Convex Functions (II)

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Outline

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Conjugate Function

☐ $f: \mathbf{R}^n \to \mathbf{R}$. Its conjugate function is
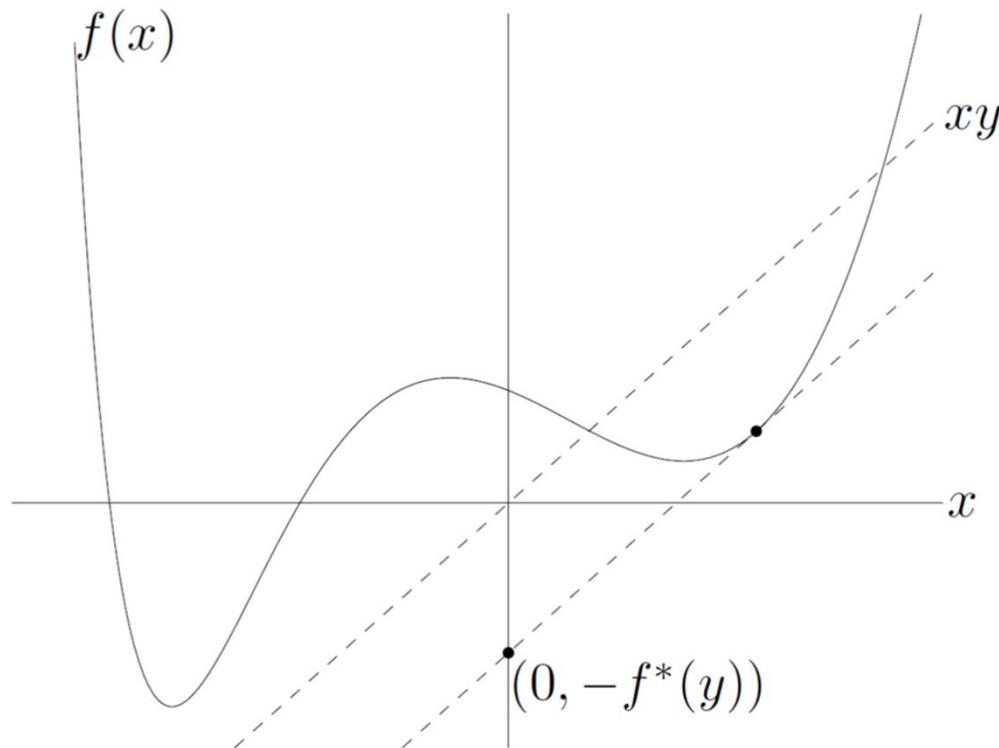
$$f^*(y) = \sup_{x \in \mathrm{dom}\, f} \left(y^\top x - f(x)\right)$$

■ $\mathrm{dom}\, f^* = \{y | f^*(y) < \infty\}$

■ f^* is always convex

Conjugate Function

□ $f: \mathbf{R}^n \to \mathbf{R}$. Its conjugate function is

$$f^*(y) = \sup_{x \in \text{dom} f} \left(y^\top x - f(x) \right)$$

Conjugate examples

- ☐ Affine function
  - ■ $f(x) = ax + b$
  - ■ $f^*(y) = \sup_{x \in \mathbf{R}}(yx - ax - b)$
  - ■ $\operatorname{dom} f^* = \{a\}, f^*(a) = -b$

- ☐ Negative logarithm
  - ■ $f(x) = -\log x$
  - ■ $f^*(y) = \sup_{x \in \mathbf{R}_{++}}(yx + \log x)$
  - ■ $\operatorname{dom} f^* = -\mathbf{R}_{++}, f^*(y) = -\log(-y) - 1$

Conjugate examples

- Exponential
  - $f(x) = e^x$
  - $f^*(y) = \sup_{x \in \mathbf{R}} (xy - e^x)$
  - $\operatorname{dom} f^* = \mathbf{R}_+, f^*(y) = y \log y - y$

- Negative entropy
  - $f(x) = x \log x$
  - $f^*(y) = \sup_{x \in \mathbf{R}_+} (yx - x \log x)$
  - $\operatorname{dom} f^* = \mathbf{R}, f^*(y) = e^{y-1}$

