Instantaneous location determination by Celestial Navigation.

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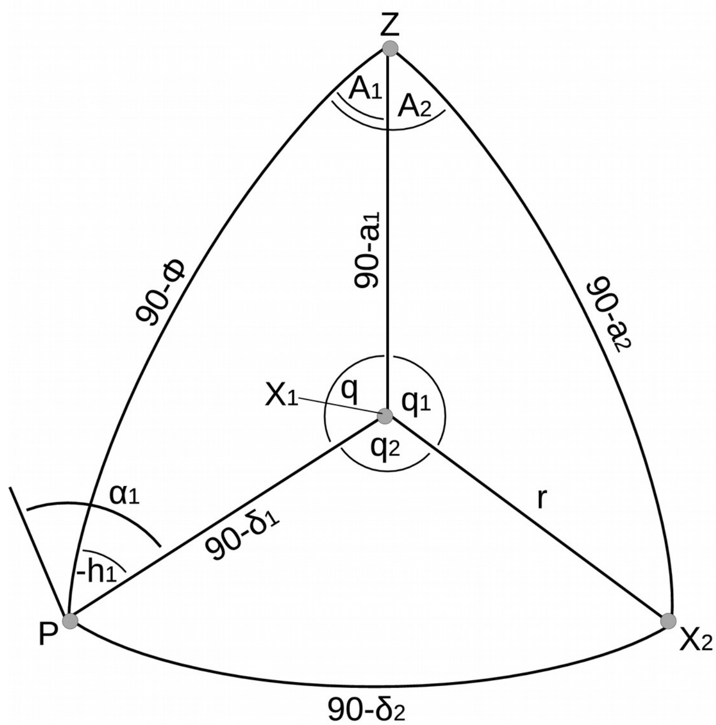

In the image above

- Z is the zenith of the observer
- P is the north pole
- X_1 and X_2 are two selected stars with known equatorial coordinates.
- Φ is the latitude of the observer
- δ_1 and δ_2 are the declinations from the equator of X_1 and X_2 respectively
- a_1 and a_2 are the altitudes above the observer's horizon of X_1 and X_2 respectively
- α_1 is the right ascension of X_1. α_2 is similar but not shown
- A_1 and A_2 are the azimuths, measured east of north, of X_1 and X_2 respectively
- h_1 is the hour-angle to the observer's meridian (PZ) for X_1
- q is the parallactic angle of X_1

The object is to determine the location of the observer assuming the altitude and azimuth measurements (a_1, a_2, A_1, A_2) had zero error. By using this new location, subsequent conversions between horizontal and equatorial frames of reference can be done with increased accuracy.
It is also assumed that the declinations and right ascensions (δ_1, δ_2, α_1, α_2) are known with zero error.
Lastly it as assumed that the altitude and azimuth differences between X_1 and X_2 can be measured more accurately than their absolute angles (that the largest errors are common to both).

Using the cosine identity In PX_1X_2 let

$$r_c = \cos(r) = \cos(90-\delta_1)\cdot\cos(90-\delta_2) + \sin(90-\delta_1)\cdot\sin(90-\delta_2)\cdot\cos(\alpha_2-\alpha_1)$$

$$r_c = \sin(\delta_1)\cdot\sin(\delta_2) + \cos(\delta_1)\cdot\cos(\delta_2)\cdot\cos(\alpha_2-\alpha_1) \tag{1}$$

then let

$$r_s^2 = \sin^2(r) = 1-\cos^2(r) = 1-r_c^2 \tag{2}$$

In ZX_1X_2 the sine rule gives

$$\frac{\sin(q_1)}{\sin(90-a_2)} = \frac{\sin(A_2-A_1)}{\sin(r)}$$

$$\frac{\sin(q_1)}{\cos(a_2)} = \frac{\sin(A_2-A_1)}{\sin(r)}$$

then let

$$Q_1 = \sin(q_1)\cdot\sin(r) = \cos(a_2)\cdot\sin(A_2-A_1) \tag{3}$$

similarly in PX_1X_2, let

$$Q_2 = \sin(q_2)\cdot\sin(r) = \cos(\delta_2)\cdot\sin(\alpha_2-\alpha_1) \tag{4}$$

