- x^* is feasible and $f_0(x^*) = p^*$

□ Optimal Set

$$X_{\text{opt}} = \{x | f_i(x) \leq 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, p, f_0(x) = p^*\}$$

□ p^* is achieved if X_{opt} is nonempty



Basic Terminology

- ε -suboptimal Points
 - a feasible x with $f_0(x) \leq p^* + \varepsilon$
- ε -suboptimal Set
 - the set of all ε -suboptimal points
- Locally Optimal Points
$$\begin{aligned} & \min f_0(z) \\ \text{s. t. } & f_i(z) \leq 0, \quad i = 1, \dots, m \\ & h_i(z) = 0, \quad i = 1, \dots, p \\ & \|z - x\|_2 \leq R \end{aligned}$$
 - x is feasible and solves this above problem
- Globally Optimal Points



Basic Terminology

□ Types of Constraints

- If $f_i(x) = 0$, $f_i(x) \leq 0$ is **active** at x
- If $f_i(x) < 0$, $f_i(x) \leq 0$ is **inactive** at x
- $h_i(x) = 0$ is active at all feasible points
- **Redundant** constraint: deleting it does not change the feasible set

□ Examples on $x \in \mathbf{R}$ and $\text{dom } f_0 = \mathbf{R}_{++}$

- $f_0(x) = 1/x$: $p^* = 0$, the optimal value is not achieved
- $f_0(x) = -\log x$: $p^* = -\infty$, unbounded blow
- $f_0(x) = x \log x$: $p^* = -1/e$, $x^* = 1/e$ is optimal



Basic Terminology

□ Feasibility Problems

$$\begin{aligned} & \text{find } x \\ \text{s.t. } & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{aligned}$$

- Determine whether constraints are consistent

□ Maximization Problems

$$\begin{aligned} & \max f_0(x) \\ \text{s.t. } & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{aligned}$$

- It can be solved by minimizing $-f_0$
- Optimal Value p^*

$$p^* = \sup \{f_0(x) | f_i(x) \leq 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, p\}$$



Basic Terminology

□ Standard Form

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s. t.} \quad & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{aligned}$$

□ Box constraints

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s. t.} \quad & l_i \leq x_i \leq u_i, \quad i = 1, \dots, n \end{aligned}$$

□ Reformulation

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s. t.} \quad & l_i - x_i \leq 0, \quad i = 1, \dots, n \\ & x_i - u_i \leq 0, \quad i = 1, \dots, n \end{aligned}$$



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Equivalent Problems

- Two Equivalent Problems
 - If from a solution of one, a solution of the other is readily found, and vice versa
- A Simple Example

$$\begin{aligned} \min \quad & \tilde{f}(x) = \alpha_0 f_0(x) \\ \text{s. t.} \quad & \tilde{f}_i(x) = \alpha_i f_i(x) \leq 0, \quad i = 1, \dots, m \\ & \tilde{h}_i(x) = \beta_i h_i(x) = 0, \quad i = 1, \dots, p \end{aligned}$$

- $\alpha_i > 0, i = 0, \dots, m$
- $\beta_i \neq 0, i = 1, \dots, p$
- Equivalent to the problem (1)



Change of Variables

- $\phi: \mathbf{R}^n \rightarrow \mathbf{R}^n$ is one-to-one and $\phi(\text{dom } \phi) \supseteq \mathcal{D}$, and define

$$\begin{aligned}\tilde{f}_i(z) &= f_i(\phi(z)), & i &= 0, \dots, m \\ \tilde{h}_i(z) &= h_i(\phi(z)), & i &= 1, \dots, p\end{aligned}$$

- An Equivalent Problem

$$\begin{aligned}\min \quad & \tilde{f}_0(z) \\ \text{s.t.} \quad & \tilde{f}_i(z) \leq 0, \quad i = 1, \dots, m \\ & \tilde{h}_i(z) = 0, \quad i = 1, \dots, p\end{aligned}$$

- If z solves it, $x = \phi(z)$ solves the problem (1)
- If x solves (1), $z = \phi^{-1}(x)$ solves it