Conjugate examples

- ## Inverse
  - $f(x) = 1/x$
  - $f^*(y) = \sup_{x \in \mathbf{R}_{++}} (xy - 1/x)$
  - $\operatorname{dom} f^* = -\mathbf{R}_+, f^*(y) = -2(-y)^{1/2}$

- ## Strictly convex quadratic function
  - $f(x) = \frac{1}{2} x^\top Q x, Q \in \mathbf{S}_{++}^n$
  - $f^*(y) = \sup_{x \in \mathbf{R}^n} (y^\top x - \frac{1}{2} x^\top Q x)$
  - $\operatorname{dom} f^* = \mathbf{R}^n, f^*(y) = \frac{1}{2} y^\top Q^{-1} y$

Conjugate examples

☐ Log-determinant

- $f(X) = \log\det X^{-1}, X \in \mathbf{S}_{++}^n$
- $f^*(Y) = \sup_{X \in \mathbf{S}_{++}^n} (\mathrm{tr}(XY) + \log\det X)$
- $\mathrm{dom}\, f^* = -\mathbf{S}_{++}^n, f^*(Y) = \log\det(-Y)^{-1} - n$

☐ Indicator function

- $I_S(x) = 0, \mathrm{dom}\, I_S = S, S \subseteq \mathbf{R}^n$ is not necessarily convex
- $I_S^*(y) = \sup_{x \in S} y^\top x$
- $I_S^*(y)$ is the support function of the set S

Conjugate examples

- **Norm**
  - $f(x) = \|x\|, x \in \mathbf{R}^n$, with dual norm $\| \cdot \|_*$
  - $f^*(y) = \sup\limits_{x \in \mathbf{R}^n} (x^\top y - \|x\|)$
  - $\operatorname{dom} f^* = \{y | \|y\|_* \leq 1\}, f^*(y) = 0$

- **Norm squared**
  - $f(x) = \frac{1}{2}\|x\|^2, x \in \mathbf{R}^n$, with dual norm $\| \cdot \|_*$
  - $f^*(y) = \sup\limits_{x \in \mathbf{R}^n} (x^\top y - \frac{1}{2}\|x\|^2)$
  - $\operatorname{dom} f^* = \mathbf{R}^n, f^*(y) = \frac{1}{2}\|y\|_*^2$

Basic properties

☐ **Fenchel's inequality**

- $\forall x \in \operatorname{dom} f, y \in \operatorname{dom} f^*, f(x) + f^*(y) \geq x^\top y$
- $f^*(y) = \sup_{x \in \mathbf{R}^n} (x^\top y - f(x))$
- $f(x) = \frac{1}{2} x^\top Q x, Q \in \mathbf{S}_{++}^n$
  $$\Rightarrow x^\top y \leq \frac{1}{2} x^\top Q x + \frac{1}{2} y^\top Q^{-1} y$$

☐ **Conjugate of the conjugate**

- f is convex and closed $\Rightarrow f^{**} = f$

Basic properties

☐ Differentiable functions

■ f is convex and differentiable, $\operatorname{dom} f = \mathbf{R}^n$

■ $f^*(y) = \sup\limits_{x \in \mathbf{R}^n} (x^\top y - f(x))$

■ $x^* = \operatorname{argmax}(x^\top y - f(x)) \Rightarrow \nabla f(x^*) = y$

■ $f^*(y) = {x^*}^\top \nabla f(x^*) - f(x^*) = {x^*}^\top y - f(x^*)$

  ✓ $x^* = \nabla^{-1} f(y)$

Basic properties

☐ Scaling with affine transformation

- $a > 0, b \in \mathbf{R}, g(x) = af(x) + b$
  $$\Rightarrow g^*(y) = af^*\left(\frac{y}{a}\right) - b$$

- $A \in \mathbf{R}^{n \times n}$ is nonsingular, $b \in \mathbf{R}^n, g(x) = f(Ax + b) \Rightarrow g^*(y) = f^*(A^{-\top}y) - b^\top A^{-\top}y, \operatorname{dom} g^* = A^\top \operatorname{dom} f^*$

☐ Sums of independent functions

- $f(u, v) = f_1(u) + f_2(v), f_1, f_2$ are convex $\Rightarrow f^*(w, z) = f_1^*(w) + f_2^*(z)$

