

# Convex Sets

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# Outline

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- Affine and Convex Sets
- Operations That Preserve Convexity
- Generalized Inequalities
- Separating and Supporting Hyperplanes
- Dual Cones and Generalized Inequalities
- Summary



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# Line

## □ Lines

$$y = \theta x_1 + (1 - \theta)x_2$$

$$y = x_2 + \theta(x_1 - x_2)$$

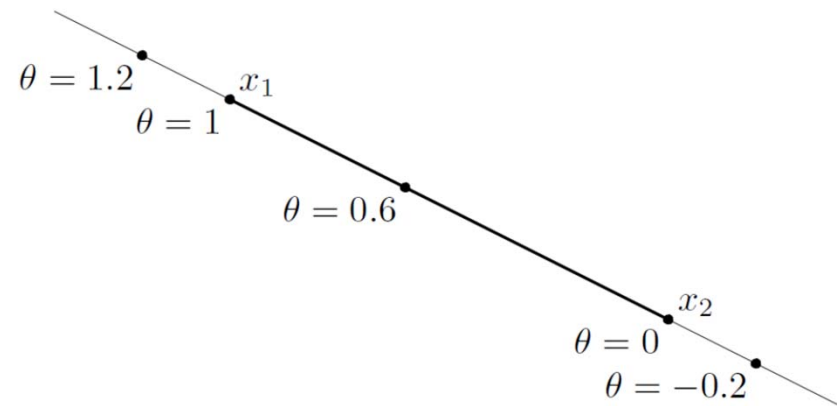
- $\theta \in \mathbf{R}$

- $x_1 \neq x_2$

## □ Line segments

- $\theta \in [0,1]$

- $x_1 \neq x_2$





# Affine Sets (1)

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## □ Definition

- $C \in \mathbf{R}^n$  is affine, if

$$\theta x_1 + (1 - \theta)x_2 \in C$$

for any  $x_1, x_2 \in C$  and  $\theta \in \mathbf{R}$

## □ Generalized form

- Affine Combination

$$\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k \in C$$

- $\theta_1 + \theta_2 + \cdots + \theta_k = 1$



# Affine Sets (2)

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## □ Subspace

$$V = C - x_0 = \{x - x_0 | x \in C\}$$

- $C \in \mathbf{R}^n$  is an affine set,  $x_0 \in C$
- Subspace is closed under sums and scalar multiplication

$$\alpha v_1 + \beta v_2 \in V, \quad \forall v_1, v_2 \in V$$

- $C$  can be expressed as a subspace plus an offset  $x_0 \in C$

$$C = V + x_0$$

- Dimension of  $C$ : dimension of  $V$



# Affine Sets (3)

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- Solution set of linear equations is affine

$$C = \{x | Ax = b\}$$

- Suppose  $x_1, x_2 \in C$

$$\begin{aligned} A(\theta x_1 + (1 - \theta)x_2) &= \theta Ax_1 + (1 - \theta)Ax_2 \\ &= \theta b + (1 - \theta)b \\ &= b \end{aligned}$$

- Every affine set can be expressed as the solution set of a system of linear equations.



# Affine Sets (4)

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## □ Affine Hull of Set $C$

$$\text{aff } C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \dots, x_k \in C, \theta_1 + \cdots + \theta_k = 1\}$$

- Affine hull is the smallest affine set that contains  $C$

## □ Affine dimension

- Affine dimension of a set  $C$  as the dimension of its affine hull  $\text{aff } C$
- Consider the unit circle  $B = \{x \in \mathbf{R}^2 \mid x_1^2 + x_2^2 = 1\}$ ,  $\text{aff } B$  is  $\mathbf{R}^2$ . So affine dimension is 2.





# Affine Sets (5)

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## □ Relative interior

$\text{relint } C = \{x \in C \mid B(x, r) \cap \text{aff } C \subseteq C \text{ for some } r > 0\}$

- $B(x, r) = \{y \mid \|y - x\| \leq r\}$ , the ball of radius  $r$  and center  $x$  in the norm  $\|\cdot\|$ .

## □ Relative boundary

$\text{cl } C \setminus \text{relint } C$

- $\text{cl } C$  is the closure of  $C$



# Affine Sets (5)

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□ A Square in  $(x_1, x_2)$ -plane in  $\mathbf{R}^3$

$$C = \{x \in \mathbf{R}^3 \mid -1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1, x_3 = 0\}$$

- Interior is empty
- Boundary is itself
- Affine hull is the  $(x_1, x_2)$ -plane
- Relative interior

$$\text{relint } C = \{x \in \mathbf{R}^3 \mid -1 < x_1 < 1, -1 < x_2 < 1, x_3 = 0\}$$

- Relative boundary

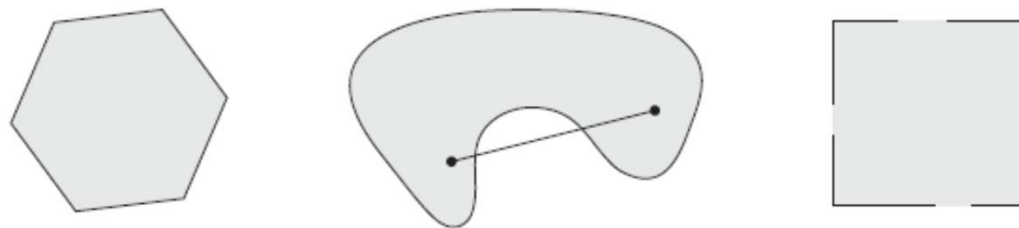
$$\{x \in \mathbf{R}^3 \mid \max\{|x_1|, |x_2|\} = 1, x_3 = 0\}$$

# Convex Sets (1)

## □ Convex sets

- A set  $\mathcal{C}$  is convex if for any  $x_1, x_2 \in \mathcal{C}$ , any  $\theta \in [0,1]$ , we have

$$\theta x_1 + (1 - \theta)x_2 \in \mathcal{C}$$



## □ Generalized Form

- Convex Combination

$$\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k \in \mathcal{C}$$

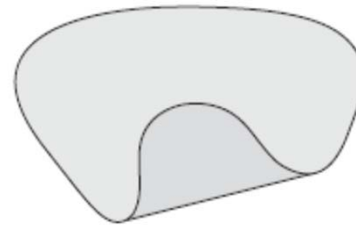
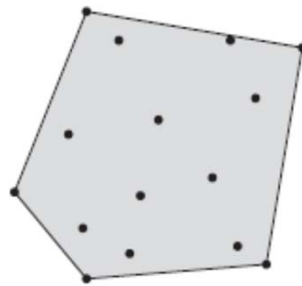
$$\theta_1 + \theta_2 + \cdots + \theta_k = 1, \theta_i \geq 0, i = 1, \cdots, k$$

# Convex Sets (2)

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## □ Convex hull

$$\text{conv } C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_i \in C, \theta_1 + \theta_2 + \cdots + \theta_k = 1, \theta_i \geq 0, i = 1, \dots, k\}$$