Transformation of Functions

- $\psi_0: \mathbf{R} \rightarrow \mathbf{R}$ is monotone increasing
- $\psi_1, \dots, \psi_m: \mathbf{R} \rightarrow \mathbf{R}$ satisfy $\psi_i(u) \leq 0$ if and only if $u \leq 0$
- $\psi_{m+1}, \dots, \psi_{m+p}: \mathbf{R} \rightarrow \mathbf{R}$ satisfy $\psi_i(u) = 0$ if and only if $u = 0$
- Define $\tilde{f}_i(x) = \psi_i(f_i(x)), \quad i = 0, \dots, m$
 $\tilde{h}_i(x) = \psi_{m+i}(h_i(x)), \quad i = 1, \dots, p$
- An Equivalent Problem

$$\begin{aligned} & \min \quad \tilde{f}_0(x) \\ \text{s. t. } & \tilde{f}_i(x) \leq 0, \quad i = 1, \dots, m \\ & \tilde{h}_i(x) = 0, \quad i = 1, \dots, p \end{aligned}$$



Example

□ Least-norm Problems

$$\min \|Ax - b\|_2$$

- Not differentiable at any x with $Ax - b = 0$

□ Least-norm-squared Problems

$$\min \|Ax - b\|_2^2 = (Ax - b)^T(Ax - b)$$

- Differentiable for all x



Slack Variables

- $f_i(x) \leq 0$ if and only if there is an $s_i \geq 0$ that satisfies $f_i(x) + s_i = 0$
- An Equivalent Problem

$$\begin{aligned} & \min \quad f_0(x) \\ \text{s.t.} \quad & s_i \geq 0, \quad i = 1, \dots, m \\ & f_i(x) + s_i = 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{aligned}$$

- s_i is the slack variable associated with the inequality constraint $f_i(x) \leq 0$
- x is optimal for the problem (1) if and only if (x, s) is optimal for the above problem, where $s_i = -f_i(x)$



Eliminating Equality Constraints

- Assume $\phi: \mathbf{R}^k \rightarrow \mathbf{R}^n$ is such that x satisfies

$$h_i(x) = 0, \quad i = 1, \dots, p$$

if and only if there is some $z \in \mathbf{R}^k$ such that

$$x = \phi(z)$$

- An Equivalent Problem

$$\begin{aligned} \min \quad & \tilde{f}_0(z) = f_0(\phi(z)) \\ \text{s.t.} \quad & \tilde{f}_i(z) = f_i(\phi(z)) \leq 0, \quad i = 1, \dots, m \end{aligned}$$

- If z is optimal for this problem, $x = \phi(z)$ is optimal for the problem (1)
- If x is optimal for (1), there is at least one z which is optimal for this problem



Eliminating linear equality constraints

- Assume the equality constraints are all linear $Ax = b$, and x_0 is one solution

- Let $F \in \mathbf{R}^{n \times k}$ be any matrix with $\mathcal{R}(F) = \mathcal{N}(A)$, then

$$\{x | Ax = b\} = \{Fz + x_0 | z \in \mathbf{R}^k\}$$

- An Equivalent Problem ($x = Fz + x_0$)

$$\begin{aligned} \min \quad & f_0(Fz + x_0) \\ \text{s. t.} \quad & f_i(Fz + x_0) \leq 0, \quad i = 1, \dots, m \end{aligned}$$

- $k = n - \text{rank}(A)$



Introducing Equality Constraints

□ Consider the problem

$$\begin{aligned} \min \quad & f_0(A_0x + b_0) \\ \text{s.t.} \quad & f_i(A_i x + b_i) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{aligned}$$

■ $x \in \mathbf{R}^n, A_i \in \mathbf{R}^{k_i \times n}$ and $f_i: \mathbf{R}^{k_i} \rightarrow \mathbf{R}$

□ An Equivalent Problem

$$\begin{aligned} \min \quad & f_0(y_0) \\ \text{s.t.} \quad & f_i(y_i) \leq 0, \quad i = 1, \dots, m \\ & y_i = A_i x + b_i, \quad i = 0, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{aligned}$$