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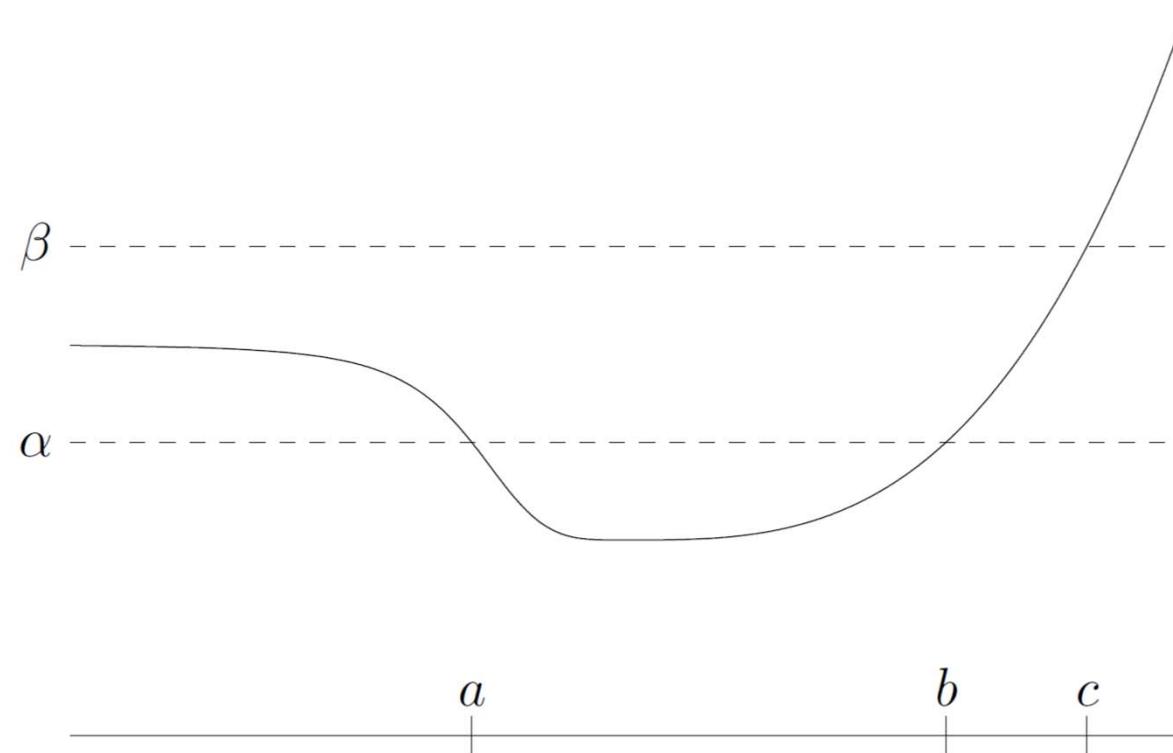
□ The Conjugate Function

□ Quasiconvex Functions

□ Log-concave and Log-convex Functions

□ Convexity with Respect to Generalized Inequalities

□ Summary

Quasiconvex functions

□ **Quasiconvex**

- $f: \mathbf{R}^n \to \mathbf{R}$
- $S_\alpha = \{x \in \operatorname{dom} f \mid f(x) \le \alpha\}, \forall \alpha \in \mathbf{R}$ is convex

Quasiconvex functions

☐ **Quasiconvex**
  - ■ $f: \mathbf{R}^n \to \mathbf{R}$
  - ■ $S_\alpha = \{x \in \mathrm{dom}\, f \mid f(x) \leq \alpha\}, \forall \alpha \in \mathbf{R}$ is convex

☐ **Quasiconcave**
  - ■ $-f$ is quasiconvex $\Rightarrow f$ is quasiconcave

☐ **Quasilinear**
  - ■ f is quasiconvex and quasiconcave $\Rightarrow f$ is quasilinear

