

# Linear Algebra Study Guide

Ziyad Rahman

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# 1 An Introduction to Vector Spaces

## 1.1 The Field

Before we begin discussing what the field is, we need to make sure we understand what complex numbers are.

**Definition.** The complex numbers is defined as follows.

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\} \quad (1)$$

**Definition.** A **Field** refers to the space over which a vector space is defined. In this class we will often refer to a special field (which we may refer to simply as "the field"). It is defined as follows.

$$\mathbb{F} = \{\mathbb{R} \cup \mathbb{C}\} \quad (2)$$

That is, the vector space is either defined over the real or complex numbers.

**Remark.** If  $\mathbb{F} = \mathbb{R}$ , then  $V$  is called a real vector space.

If  $\mathbb{F} = \mathbb{C}$ , then  $V$  is called a complex vector space.

## 1.2 Defining Vector Spaces

**Definition.** A **vector space**  $V$  over  $\mathbb{F}$  is a set of elements called **vectors** that have two operations, closed addition and closed scalar multiplication. These operations satisfy 8 axioms. That is,

$$\text{Addition: } f : V \times V \rightarrow V$$

$$\text{Scalar Multiplication: } g : \mathbb{F} \times V \rightarrow V$$

**Theorem 1.1.** If  $V$  is a vector space, then it must have closed addition and closed scalar multiplication such that they fulfill the following vector space axioms.

1. **Commutativity:**  $v + w = w + v$  for all  $v, w \in V$ .
2. **Associativity:**  $(v + w) + u = v + (w + u)$  for all  $v, w, u \in V$ .
3. **Additive Identity:** There exists (a unique)  $\mathbf{0} \in V$  such that  $v + \mathbf{0} = v$  for all  $v \in V$ .
4. **Additive Inverse:** For all  $v \in V$ , there exists (a unique)  $v' \in V$  such that  $v + v' = \mathbf{0}$ .
5. **Multiplicative Identity:**  $1 \times v = v$  for all  $v \in V$  and  $1 \in \mathbb{F}$ .
6. **Multiplicative Associativity:**  $(\alpha\beta)v = \alpha(\beta v)$  for all  $v \in V$  and  $\alpha, \beta \in \mathbb{F}$ .
7. **Scalar Distribution:**  $\alpha(v + w) = \alpha v + \alpha w$  for all  $v, w \in V$  and  $\alpha \in \mathbb{F}$ .
8. **Vector Distribution:**  $v(\alpha + \beta) = \alpha v + \beta v$  for all  $v \in V$  and  $\alpha, \beta \in \mathbb{F}$ .

**Remark.** The first 4 axioms relate to addition. The next 2 are about multiplication. The final two connect addition and multiplication through the distributive property.

**Lemma 1.2.** If  $V$  is a vector space, then  $\mathbf{0} \times v = \mathbf{0}$  for all  $v \in V$ .

## 1.3 Important Vector Spaces and Vector Space Properties

This section will basically be a bunch of lemmas and theorems that state "this thing" is a vector space.

**Theorem 1.3.**  $\mathbb{R}^n$  is a vector space over  $\mathbb{R}$ .

**Theorem 1.4.**  $\mathbb{C}^n$  is a vector space over  $\mathbb{C}$ .

**Theorem 1.5.** If  $M_{m \times n}^{\mathbb{F}}$  denotes the set of all  $m \times n$  matrices with entries in  $\mathbb{F}$ , then  $M_{m \times n}^{\mathbb{F}}$  is a vector space over  $\mathbb{F}$ .

**Theorem 1.6.** Let  $\mathbb{P}_n^{\mathbb{F}}$  denote the set of all polynomials of degree  $\leq n$  and coefficients in  $\mathbb{F}$ . That is,

$$\mathbb{P}_n^{\mathbb{F}} = \{p(t) = a_0 + a_1t + a_2t^2 + \cdots + a_nt^n \mid a_i \in \mathbb{F} \text{ for } i \in \{0, \dots, n\}\}$$

$\mathbb{P}_n^{\mathbb{F}}$  is a vector space over  $\mathbb{F}$ .

## 2 Bases

### 2.1 Linear Combinations

**Definition.** Let  $V$  be a vector space over  $\mathbb{F}$  and  $v_1, \dots, v_p \in V$  be a list of vectors. A **linear combination** of these vectors is a sum of the form,

$$\alpha_1 v_1 + \dots + \alpha_p v_p = \sum_{i=1}^p \alpha_i v_i \text{ where } \alpha_i \in \mathbb{F}$$

### 2.2 Linear Independence

**Definition.** A linear combination,  $\sum_{i=1}^p \alpha_i v_i$  is said to be the **trivial combination (or trivial case)** if  $\alpha_i = 0$  for all  $i$ .