■ Introduce $y_i \in \mathbf{R}^{k_i}$ and $y_i = A_i x + b_i$



Optimizing over Some Variables

- Suppose $x \in \mathbf{R}^n$ is partitioned as $x = (x_1, x_2)$, with $x_1 \in \mathbf{R}^{n_1}, x_2 \in \mathbf{R}^{n_2}$ and $n_1 + n_2 = n$
- Consider the problem

$$\begin{aligned}\min \quad & f_0(x_1, x_2) \\ \text{s.t.} \quad & f_i(\textcolor{red}{x}_1) \leq 0, \quad i = 1, \dots, m_1 \\ & \tilde{f}_i(\textcolor{red}{x}_2) \leq 0, \quad i = 1, \dots, m_2\end{aligned}$$

- An Equivalent Problem

$$\begin{aligned}\min \quad & \tilde{f}_0(x_1) \\ \text{s.t.} \quad & f_i(x_1) \leq 0, \quad i = 1, \dots, m_1\end{aligned}$$

■ where

$$\tilde{f}_0(x_1) = \inf \{f_0(x_1, z) | \tilde{f}_i(z) \leq 0, i = 1, \dots, m_2\}$$



Example

□ Minimize a Quadratic Function

$$\begin{aligned} \min \quad & x_1^\top P_{11} x_1 + 2x_1^\top P_{12} x_2 + x_2^\top P_{22} x_2 \\ \text{s.t.} \quad & f_i(x_1) \leq 0, \quad i = 1, \dots, m \end{aligned}$$

□ Minimize over x_2

$$\begin{aligned} \inf_{x_2} \quad & (x_1^\top P_{11} x_1 + 2x_1^\top P_{12} x_2 + x_2^\top P_{22} x_2) \\ & = x_1^\top (P_{11} - P_{12} P_{22}^{-1} P_{12}^\top) x_1 \end{aligned}$$

□ An Equivalent Problem

$$\begin{aligned} \min \quad & x_1^\top (P_{11} - P_{12} P_{22}^{-1} P_{12}^\top) x_1 \\ \text{s.t.} \quad & f_i(x_1) \leq 0, \quad i = 1, \dots, m \end{aligned}$$



Epigraph Problem Form

□ Epigraph Form

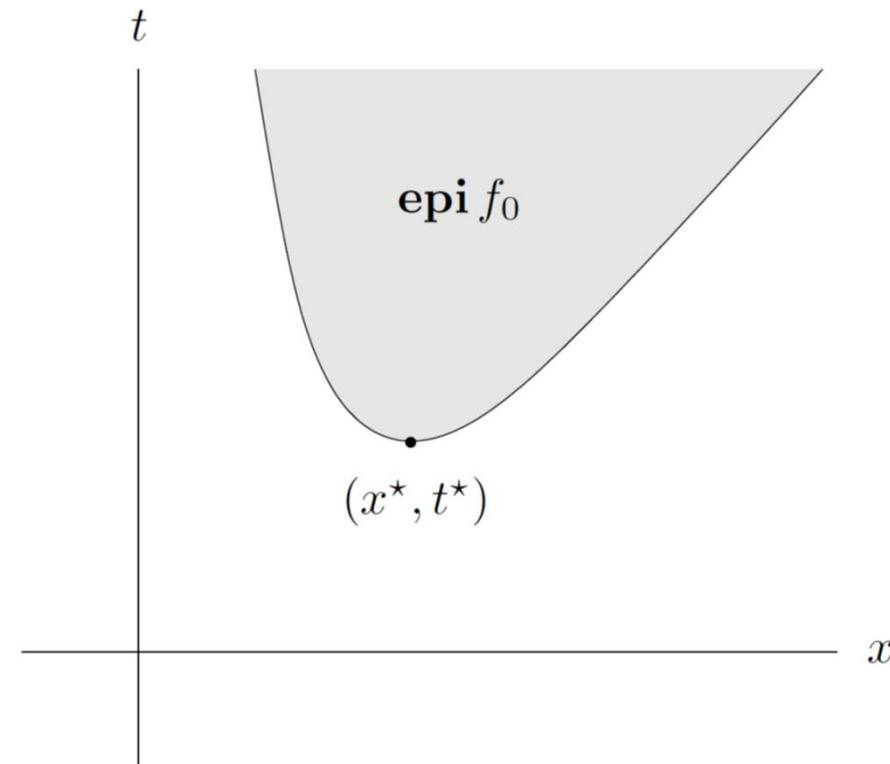
$$\begin{aligned} \min \quad & t \\ \text{s. t. } & f_0(x) - t \leq 0 \\ & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{aligned}$$

