Linear Algebra Study Guide

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Contents

1 An Introduction to Vector Spaces

1.1 The Field

Before we begin discussing what the field is, we need to makes sure we understand what complex numbers are.

Definition. **The complex numbers** is defined as follows.

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\} \tag{1}$$

Definition. **A Field** refers to the space over which a vector space is defined. In this class we will often refer to a special field (which we may refer to simply as "the field). It is defined as follows.

$$\mathbb{F} = \{\mathbb{R} \cup \mathbb{C}\} \tag{2}$$

That is, the vector space is either defined over the real or complex numbers.

Remark. If $\mathbb{F} = \mathbb{R}$, then V is called a real vector space.
If $\mathbb{F} = \mathbb{C}$, then V is called a complex vector space.

1.2 Defining Vector Spaces

Definition. A **vector space** V over $\mathbb{F}$ is a set of elements called **vectors** that have two operations, closed addition and closed scalar multiplication. These operations satisfy 8 axioms. That is,

$$\text{Addition: } f : V \times V \to V$$
$$\text{Scalar Multiplication: } g : \mathbb{F} \times V \to V$$

Theorem 1.1. *If V is a vector space, then it must have closed addition and closed scalar multiplication such that they fulfill the following vector space axioms.*

1. ***Commutativity:*** *$v + w = w + v$ for all $v, w \in V$.*
2. ***Associativity:*** *$(v + w) + u = v + (u + w)$ for all $v, w, u \in V$.*
3. ***Additive Identity:*** *There exists (a unique) $\mathbf{0} \in V$ such that $v + \mathbf{0} = v$ for all $v \in V$.*
4. ***Additive Inverse:*** *For all $v \in V$, there exists (a unique) $v' \in V$ such that $v + v' = \mathbf{0}$.*
5. ***Multiplicative Identity:*** *$1 \times v = v$ for all $v \in V$ and $1 \in \mathbb{F}$.*
6. ***Multiplicative Associativity:*** *$(\alpha\beta)v = \alpha(\beta v)$ for all $v \in V$ and $\alpha, \beta \in \mathbb{F}$.*
7. ***Scalar Distribution:*** *$\alpha(v + w) = \alpha v + \alpha w$ for all $v, w \in V$ and $\alpha \in \mathbb{F}$.*
8. ***Vector Distribution:*** *$v(\alpha + \beta) = \alpha v + \beta v$ for all $v \in V$ and $\alpha, \beta \in \mathbb{F}$.*

Remark. The first 4 axioms relate to addition. The next 2 are about multiplication. The final two connect addition and multiplication through the distributive property.

Lemma 1.2. *If V is a vector space, then $\mathbf{0} \times v = \mathbf{0}$ for all $v \in V$.*

1.3 Important Vector Spaces and Vector Space Properties

This section will basically be a bunch of lemmas and theorems that state "this thing" is a vector space.

Theorem 1.3. *$\mathbb{R}^n$ is a vector space over $\mathbb{R}$.*

Theorem 1.4. *$\mathbb{C}^n$ is a vector space over $\mathbb{C}$.*

Theorem 1.5. *If $M^{\mathbb{F}}_{m \times n}$ denotes the set of all $m \times n$ matrices with entries in $\mathbb{F}$, then $M^{\mathbb{F}}_{m \times n}$ is a vector space over $\mathbb{F}$.*

Theorem 1.6. *Let $\mathbb{P}^{\mathbb{F}}_n$ denote the set of all polynomials of degree $\leqslant n$ and coefficients in $\mathbb{F}$. That is,*

$$\mathbb{P}^{\mathbb{F}}_n = \{p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \mid a_i \in \mathbb{F} \text{ for } i \in \{0, ..., n\}\}$$

$\mathbb{P}^{\mathbb{F}}_n$ is a vector space over $\mathbb{F}$.

2 Bases

2.1 Linear Combinations

Definition. Let V be a vector space over $\mathbb{F}$ and $v_1, ..., v_p \in V$ be a list of vectors. A **linear combination** of these vectors is a sum of the form,

$$\alpha_1 v_1 + \cdots \alpha_p v_p = \sum_{i=1}^{p} \alpha_i v_i \text{ where } \alpha_i \in \mathbb{F}$$

2.2 Linear Independence

Definition. A linear combination, $\sum_{i=1}^{p} \alpha_i v_i$ is said to be the **trivial combination (or trivial case)** if $a_i = 0$ for all i.