**Definition.** A system of vectors is **linearly independent** if only the trivial linear combination of  $v_1, \dots, v_p$  is equal to the zero vector. That is, if

$$\sum_{i=1}^p \alpha_i v_i = \mathbf{0}$$

Then,  $\alpha_1 = \alpha_2 = \dots = \alpha_p = 0 \in \mathbb{F}$ .

**Definition.** A system of vectors is **linearly dependent** if it is not linearly independent. We can represent this in two ways. First, there exists a linear combination of this list for the zero vector such that the linear combination is not the trivial one.

$$\sum_{i=1}^p \alpha_i v_i = \mathbf{0} \text{ but there exists a } \alpha_i \text{ such that } \alpha_i \neq 0$$

Second, that a list is linearly dependent if and only if one of the vectors in the list can be written as a linear combination of the other vectors in that list.

$$\begin{aligned} v_k &= \sum_{\substack{i=1 \\ i \neq k}}^p \alpha_i v_i \\ &= \alpha_1 v_1 + \dots + \alpha_{k-1} v_{k-1} + \alpha_{k+1} v_{k+1} + \dots + \alpha_p v_p \end{aligned}$$

Technically, this second definition follows from Proposition 2.6 from the textbook, but its so fundamental (and rather easy to prove, since it directly follows the first definition) that I've decided to include it here.

### 2.3 Spanning

**Definition.** A collection of vectors is said to **span** (or **generate**, **be a spanning system of**, **be a complete system of**)  $V$  if any vector  $v \in V$  can be written as a linear combination of that collection.

More formally, suppose you have a list  $v_1, \dots, v_p \in V$  and an arbitrary  $v \in V$ . This list spans only if

$$\sum_{i=1}^p \alpha_i v_i = v$$

### 2.4 Defining Bases

**Definition.** A system of vectors  $v_1, \dots, v_p \in V$  is a **basis** for  $V$  if every vector  $v \in V$  admits a unique representation as a linear combination of  $v_1, \dots, v_p$ .

That is, a system of vectors is a basis if that list is linearly independent and spans  $V$ .

**Remark.** Let  $v \in V$  be arbitrary. If the list  $v_1, \dots, v_p \in V$  is a basis, then

$$\sum_{i=1}^p \alpha_i v_i = v$$

We would refer to the  $\alpha_i$  as the **coordinates** of  $v$  with respect to this basis.

**Theorem 2.1.** *Any finite set of vectors that span a vector space contains a basis.*

**Remark.** The proof is "recursive" in that the idea is you can remove elements of that list until it becomes linearly independent. Removing elements doesn't impact the list's ability to span.

**Definition.** The **standard basis vectors** refers to a simple, reliable basis for various vector spaces. The two important ones are for  $\mathbb{F}^n$  and  $\mathbb{P}_n^{\mathbb{R}}$ .

The standard basis vectors for  $\mathbb{F}^n$  are

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{F}^n$$

The standard basis vectors for  $\mathbb{P}_n^{\mathbb{R}}$  are

$$1, t, t^2, \dots, t^n \in \mathbb{P}_n^{\mathbb{R}}$$

## 3 Linear Transformations

### 3.1 Introducing Linear Transformations

**Definition.** A **transformation**  $T$  from a set  $X$  to  $Y$  is a function from  $X$  to  $Y$ . We denote it as

$$T : X \rightarrow Y$$

**Definition.** Suppose there are two vector spaces  $V$  and  $W$  both over  $\mathbb{F}$ . A transformation between them is **linear** if it holds for two properties.

1. **Additivity:**  $T(v + u) = T(v) + T(u)$  for all  $v, u \in V$ .
2. **Homogeneity:**  $T(\alpha v) = \alpha T(v)$  for all  $v \in V$  and  $\alpha \in \mathbb{F}$ .

**Lemma 3.1.** *Let  $T : \mathbb{P}_n^{\mathbb{R}} \rightarrow \mathbb{P}_{n-1}^{\mathbb{R}}$  be the differentiation of polynomials.  $T$  is a linear map.*

### 3.2 Representing Transformations as Matrices

#### 3.2.1 Transformations as Matrices

**Definition.** Suppose

$$T(e_k) = \begin{pmatrix} a_{1,k} \\ a_{2,k} \\ \vdots \\ a_{n,k} \end{pmatrix} \in \mathbb{F}^n$$

Then, the **matrix representing**  $T$  is defined as the following.