- Introduce a variable $t \in \mathbf{R}$
- (x, t) is optimal for this problem if and only if x is optimal for (1) and $t = f_0(x)$
- The objective function of the epigraph form problem is a **linear function** of x, t



Epigraph Problem Form

□ Geometric Interpretation



- Find the point in the epigraph that minimizes t



Making Constraints Implicit

□ Unconstrained problem

$$\min F(x)$$

- $\text{dom } F = \{x \in \text{dom } f_0 | f_i(x) \leq 0, i = 1, \dots, m,$
 $h_i(x) = 0, i = 1, \dots, p\}$
- $F(x) = f_0(x)$ for $x \in \text{dom } F$
- It has not make the problem any easier
- It could make the problem more difficult,
because F is probably not differentiable



Making Constraints Explicit

□ A Unconstrained Problem

$$\min f(x)$$

■ where

$$f(x) = \begin{cases} x^\top x & Ax = b \\ \infty & \text{otherwise} \end{cases}$$

■ An implicit equality constraint $Ax = b$

□ An Equivalent Problem

$$\begin{aligned} \min \quad & x^\top x \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

■ Objective and constraint functions are differentiable



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Problem Descriptions

□ Parameter Problem Description

- Functions have some analytical or closed form
- Example: $f_0(x) = x^\top Px + q^\top x + r$, where $P \in \mathbb{S}^n, q \in \mathbb{R}^n$ and $r \in \mathbb{R}$
- Give the values of the parameters

□ Oracle Model (Black-box Model)

- Can only query the objective and constraint functions by oracle
- Evaluate $f(x)$ and its gradient $\nabla f(x)$
- Know some prior information



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Convex Optimization Problems

□ Standard Form

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s. t.} \quad & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & a_i^T x = b_i, \quad i = 1, \dots, p \end{aligned}$$

- The objective function must be **convex**
- The inequality constraint functions must be **convex**
- The equality constraint functions $h_i(x) = a_i^T x - b_i$ must be **affine**



Convex Optimization Problems

□ Properties

- Feasible set of a convex optimization problem is convex

$$\bigcap_{i=0}^m \text{dom } f_i \cap \bigcap_{i=1}^m \{x | f_i(x) \leq 0\} \cap \bigcap_{i=1}^p \{x | a_i^\top x = b_i\}$$

- ✓ Minimize a convex function over a convex set
- ε -suboptimal set is convex
- The optimal set is convex
- If the objective is strictly convex, then the optimal set contains at most one point



Concave Maximization Problems

□ Standard Form

$$\begin{aligned} \max \quad & f_0(x) \\ \text{s. t.} \quad & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & a_i^\top x = b_i, \quad i = 1, \dots, p \end{aligned}$$

- It is referred as a convex optimization problem if f_0 is concave and f_1, \dots, f_m are convex

- It is readily solved by minimizing the convex objective function $-f_0$



Abstract From Convex Optimization Problem

□ Consider the Problem

$$\begin{aligned} \min \quad & f_0(x) = x_1^2 + x_2^2 \\ \text{s. t.} \quad & f_1(x) = x_1/(1+x_2^2) \leq 0 \\ & h_1(x) = (x_1 + x_2)^2 = 0 \end{aligned}$$

- Not a convex optimization problem
 - ✓ f_1 is not convex and h_1 is not affine
- But the **feasible set** is indeed convex
- Abstract convex optimization problem

□ An Equivalent Convex Problem

$$\begin{aligned} \min \quad & f_0(x) = x_1^2 + x_2^2 \\ \text{s. t.} \quad & f_1(x) = x_1 \leq 0 \\ & h_1(x) = x_1 + x_2 = 0 \end{aligned}$$