Examples

☐ **Some example on $\mathbf{R}$**

■ Logarithm: $\log x$ on $\mathbf{R}_{++}$

■ Ceiling function： $\mathrm{ceil}(x) = \inf\{z \in \mathbf{Z} \mid z \geq x\}$
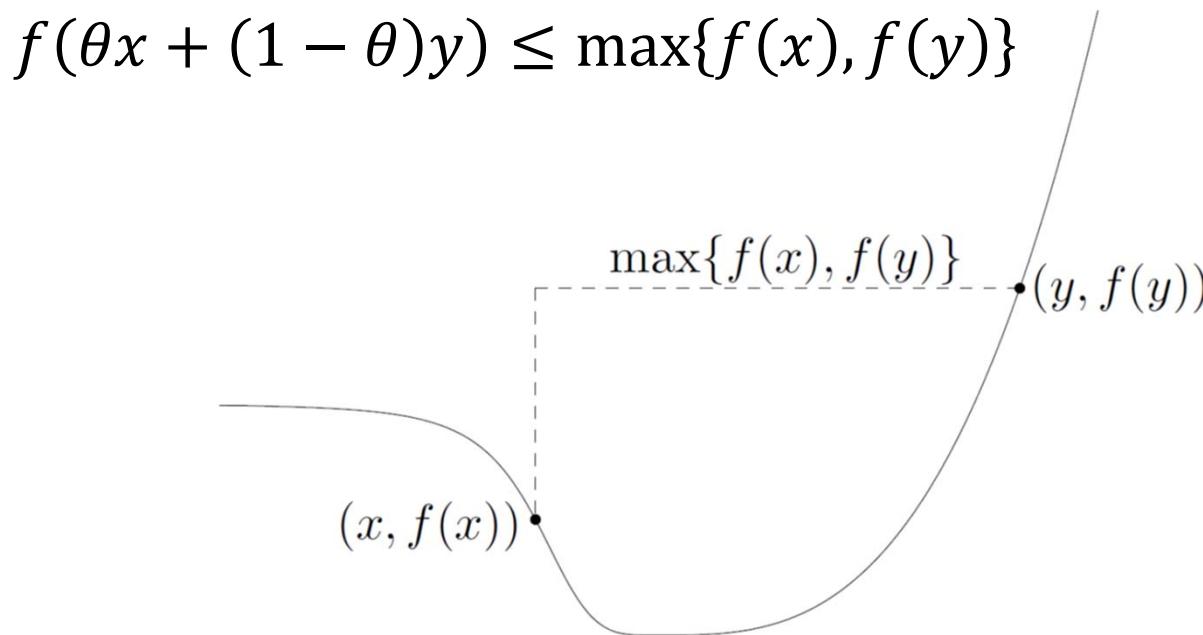
☐ **Linear-fractional function**

■ $f(x) = \dfrac{a^\top x + b}{c^\top x + d}, \mathrm{dom}\, f = \{x \mid c^\top x + d > 0\}$

■ $\left\{ x \;\middle|\; c^\top x + d > 0, \dfrac{a^\top x + b}{c^\top x + d} \geq \alpha \right\}$ and

$\left\{ x \;\middle|\; c^\top x + d > 0, \dfrac{a^\top x + b}{c^\top x + d} \leq \alpha \right\}$ is convex

$\Rightarrow f$ is Quasilinear

Basic properties

□ Jensen's inequality for quasiconvex functions

- f is quasiconvex $\Leftrightarrow \mathrm{dom}\, f$ is convex and $\forall x, y \in \mathrm{dom}\, f, 0 \leq \theta \leq 1$

$$f(\theta x + (1 - \theta)y) \leq \max\{f(x), f(y)\}$$

Basic properties

☐ **Condition**

- ■ f is quasiconvex ⇔ its restriction to any line intersecting its domain is quasiconvex

☐ **Quasiconvex functions on $\mathbf{R}$**

- ■ A continuous function $f: \mathbf{R} \to \mathbf{R}$ is quasiconvex ⇔ one of the following conditions holds

  - f is nondecreasing

  - f is nonincreasing

  - $\exists c \in \operatorname{dom} f, \forall t \in \operatorname{dom} f, t \leq c, f$ is nonincreasing, and $t \geq c, f$ is nondecreasing

Differentiable quasiconvex functions

☐ **First-order conditions**

  ■ f is differentiable

  ■ f is quasiconvex $\Leftrightarrow$ dom f is convex, $\forall x, y \in$ dom $f, f(y) \leq f(x) \Rightarrow \nabla f(x)^\top (y - x) \leq 0$

  ■ It is possible that $\nabla f(x) = 0$, but x is not a global minimizer of f.