Definition. A system of vectors is **linearly independent** if only the trivial linear combination of $v_1, ..., v_p$ is equal to the zero vector. That is, if

$$\sum_{i=1}^{p} \alpha_i v_i = \mathbf{0}$$

Then, $\alpha_1 = \alpha_2 = \cdots = \alpha_p = 0 \in \mathbb{F}$.

Definition. A system of vectors is **linearly dependent** if it is not linearly independent. We can represent this in two ways. First, there exists a linear combination of this list for the zero vector such that the linear combination is not the trivial one.

$$\sum_{i=1}^{p} \alpha_i v_i = \mathbf{0} \text{ but there exists a } \alpha_i \text{ such that } \alpha_i \neq 0$$

Second, that a list is linearly dependent if and only if one of the vectors in the list can be written as a linear combination of the other vectors in that list.

$$\begin{aligned} v_k &= \sum_{\substack{i=1 \\ i \neq k}}^{p} \alpha_i v_i \\ &= \alpha_1 v_1 + \cdots + \alpha_{k-1} v_{k-1} + \alpha_{k+1} v_{k+1} + \cdots + \alpha_p v_p \end{aligned}$$

Technically, this second definition follows from Proposition 2.6 from the textbook, but its so fundamental (and rather easy to prove, since it directly follows the first definition) that I've decided to include it here.

2.3 Spanning

Definition. A collection of vectors is said to **span** (or **generate, be a spanning system of, be a complete system of**) V if any vector $v \in V$ can be written as a linear combination of that collection.

More formally, suppose you have a list $v_1, ..., v_p \in V$ and an arbitrary $v \in V$. This list spans only if

$$\sum_{i=1}^{p} \alpha_i v_i = v$$

2.4 Defining Bases

Definition. A system of vectors $v_1, ..., v_p \in V$ is a **basis** for V if every vector $v \in V$ admits a unique representation as a linear combination of $v_1, ..., v_p$.

That is, a system of vectors is a basis if that list is linearly independent and spans V.

Remark. Let $v \in V$ be arbitrary. If the list $v_1, ..., v_p \in V$ is a basis, then

$$\sum_{i=1}^{p} \alpha_i v_i = v$$

We would refer to the α_i as the **coordinates** of v with respect to this basis.

Theorem 2.1. *Any finite set of vectors that span a vector space contains a basis.*

Remark. The proof is "recursive" in that the idea is you can remove elements of that list until it becomes linearly independent. Removing elements doesn't impact the list's ability to span.

Definition. The **standard basis vectors** refers to a simple, reliable basis for various vector spaces. The two important ones are for $\mathbb{F}^n$ and $\mathbb{P}_n^{\mathbb{R}}$.

The standard basis vectors for $\mathbb{F}^n$ are

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_1 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{F}^n$$

The standard basis vectors for $\mathbb{P}_n^{\mathbb{R}}$ are

$$1, t, t^2, ..., t^n \in \mathbb{P}_n^{\mathbb{R}}$$

3 Linear Transformations

3.1 Introducing Linear Transformations

Definition. A **transformation** T form a set X to Y is a function from X to Y. We denote it as

$$T : X \to Y$$

Definition. Suppose there are two vector spaces V and W both over $\mathbb{F}$. A transformation between them is **linear** if it holds for two properties.

1. **Additivity:** $T(v + u) = T(v) + T(u)$ for all $v, u \in V$.
2. **Homogeneity:** $T(\alpha v) = \alpha T(v)$ for all $v \in V$ and $\alpha \in \mathbb{F}$.

Lemma 3.1. *Let* $T : \mathbb{P}_n^{\mathbb{R}} \to \mathbb{P}_{n-1}^{\mathbb{R}}$ *be the differentiation of polynomials.* T *is a linear map.*

3.2 Representing Transformations as Matrices

3.2.1 Transformations as Matrices

Definition. Suppose

$$T(e_k) = \begin{pmatrix} a_{1,k} \\ a_{2,k} \\ \vdots \\ a_{n,k} \end{pmatrix} \in \mathbb{F}^n$$

Then, the **matrix representing** T is defined as the following.

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

Basically, the first column is just $T(e_1)$. The second column is $T(e_2)$, so the *nth* column is $T(e_n)$.

3.2.2 Matrix-Vector Mutliplication

The concept of matrix-vector multiplication is the ability to find $T(v)$ through the use of a matrix.

Definition. Let A be the matrix representing T and $x \in \mathbb{F}^n$. Then,

$$Ax = T(x)$$

More explicitly,

$$\begin{aligned} Ax &= \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= \begin{pmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \end{pmatrix} \end{aligned}$$

3.2.3 Composing Linear Maps

3.2.4 Matrix-Matrix Multiplication

Definition. Let A be a $m \times n$ matrix and B be a $n \times r$ matrix. We have two definitions for **matrix-matrix multiplication**.