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Local and Global Optima

- Any locally optimal point of a convex problem is also (globally) optimal
- Proof by Contradiction
 - x is locally optimal implies
$$f_0(x) = \inf\{f_0(z) \mid z \text{ feasible}, \|z - x\|_2 \leq R\}$$
for some R
 - Suppose x is not globally optimal, i.e., there exists y such that $f_0(y) < f_0(x)$ and $\|y - x\|_2 > R$
 - Define
$$z = (1 - \theta)x + \theta y, \theta = \frac{R}{2\|y - x\|_2} \in (0,1)$$



Local and Global Optima

- By convexity of the feasible set

z is feasible

- It is easy to check

$$\|z - x\|_2 = \|\theta(y - x)\|_2 = \left\| \frac{R(y - x)}{2\|y - x\|_2} \right\|_2 = \frac{R}{2} < R$$

- By convexity of f_0

$$f_0(z) \leq (1 - \theta)f_0(x) + \theta f_0(y) < f_0(x)$$

which contradicts

$$f_0(x) = \inf\{f_0(z) \mid z \text{ feasible}, \|z - x\|_2 \leq R\}$$



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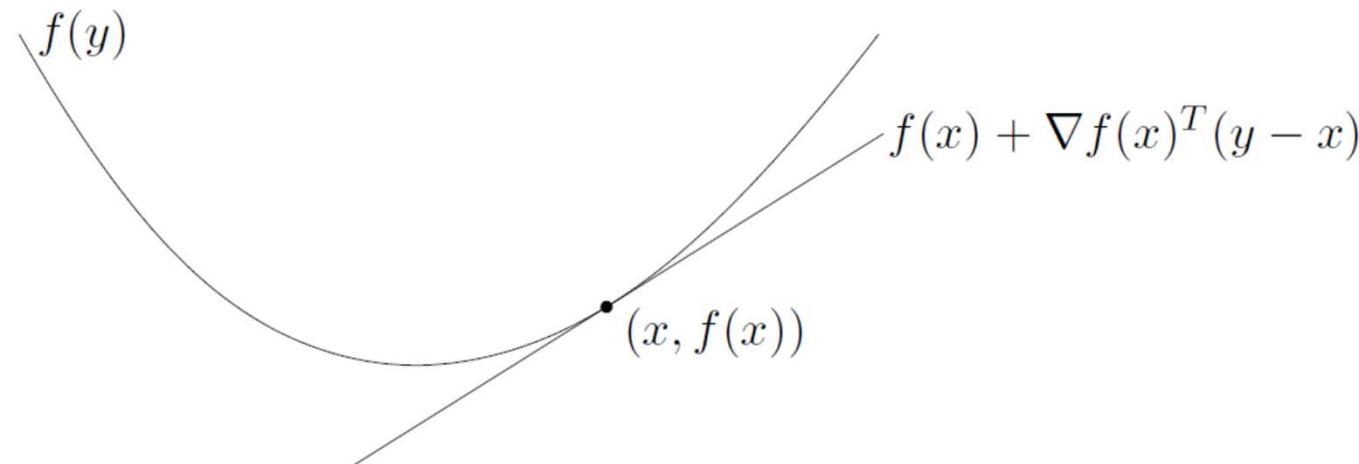
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An Optimality Criterion for Differentiable f_0

□ Suppose f_0 is differentiable

$$f_0(y) \geq f_0(x) + \nabla f_0(x)^T(y - x), \forall x, y \in \text{dom } f_0$$





An Optimality Criterion for Differentiable f_0

- Suppose f_0 is differentiable

$$f_0(y) \geq f_0(x) + \nabla f_0(x)^\top (y - x), \forall x, y \in \text{dom } f_0$$

- Let X denote the feasible set

$$X = \{x | f_i(x) \leq 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, p\}$$