☐ **Second-order conditions**

  ■ f is twice differentiable

  ■ $\forall x \in$ dom $f, \forall y \in \mathbf{R}^n, y^\top \nabla f(x) = 0 \Rightarrow y^\top \nabla^2 f(x) y > 0 \Rightarrow f$ is quasiconvex

Operations that preserve quasiconvexity

☐ **Nonnegative weighted maximum**
- f_i is quasicovex, $w_i \geq 0 \Rightarrow f = \max\{w_1 f_1, \dots, w_n f_n\}$ is quasiconvex

- $g(x, y)$ is quasiconvex in x for each y, $w(y) \geq 0 \Rightarrow f(x) = \sup_{y \in C}(w(y)f(x, y))$ is quasiconvex

Operations that preserve quasiconvexity

☐ **Composition**

■ $g: \mathbf{R}^n \to \mathbf{R}$ is quasiconvex, $h: \mathbf{R} \to \mathbf{R}$ is nondecreasing $\Rightarrow f = h \circ g$ is quasiconvex

■ $f: \mathbf{R}^n \to \mathbf{R}$ is quasiconvex $\Rightarrow g(x) = f(Ax + b)$ is quasiconvex

■ $f: \mathbf{R}^n \to \mathbf{R}$ is quasiconvex $\Rightarrow g(x) = f(\frac{Ax+b}{c^\top x+d})$ is quasiconvex, dom $g = \{x | c^\top x + d > 0, (Ax + b)/(c^\top x + d) \in \text{dom } f\}$

☐ **Minimization**

■ $f(x, y)$ is quasicovex in x and y, C is a convex set $\Rightarrow g(x) = \inf_{y \in C} f(x, y)$ is quasiconvex

Outline

☐ The Conjugate Function

☐ Quasiconvex Functions

☐ Log-concave and Log-convex Functions

☐ Convexity with Respect to Generalized Inequalities

☐ Summary

Log-concave and log-convex functions

- □ Definition
  - $f: \mathbf{R}^n \to \mathbf{R}, f(x) > 0, \forall x \in \text{dom } f, \log f(x)$ is concave (convex) $\Rightarrow f$ is log-concave (convex)

- □ Condition
  - $f: \mathbf{R}^n \to \mathbf{R}, f(x) > 0, \forall x \in \text{dom } f, f$ is log-concave $\Leftrightarrow \forall x, y \in \text{dom } f, 0 \leq \theta \leq 1$
    $$f(\theta x + (1 - \theta)y) \leq f(x)^\theta f(y)^{1-\theta}$$

Examples

- $f(x) = a^\top x + b, \operatorname{dom} f = \{x | a^\top x + b > 0\}$ is log-concave

- $f(x) = x^a, \operatorname{dom} f = \mathbf{R}_{++}, a \leq 0 \Rightarrow f$ is log-convex, $a \geq 0 \Rightarrow f$ is log-concave

- $\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} \, du$ is log-concave

- $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} \, du$ is log-convex for $x \geq 1$

- $\det X$ and $\frac{\det X}{\operatorname{tr} X}$ are log-concave on $\mathbf{S}_{++}^n$

Properties

☐ Twice differentiable log convex/concave functions

- ■ f is twice differentiable, $\operatorname{dom} f$ is convex

- ■ $\nabla^2 \log f(x) = \frac{1}{f(x)} \nabla^2 f(x) - \frac{1}{f(x)^2} \nabla f(x) \nabla f(x)^\top$

- ■ f is log convex $\Leftrightarrow f(x) \nabla^2 f(x) \succcurlyeq \nabla f(x) \nabla f(x)^\top$

- ■ f is log concave $\Leftrightarrow f(x) \nabla^2 f(x) \preccurlyeq \nabla f(x) \nabla f(x)^\top$

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- ☐ The Conjugate Function

- ☐ Quasiconvex Functions

- ☐ Log-concave and Log-convex Functions

- ☐ Convexity with Respect to Generalized Inequalities

- ☐ Summary

Convexity with respect to a generalized inequality

- ☐ K-convex
  - ■ $K \subseteq \mathbf{R}^m$ is a proper cone with associated generalized inequality $\preccurlyeq_K$
  - ■ $f: \mathbf{R}^n \to \mathbf{R}^m$ is K-convex if $\forall x, y \in \text{dom } f, 0 \le \theta \le 1$
    $$f(\theta x + (1-\theta)y) \preccurlyeq_K \theta f(x) + (1-\theta)f(y)$$
  - ■ $f: \mathbf{R}^n \to \mathbf{R}^m$ is stricly $K-$convex if $\forall x \ne y \in \text{dom } f, 0 < \theta < 1$
    $$f(\theta x + (1-\theta)y) \prec_K \theta f(x) + (1-\theta)f(y)$$