1. Suppose,

$$B = \begin{pmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_r \\ | & | & & | \end{pmatrix}$$

   Where b_i is the ith column of B. Then,

$$AB := \begin{pmatrix} | & | & & | \\ Ab_1 & Ab_2 & \cdots & Ab_r \\ | & | & & | \end{pmatrix}$$

2. The entry $(AB)_{j,k}$ is

$$(jth \text{ row of } A) \times (ith \text{ column of } B) = \sum_{l=1}^{n} a_{j,l} \times b_{l,k}$$

Remark. Matrix multiplication holds for the following properties.

1. **Associativity:** $A(BC) = (AB)C$
2. **Matrix Distribution:** $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$
3. **Scalar Distribution:** $A(\alpha B) = \alpha(AB)$

Very importantly, **matrix multiplication is NOT commutative.**

Theorem 3.2. *Let A be the matrix representing T_1 and B be the matrix representing T_2. Then, the matrix BA represents the composition $T_2 \circ T_1$.*

3.2.5 The Rotational Matrix

Definition. Let the **rotational matrix** be defined as follows.

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

When multiplied by a vector, it rotates the vectors θ clockwise. You are functionally composing a vector with a rotation.

3.2.6 Transposing and Tracing Matricies

Two more useful tools to have are as follows.

Definition. The **transpose**of A is a matrix

$$A^T = (a'_{i,j})_{\substack{1\leqslant j\leqslant n \\ 1\leqslant i\leqslant m}} \text{ and } a'_{i,j} = a_{j,i}$$

Definition. The **trace** of an $n \times n$ matrix $A = (a_{j,k})$ is the sum of the diagonal entries.

$$tr(A) = a_{1,1} + a_{2,2} + \cdots + a_{n,n}$$

Equivalently,

$$tr(A) = \sum_{i=1}^{n} a_{i,i}$$

3.3 Subspaces

Definition. Suppose V is a vector space. V_0 is a **subspace** of V, if the following properties hold.

1. $V_0 \subset V$
2. $V_0 \neq \varnothing$
3. $v + w \in V_0$ for all $v, w \in V_0$
4. $\alpha v \in V$ for all $v \in V$ and $\alpha \in \mathbb{F}$

3.3.1 Null Spaces and Range

Definition. Suppose $T : V \to W$. The **null space or kernel** of T is the set of all the values $v_0 \in V$ such that

$$T(v_0) = \mathbf{0}_W$$

Definition. Suppose $T : V \to W$. The **range** of T is the set of all values $w \in W$ such w is in the image of T.

3.4 Isomorphisms

3.4.1 Inverses

Definition. Suppose $A : T \to W$ is a linear transformation. A is **right invertible** if there exists a $B : W \to V$ such that $B \circ A = I_V$. In matrix multiplication, BA is the identity matrix.

Definition. Suppose $A : T \to W$ is a linear transformation. A is **left invertible** if there exists a $B : W \to V$ such that $B \circ A = I_W$. In matrix multiplication, AB is the identity matrix.

Definition. A linear transformation $A : V \to W$ is **invertible** only if it is right invertible and left invertible.

From that, we have the following corollary.

Corollary 3.3. *A linear transformation $A : V \to W$ is invertible if and only if there exists a unique linear transformation $A^{-1} : W \to V$ such that*

$$A^{-1} \circ A = I_V \text{ and } A \circ A^{-1} = I_W$$

We call A^{-1} the inverse of A.

Lemma 3.4. *If A and B are invertible linear transformations, and AB is defined, then $A \circ B$ is invertible and*

$$(A \circ B)^{-1} = B^{-1} \circ A^{-1}$$

Theorem 3.5. *Let $A : X \to Y$ be a linear transformation. Then, A is invertible if and only if for all $b \in Y$, the equation*

$$Ax = b$$

has a unique solution.

3.4.2 Defining Isomorphisms

Definition. An **Isomorphism** from a vector space V to a vector space W is an invertible linear map from V to W. If there exists an isomorphism between two vector spaces, then we would say that those vector spaces are **isomorphic**.

We will now list the foundational theorems for isomorphisms.

Theorem 3.6. *Let $A : V \to W$ be an isomorphism. If $v_1, ..., v_n$ is a basis for V, then $A(v_1), ..., A(v_n)$ is a basis for W.*

Theorem 3.7. *Let $v_1, ..., v_n$ be a basis for V and $w_1, ..., w_n$ be a basis for W. Let $A : V \to W$ be the map defined by*

$$A(v_i) = w_i \text{ for } 1 \leqslant i \leqslant n.$$

A is an isomorphism.