- x is optimal if and only if $x \in X$ and

$$\nabla f_0(x)^\top (y - x) \geq 0 \text{ for all } y \in X$$

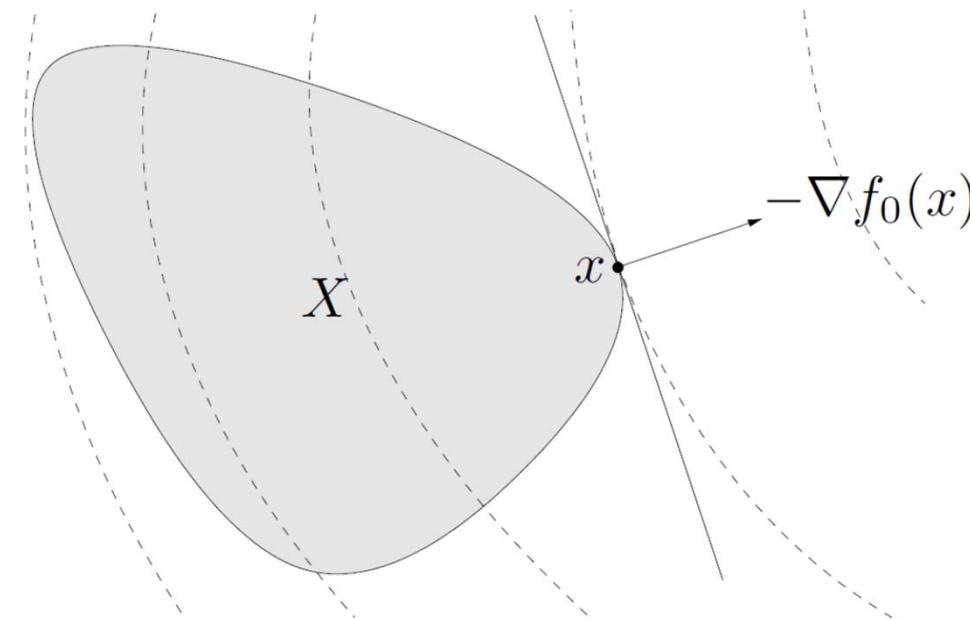


An Optimality Criterion for Differentiable f_0

- x is optimal if and only if $x \in X$ and

$$\nabla f_0(x)^\top (y - x) \geq 0 \text{ for all } y \in X$$

- $-\nabla f_0(x)$ defines a supporting hyperplane to the feasible set at x





Proof of Optimality Condition

□ Sufficient Condition

$$\left. \begin{array}{l} \nabla f_0(x)^\top (y - x) \geq 0 \\ f_0(y) \geq f_0(x) + \nabla f_0(x)^\top (y - x) \end{array} \right\} \Rightarrow f_0(y) \geq f_0(x)$$

□ Necessary Condition

- Suppose x is optimal but

$$\exists y \in X, \nabla f_0(x)^\top (y - x) < 0$$

- Define $z(t) = ty + (1 - t)x, t \in [0,1]$

$$f_0(z(0)) = f_0(x), \quad \frac{d}{dt} f_0(z(t)) \Big|_{t=0} = \nabla f_0(x)^\top (y - x) < 0$$

- So, for small positive t , $f_0(z(t)) < f_0(x)$



Unconstrained Problems

□ x is optimal if and only if $\nabla f_0(x) = 0$

- Consider $y = x - t\nabla f_0(x)$ and $t > 0$
- When t is small, y is feasible

$$\nabla f_0(x)^\top (y - x) = -t \|\nabla f_0(x)\|_2^2 \geq 0 \Leftrightarrow \nabla f_0(x) = 0$$

□ Unconstrained Quadratic Optimization

$$\min f_0(x) = (1/2)x^\top Px + q^\top x + r, \quad \text{where } P \in \mathbf{S}_+^n$$

- x is optimal if and only if $\nabla f_0(x) = Px + q = 0$
- If $q \notin \mathcal{R}(P)$, no solution, f_0 is unbound below
- If $P > 0$, unique minimizer $x^* = -P^{-1}q$
- If P is singular, but $q \in \mathcal{R}(P)$, $X_{\text{opt}} = -P^\dagger q + \mathcal{N}(P)$



Problems with Equality Constraints Only

- Consider the Problem

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s. t.} \quad & Ax = b \end{aligned}$$