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

Basically, the first column is just  $T(e_1)$ . The second column is  $T(e_2)$ , so the  $n$ th column is  $T(e_n)$ .

### 3.2.2 Matrix-Vector Multiplication

The concept of matrix-vector multiplication is the ability to find  $T(v)$  through the use of a matrix.

**Definition.** Let  $A$  be the matrix representing  $T$  and  $x \in \mathbb{F}^n$ . Then,

$$Ax = T(x)$$

More explicitly,

$$\begin{aligned} Ax &= \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= \begin{pmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \end{pmatrix} \end{aligned}$$

### 3.2.3 Composing Linear Maps

### 3.2.4 Matrix-Matrix Multiplication

**Definition.** Let  $A$  be a  $m \times n$  matrix and  $B$  be a  $n \times r$  matrix. We have two definitions for **matrix-matrix multiplication**.

1. Suppose,

$$B = \left( \begin{array}{c|c|c|c} | & | & & | \\ b_1 & b_2 & \cdots & b_r \\ | & | & & | \end{array} \right)$$

Where  $b_i$  is the  $i$ th column of  $B$ . Then,

$$AB := \left( \begin{array}{c|c|c|c} | & | & & | \\ Ab_1 & Ab_2 & \cdots & Ab_r \\ | & | & & | \end{array} \right)$$

2. The entry  $(AB)_{j,k}$  is

$$(j\text{th row of } A) \times (i\text{th column of } B) = \sum_{l=1}^n a_{j,l} \times b_{l,k}$$

**Remark.** Matrix multiplication holds for the following properties.

1. **Associativity:**  $A(BC) = (AB)C$
2. **Matrix Distribution:**  $A(B + C) = AB + AC$  and  $(A + B)C = AC + BC$
3. **Scalar Distribution:**  $A(\alpha B) = \alpha(AB)$

Very importantly, **matrix multiplication is NOT commutative**.

**Theorem 3.2.** Let  $A$  be the matrix representing  $T_1$  and  $B$  be the matrix representing  $T_2$ . Then, the matrix  $BA$  represents the composition  $T_2 \circ T_1$ .

### 3.2.5 The Rotational Matrix

**Definition.** Let the **rotational matrix** be defined as follows.

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

When multiplied by a vector, it rotates the vectors  $\theta$  clockwise. You are functionally composing a vector with a rotation.

### 3.2.6 Transposing and Tracing Matrices

Two more useful tools to have are as follows.

**Definition.** The **transpose** of  $A$  is a matrix

$$A^T = (a'_{i,j})_{\substack{1 \leq j \leq n \\ 1 \leq i \leq m}} \text{ and } a'_{i,j} = a_{j,i}$$

**Definition.** The **trace** of an  $n \times n$  matrix  $A = (a_{j,k})$  is the sum of the diagonal entries.

$$tr(A) = a_{1,1} + a_{2,2} + \cdots + a_{n,n}$$

Equivalently,

$$tr(A) = \sum_{i=1}^n a_{i,i}$$

## 3.3 Subspaces

**Definition.** Suppose  $V$  is a vector space.  $V_0$  is a **subspace** of  $V$ , if the following properties hold.

1.  $V_0 \subset V$
2.  $V_0 \neq \emptyset$
3.  $v + w \in V_0$  for all  $v, w \in V_0$
4.  $\alpha v \in V_0$  for all  $v \in V_0$  and  $\alpha \in \mathbb{F}$

### 3.3.1 Null Spaces and Range

**Definition.** Suppose  $T : V \rightarrow W$ . The **null space or kernel** of  $T$  is the set of all the values  $v_0 \in V$  such that

$$T(v_0) = \mathbf{0}_W$$

**Definition.** Suppose  $T : V \rightarrow W$ . The **range** of  $T$  is the set of all values  $w \in W$  such  $w$  is in the image of  $T$ .

## 3.4 Isomorphisms

### 3.4.1 Inverses

**Definition.** Suppose  $A : T \rightarrow W$  is a linear transformation.  $A$  is **right invertible** if there exists a  $B : W \rightarrow V$  such that  $B \circ A = I_V$ . In matrix multiplication,  $BA$  is the identity matrix.

**Definition.** Suppose  $A : T \rightarrow W$  is a linear transformation.  $A$  is **left invertible** if there exists a  $B : W \rightarrow V$  such that  $B \circ A = I_W$ . In matrix multiplication,  $AB$  is the identity matrix.