Corollary 3.8. *If vector spaces have bases of the same size, then they are isomorphic.*

4 Linear Equations

4.1 Introduction to Linear Systems

Definition. A **Linear System** is a collection of m linear equations with n unknowns.

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m \end{cases}$$

Where $a_{i,j}, b_i \in \mathbb{F}$.

Notation. Given the following arbitrary linear system,

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m \end{cases}$$

We can write this as a system of matrices.

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

We can further simplify writing such an equation through an **augmented matrix**.

$$\left(\begin{array}{ccc|c} a_{1,1} & \cdots & a_{1,n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & \cdots & a_{m,n} & b_m \end{array} \right)$$

Also denoted as $(A|b)$.

4.2 Finding Solutions: Gauss-Jordan Elimination

Definition. Let $(A|b)$ be an augmented matrix. You can apply three types of **row operations** to find the solution to such a system.

1. **Row Exchange**: Interchange two rows.
2. **Scaling**: Multiply a row by a non-zero number.
3. **Row Replacement**: Replace row k by its sum with a constant multiple of row j. That is, $R_k + \alpha R_j$ such that $\alpha \neq 0$.

Theorem 4.1. *Row operations do not change the solutions to a linear system.*

Proposition 4.2. *Each row operation can be realized by multiplication of an invertible matrix.*

1. ***Row Exchange****: Suppose you are changing the kth and jth rows. This can be expressed as the identity matrix with the kth and jth rows swapped.*
2. ***Scaling****: Suppose you are scaling the kth row by $\alpha \in \mathbb{F}$. You can represent this as the identity matrix where the kth row is scaled by α.*

4.3 Finding the Inverse of a Function via Row Reduction

Strategy. Suppose you have a $n \times n$ matrix,

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}$$

You can create an augmented matrix, $(A|b)$, where b is the identity matrix.

$$\left(\begin{array}{ccc|cccc} a_{1,1} & \cdots & a_{1,n} & 1 & 0 & \cdots & 0 \\ a_{2,1} & \cdots & a_{2,n} & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} & 0 & 0 & \cdots & 0 \end{array} \right)$$

Now, make A into its row reduced echelon form. The right side (what was originally the identity matrix) will be transformed into A's inverse matrix.

4.4 Echelon Form

Definition. A matrix or augmented matrix is in **echelon form** if it satisfies two conditions.

1. All zero rows are below all non-zero rows.
2. For a non-zero row, its leftmost non-zero entry (which is called the leading entry or **pivot**) is strictly to the right of the leading entry in the row before it.

Example. The following matrix is in echelon form.

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 8 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{array}\right)$$

Theorem 4.3. *Every matrix can be transformed into echelon form by applying a finite sequence of row operations.*

Strategy. (Echelon Form)
Main Step:

1. Find leftmost non-zero column matrix.
2. Apply row-exchange to ensure entry in the first row of the column is non-zero.
3. Apply scaling so the pivot in the first row equals 1.
4. Apply row replacement by adding an appropriate multiple of row 1 to each other row, so all entries below the pivot $= 0$.

Next:

1. Leave the first row alone.
2. Apply the main step to columns 2 through m.
3. Apply the main step to columns 3 through m.
4. Continue applying the main step to each subsequent column.

Then, you have echelon form.

4.5 Row Reduced Echelon Form

Definition. A matrix or augmented matrix is in **row reduced echelon form** if it fulfills the requirements of echelon form and two more constrains. In total, these requirements are summarized below.

1. All zero rows are below all non-zero rows.
2. The left most non-zero entry in each column is a pivot and every pivot is strictly to the right of every pivot above it.
3. Every pivot is equal to 1.
4. All entries above a pivot are zero.

Theorem 4.4. *Every matrix can be put into reduced echelon form by a finite sequence of row operations.*

Definition. The equation $Ax = b$ is **inconsistent** if it does not have a solution.

Theorem 4.5. *The system $Ax = b$ is inconsistent if and only if there exists a row of the form,*

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & | & b \end{pmatrix}$$

Such that $b \neq 0$.

