Data Structures and Algorithms

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- Questions (Tu Cao)

Ex. (8.6 Lower bound on merging sorted lists (c)(d))

- *Show that if two elements are consecutive in the sorted order and from different lists, then they must be compared.*
- *Show a lower bound of $2n - 1$ comparsions for merging two sorted lists.*

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$$A' = (a_1, \dots, a_{s-1}, b_t, a_{s+1}, \dots, a_n)$$

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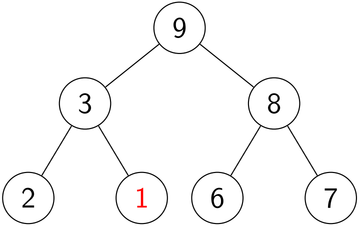
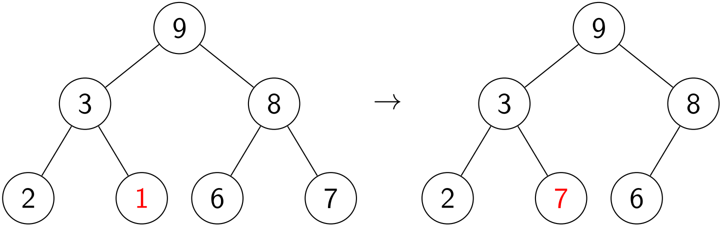
Ex. (6.5.8 Heap Delete)

The operation HEAP-DELETE(A, i) deletes the item in node i from heap A. Give an implementation of HEAP-DELETE that runs in $O(\lg n)$ time for an n-element max-heap.

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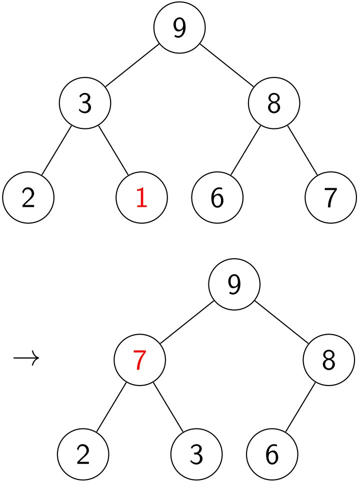
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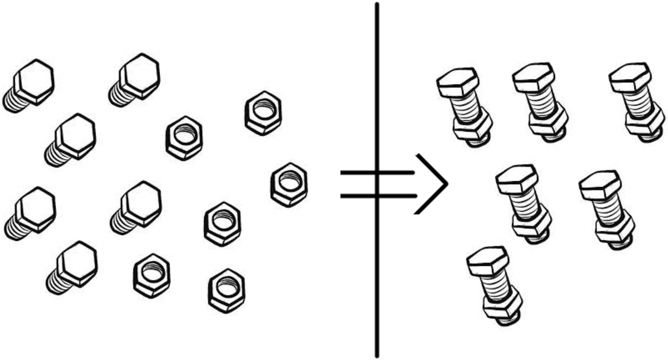
Ex. (Problem set 4, Additional Problem One, Nuts and Bolts)



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$\log \frac{n!}{(n/2)!} \geq \log(n/2)! = \Omega(n \log n)$ $\square$

- Prove that in the worst case, $\Omega(n + k \log n)$ nut-bolt tests are required to find k arbitrary matching pairs.

Proof.

$\log \frac{n!}{(n-k)!} = \cdots = \Omega(n + k \log n)$ □

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When $k = 1 \rightarrow \log \frac{n!}{(n-1)!} = \log n = o(n + \log n)$

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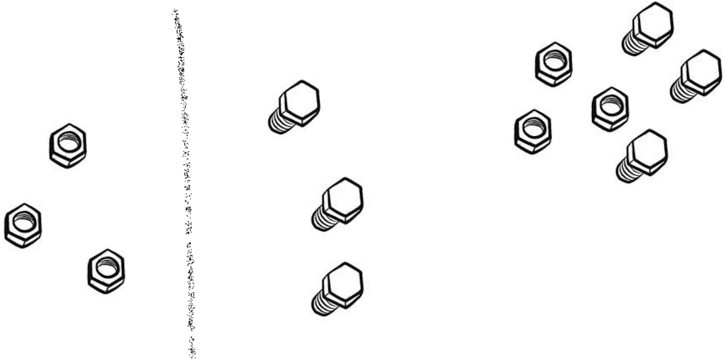

- Describe a randomized algorithm that finds k matching nut-bolt pairs in $O(n + k \log n)$ expected time.