**Definition.** A linear transformation  $A : V \rightarrow W$  is **invertible** only if it is right invertible and left invertible.

From that, we have the following corollary.

**Corollary 3.3.** *A linear transformation  $A : V \rightarrow W$  is invertible if and only if there exists a unique linear transformation  $A^{-1} : W \rightarrow V$  such that*

$$A^{-1} \circ A = I_V \text{ and } A \circ A^{-1} = I_W$$

*We call  $A^{-1}$  the inverse of  $A$ .*

**Lemma 3.4.** *If  $A$  and  $B$  are invertible linear transformations, and  $AB$  is defined, then  $A \circ B$  is invertible and*

$$(A \circ B)^{-1} = B^{-1} \circ A^{-1}$$

**Theorem 3.5.** *Let  $A : X \rightarrow Y$  be a linear transformation. Then,  $A$  is invertible if and only if for all  $b \in Y$ , the equation*

$$Ax = b$$

*has a unique solution.*

### 3.4.2 Defining Isomorphisms

**Definition.** An **Isomorphism** from a vector space  $V$  to a vector space  $W$  is an invertible linear map from  $V$  to  $W$ . If there exists an isomorphism between two vector spaces, then we would say that those vector spaces are **isomorphic**.

We will now list the foundational theorems for isomorphisms.

**Theorem 3.6.** *Let  $A : V \rightarrow W$  be an isomorphism. If  $v_1, \dots, v_n$  is a basis for  $V$ , then  $A(v_1), \dots, A(v_n)$  is a basis for  $W$ .*

**Theorem 3.7.** *Let  $v_1, \dots, v_n$  be a basis for  $V$  and  $w_1, \dots, w_n$  be a basis for  $W$ . Let  $A : V \rightarrow W$  be the map defined by*

$$A(v_i) = w_i \text{ for } 1 \leq i \leq n.$$

*$A$  is an isomorphism.*

**Corollary 3.8.** *If vector spaces have bases of the same size, then they are isomorphic.*

## 4 Linear Equations

### 4.1 Introduction to Linear Systems

**Definition.** A **Linear System** is a collection of  $m$  linear equations with  $n$  unknowns.

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m \end{cases}$$

Where  $a_{i,j}, b_i \in \mathbb{F}$ .



**Notation.** Given the following arbitrary linear system,

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m \end{cases}$$

We can write this as a system of matrices.

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

We can further simplify writing such an equation through an **augmented matrix**.

$$\left( \begin{array}{ccc|c} a_{1,1} & \cdots & a_{1,n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & \cdots & a_{m,n} & b_m \end{array} \right)$$

Also denoted as  $(A|b)$ .

## 4.2 Finding Solutions: Gauss-Jordan Elimination

**Definition.** Let  $(A|b)$  be an augmented matrix. You can apply three types of **row operations** to find the solution to such a system.

1. **Row Exchange:** Interchange two rows.
2. **Scaling:** Multiply a row by a non-zero number.
3. **Row Replacement:** Replace row  $k$  by its sum with a constant multiple of row  $j$ . That is,  $R_k + \alpha R_j$  such that  $\alpha \neq 0$ .

**Theorem 4.1.** Row operations do not change the solutions to a linear system.

**Proposition 4.2.** Each row operation can be realized by multiplication of an invertible matrix.

1. **Row Exchange:** Suppose you are changing the  $k$ th and  $j$ th rows. This can be expressed as the identity matrix with the  $k$ th and  $j$ th rows swapped.
2. **Scaling:** Suppose you are scaling the  $k$ th row by  $\alpha \in \mathbb{F}$ . You can represent this as the identity matrix where the  $k$ th row is scaled by  $\alpha$ .

## 4.3 Finding the Inverse of a Function via Row Reduction

**Strategy.** Suppose you have a  $n \times n$  matrix,

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}$$

You can create an augmented matrix,  $(A|b)$ , where  $b$  is the identity matrix.

$$\left( \begin{array}{ccc|cccc} a_{1,1} & \cdots & a_{1,n} & 1 & 0 & \cdots & 0 \\ a_{2,1} & \cdots & a_{2,n} & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} & 0 & 0 & \cdots & 0 \end{array} \right)$$

Now, make  $A$  into its row reduced echelon form. The right side (what was originally the identity matrix) will be transformed into  $A$ 's inverse matrix.

## 4.4 Echelon Form

**Definition.** A matrix or augmented matrix is in **echelon form** if it satisfies two conditions.

1. All zero rows are below all non-zero rows.
2. For a non-zero row, its leftmost non-zero entry (which is called the leading entry or **pivot**) is strictly to the right of the leading entry in the row before it.