Corollary 4.6. *A system $Ax = b$ has a solution if there exists a pivot in every row.*

Corollary 4.7. *A system $Ax = b$ with a solution has a unique solution for b if and only if there exists a pivot in every column of the reduced echelon form of A.*

Theorem 4.8. *Let $v_1, ..., v_m \in \mathbb{F}^n$ be a system of vectors. Let,*

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{pmatrix}$$

be the $m \times n$ matrix. The system is

1. ***Linearly Independent*** *if and only if A_e has a pivot in every column (no free variables).*

2. ***Spanning*** *if and only if A_e has a pivot in every row (solutions exist) .*

5 Dimensions

First, we talk about bases.

Proposition 5.1. *Suppose a vector space V has a finite basis. Then, any two bases have the same number of elements in them.*

Corollary 5.2. *The basis of $\mathbb{F}^n$ has n vectors in it.*

Definition. The **dimension** of a vector space V, denoted by $\dim(V)$, is the number of vectors in a basis for V if V has a finite basis.

- The dimensions of $\{\mathbf{0}\}$ is 0.

- If $V$is not finite-dimensional, then V is infinitely-dimensional.

Using this definition, we can construct bases.

Proposition 5.3. *Any linearly independent list of vectors in a finite-dimensional vector space V has at most $dim(V)$ elements.*

Proposition 5.4. *Any spanning list of vectors in a finite-dimensional vector space V has at least $dim(V)$ elements.*

Following this,

Theorem 5.5. *If $v_1, ..., v_r$ are linearly independent vectors in a finite-dimensional vector space V with $dim(V) = n$, then there exists vectors $v_{r+1}, ..., v_n$ so that $v_1, ..., v_n$ are a basis for V.*

Theorem 5.6. *Let V be a subspace of a vector space W. Suppose $dim(W) \leqslant \infty$. Then,*

- *V is finite dimensional.*

- *$dim(V) \leqslant dim(W)$.*

- *If $dim(V) = dim(W)$, then $V = W$*

6 Fundamental Subspaces Associated To Linear Maps

We'll begin by redefining the null space and range of a linear transformation.

Definition. Suppose $T : V \to W$. The **null space or kernel** of T is the set of all the values $v_0 \in V$ such that

$$T(v_0) = \mathbf{0}_W$$

Note: $\dim(\text{null}(A)) \leqslant \dim(V)$.

Definition. Suppose $T : V \to W$. The **range** of T is the set of all values $w \in W$ such w is in the image of T.

Note: $\dim(\text{Ran}(A)) \leqslant \dim(W)$.

From here, we can make a relationship between reduced row echelon form and these subspaces.

Theorem 6.1. *A basis for null(A) is made of the vectors at the free variables in the vector form of the set of solutions to $Ax = \mathbf{0}$. In particular, dim(null(A)) equals the number of free variables in A.*

Definition. Let A be a matrix. Let A_e be the echelon form of A. The ith column is a **pivot column** if the ith column of A_e contains a pivot.

Theorem 6.2. *Let A be a matrix and A_e be the echelon form. The pivot columns of A form a basis for Ran(A). In particular, dim(Ran(A)) equals the number of pivot columns in A_e.*

Theorem 6.3. ***(Rank's Theorem)*** *Let A be a $m \times n$ matrix representing the linear map $\mathbb{F}^n \to \mathbb{F}^m$. Let A_e be the echelon form of A. Then,*

$$dim(\mathbb{F}^n) = dim(null(A)) + dim(Ran(A))$$

7 Arbitrary Bases and Coordinate Change

Definition. Suppose you have a basis $\mathcal{B} = \{v_1, ..., v_n\} \in V$. Let $v \in V$ be arbitrary. You can express v as a linear combination of the basis.

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = v \text{ for some } \alpha_i \in \mathbb{F}$$

The the values $\alpha_1, ..., \alpha_n$ are called the **coordinates**. You can write these as a **coordinate vector**. Since these are in terms of the basis $\mathcal{B}$, we say this is the **coordinate vector with respect to $\mathcal{B}$**.

$$[v]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Definition. Let $T : V \to W$ be a linear transformation. Let $\mathcal{A} = \{v_1, ..., v_n\}$ be a basis for V. Let $\mathcal{B} = \{w_1, ..., w_m\}$ be a basis for W. **The matrix of the linear transformation with respect to $\mathcal{A}$ and $\mathcal{B}$** is the $m \times n$ matrix $[T]_{\mathcal{B}\mathcal{A}}$, whose kth column is the coordinate vector of $T(v_k)$ relative to $\mathcal{B}$.

$$\begin{pmatrix} | & & | \\ [T(v_1)]_{\mathcal{B}} & \cdots & [T(v_n)]_{\mathcal{B}} \\ | & & | \end{pmatrix}$$

Lemma 7.1. *Let $T : V \to W$ be a linear transformation. Let $\mathcal{A} = \{v_1, ..., v_n\}$ be a basis for V. Let $\mathcal{B} = \{w_1, ..., w_m\}$ be a basis for W. Let $v \in V$ be arbitrary. Then,*

$$[T]_{\mathcal{B}\mathcal{A}}[v]_{\mathcal{A}} = [T(v)]_{\mathcal{B}}$$

Definition. The matrix $[I]_{\mathcal{B}}\mathcal{A}$ is called the **change in coordinates matrix** from basis $\mathcal{A}$ to $\mathcal{B}$. This is just the linear transformation of the identity matrix with respect to $\mathcal{B}$ and $\mathcal{A}$.