- x is optimal if and only if

$$\nabla f_0(x)^\top (y - x) \geq 0, \forall Ay = b$$



Problems with Equality Constraints Only

□ Consider the Problem

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

Lagrange Multiplier Optimality Condition

$$\begin{aligned} Ax = b \\ \nabla f_0(x) + A^\top v = 0 \end{aligned}$$

□ x is optimal if and only if

$$\left. \begin{aligned} \nabla f_0(x)^\top (y - x) \geq 0, \forall Ay = b \\ \{y | Ay = b\} = \{x + v | v \in \mathcal{N}(A)\} \end{aligned} \right\}$$

$$\Leftrightarrow \nabla f_0(x)^\top v \geq 0, \forall v \in \mathcal{N}(A)$$

$$\Leftrightarrow \nabla f_0(x)^\top v = 0, \forall v \in \mathcal{N}(A)$$

$$\Leftrightarrow \nabla f_0(x) \perp \mathcal{N}(A) \Leftrightarrow \nabla f_0(x) \in \mathcal{N}(A)^\perp = \mathcal{R}(A^\top)$$

$$\Leftrightarrow \exists v \in \mathbf{R}^p, \nabla f_0(x) + A^\top v = 0$$



Minimization over the Nonnegative Orthant

□ Consider the Problem

$$\begin{aligned} & \min f_0(x) \\ & \text{s.t. } x \geq 0 \end{aligned}$$

□ x is optimal if and only if

$$\begin{aligned} & \nabla f_0(x)^\top (y - x) \geq 0, \forall y \geq 0 \\ \Leftrightarrow & \begin{cases} \nabla f_0(x) \geq 0 \\ -\nabla f_0(x)^\top x \geq 0 \end{cases} \Leftrightarrow \begin{cases} \nabla f_0(x) \geq 0 \\ \nabla f_0(x)^\top x = 0 \end{cases} \end{aligned}$$

□ The Optimality Condition

$$x \geq 0, \quad \nabla f_0(x) \geq 0, \quad x_i (\nabla f_0(x))_i = 0, i = 1, \dots, n$$

■ The last condition is called **complementarity**



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Equivalent Convex Problems

□ Standard Form

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s. t.} \quad & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & a_i^\top x = b_i, \quad i = 1, \dots, p \end{aligned}$$

□ Eliminating Equality Constraints

$$\begin{aligned} \min \quad & f_0(Fz + x_0) \\ \text{s. t.} \quad & f_i(Fz + x_0) \leq 0, \quad i = 1, \dots, m \end{aligned}$$

- $A = [a_1^\top; \dots; a_p^\top]$, $b = (b_1; \dots; b_p)$
- $Ax_0 = b$, $\mathcal{R}(F) = \mathcal{N}(A)$
- The composition of a convex function with an affine function is convex



Equivalent Convex Problems

□ Introducing Equality Constraints

- If an objective or constraint function has the form $f_i(A_i x + b_i)$, where $A_i \in \mathbf{R}^{k_i \times n}$, we can replace it with $f_i(y_i)$ and add the constraint $y_i = A_i x + b_i$, where $y_i \in \mathbf{R}^{k_i}$

□ Slack Variables

- Introduce new constraint $f_i(x) + s_i = 0$ and requiring that f_i is affine
- Introduce slack variables for linear inequalities preserves convexity of a problem

□ Minimizing over Some Variables

- It preserves convexity. $f_0(x_1, x_2)$ needs to be jointly convex in x_1 and x_2



Equivalent Convex Problems

□ Epigraph Problem Form

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & f_0(x) - t \leq 0 \\ & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & a_i^\top x = b_i, \quad i = 1, \dots, p \end{aligned}$$

- The objective is linear (hence convex)
- The new constraint function $f_0(x) - t$ is also convex in (x, t)
- This problem is convex
- Any convex optimization problem is readily transformed to one with linear objective.