**Example.** The following matrix is in echelon form.

$$\left( \begin{array}{ccc|c} 1 & 2 & 0 & 8 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{array} \right)$$

**Theorem 4.3.** *Every matrix can be transformed into echelon form by applying a finite sequence of row operations.*

**Strategy. (Echelon Form)**

**Main Step:**

1. Find leftmost non-zero column matrix.
2. Apply row-exchange to ensure entry in the first row of the column is non-zero.
3. Apply scaling so the pivot in the first row equals 1.
4. Apply row replacement by adding an appropriate multiple of row 1 to each other row, so all entries below the pivot = 0.

**Next:**

1. Leave the first row alone.
2. Apply the main step to columns 2 through m.
3. Apply the main step to columns 3 through m.
4. Continue applying the main step to each subsequent column.

Then, you have echelon form.

## 4.5 Row Reduced Echelon Form

**Definition.** A matrix or augmented matrix is in **row reduced echelon form** if it fulfills the requirements of echelon form and two more constraints. In total, these requirements are summarized below.

1. All zero rows are below all non-zero rows.
2. The left most non-zero entry in each column is a pivot and every pivot is strictly to the right of every pivot above it.
3. Every pivot is equal to 1.
4. All entries above a pivot are zero.

**Theorem 4.4.** *Every matrix can be put into reduced echelon form by a finite sequence of row operations.*

**Definition.** The equation  $Ax = b$  is **inconsistent** if it does not have a solution.

**Theorem 4.5.** *The system  $Ax = b$  is inconsistent if and only if there exists a row of the form,*

$$(0 \ 0 \ \cdots \ 0 \ | \ b)$$

*Such that  $b \neq 0$ .*

**Corollary 4.6.** A system  $Ax = b$  has a solution if there exists a pivot in every row.

**Corollary 4.7.** A system  $Ax = b$  with a solution has a unique solution for  $b$  if and only if there exists a pivot in every column of the reduced echelon form of  $A$ .

**Theorem 4.8.** Let  $v_1, \dots, v_m \in \mathbb{F}^n$  be a system of vectors. Let,

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{pmatrix}$$

be the  $m \times n$  matrix. The system is

1. **Linearly Independent** if and only if  $A_e$  has a pivot in every column (no free variables).
2. **Spanning** if and only if  $A_e$  has a pivot in every row (solutions exist) .

## 5 Dimensions

First, we talk about bases.

**Proposition 5.1.** Suppose a vector space  $V$  has a finite basis. Then, any two bases have the same number of elements in them.

**Corollary 5.2.** The basis of  $\mathbb{F}^n$  has  $n$  vectors in it.

**Definition.** The **dimension** of a vector space  $V$ , denoted by  $\dim(V)$ , is the number of vectors in a basis for  $V$  if  $V$  has a finite basis.

- The dimensions of  $\{0\}$  is 0.
- If  $V$  is not finite-dimensional, then  $V$  is infinitely-dimensional.

Using this definition, we can construct bases.

**Proposition 5.3.** Any linearly independent list of vectors in a finite-dimensional vector space  $V$  has at most  $\dim(V)$  elements.

**Proposition 5.4.** Any spanning list of vectors in a finite-dimensional vector space  $V$  has at least  $\dim(V)$  elements.

Following this,

**Theorem 5.5.** If  $v_1, \dots, v_r$  are linearly independent vectors in a finite-dimensional vector space  $V$  with  $\dim(V) = n$ , then there exists vectors  $v_{r+1}, \dots, v_n$  so that  $v_1, \dots, v_n$  are a basis for  $V$ .

**Theorem 5.6.** Let  $V$  be a subspace of a vector space  $W$ . Suppose  $\dim(W) \leq \infty$ . Then,

- $V$  is finite dimensional.
- $\dim(V) \leq \dim(W)$ .
- If  $\dim(V) = \dim(W)$ , then  $V = W$

## 6 Fundamental Subspaces Associated To Linear Maps

We'll begin by redefining the null space and range of a linear transformation.

**Definition.** Suppose  $T : V \rightarrow W$ . The **null space or kernel** of  $T$  is the set of all the values  $v_0 \in V$  such that

$$T(v_0) = \mathbf{0}_W$$

Note:  $\dim(\text{null}(A)) \leq \dim(V)$ .

**Definition.** Suppose  $T : V \rightarrow W$ . The **range** of  $T$  is the set of all values  $w \in W$  such  $w$  is in the image of  $T$ .