Corollary 7.2. *If $[I]_{\mathcal{B}}\mathcal{A}$ is a change in coordinates matrix and $v \in V$ is arbitrary, then*

$$[I]_{\mathcal{B}\mathcal{A}}[v]_{\mathcal{A}} = [I(v)]_{\mathcal{B}} = [v]_{\mathcal{B}}$$

Lemma 7.3. ***(Inverse Change in Coordinates)***

$$[I]_{\mathcal{B}\mathcal{A}} = ([I]_{\mathcal{A}\mathcal{B}})^{-1}$$

Lemma 7.4. ***(Composition of Change in Coordinates)*** *Let $T_1 : X \to Y$ and $T_2 : Y \to Z$ be linear transformations. Let A, B, C be bases for X, Y, Z respectively. Then, $T = T_1T_2 : X \to Z$, then,*

$$[T]_{CA} = [T_2T_1]_{CA} = [T_2]_{CB}[T_1]_{BA}$$

Corollary 7.5. *It follows that,*

$$[I]_{CA} = [I]_{CB}[I]_{BA}$$

We can use these two lemmas to find the change in coordinate matrices for more difficult examples.

Strategy. Let A and B be bases. Let S denote the standard basis vector. Suppose you want to find $[T]_{AB}$, but it is difficult to do. Then,

1. Find $[T]_{SA}$ and $[T]_{SB}$.
2. Use the inverse lemma to get $[T]_{AS} = ([T_{SA}])^{-1}$.
3. Use the composition lemma to get $[T]_{AB} = [T]_{AS}[T]_{SB}$.

Note that you can replace T with I and stay in the same vector space.

Example. Let

$$A = \{1, 1+t\} \qquad\qquad B = \{1+2t, 1-2t\}$$

1. Write each vector as a linear combination of the standard basis, $S = \{e_1, e_2\} = \{1, t\}$.

$$\begin{aligned} 1 &= e_1 + 0e_2 \\ 1+t &= e_1 + e_2 \\ 1+2t &= e_1 + 2e_2 \\ 1-2t &= e_1 - 2e_2 \end{aligned}$$

Now, we have,

$$[I]_{SA} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$[I]_{SB} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$$

Now, find $[I]_{BS} = ([I]_{SB})^{-1}$.

Use the composition lemma to get your final answer.

8 Determinants

8.1 Constructing the Determinant

Definition. The **determinant** of the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by

$$detA = det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \in \mathbb{F}$$

Remark. If the 2×2 matrix $A = \begin{pmatrix} v_1 v_2 \end{pmatrix}$, then the determinant of A equals the signed area of a parallelogram spanned by the vector columns. The sign is positive if the angle from v_1 to v_2 is less than π.

If A has a dimension higher than 2×2, then the determinant is the signed volume of the parallepiped spanned by the columns vectors of A.

8.2 Expanding Along Columns and Rows

Notation. The (i, j)**-minor** of A, denoted $A_{(i,j)}$ is the $(n-1) \times (n-1)$ matrix obtained from deleting the ith row and jth column.

Definition. Let A be an $n \times n$ matrix. The number

$$C_{(i,j)} = (-1)^{i+j} A_{(i,j)}$$

Is called the $(i, j)-$cofactor of A.

Definition. Suppose $A = (a_{(j,k)})$ is an $n \times n$ matrix. We can compute the determinant of A via a process called **Cofactor Expansion**. We can either **expand along the jth row** or **expand along the kth column**

Expanding Along the jth Row:

$$\begin{aligned} \det A &= \sum_{k=1}^{n} a_{j,k} C_{(j,k)} \\ &= \sum_{k=1}^{n} a_{j,k} (-1)^{j+k} \det(A_{(j,k)}) \end{aligned}$$

Note that you're expanding along j, so k is in the sum notation. That is, j remains fixed (for example, you might be expanding along the first row, so $j = 1$).