## □ Infinite sums, integrals

# Cone (1)

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## □ Cone

- Cone is a set that

$$x \in C, \theta \geq 0 \Rightarrow \theta x \in C$$

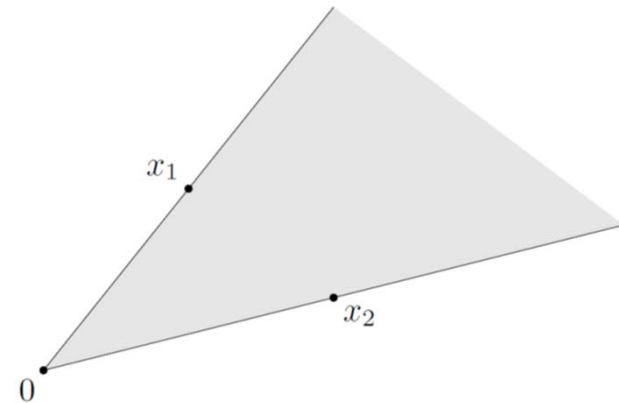
## □ Convex cone

- For any  $x_1, x_2 \in C, \theta_1, \theta_2 \geq 0$

$$\theta_1 x_1 + \theta_2 x_2 \in C$$

## □ Conic combination

- $\theta_1 x_1 + \cdots + \theta_k x_k, \theta_i > 0, i = 1, \cdots, k$

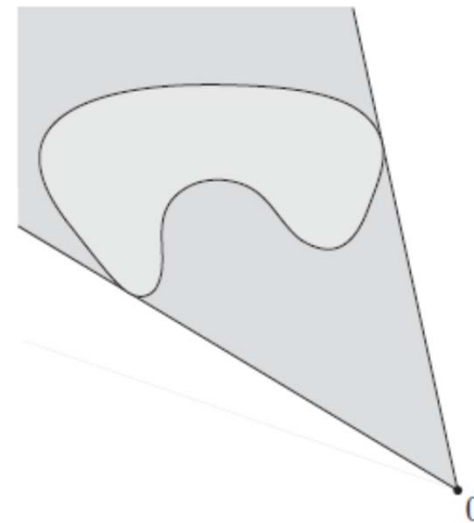
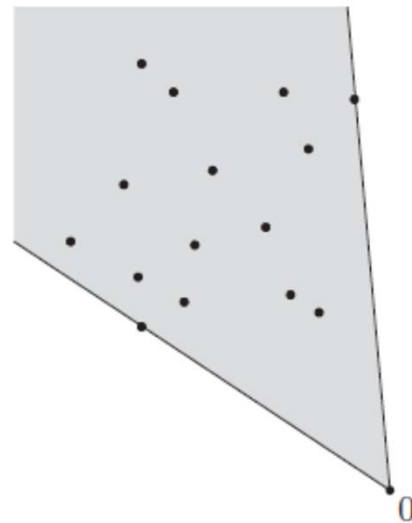


# Cone (2)

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## □ Conic hull

$$\{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_i \in C, \theta_i \geq 0, i = 1, \dots, k\}$$





# Some Examples

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- The empty set  $\emptyset$ , any single point  $\{x_0\}$ , and the whole space  $\mathbf{R}^n$  are affine (hence, convex) subsets of  $\mathbf{R}^n$
- Any line is affine. If it passes through zero, it is a subspace, hence also a convex cone.
- A line segment is convex, but not affine (unless it reduces to a point).
- A ray, which has the form  $\{x_0 + \theta v \mid \theta \geq 0\}$ , where  $v \neq 0$ , is convex, but not affine. It is a convex cone if its base  $x_0$  is 0.
- Any subspace is affine, and a convex cone (hence convex).



# Hyperplanes

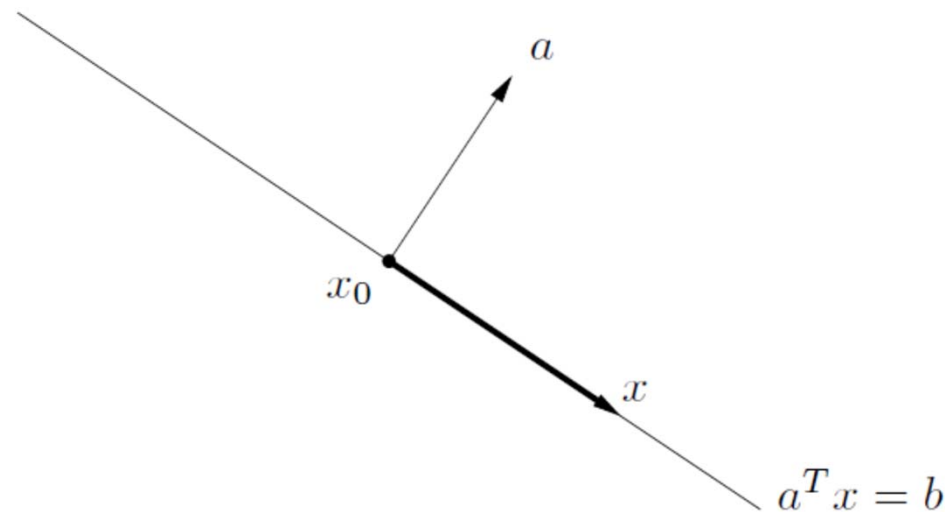
$$\{x | a^T x = b\}$$

- $a \in \mathbf{R}^n$ ,  $a \neq 0$  and  $b \in \mathbf{R}$

## □ Other Forms

$$\{x | a^T (x - x_0) = 0\}$$

- $x_0$  is any point such that  $a^T x_0 = b$







# Hyperplanes

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$$\{x | a^T x = b\}$$

- $a \in \mathbf{R}^n$ ,  $a \neq 0$  and  $b \in \mathbf{R}$

## □ Other Forms

$$\{x | a^T (x - x_0) = 0\}$$

- $x_0$  is any point such that  $a^T x_0 = b$

$$\{x | a^T (x - x_0) = 0\} = x_0 + a^\perp$$

- $a^\perp = \{v | a^T v = 0\}$



# Halfspaces

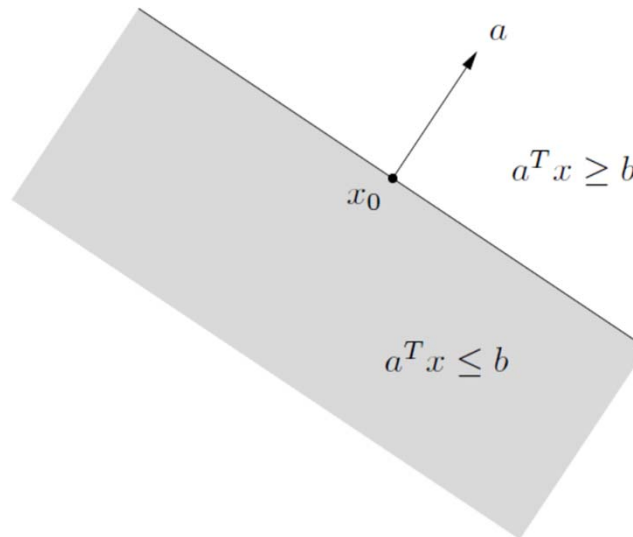
$$\{x | a^T x \leq b\}$$

- $a \in \mathbf{R}^n$ ,  $a \neq 0$  and  $b \in \mathbf{R}$

## □ Other Forms

$$\{x | a^T (x - x_0) \leq 0\}$$

- $x_0$  is any point such that  $a^T x_0 = b$



- Convex
- Not affine

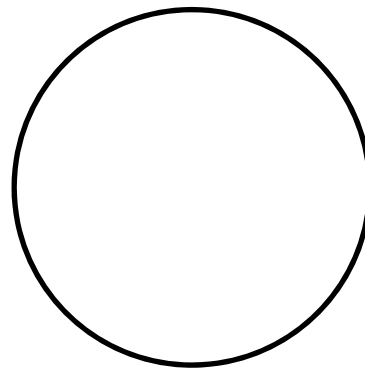
# Balls

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## □ Definition

$$\begin{aligned} B(x_c, r) &= \{x \mid \|x - x_c\|_2 \leq r\} \\ &= \{x \mid (x - x_c)^\top (x - x_c) \leq r^2\} \\ &= \{x_c + ru \mid \|u\|_2 \leq 1\} \end{aligned}$$

- $r > 0$  , and  $\|\cdot\|_2$  denotes the Euclidean norm
- Convex

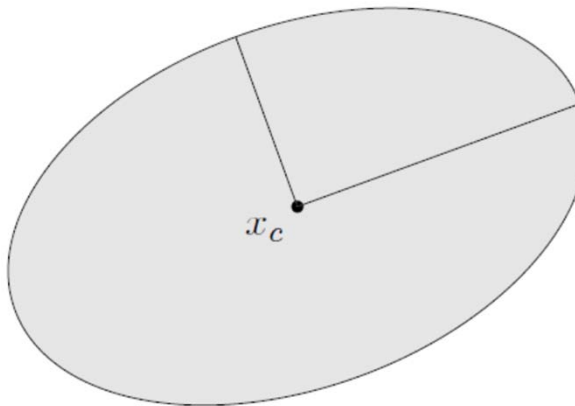


# Ellipsoids

## □ Definition

$$\begin{aligned}\mathcal{E} &= \{x | (x - x_c)^T P^{-1} (x - x_c) \leq 1\} \\ &= \{x_c + Au | \|u\|_2 \leq 1\}\end{aligned}$$

- $P \in \mathbf{S}_{++}^n$  determines how far the ellipsoid extends in every direction from  $x_c$ ;
- Lengths of semi-axes are  $\sqrt{\lambda_i}$
- Convex





