Discrete Math Study Guide

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1 Logical Symbols and Deductive Reasoning

1.1 Variables and Statements

A **variable** is a symbol that stands in for some specific value, be it a person, number, etc.

A **statement** is a something that may evaluate to true or false. It is usually either in the form P if it does not depend on a variable or $P(x)$ if the statement's truth depends on what the input is.

1.2 Connective Symbols

1.3 Logical Laws

1.3.1 Associative Law

$$\begin{aligned} (P \wedge Q) \wedge R &= P \wedge (Q \wedge R) && (1) \\ (P \vee Q) \vee R &= P \vee (Q \vee R) && (2) \end{aligned}$$

1.3.2 Communative Law

$$\begin{aligned} P \wedge Q &= Q \wedge P && (3) \\ P \vee Q &= Q \vee P && (4) \end{aligned}$$

1.3.3 Distributive Law

$$\begin{aligned} P \wedge (Q \vee R) &= (P \wedge Q) \vee (P \wedge R) && (5) \\ P \vee (Q \wedge R) &= (P \vee Q) \wedge (P \vee R) && (6) \end{aligned}$$

1.3.4 Double Negation Law

$$\neg\neg P = P \quad (7)$$

1.3.5 De Morgan's Law

$$\begin{aligned} \neg(P \wedge Q) &= (\neg P \vee \neg Q) && (8) \\ \neg(P \vee Q) &= (\neg P \wedge \neg Q) && (9) \end{aligned}$$

1.3.6 Idempotent Law

$$\begin{aligned} P \wedge P &= P && (10) \\ P \vee P &= P && (11) \end{aligned}$$

1.3.7 Absorption Law

$$P \wedge (P \vee Q) = P \quad (12)$$
$$P \wedge (P \vee Q) = P \quad (13)$$

1.4 Truth Tables

Truth tables are are relative straight forward concept. The aim is to evaluate the truth of statement by breaking it down into its smallest parts, then seeing if the final statement is true or false based on the truth of the sub-statements. Here is a simple example,

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

1.5 Tautologies and Contradictions

1.6 The Conditional

1.6.1 Definition

The conditional can be thought of as an "if, then" statement. It primarily demonstrates some relationship between two statements. In symbols, it is represented as,

$$P \to Q$$

This statement can be read several ways in English:

1. P implies Q
2. P only if Q
3. P is a sufficient condition for Q
4. Q, if P
5. Q is a necessary condition for P

1.6.2 The Truth of a Conditional

We can demonstrate the truth of the conditional via a truth table.

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

To put the truth table into plain words, the conditional is true only if both Q is true or if P and Q are both false. In other words, the conditional is only false if only Q is false.

1.6.3 The Conditonal in Logical Connectives

We can write the conditional in terms of basic logical connectives. The definition of conditional in these terms is as follows.

$$P \to Q \quad \equiv \quad \neg P \vee Q \quad \equiv \quad P \wedge \neg Q$$

Note that the rightmost statement is the same as the middle statement, but De Morgan's Law was applied.

We can verify that these statements are equivalent via another truth table.

P	Q	$P \to Q$	$\neg P \vee Q$	$P \wedge \neg Q$
T	T	T	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

1.6.4 The Converse

The converse of a conditional is simply the conditional, but the statements have been swapped around.

$$P \to Q \quad \not\equiv \quad Q \to P$$

We could write a truth table to demonstrate that these statements are **NOT** equivalent, but we will use the definition of the conditional (the logical symbols version) to demonstrate intuitively that these are not the same.

$$\begin{aligned} P \to Q \quad &\equiv \quad \neg P \vee Q \\ Q \to P \quad &\equiv \quad \neg Q \vee P \qquad (14) \\ \neg P \vee Q \quad &\not\equiv \quad \neg Q \vee P \qquad (15) \end{aligned}$$

1.6.5 The Contrapositive

The contrapositive of a conditional is a negated version of the original statement. Unlike the converse of conditional, the contrapositive is equivalent to the original statement.

$$P \to Q \quad \equiv \quad \neg Q \to P$$

We could use a truth table to show that these statements are equivalent, but we can also use the logical forms of the conditionals achieve the same end.

$$\begin{aligned} P \to Q &\equiv \neg P \lor Q \\ \neg Q \to P &\equiv \neg\neg Q \lor \neg P \\ \neg P \lor Q &\equiv \neg\neg Q \lor P \end{aligned}$$

1.7 The Biconditional

1.7.1 Definition

The biconditional is often read as "if an only if". It can be written in terms of the conditional or logical connectors. It is written as follows.

$$P \leftrightarrow Q$$

In terms of the conditional,

$$P \leftrightarrow Q \quad \equiv \quad (P \to Q) \land (Q \to P) \tag{16}$$

The final definition is in terms of logical connectors.

$$P \leftrightarrow \quad \equiv \quad (\neg P \lor Q) \land (\neg Q \lor P) \tag{17}$$

Any statements that are equivalent to those above are valid definition of the biconditional.