Expanding Along the kth Column:

$$\begin{aligned} \det A &= \sum_{j=1}^{n} a_{j,k} C_{(j,k)} \\ &= \sum_{j=1}^{n} a_{j,k} (-1)^{j+k} \det(A_{(j,k)}) \end{aligned}$$

Note that you're expanding along k, so j is in the sum notation. That is, k is a fixed value.

Theorem 8.1. *Let A be an $n \times n$ matrix. It doesn't matter if you do cofactor expansion along a row or column (or even which row or column). The result will always be the same. That is, you can do cofactor expansion along any row and column to find the determinant of a matrix.*

Remark. In practice, its often easier to expand along the row or column with the most amount of 0 entries. This will allow you to cancel out terms early on. Expanding along other rows will yield the same result, but you will have computed more determinants than you would have otherwise had to.

8.3 Special Cases of the Determinant

Lemma 8.2. *If A is an $n \times n$ matrix with a zero row or zero column, then $detA = 0$.*

Definition. If A is an $n \times n$ matrix, we call it **diagonal** if $a_{j,k} = 0$ for all $a_{j,k}$ where $j \neq k$.

Definition. Suppose A is an $n \times n$ matrix.
A is **upper triangular** if $a_{j,k} = 0$ for all $a_{j,k}$ where $k < j$.
A is **lower triangular** if $a_{j,k} = 0$ for all $a_{j,k}$ where $k > j$.
A is a **triangular matrix** if it is upper triangular or lower triangular.

Remark. A diagonal matrix is a type of triangular matrix. So, any theorem that applies to triangular matrices applies to diagonal ones. Although, not all theorem related to diagonal matrices apply to triangular ones.

Theorem 8.3. *If A is a triangular matrix, then $detA$ is the produce of the entries in A's main diagonal. So,*

$$detA = a_{1,1} \times a_{2,2} \times \cdots \times a_{n,n}$$

Theorem 8.4. *Suppose A is an $n \times n$ matrix. Let A' be the result of switching any two rows or columns of A. Then,*

$$detA' = -detA$$

8.4 Determinants and Invertibility

Setup. Suppose A is an $n \times n$ matrix. We will work towards showing that if and only if $\det A \neq 0$.
What we know:

1. A is invertible if and only if its reduced row echelon, A_{re} form has a pivot in every row and column.

2. A_{re} is a triangular matrix, so its determinant is the product of its diagonal.

From these facts, we can deduced that A is invertible if and only if $\det A_{re} = 1$. Now, we will give the rest of the building blocks.

Lemma 8.5. *Suppose A and B are $n \times n$ matrices.*

$$detAB = detA \times detB$$

We will now address the determinants of each elementary matrix.

Lemma 8.6. *Let A be an $n \times n$ matrix.*

Row Exchange: *Suppose $E_{j,k}$ is obtained from swapping the jth and kth row of I. Then,*

$$detE_{j,k} = -1$$

Row Scaling: *Suppose E is obtained by scaling the jth row of I by c. Note that by our construction on row scaling, $c \neq 0$.*

$$detE = c \neq 0$$

Row Replacement: *Suppose E is obtained from replacing row k with its sum with $c \times$row j. Then,*

$$detE = 1$$

Corollary 8.7. *The determinants of the elementary matrices are not zero.*

So, it follows that,

Theorem 8.8. *If A is an $n \times n$ matrix, then A is invertible if and only if $detA \neq 0$.*

From this, we find three corollaries.

Corollary 8.9. *If A has 2 identical columns, then A is not invertible.*

Corollary 8.10. *If A has 2 identical rows, then A is not invertible.*

Corollary 8.11. *Let A be invertible, then*

$$det(A^{-1}) = \frac{1}{detA}$$

9 Eigenbases and Eigenspaces

9.1 An Introduction to Spectral Theory

Definition. A scalar $\lambda \in \mathbb{F}$ is a **eigenvalue** of a linear map $A : V \to V$ if there exists a nonzero vector such that

$$Ax = \lambda x$$

Definition. A vector $v \in V$ is an **eigenvector** of a corresponding eigenvalue λ if

$$Av = \lambda v$$

Definition. The set of all eigenvalues of a linear map $A : V \to V$ is called the **spectrum** of A.

Definition. The set of all eigenvectors associated to an eigenvalue λ together with the $\{\mathbf{0}\}$ is the **eigenspace** of the linear map A associated with λ.