# Norm Balls and Norm Cones

## □ Norm balls

$$\mathcal{C} = \{x \mid \|x - x_c\| \leq r\}$$

- $\|\cdot\|$  is any norm on  $\mathbf{R}^n$ ,  $x_c$  is the center

## □ Norm cones

$$\mathcal{C} = \{(x, t) \mid \|x\| \leq t\} \subseteq \mathbf{R}^{n+1}$$

- Second-order Cone

$$\begin{aligned}\mathcal{C} &= \{(x, t) \in \mathbf{R}^{n+1} \mid \|x\|_2 \leq t\} \\ &= \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \mid \begin{bmatrix} x \\ t \end{bmatrix}^\top \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \leq 0, t \geq 0 \right\}\end{aligned}$$

# Norm Balls and Norm Cones

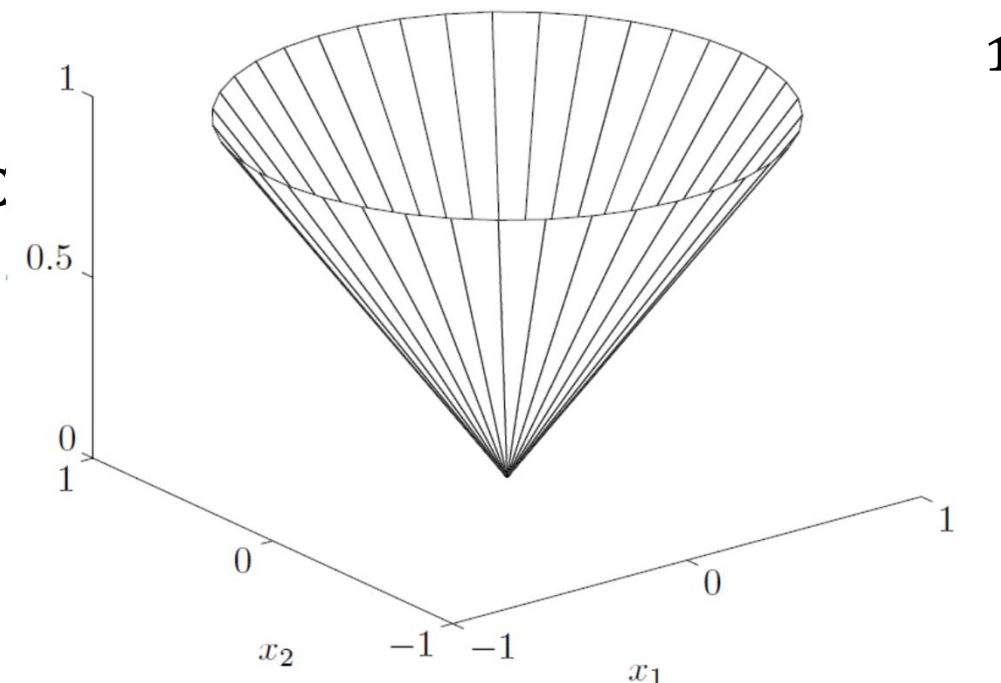
## □ Norm balls

$$C = \{x \mid \|x - x_c\| \leq r\}$$

- $\|\cdot\|$  is any norm on  $R^n$ ,  $x_c$  is the center

## □ Norm cones

- Sec





# Polyhedra (1)

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## □ Polyhedron

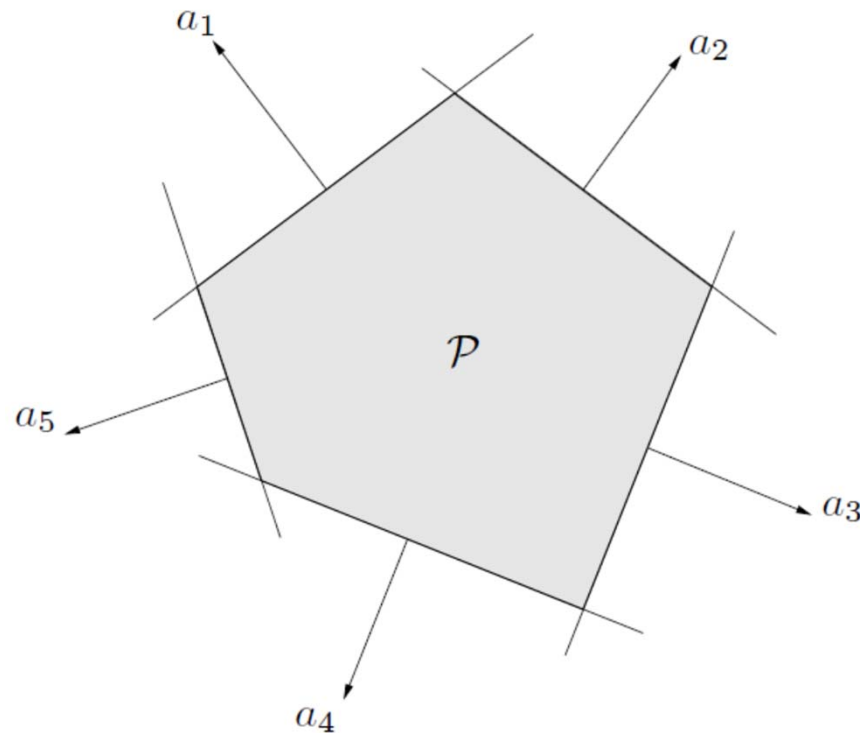
$$\mathcal{P} = \{x \mid a_j^\top x \leq b_j, j = 1, \dots, m, c_j^\top x = d_j, j = 1, \dots, p\}$$

- Solution set of a finite number of linear equalities and inequalities
- Intersection of a finite number of halfspaces and hyperplanes

# Polyhedra (2)

## □ Polyhedron

$$\mathcal{P} = \{x \mid a_j^\top x \leq b_j, j = 1, \dots, m, c_j^\top x = d_j, j = 1, \dots, p\}$$







# Polyhedra

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## □ Polyhedron

$$\mathcal{P} = \{x \mid a_j^\top x \leq b_j, j = 1, \dots, m, c_j^\top x = d_j, j = 1, \dots, p\}$$

### ■ Matrix Form

$$\mathcal{P} = \{x \mid Ax \preccurlyeq b, Cx = d\}$$

$$A = \begin{bmatrix} a_1^\top \\ \vdots \\ a_m^\top \end{bmatrix}, \quad C = \begin{bmatrix} c_1^\top \\ \vdots \\ c_m^\top \end{bmatrix}$$

$u \preccurlyeq v$  means  $u_i \leq v_i$  for all  $i$



Polyhedron?

# Simplexes

□ An important family of **polyhedral**

$$C = \text{conv}\{v_0, \dots, v_k\} = \{\theta_0 v_0 + \dots + \theta_k v_k \mid \theta \geq 0, 1^T \theta = 1\}$$

- $k + 1$  points  $v_0, \dots, v_k$  are affinely independent
- The affine dimension of this simplex is  $k$

□ 1-dimensional simplex: line segment

□ 2-dimensional simplex: triangle

□ Unit simplex:  $x \geq 0, 1^T x \leq 1$

- $n$ -dimensional

□ Probability simplex:  $x \geq 0, 1^T x = 1$

- $(n - 1)$ -dimensional



# The positive semidefinite cone

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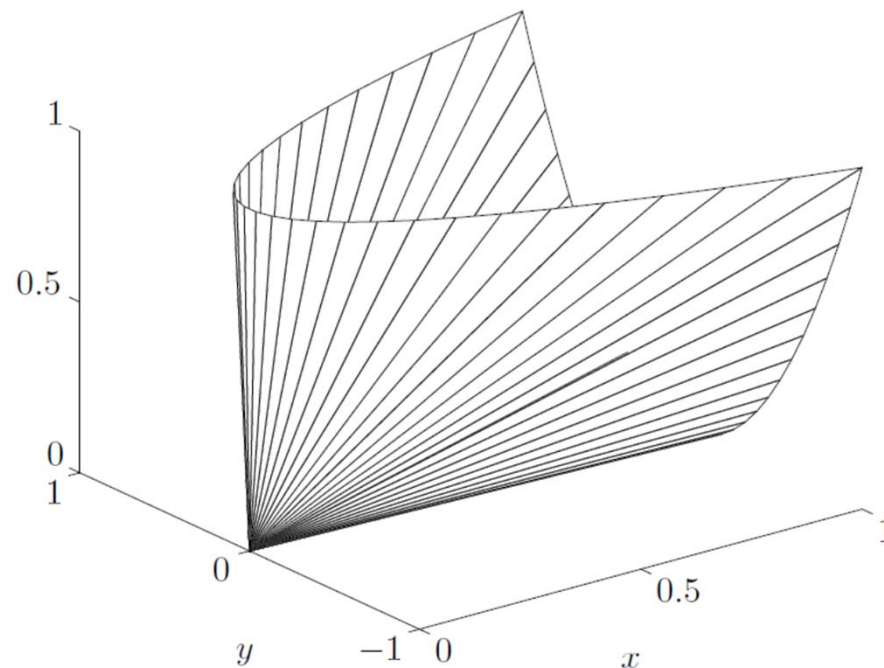
- $\mathbf{S}^n = \{X \in \mathbf{R}^{n \times n} | X = X^T\}$  is the set of symmetric  $n \times n$  matrices
  - Vector space with dimension  $n(n + 1)/2$
- $\mathbf{S}_+^n = \{X \in \mathbf{S}^n | X \succcurlyeq 0\}$  is the set of symmetric positive semidefinite matrices
  - Convex cone
- $\mathbf{S}_{++}^n = \{X \in \mathbf{S}^n | X \succ 0\}$  is the set of symmetric positive definite