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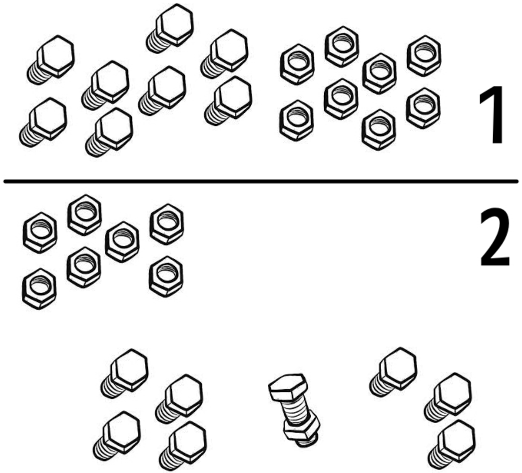

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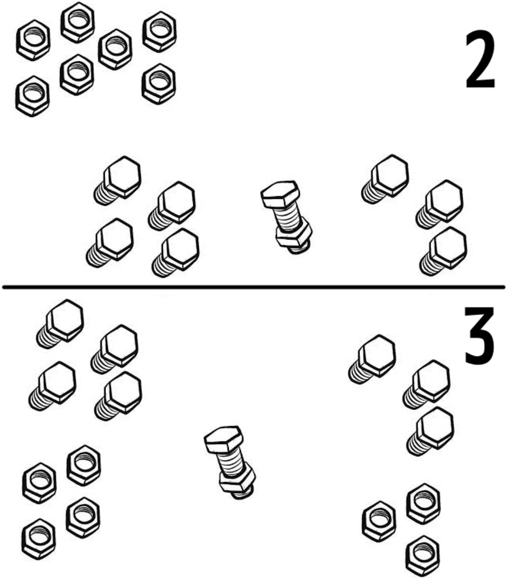

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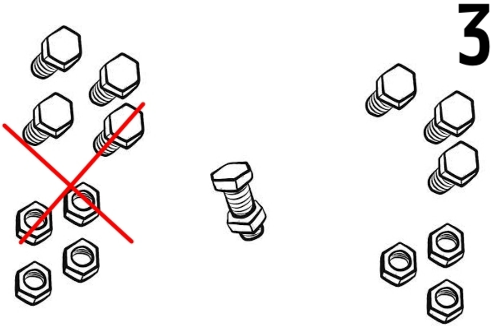

$$T(n) = \begin{cases} n \log n & \text{if } n < 2k \\ T(n/2) + 2n & \text{otherwise} \end{cases} \implies T(n) = \Theta(n + k \log k)$$

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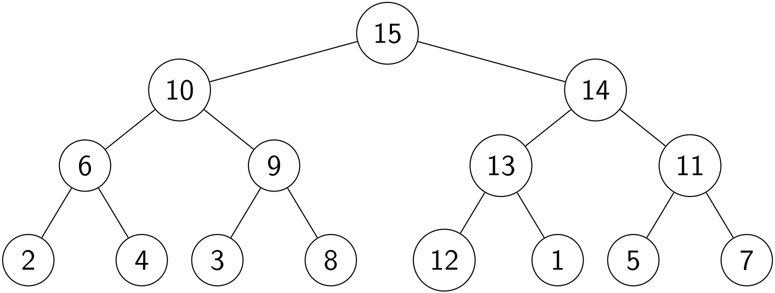
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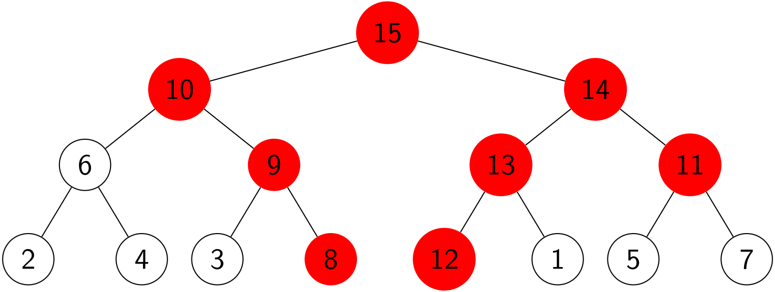
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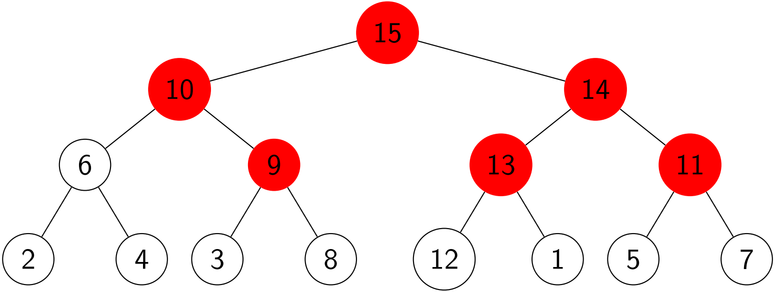
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