Note:  $\dim(\text{Ran}(A)) \leq \dim(W)$ .

From here, we can make a relationship between reduced row echelon form and these subspaces.

**Theorem 6.1.** *A basis for  $\text{null}(A)$  is made of the vectors at the free variables in the vector form of the set of solutions to  $Ax = \mathbf{0}$ . In particular,  $\dim(\text{null}(A))$  equals the number of free variables in  $A$ .*

**Definition.** Let  $A$  be a matrix. Let  $A_e$  be the echelon form of  $A$ . The  $i$ th column is a **pivot column** if the  $i$ th column of  $A_e$  contains a pivot.

**Theorem 6.2.** *Let  $A$  be a matrix and  $A_e$  be the echelon form. The pivot columns of  $A$  form a basis for  $\text{Ran}(A)$ . In particular,  $\dim(\text{Ran}(A))$  equals the number of pivot columns in  $A_e$ .*

**Theorem 6.3. (Rank's Theorem)** *Let  $A$  be a  $m \times n$  matrix representing the linear map  $\mathbb{F}^n \rightarrow \mathbb{F}^m$ . Let  $A_e$  be the echelon form of  $A$ . Then,*

$$\dim(\mathbb{F}^n) = \dim(\text{null}(A)) + \dim(\text{Ran}(A))$$

## 7 Arbitrary Bases and Coordinate Change

**Definition.** Suppose you have a basis  $\mathcal{B} = \{v_1, \dots, v_n\} \in V$ . Let  $v \in V$  be arbitrary. You can express  $v$  as a linear combination of the basis.

$$\alpha_1 v_1 + \dots + \alpha_n v_n = v \text{ for some } \alpha_i \in \mathbb{F}$$

The the values  $\alpha_1, \dots, \alpha_n$  are called the **coordinates**. You can write these as a **coordinate vector**. Since these are in terms of the basis  $\mathcal{B}$ , we say this is the **coordinate vector with respect to  $\mathcal{B}$** .

$$[v]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

**Definition.** Let  $T : V \rightarrow W$  be a linear transformation. Let  $\mathcal{A} = \{v_1, \dots, v_n\}$  be a basis for  $V$ . Let  $\mathcal{B} = \{w_1, \dots, w_m\}$  be a basis for  $W$ . **The matrix of the linear transformation with respect to  $\mathcal{A}$  and  $\mathcal{B}$**  is the  $m \times n$  matrix  $[T]_{\mathcal{B}\mathcal{A}}$ , whose  $k$ th column is the coordinate vector of  $T(v_k)$  relative to  $\mathcal{B}$ .

$$\left( \begin{array}{c|ccc|c} & & & & \\ \hline [T(v_1)]_{\mathcal{B}} & \cdots & [T(v_n)]_{\mathcal{B}} & \\ \hline & & & \end{array} \right)$$

**Lemma 7.1.** *Let  $T : V \rightarrow W$  be a linear transformation. Let  $\mathcal{A} = \{v_1, \dots, v_n\}$  be a basis for  $V$ . Let  $\mathcal{B} = \{w_1, \dots, w_m\}$  be a basis for  $W$ . Let  $v \in V$  be arbitrary. Then,*

$$[T]_{\mathcal{B}\mathcal{A}}[v]_{\mathcal{A}} = [T(v)]_{\mathcal{B}}$$

**Definition.** The matrix  $[I]_{\mathcal{B}\mathcal{A}}$  is called the **change in coordinates matrix** from basis  $\mathcal{A}$  to  $\mathcal{B}$ . This is just the linear transformation of the identity matrix with respect to  $\mathcal{B}$  and  $\mathcal{A}$ .

**Corollary 7.2.** If  $[I]_{\mathcal{B}\mathcal{A}}$  is a change in coordinates matrix and  $v \in V$  is arbitrary, then

$$[I]_{\mathcal{B}\mathcal{A}}[v]_{\mathcal{A}} = [I(v)]_{\mathcal{B}} = [v]_{\mathcal{B}}$$

**Lemma 7.3. (Inverse Change in Coordinates)**

$$[I]_{\mathcal{B}\mathcal{A}} = ([I]_{\mathcal{A}\mathcal{B}})^{-1}$$

**Lemma 7.4. (Composition of Change in Coordinates)** Let  $T_1 : X \rightarrow Y$  and  $T_2 : Y \rightarrow Z$  be linear transformations. Let  $A, B, C$  be bases for  $X, Y, Z$  respectively. Then,  $T = T_1 T_2 : X \rightarrow Z$ , then,

$$[T]_{CA} = [T_2 T_1]_{CA} = [T_2]_{CB} [T_1]_{BA}$$

**Corollary 7.5.** It follows that,

$$[I]_{CA} = [I]_{CB} [I]_{BA}$$

We can use these two lemmas to find the change in coordinate matrices for more difficult examples.