In ZX_1X_2 the cosine rule gives

$$\cos(90-a_2) = \cos(90-a_1)\cdot\cos(r) + \sin(90-a_1)\cdot\sin(r)\cdot\cos(q_1)$$

$$\sin(a_2) = \sin(a_1)\cdot\cos(r) + \cos(a_1)\cdot\sin(r)\cdot\cos(q_1)$$

then let

$$Q_1{}' = \cos(q_1)\cdot\sin(r) = \frac{\sin(a_2)-\sin(a_1)\cdot\cos(r)}{\cos(a_1)} \tag{5}$$

and similarly in PX_1X_2, let

$$Q_2{}' = \cos(q_2)\cdot\sin(r) = \frac{\sin(\delta_2)-\sin(\delta_1)\cdot\cos(r)}{\cos(\delta_1)} \tag{6}$$

From the angle sum identity

$$\cos(q_1+q_2)=\cos(-q)=\cos(q)=\cos(q_1)\cdot\cos(q_2)-\sin(q_1)\cdot\sin(q_2)$$

or using the equations above

$$\cos(q)=\frac{Q_1'\cdot Q_2'-Q_1\cdot Q_2}{r_s^2} \tag{7}$$

Using (7) and the cosine rule in PZX_1, it is now possible to determine Φ

$$\cos(90-\Phi)=\cos(90-\delta_1)\cdot\cos(90-a_1)+\sin(90-\delta_1)\cdot\sin(90-a_1)\cdot\cos(q)$$

then let

$$\Phi_s=\sin(\Phi)=\sin(\delta_1)\cdot\sin(a_1)+\cos(\delta_1)\cdot\cos(a_1)\cdot\cos(q) \tag{8}$$

and because $-90° < \Phi < 90°$

$$\Phi=\sin^{-1}(\Phi_s) \tag{9}$$

Applying the sine rule in PZX_1 gives

$$\frac{\sin(-h_1)}{\sin(90-a_1)}=\frac{\sin(q)}{\sin(90-\Phi)}$$

then let

$$H=\sin(h_1)\cdot\cos(\Phi)=-\cos(a_1)\cdot\sin(q) \tag{10}$$

The cos rule in PZX_1 gives

$$\cos(90-a_1)=\cos(90-\delta_1)\cdot\cos(90-\Phi)+\sin(90-\delta_1)\cdot\sin(90-\Phi)\cdot\cos(-h_1)$$

$$\sin(a_1)=\sin(\delta_1)\cdot\sin(\Phi)+\cos(\delta_1)\cdot\cos(\Phi)\cdot\cos(h_1)$$

then let

$$H'=\cos(h_1)\cdot\cos(\Phi)=\frac{\sin(a_1)-\sin(\delta_1)\cdot\sin(\Phi)}{\cos(\delta_1)} \tag{11}$$

then, from (10) and (11)

$$h_1 = \tan^{-1}\left(\frac{H}{H'}\right)$$

the Local Sidereal Time (θ_L) is given by

$$\theta_L = \alpha_1 + h_1$$

if θ_G is the Greenwhich Sidereal Time, then the observer's longitude (λ_0) is given by

$$\lambda_0 = \theta_L - \theta_G$$

And so both latitude and longitude are determined without reference to an absolute azimuth measurement. If the observation platform is level, altitude measurements will be accurate and the computed location will match the observer's. Regardless of altitude measurement errors (common to X_1 and X_2) the relative positions of all celestial objects can be accurately determined.

By using these derived coordinates for conversion between horizontal and equatorial coordinate spaces, any errors in alignment between the observation platform and the horizon/zenith are eliminated in future calculations, enabling accurate redirection to any target in the celestial sky.

The equations below are adjusted for azimuth measurements east of north.
These standard equations are not derived here. See 'Celestial Coordinate System' on Wikipedia.

Conversion from Equatorial to Horizontal

$$A = \tan^{-1}\left(\frac{-\sin(h)}{-\cos(h)\cdot\sin(\Phi) + \tan(\delta)\cdot\cos(\Phi)}\right)$$

$$a = \sin^{-1}(\sin(\Phi)\cdot\sin(\delta) + \cos(\Phi)\cdot\cos(\delta)\cdot\cos(h))$$

Conversion from Horizontal to Equatorial

$$h = \tan^{-1}\left(\frac{-\sin(A)}{-\cos(A)\cdot\sin(\Phi) + \tan(a)\cdot\cos(\Phi)}\right)$$

$$\delta = \sin^{-1}(\sin(\Phi)\cdot\sin(a) + \cos(\Phi)\cdot\cos(a)\cdot\cos(A))$$

For a practical software implementation of the above equations see *github.com/mmoller2k/godob*