# The positive semidefinite cone

## □ PSD Cone in $\mathbf{S}^2$

$$X = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}_+^2 \iff x \geq 0, z \geq 0, xz \geq y^2$$





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# Intersection

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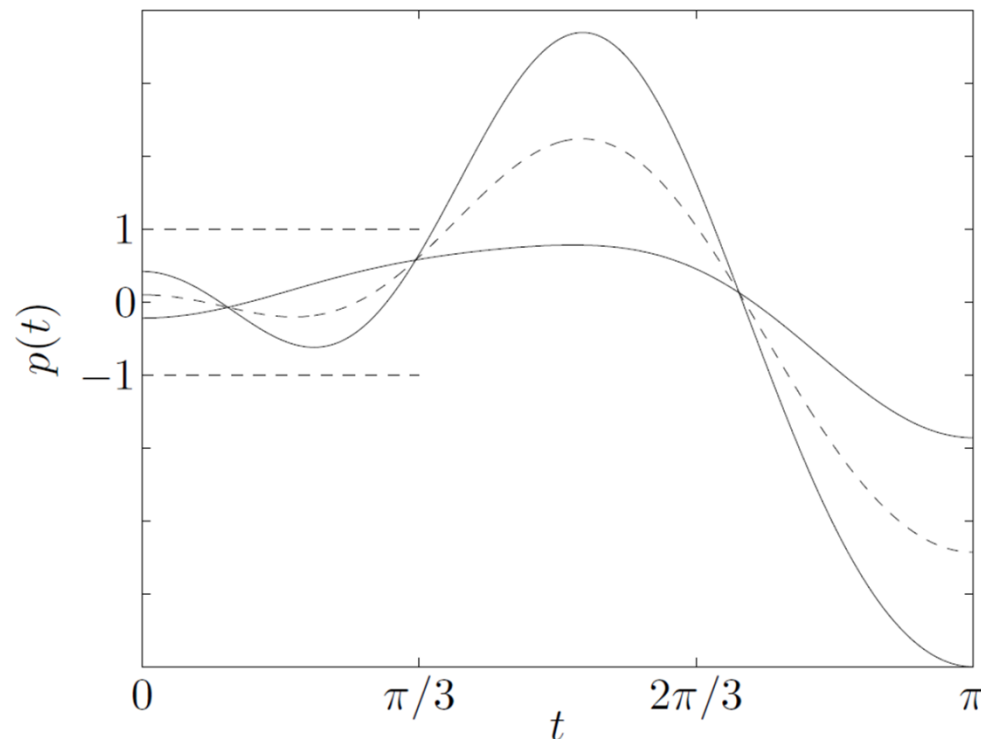
- If  $S_1$  and  $S_2$  are convex, then  $S_1 \cap S_2$  is convex.
- A polyhedron is the intersection of halfspaces and hyperplanes
- if  $S_\alpha$  is convex for every  $\alpha \in \mathcal{A}$ , then  $\bigcap_{\alpha \in \mathcal{A}} S_\alpha$  is convex.
- Positive semidefinite cone

$$\mathbf{S}_+^n = \bigcap_{z \neq 0} \{X \in \mathbf{S}^n \mid z^\top X z \geq 0\}$$

# A Complicated Example (1)

$$S = \left\{ x \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \frac{\pi}{3} \right\}$$

■  $p(t) = \sum_{k=1}^m x_k \cos kt$





# A Complicated Example (2)

---

$$S = \left\{ x \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \frac{\pi}{3} \right\}$$

■  $p(t) = \sum_{k=1}^m x_k \cos kt$

$$S = \bigcap_{|t| \leq \pi/3} S_t$$

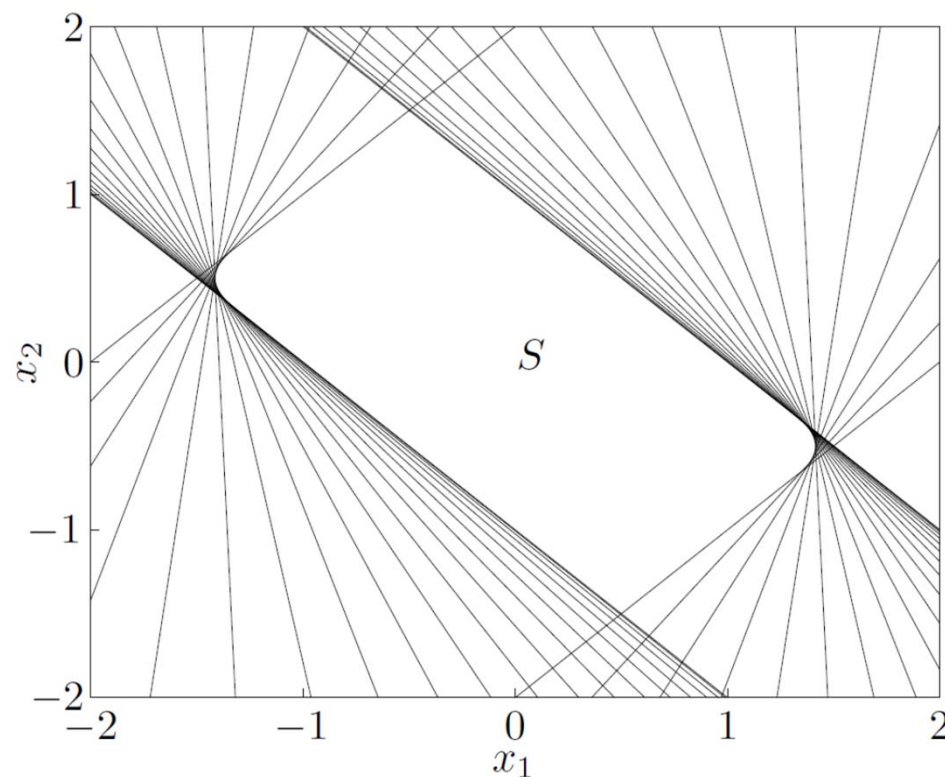
■  $S_t = \{x \mid -1 \leq (\cos t, \dots, \cos mt)^\top x \leq 1\}$





# A Complicated Example (3)

$$S = \bigcap_{|t| \leq \pi/3} S_t = \bigcap_{|t| \leq \pi/3} \{x \mid -1 \leq (\cos t, \dots, \cos mt)^\top x \leq 1\}$$





# Affine Functions

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□ Affine function  $f: \mathbf{R}^n \rightarrow \mathbf{R}^m$

$$f(x) = Ax + b, A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^m$$

□  $S \subseteq \mathbf{R}^n$  is convex

□ Then, the **image** of  $S$  under  $f$

$$f(S) = \{f(x) \mid x \in S\}$$

and the **inverse image** of  $S$  under  $f$

$$f^{-1}(S) = \{x \mid f(x) \in S\}$$

are convex



# Examples (1)

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## □ Scaling

$$\alpha S = \{\alpha x \mid x \in S\}$$

## □ Translation

$$S + a = \{x + a \mid x \in S\}$$

## □ Projection of a convex set onto some of its coordinates

$$T = \{x_1 \in \mathbf{R}^m \mid (x_1, x_2) \in S \text{ for some } x_2 \in \mathbf{R}^n\}$$

■ where  $S \subseteq \mathbf{R}^m \times \mathbf{R}^n$  is convex



## Examples (2)

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### □ Sum of two sets

$$S_1 + S_2 = \{x + y | x \in S_1, y \in S_2\}$$

- Cartesian product:  $S_1 \times S_2 = \{(x_1, x_2) | x_1 \in S_1, x_2 \in S_2\}$
- Linear function:  $f(x_1, x_2) = x_1 + x_2$