Lemma 9.1. *Let $A : \mathbb{F}^n \to \mathbb{F}^n$ be a linear map. Let $\lambda \in \mathbb{F}$ be an eigenvalue of A. Then,*

$$x \text{ is an eigenvector associated to } \lambda \iff x \in null(A - \lambda I)$$

Corollary 9.2.

$$\begin{aligned} (A - \lambda I)x &= \mathbf{0} \\ &\Updownarrow \\ Ax - \lambda I x &= \mathbf{0} \\ &\Updownarrow \\ Ax - \lambda x &= \mathbf{0} \\ &\Updownarrow \\ Ax &= \lambda x \end{aligned}$$

Theorem 9.3. *A scalar λ is an eigenvalue of $A : \mathbb{F}^n \to \mathbb{F}^n \iff det(A - \lambda I) = 0$*

Definition. The polynomial (with variable λ), $\det(A\lambda I)$ is called the **characteristic polynomial** of A. By the theorem, to find the eigenvalues, compute the characteristic polynomial and find its roots.

Theorem 9.4. ***(The Fundamental Theorem of Algebra)*** *Any polynomial can be factored across complex roots.*

9.2 Working in Arbitrary Bases

Setup. Let $A : V \to V$ be a linear transformation. Suppose $\dim(V) = n$. Let $\mathcal{B} = \{v_1, ..., v_n\}$ be a basis for V.

1.
$$[A]_{\mathcal{B}\mathcal{B}} = \begin{pmatrix} | & & | \\ [A(v_1)]_{\mathcal{B}} & \cdots & [A(v_n)]_{\mathcal{B}} \\ | & & | \end{pmatrix}$$

2. Find the eigenvalues of $[A]_{\mathcal{B}\mathcal{B}}$.

   We will show that det$([A]_{\mathcal{B}} - \lambda I) = 0$ *is independent of the basis.*

Definition. Two square $n \times n$ matrices A and B are **similar** or **conjugate** if there exists an invertible matrix Q such that,

$$A = SQS^{-1}$$

Lemma 9.5. *If $\mathcal{B}$ and $\mathcal{C}$ are 2 bases for V and $A : V \to V$ is a linear map, then $[A]_{\mathcal{B}\mathcal{B}}$ and $[A]_{\mathcal{C}\mathcal{C}}$ are similar.*

Lemma 9.6. *If A and B are similar, they have the same determinant. That is,*

$$detA = detB$$

Proposition 9.7. *The characteristic polynomials of $[A]_{\mathcal{B}\mathcal{B}}$ and $[A]_{\mathcal{C}\mathcal{C}}$ are equal.*

Proof. Let $R = [A]_{\mathcal{B}\mathcal{B}}$ and $T = [A]_{\mathcal{C}\mathcal{C}}$. Since these are similar matrices, there exists an invertible matrix Q such that,

$$R = STS^{-1}.$$

To show $\det(T - \lambda I) = \det(R - \lambda I)$, it is to show that $R - \lambda I$ and $T - \lambda I$ are similar matrices (since similar matrices have the same determinant). It follows that,

$$\begin{aligned} R - \lambda I &= STS^{-1} - \lambda I \\ &= STS^{-1} - \lambda SIS^{-1} \\ &= S(TS^{-1} - \lambda IS^{-1}) \\ &= S(T - \lambda I)S^{-1} \end{aligned}$$

So, $R - \lambda I$ and $T - \lambda I$ are similar matrices as desired. □

Remark. Consequently, the spectrum $A : V \to V$ can be computed with respect to any basis.

9.3 Special Cases

Lemma 9.8. *The eigenvalues of a triangular matrix are its diagonal entries.*

Corollary 9.9. *The eigenvalues of a diagonal matrix are its diagonal entries.*

Lemma 9.10. *Let A be an $n \times n$ matrix. Let $\lambda_1, ..., \lambda_n$ be its complex eigenvalues, listed with multiplicity. Then,*

$$detA = \lambda_1 \lambda_2 \cdots \lambda_n$$

The following definitions were not covered in class, but were used heavily on homework 9.

Definition. Suppose A is an $n \times n$ matrix. A is a **nilpotent matrix** if there exists a *positive integer* k such that,

$$A^k = 0$$

Note that this is *not* for all k. Only a specific k and all the positive integers above that.

Definition. Suppose $A : V \to V$ is a linear map. Let U be a subspace of V. We call U an invariant under A if for any arbitrary $u \in U$, $Au \in U$. More concisely,

$$\forall u \in U, \ Au \in U.$$

The following lemma was not explicitly done in class or on homework, but is useful to know.

Lemma 9.11. *If A is an $n \times n$ matrix, it has n eigenvalues by the fundamental theorem of algebra.*