**Strategy.** Let  $A$  and  $B$  be bases. Let  $S$  denote the standard basis vector. Suppose you want to find  $[T]_{AB}$ , but it is difficult to do. Then,

1. Find  $[T]_{SA}$  and  $[T]_{SB}$ .
2. Use the inverse lemma to get  $[T]_{AS} = ([T]_{SA})^{-1}$ .
3. Use the composition lemma to get  $[T]_{AB} = [T]_{AS} [T]_{SB}$ .

Note that you can replace  $T$  with  $I$  and stay in the same vector space.

**Example.** Let

$$A = \{1, 1+t\}$$

$$B = \{1+2t, 1-2t\}$$

1. Write each vector as a linear combination of the standard basis,  $S = \{e_1, e_2\} = \{1, t\}$ .

$$1 = e_1 + 0e_2$$

$$1+t = e_1 + e_2$$

$$1+2t = e_1 + 2e_2$$

$$1-2t = e_1 - 2e_2$$

Now, we have,

$$[I]_{SA} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$[I]_{SB} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$$

Now, find  $[I]_{BS} = ([I]_{SB})^{-1}$ .

Use the composition lemma to get your final answer.

## 8 Determinants

### 8.1 Constructing the Determinant

**Definition.** The **determinant** of the  $2 \times 2$  matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is given by

$$\det A = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \in \mathbb{F}$$

**Remark.** If the  $2 \times 2$  matrix  $A = (v_1 v_2)$ , then the determinant of  $A$  equals the signed area of a parallelogram spanned by the vector columns. The sign is positive if the angle from  $v_1$  to  $v_2$  is less than  $\pi$ .

If  $A$  has a dimension higher than  $2 \times 2$ , then the determinant is the signed volume of the parallelepiped spanned by the columns vectors of  $A$ .

### 8.2 Expanding Along Columns and Rows

**Notation.** The  $(i, j)$ -**minor** of  $A$ , denoted  $A_{(i,j)}$  is the  $(n-1) \times (n-1)$  matrix obtained from deleting the  $i$ th row and  $j$ th column.

**Definition.** Let  $A$  be an  $n \times n$  matrix. The number

$$C_{(i,j)} = (-1)^{i+j} A_{(i,j)}$$

Is called the  $(i, j)$ -cofactor of  $A$ .

**Definition.** Suppose  $A = (a_{(j,k)})$  is an  $n \times n$  matrix. We can compute the determinant of  $A$  via a process called **Cofactor Expansion**. We can either **expand along the  $j$ th row** or **expand along the  $k$ th column**

**Expanding Along the  $j$ th Row:**

$$\begin{aligned} \det A &= \sum_{k=1}^n a_{j,k} C_{(j,k)} \\ &= \sum_{k=1}^n a_{j,k} (-1)^{j+k} \det(A_{(j,k)}) \end{aligned}$$

Note that you're expanding along  $j$ , so  $k$  is in the sum notation. That is,  $j$  remains fixed (for example, you might be expanding along the first row, so  $j = 1$ ).

**Expanding Along the  $k$ th Column:**

$$\begin{aligned} \det A &= \sum_{j=1}^n a_{j,k} C_{(j,k)} \\ &= \sum_{j=1}^n a_{j,k} (-1)^{j+k} \det(A_{(j,k)}) \end{aligned}$$

Note that you're expanding along  $k$ , so  $j$  is in the sum notation. That is,  $k$  is a fixed value.

**Theorem 8.1.** *Let  $A$  be an  $n \times n$  matrix. It doesn't matter if you do cofactor expansion along a row or column (or even which row or column). The result will always be the same. That is, you can do cofactor expansion along any row and column to find the determinant of a matrix.*

**Remark.** In practice, it's often easier to expand along the row or column with the most amount of 0 entries. This will allow you to cancel out terms early on. Expanding along other rows will yield the same result, but you will have computed more determinants than you would have otherwise had to.

### 8.3 Special Cases of the Determinant

**Lemma 8.2.** *If  $A$  is an  $n \times n$  matrix with a zero row or zero column, then  $\det A = 0$ .*

**Definition.** If  $A$  is an  $n \times n$  matrix, we call it **diagonal** if  $a_{j,k} = 0$  for all  $a_{j,k}$  where  $j \neq k$ .