### □ Partial sum of $S_1, S_2 \in \mathbf{R}^n \times \mathbf{R}^m$

$$S = \{(x, y_1 + y_2) | (x, y_1) \in S_1, (x, y_2) \in S_2\}$$

- $m = 0$ , intersection of  $S_1$  and  $S_2$
- $n = 0$ , set addition



## Examples (3)

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### □ Polyhedron

$$\{x | Ax \preceq b, Cx = d\} = \{x | f(x) \in \mathbf{R}_+^m \times \{0\}\}$$

■  $f(x) = (b - Ax, d - Cx)$

### □ Linear Matrix Inequality

$$A(x) = x_1 A_1 + \cdots + x_n A_n \preceq B$$

■ The solution set  $\{x | A(x) \preceq B\}$

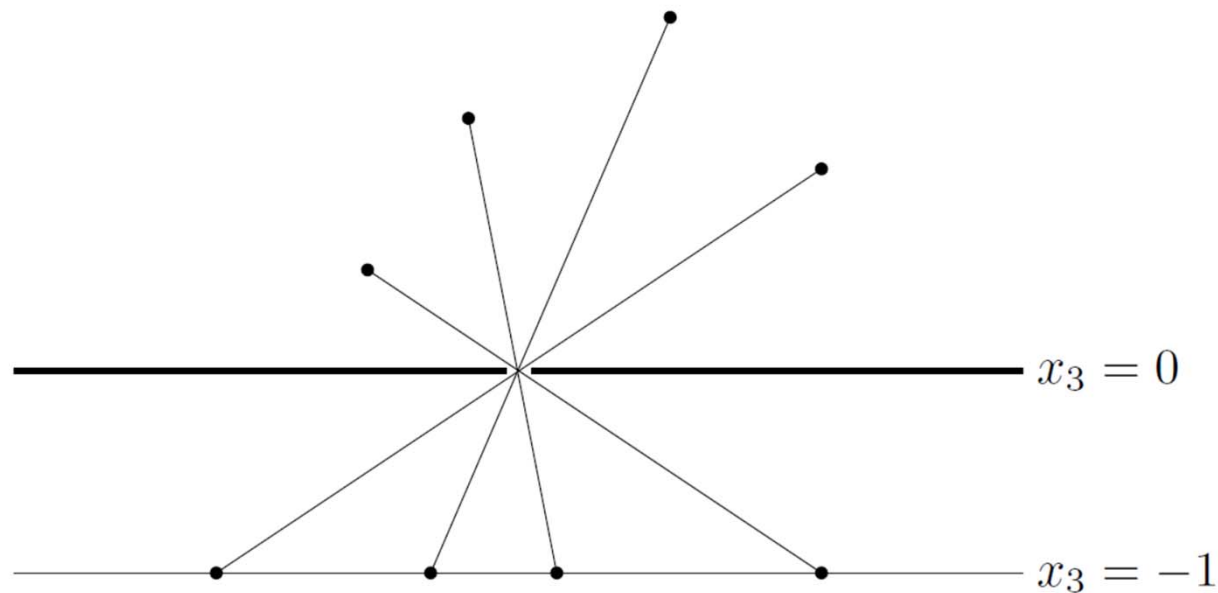
$$\{x | A(x) \preceq B\} = \{x | B - A(x) \in \mathbf{S}_+^m\}$$



# Perspective Functions (1)

□ Perspective function  $P: \mathbf{R}^{n+1} \rightarrow \mathbf{R}^n$

$$P(z, t) = \frac{z}{t}, \text{ dom } P = \mathbf{R}^n \times \mathbf{R}_{++}$$





## Perspective Functions (2)

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□ Perspective function  $P: \mathbf{R}^{n+1} \rightarrow \mathbf{R}^n$

$$P(z, t) = \frac{z}{t}, \text{ dom } P = \mathbf{R}^n \times \mathbf{R}_{++}$$

□ If  $C \in \text{dom } P$  is convex, then its image

$$P(C) = \{P(x) | x \in C\}$$

is convex

□ If  $C \in \mathbf{R}^n$  is convex, the inverse image

$$P^{-1}(C) = \left\{ (x, t) \in \mathbf{R}^{n+1} \mid \frac{x}{t} \in C, t \geq 0 \right\}$$

is convex



# Linear-fractional Functions (1)

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□ Suppose  $g: \mathbf{R}^n \rightarrow \mathbf{R}^{m+1}$  is affine

$$g(x) = \begin{bmatrix} A \\ c^\top \end{bmatrix} x + \begin{bmatrix} b \\ d \end{bmatrix}$$

□ The function  $f: \mathbf{R}^n \rightarrow \mathbf{R}^m$  given by  $P \circ g$

$$f(x) = \frac{Ax + b}{c^\top x + d}, \text{ dom } f = \{c^\top x + d > 0\}$$





## Linear-fractional Functions (2)

- If  $C$  is convex and  $\{c^\top x + d > 0 \text{ for } x \in C\}$ , then

$$f(C) = \left\{ \frac{Ax + b}{c^\top x + d} \mid x \in C \right\}$$

is convex

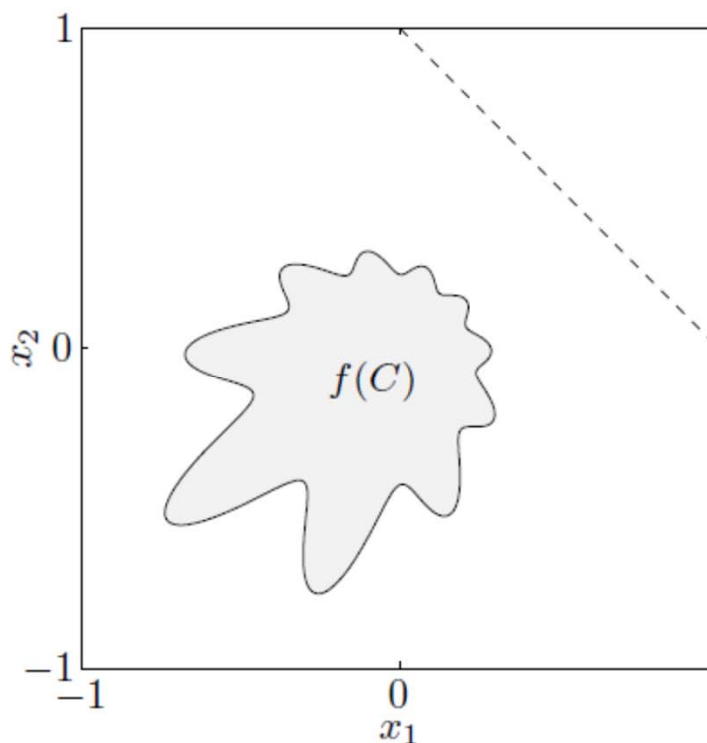
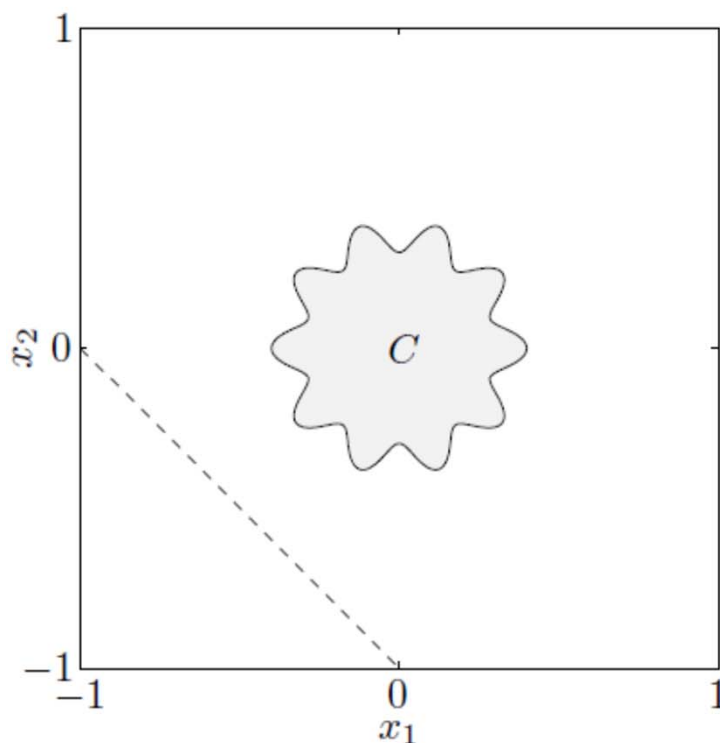
- If  $C \in \mathbf{R}^m$  is convex, then the inverse image

$$f^{-1}(C) = \left\{ x \mid \frac{Ax + b}{c^\top x + d} \in C \right\}$$

is convex

# Example

$$f(x) = \frac{1}{x_1 + x_2 + 1} x, \text{ dom } f = \{(x_1, x_2) | x_1 + x_2 + 1 > 0\}$$