Examples

□ Componentwise inequality

- $K = \mathbf{R}_+^m$

- $f: \mathbf{R}^n \to \mathbf{R}^m$ is convex with respect to componentwise inequality $\Leftrightarrow \forall x, y \in \operatorname{dom} f, 0 \le \theta \le 1,$

$$f(\theta x + (1 - \theta)y) \preccurlyeq \theta f(x) + (1 - \theta)f(y)$$

- Each f_i is a convex function

Examples

□ **Matrix convexity**

■ $f: \mathbf{R}^n \to \boldsymbol{S}^m$ is convex with respect to matrix inequality $\Leftrightarrow \forall x, y \in \operatorname{dom} f, 0 \leq \theta \leq 1$

$$f(\theta x + (1 - \theta)y) \preccurlyeq \theta f(x) + (1 - \theta)f(y)$$

■ $f(X) = XX^\top, X \in \mathbf{R}^{m \times n}$ is matrix convex

■ X^p is matrix convex on $\boldsymbol{S}^n_{++}$ for $1 \leq p \leq 2$ or $-1 \leq p \leq 0$, and matrix concave for $0 \leq p \leq 1$

Convexity with respect to generalized inequalities

- ☐ **Dual characterization of K-convexity**
  - ■ A function f is (strictly) K-convex $\Leftrightarrow$ For every $w \succcurlyeq_{K^*} 0$, the real-valued function $w^\top f$ is (strictly) convex in the ordinary sense.

- ☐ **Differentiable K-convex functions**
  - ■ A differentiable function f is K-convex $\Leftrightarrow$ $\operatorname{dom} f$ is convex, $\forall x, y \in \operatorname{dom} f$,
  $$f(y) \succcurlyeq_K f(x) + Df(x)(y - x)$$
  - ■ A differentiable function f is strictly K-convex $\Leftrightarrow$ $\operatorname{dom} f$ is convex, $\forall x, y \in \operatorname{dom} f, x \neq y$,
  $$f(y) \succ_K f(x) + Df(x)(y - x)$$

Convexity with respect to generalized inequalities

- ☐ Composition theorem
  - ■ $g \colon \mathbf{R}^n \to \mathbf{R}^p$ is K-convex, $h \colon \mathbf{R}^p \to \mathbf{R}$ is convex, and $\tilde{h}$ (the extended-value extension of h) is K-nondecreasing $\Rightarrow h \circ g$ is convex.
- ☐ Example
  - ■ $g \colon \mathbf{R}^{m \times n} \to \mathbf{S}^n, g(X) = X^\top A X + B^\top X + X^\top B + C$ is convex, where $A \succeq 0, B \in \mathbf{R}^{m \times n}$ and $C \in \mathbf{S}^n$
  - ■ $h \colon \mathbf{S}^n \to \mathbf{R}, h(Y) = -\log \det(-Y)$ is convex and increasing on $\operatorname{dom} h = -\mathbf{S}^n_{++}$
  - ■ $f(X) = -\log \det(-(X^\top A X + B^\top X + X^\top B + C))$ is convex on $\operatorname{dom} f = \{X \in \mathbf{R}^{m \times n} | X^\top A X + B^\top X + X^\top B + C \prec 0\}$

Monotonicity with respect to a generalized inequality

□ $K \subseteq \mathbf{R}^n$ is a proper cone with associated generalized inequality $\preccurlyeq_K$

  ■ $f: \mathbf{R}^n \to \mathbf{R}$ is K-nondecreasing if

$$x \preccurlyeq_K y \Rightarrow f(x) \leq f(y)$$

  ■ $f: \mathbf{R}^n \to \mathbf{R}$ is K-increasing if

$$x \preccurlyeq_K y, x \neq y \Rightarrow f(x) < f(y)$$

Summary

- ☐ The Conjugate Function

    - ■ Definitions, Basic properties

- ☐ Quasiconvex Functions

- ☐ Log-concave and Log-convex Functions

- ☐ Convexity with Respect to Generalized Inequalities