1.7.2 The Truth of a Biconditional

We can determine the truth of a biconditional via a truth table.

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

As we can see, the biconditional only evaluates to true if (and only if) both statements involved are true.

1.8 Arguments

2 Quantifiers

2.1 Motivating Quantifiers

2.2 The Universe of Discourse

2.3 The Universal Quantifier

2.4 The Existential Quantifier

2.4.1 Uniqueness

2.5 Bound Variables

2.6 Quantifier Negation

3 Set Theory

3.1 Defining Sets

Set: A collection of objects. ex. $\{0, 1, 2, 3\}$
Object: An element of a set. ex. 52.

3.1.1 Important Sets

There are a few very important important sets that we must know. They are as follow:

$$\begin{aligned}
\varnothing &= \{\}\text{; a set with no objects.} \\
\mathbb{N} &= \{x \mid x \text{ is a natural number}\} \\
\mathbb{Z} &= \{x \mid x \text{ is an integer}\} \\
\mathbb{R} &= \{x \mid x \text{ is a real number}\} \\
\mathbb{Q} &= \{x \mid x \text{ is a rational number of form } \frac{p}{q} \text{ where } p, q \in \mathbb{R} \text{ and } q \neq 0\} \\
\text{Truth Set} &= \{x \mid P(x)\}\text{; the set of all objects that makes the statement true}
\end{aligned}$$

3.2 Basic Set Operations

3.2.1 Intersection

The intersection of two sets is the set of elements in both sets.

$$A \cup B = \{x \mid x \in A \land x \in B\} \tag{18}$$

3.2.2 Union

The union of two sets is the set of all elements in the sets.

$$A \cap B = \{x \mid x \in A \vee x \in B\} \tag{19}$$

3.2.3 Difference

The difference of two sets is the set of elements in the first but not the second.

$$A \backslash B = \{x \mid x \in A \wedge x \notin B\} \tag{20}$$

3.2.4 Symmetric Difference

The symmetric difference of two sets is the set of all elements not in their intersection.

$$A \triangle B = \{x \mid (A \backslash B) \cup (B \backslash A)\} = \{x \mid (A \cup B) \backslash (A \cap B)\} \tag{21}$$

3.2.5 Subsets, Proper and Improper

An (improper) subset is when all elements in some set is contained in some other set.

$$A \subseteq B \quad = \quad \{x \mid \forall x(x \in A \rightarrow x \in B)\} \tag{22}$$

A proper subset is when all elements in some set are equal to all elements of another set.

$$A \subset B \quad = \quad \{x \mid \forall x(x \in A \leftrightarrow x \in B)\} \tag{23}$$

3.3 Families of Sets

Family of Sets: Any set that is a set of sets, often denoted as $\mathcal{F}$. For example,

$$\mathcal{F} = \{A, B, C\} = \{\{1, 2, 3\}, \{4, 5, 6\}, \{5, 6, 7\}\}$$

3.3.1 Index Sets

Index Set: A set that indexes another set.

An index set does not need to be strictly consecutive integers. For example, both of the following sets are valid index sets.

$$\begin{aligned} I &= \{1, 2, 3\} \\ J &= \{3, 42, 54\} \end{aligned}$$

Indexed Set: A family of sets that has been indexed by an indexed set. It is often defined as the following.

$$\mathcal{A} \quad = \quad \{A_i \mid i \in I\} \tag{24}$$

3.3.2 The Power Set

Power Set: A set whose elements are all subsets of some other set.

$$\mathcal{P}(A) = \{S \mid S \subseteq A\} \tag{25}$$

Example:

$$\begin{aligned} A &= \{1,2\} \\ \mathcal{P}(x) &= \{\{1\},\{2\},\{1,2\},\varnothing\} \end{aligned}$$

3.3.3 Operations on Families of Sets

Intersection of Sets: A set of all elements that are common to all the sets in the family.

$$\bigcap \mathcal{F} = \{x \mid \forall A \in \mathcal{F}(x \in A)\} \tag{26}$$

Another way to think about the intersection of sets is as follows.

$$A_1 \cap A_2 \cap ... \cap A_i$$

Union of Sets: A set of all elements that are in all the sets in the family.

$$\bigcup \mathcal{F} = \{x \mid \exists A \in \mathcal{F}(x \in A)\} \tag{27}$$

Another way to think about the intersection of sets is as follows.

$$A_1 \cup A_2 \cup ... \cup A_i$$

4 Introductory Proof Strategies

4.1 Theorems, Propositions, and Lemmas

4.2 Proof Writing Basics

4.3 Direct Proofs

4.4 Proof by Contrapositive