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# Proper Cones

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- A cone  $K \subseteq \mathbf{R}^n$  is called a proper cone if it satisfies the following
  - $K$  is convex.
  - $K$  is closed.
  - $K$  is solid, which means it has nonempty interior.
  - $K$  is pointed, which means that it contains no line ( $x \in K, -x \in K \Rightarrow x = 0$ ).
- A proper cone  $K$  can be used to define a generalized inequality



# Generalized Inequalities

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- We associate with the proper cone  $K$  the partial ordering on  $\mathbf{R}^n$  defined by

$$x \preceq_K y \iff y - x \in K$$

- We define an associated strict partial ordering by

$$x \prec_K y \iff y - x \in \text{int } K$$



# Examples

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## □ Nonnegative Orthant and Componentwise Inequality

- $K = \mathbf{R}_+^n$
- $x \preceq_K y$  means that  $x_i \leq y_i, i = 1, \dots, n$ .
- $x \prec_K y$  means that  $x_i < y_i, i = 1, \dots, n$ .

## □ Positive Semidefinite Cone and Matrix Inequality

- $K = \mathbf{S}_+^n$
- $X \preceq_K Y$  means that  $Y - X$  is PSD
- $X \prec_K Y$  means that  $Y - X$  is positive definite



# Properties of Generalized Inequalities

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- $\preceq_K$  is preserved under addition: If  $x \preceq_K y$  and  $u \preceq_K v$ , then  $x + u \preceq_K y + v$ .
- $\preceq_K$  is transitive: if  $x \preceq_K y$  and  $y \preceq_K z$ , then  $x \preceq_K z$ .
- $\preceq_K$  is preserved under nonnegative scaling: if  $x \preceq_K y$  and  $\alpha \geq 0$  then  $\alpha x \preceq_K \alpha y$ .
- $\preceq_K$  is reflexive:  $x \preceq_K x$ .
- $\preceq_K$  is antisymmetric: if  $x \preceq_K y$  and  $y \preceq_K x$ , then  $x = y$ .
- $\preceq_K$  is preserved under limits: if  $x_i \preceq_K y_i$  for  $i = 1, 2, \dots$ ,  $x_i \rightarrow x$  and  $y_i \rightarrow y$  as  $i \rightarrow \infty$ , then  $x \preceq_K y$ .

# Properties of Strict Generalized Inequalities



- If  $x \prec_K y$  then  $x \preccurlyeq_K y$ .
- If  $x \prec_K y$  and  $u \preccurlyeq_K v$  then  $x + u \prec_K y + v$ .
- If  $x \prec_K y$  and  $\alpha > 0$  then  $\alpha x \prec_K \alpha y$ .
- $x \not\prec_K x$ .
- If  $x \prec_K y$ , then for  $u$  and  $v$  small enough,  $x + u \prec_K y + v$ .





# Minimum and Minimal Elements

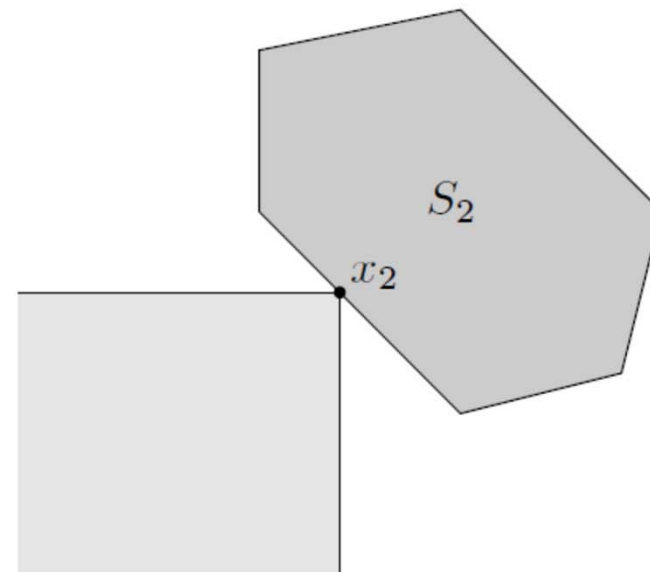
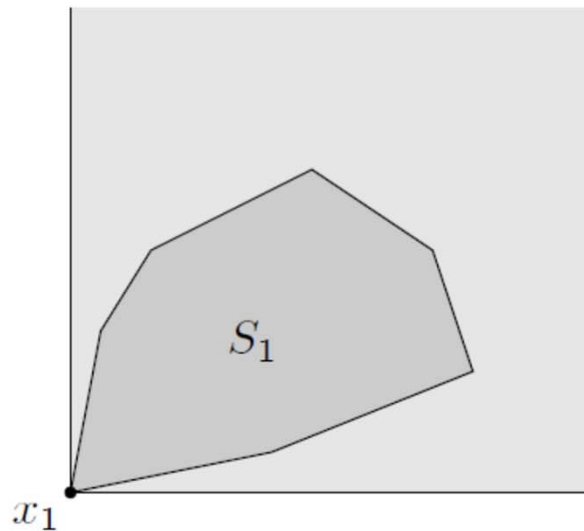
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- $x \in S$  is the **minimum** element
  - If for every  $y \in S$  we have  $x \preceq_K y$ .
  - $S \subseteq x + K$
  - Minimum element is unique, if exists
- $x \in S$  is a **minimal** element
  - if  $y \in S$ ,  $y \preceq_K x$  only if  $y = x$
  - $(x - K) \cap S = \{x\}$
  - May have different minimal elements
- **Maximum, Maximal**

# Example

## □ The Cone $\mathbf{R}_+^2$

- $x \preccurlyeq y$  means  $y$  is above and to the right of  $x$ .





# Outline

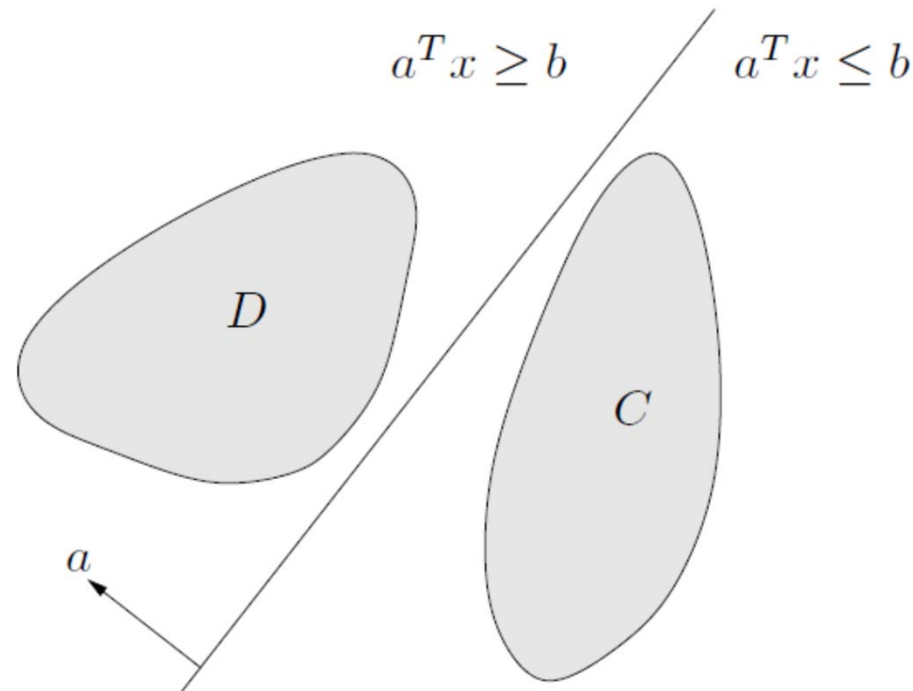
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- Affine and Convex Sets
- Operations That Preserve Convexity
- Generalized Inequalities
- Separating and Supporting Hyperplanes
- Dual Cones and Generalized Inequalities
- Summary