Outline

□ Optimization Problems

- Basic Terminology
- Equivalent Problems
- Problem Descriptions

□ Convex Optimization

- Standard Form
- Local and Global Optima
- An Optimality Criterion
- Equivalent Convex Problems
- Quasiconvex Optimization



Quasiconvex Optimization

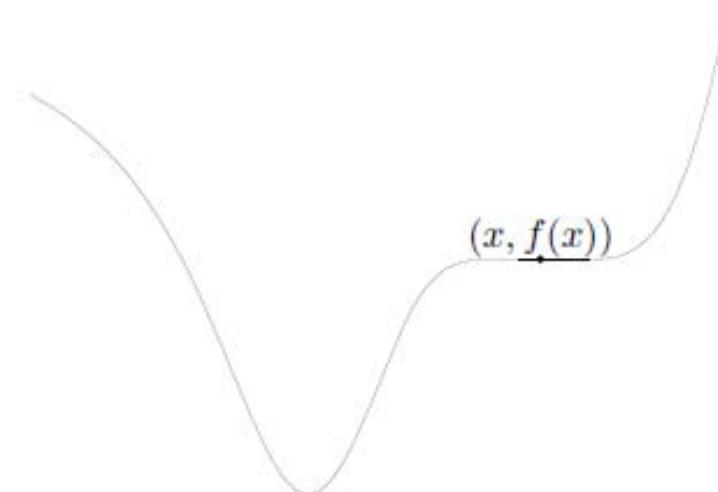
□ Standard Form

$$\min f_0(x)$$

$$\text{s. t. } f_i(x) \leq 0, \quad i = 1, \dots, m$$

$$Ax = b$$

- f_0 is quasiconvex and f_1, \dots, f_m are convex
- Have locally optimal solutions that are not (globally) optimal





Quasiconvex Optimization

□ Standard Form

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s. t.} \quad & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & Ax = b \end{aligned}$$

- f_0 is quasiconvex and f_1, \dots, f_m are convex
- Have locally optimal solutions that are not (globally) optimal

□ Optimality Conditions for Differentiable f_0

- Let X denote the feasible set, x is optimal if
 $x \in X, \quad \nabla f_0(x)^\top (y - x) > 0$ for all $y \in X \setminus \{x\}$
- 1. Only a sufficient condition
- 2. Requires $\nabla f_0(x)$ to be nonzero



Representation via family of convex functions

- Represent the sublevel sets of a quasiconvex function f via inequalities of convex functions.

- $\phi_t: \mathbf{R}^n \rightarrow \mathbf{R}$ is convex, $t \in \mathbf{R}$

$$f(x) \leq t \Leftrightarrow \phi_t(x) \leq 0$$

- ϕ_t is a nonincreasing function of t
 - Examples

$$\phi_t(x) = \begin{cases} 0 & f(x) \leq t \\ \infty & \text{otherwise} \end{cases}$$

$$\phi_t(x) = \text{dist}(x, \{z | f(z) \leq t\})$$

Quasiconvex Optimization via Convex Feasibility Problems



- Let $\phi_t: \mathbf{R}^n \rightarrow \mathbf{R}$, $t \in \mathbf{R}$, be a family of convex functions such that

$$f_0(x) \leq t \Leftrightarrow \phi_t(x) \leq 0$$

and for each x , $\phi_s(x) \leq \phi_t(x)$ whenever $s \geq t$

- Let p^* be the optimal value of quasiconvex problem
- Consider the feasibility problem

$$\begin{aligned} & \text{find } x \\ & \text{s. t. } \phi_t(x) \leq 0 \\ & \quad f_i(x) \leq 0, \quad i = 1, \dots, m \\ & \quad Ax = b \end{aligned}$$

- If it is feasible, $p^* \leq t$. Conversely, $p^* \geq t$



Bisection for Quasiconvex Optimization

□ Algorithm

given $l \leq p^*, u \geq p^*$, tolerance $\epsilon > 0$

repeat

1. $t := (l + u)/2$

2. Solve the convex feasibility problem

3. **if** it is feasible, $u := t$; **else** $l := t$

until $u - l \leq \epsilon$

- The interval $[l, u]$ is guaranteed to contain p^*
- The length of the interval after k iterations is $2^{-k}(u - l)$
- $\lceil \log_2((u - l)/\epsilon) \rceil$ iterations are required



Summary

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