# Separating Hyperplane Theorem



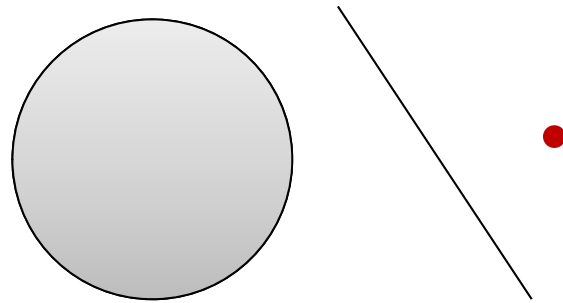
- Suppose  $C$  and  $D$  are nonempty disjoint convex sets, i.e.,  $C \cap D = \emptyset$ . Then, there exist  $a \neq 0$  and  $b$  such that



# Strict Separation

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- $a^T x < b$  for all  $x \in C$  and  $a^T x > b$  for all  $x \in D$ .
- May not be possible in general
- A Point and a Closed Convex Set



- A closed convex set is the intersection of all halfspaces that contain it

# Converse separating hyperplane theorems

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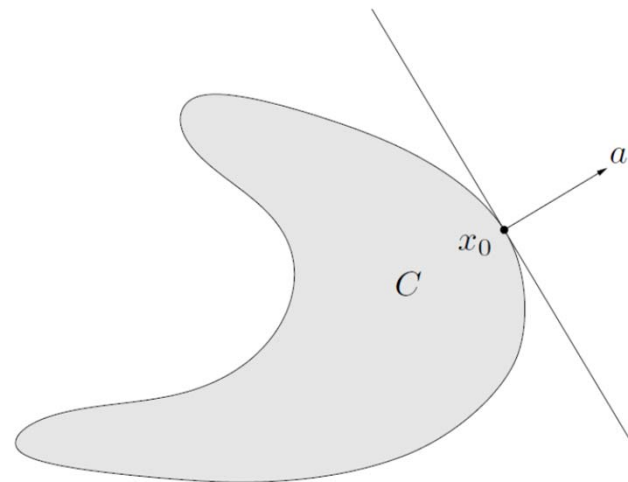
- Suppose  $C$  and  $D$  are convex sets, with  $C$  open, and there exists an affine function  $f$  that is nonpositive on  $C$  and nonnegative on  $D$ . Then  $C$  and  $D$  are disjoint.
- Any two convex sets  $C$  and  $D$ , at least one of which is open, are disjoint if and only if there exists a separating hyperplane.

# Supporting Hyperplanes

- Suppose  $C \subseteq R^n$ , and  $x_0$  is a point in its boundary  $\text{bd } C$ , i.e.,

$$x_0 \in \text{bd } C = \text{cl } C \setminus \text{int } C$$

- if  $a \neq 0$  satisfies  $a^\top x \leq a^\top x_0$  for all  $x \in C$ . The hyperplane  $\{x | a^\top x = a^\top x_0\}$  is called a **supporting** hyperplane to  $C$  at the point  $x_0$





# Two Theorems

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## □ Supporting Hyperplane Theorem

- For any nonempty convex set  $C$ , and any  $x_0 \in \text{bd } C$ , there exists a supporting hyperplane to  $C$  at  $x_0$ .

## □ Converse Theorem

- If a set is closed, has nonempty interior, and has a supporting hyperplane at every point in its boundary, then it is convex.





# Outline

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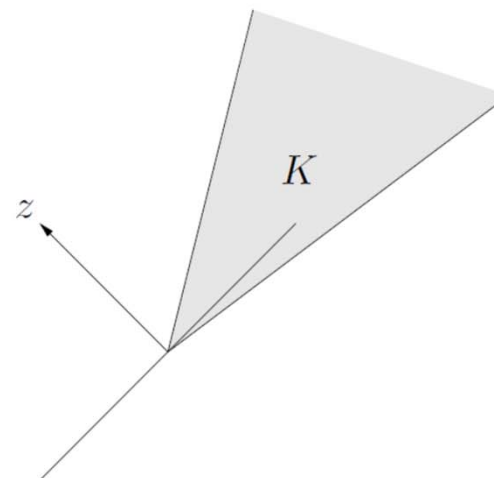
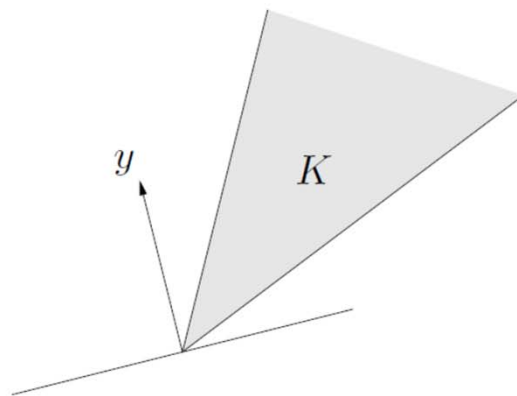
- Affine and Convex Sets
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# Dual Cone

## □ Dual Cone of a Given Cone $K$

$$K^* = \{y | x^\top y \geq 0 \text{ for all } x \in K\}$$

- $K^*$  is convex, even when  $K$  is not
- $y \in K^*$  if and only if  $-y$  is the normal of a hyperplane that supports  $K$  at the origin





# Examples

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## □ Subspace

- The dual cone of a subspace  $V \in \mathbf{R}^n$

$$V^\perp = \{y | v^\top y = 0 \text{ for all } v \in V\}$$

## □ Nonnegative Orthant

- The cone  $\mathbf{R}_+^n$  is its own dual

$$x^\top y \geq 0 \text{ for all } x \succcurlyeq 0 \iff y \succcurlyeq 0$$

## □ Positive Semidefinite Cone

- $\mathbf{S}_+^n$  is self-dual

$$\text{tr}(XY) \geq 0 \text{ for all } X \succcurlyeq 0 \iff Y \succcurlyeq 0$$



# Properties of Dual Cone

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- $K^*$  is closed and convex.
- $K_1 \subseteq K_2$  implies  $K_2^* \subseteq K_1^*$
- If  $K$  has nonempty interior, then  $K^*$  is pointed.
- If the closure of  $K$  is pointed then  $K^*$  has nonempty interior.
- $K^{**}$  is the closure of the convex hull of  $K$ . (Hence if  $K$  is convex and closed,  $K^{**} = K$ .)



# Dual Generalized Inequalities

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- Suppose that the convex cone  $K$  is proper, so it induces a generalized inequality  $\preceq_K$ .
- Its dual cone  $K^*$  is also proper. We refer to the generalized inequality  $\preceq_{K^*}$  as the dual of the generalized inequality  $\preceq_K$ .
  - $x \preceq_K y$  if and only if  $\lambda^\top x \leq \lambda^\top y$  for all  $0 \preceq_{K^*} \lambda$
  - $x \prec_K y$  if and only if  $\lambda^\top x < \lambda^\top y$  for all  $0 \preceq_{K^*} \lambda, \lambda \neq 0$



# Dual Characterization of Minimum Element

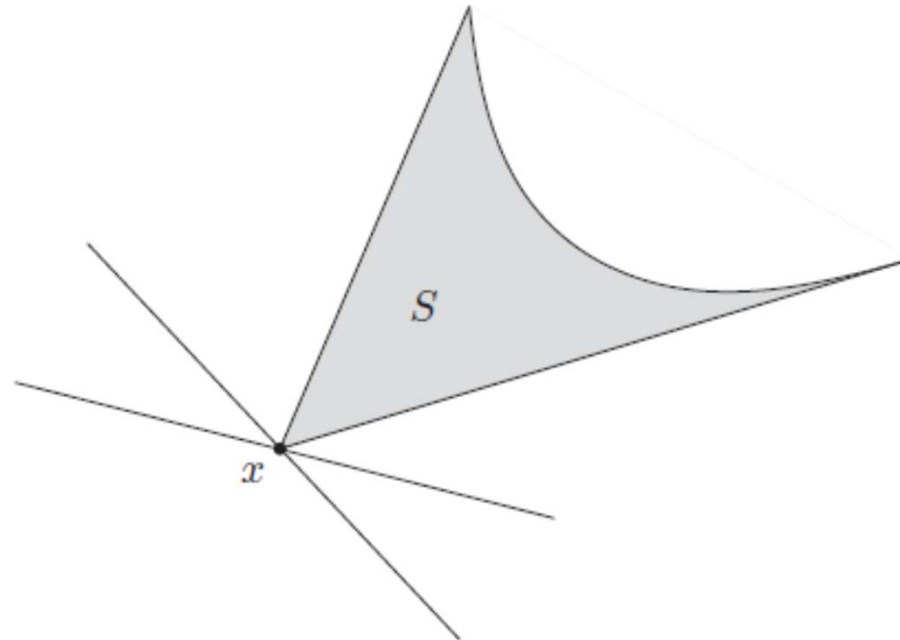
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- $x$  is the minimum element of  $S$ , with respect to the generalized inequality  $\preceq_K$ , **if and only if** for all  $\lambda \succ_{K^*} 0$ ,  $x$  is the unique minimizer of  $\lambda^T z$  over  $z \in S$ .
- That means, for **any**  $\lambda \succ_{K^*} 0$ , the hyperplane  $\{z \mid \lambda^T (z - x) = 0\}$  is a strict supporting hyperplane to  $S$  at  $x$ .

# Dual Characterization of Minimum Element



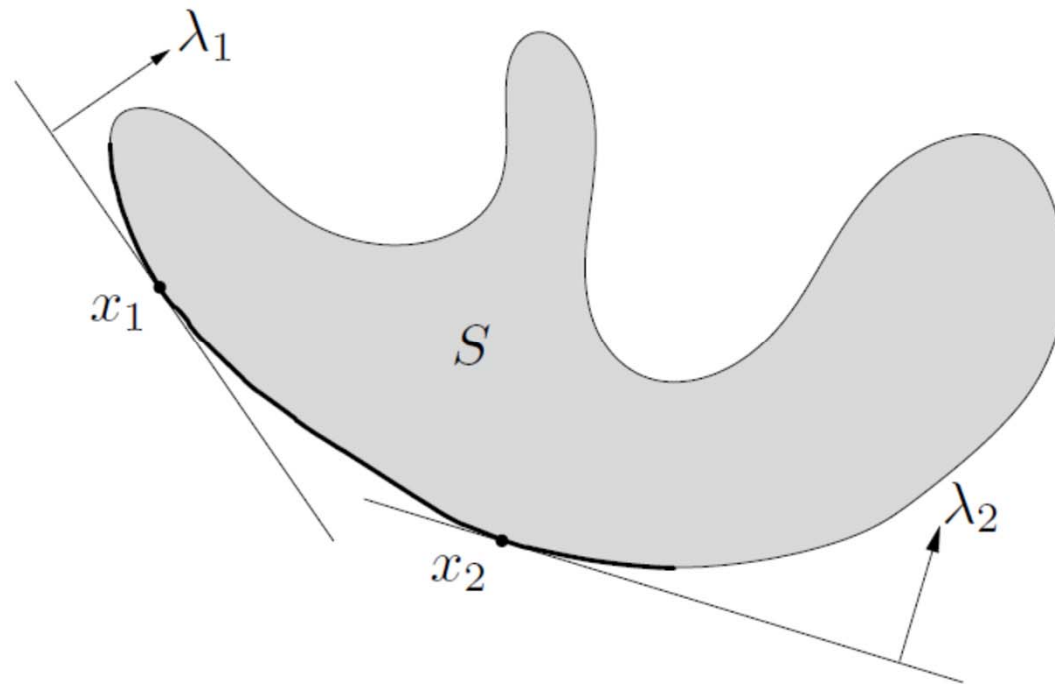
- $x$  is the minimum element of  $S$ , with respect to the generalized inequality  $\preceq_K$ , **if and only if** for all  $\lambda \succ_{K^*} 0$ ,  $x$  is the unique minimizer of  $\lambda^\top z$  over  $z \in S$ .



# Dual Characterization of Minimal Elements (1)



- If  $\lambda \succ_{K^*} 0$ , and  $x$  minimizes  $\lambda^T z$  over  $z \in S$ , then  $x$  is minimal.

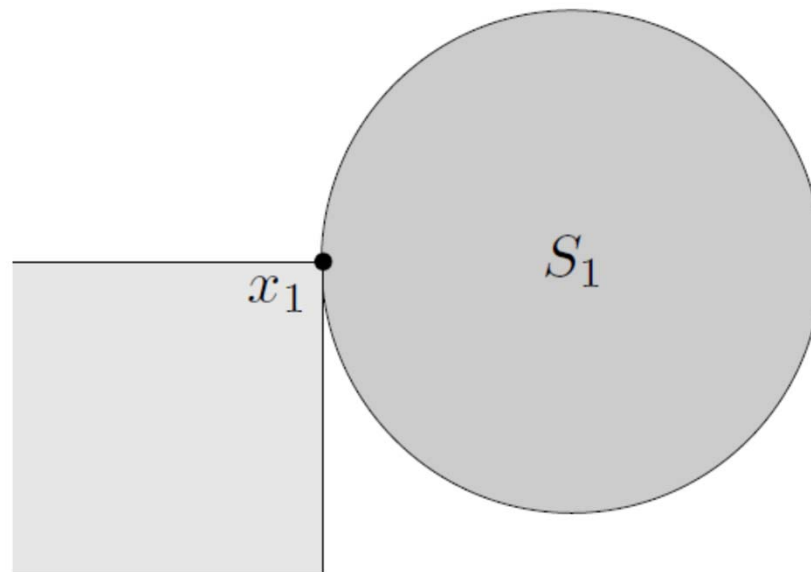




# Dual Characterization of Minimal Elements (2)



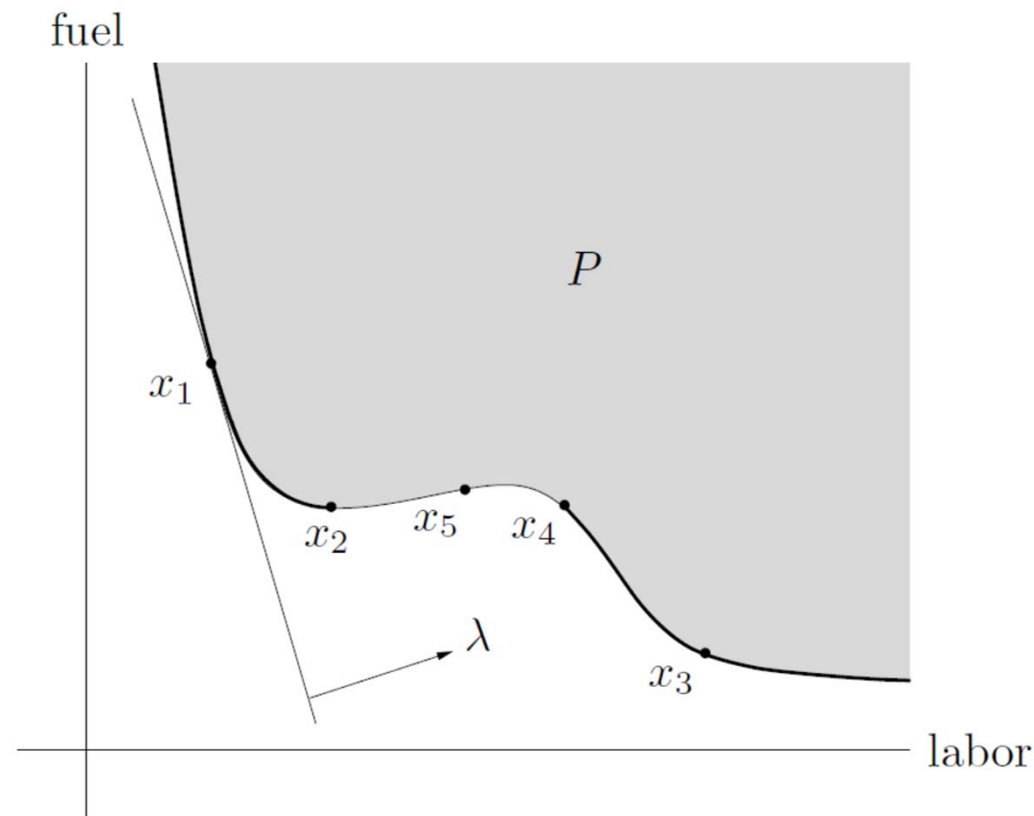
- If  $S$  is convex, for any minimal element  $x$  there exists a nonzero  $\lambda \succcurlyeq_{K^*} 0$  such that  $x$  minimizes  $\lambda^T z$  over  $z \in S$ .



# Pareto Optimal Production Frontier



- A product which requires  $n$  sources
- A resource vector  $x \in \mathbf{R}^n$





# Outline

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- Affine and Convex Sets
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# Summary

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- Affine and convex
- Operations that preserve convexity
- Generalized Inequalities
- Separating and supporting hyperplanes
  - Theorems
- Dual cones and generalized inequalities