**Definition.** Suppose  $A$  is an  $n \times n$  matrix.

$A$  is **upper triangular** if  $a_{j,k} = 0$  for all  $a_{j,k}$  where  $k < j$ .

$A$  is **lower triangular** if  $a_{j,k} = 0$  for all  $a_{j,k}$  where  $k > j$ .

$A$  is a **triangular matrix** if it is upper triangular or lower triangular.

**Remark.** A diagonal matrix is a type of triangular matrix. So, any theorem that applies to triangular matrices applies to diagonal ones. Although, not all theorem related to diagonal matrices apply to triangular ones.

**Theorem 8.3.** *If  $A$  is a triangular matrix, then  $\det A$  is the produce of the entries in  $A$ 's main diagonal. So,*

$$\det A = a_{1,1} \times a_{2,2} \times \cdots \times a_{n,n}$$

**Theorem 8.4.** *Suppose  $A$  is an  $n \times n$  matrix. Let  $A'$  be the result of switching any two rows or columns of  $A$ . Then,*

$$\det A' = -\det A$$

### 8.4 Determinants and Invertibility

**Setup.** Suppose  $A$  is an  $n \times n$  matrix. We will work towards showing that if and only if  $\det A \neq 0$ .

What we know:

1.  $A$  is invertible if and only if its reduced row echelon,  $A_{re}$  form has a pivot in every row and column.
2.  $A_{re}$  is a triangular matrix, so its determinant is the product of its diagonal.

From these facts, we can deduced that  $A$  is invertible if and only if  $\det A_{re} = 1$ . Now, we will give the rest of the building blocks.

**Lemma 8.5.** *Suppose  $A$  and  $B$  are  $n \times n$  matrices.*

$$\det AB = \det A \times \det B$$

We will now address the determinants of each elementary matrix.

**Lemma 8.6.** *Let  $A$  be an  $n \times n$  matrix.*

**Row Exchange:** *Suppose  $E_{j,k}$  is obtained from swapping the  $j$ th and  $k$ th row of  $I$ . Then,*

$$\det E_{j,k} = -1$$

**Row Scaling:** *Suppose  $E$  is obtained by scaling the  $j$ th row of  $I$  by  $c$ . Note that by our construction on row scaling,  $c \neq 0$ .*

$$\det E = c \neq 0$$

**Row Replacement:** *Suppose  $E$  is obtained from replacing row  $k$  with its sum with  $c \times$  row  $j$ . Then,*

$$\det E = 1$$

**Corollary 8.7.** *The determinants of the elementary matrices are not zero.*

So, it follows that,

**Theorem 8.8.** *If  $A$  is an  $n \times n$  matrix, then  $A$  is invertible if and only if  $\det A \neq 0$ .*

From this, we find three corollaries.

**Corollary 8.9.** *If  $A$  has 2 identical columns, then  $A$  is not invertible.*

**Corollary 8.10.** *If  $A$  has 2 identical rows, then  $A$  is not invertible.*

**Corollary 8.11.** *Let  $A$  be invertible, then*

$$\det(A^{-1}) = \frac{1}{\det A}$$

## 9 Eigenbases and Eigenspaces

### 9.1 An Introduction to Spectral Theory

**Definition.** A scalar  $\lambda \in \mathbb{F}$  is a **eigenvalue** of a linear map  $A : V \rightarrow V$  if there exists a nonzero vector such that

$$Ax = \lambda x$$

**Definition.** A vector  $v \in V$  is an **eigenvector** of a corresponding eigenvalue  $\lambda$  if

$$Av = \lambda v$$

**Definition.** The set of all eigenvalues of a linear map  $A : V \rightarrow V$  is called the **spectrum** of  $A$ .

**Definition.** The set of all eigenvectors associated to an eigenvalue  $\lambda$  together with the  $\{0\}$  is the **eigenspace** of the linear map  $A$  associated with  $\lambda$ .

**Lemma 9.1.** *Let  $A : \mathbb{F}^n \rightarrow \mathbb{F}^n$  be a linear map. Let  $\lambda \in \mathbb{F}$  be an eigenvalue of  $A$ . Then,*

$$x \text{ is an eigenvector associated to } \lambda \iff x \in \text{null}(A - \lambda I)$$

**Corollary 9.2.**

$$\begin{aligned} (A - \lambda I)x &= \mathbf{0} \\ \iff \\ Ax - \lambda Ix &= \mathbf{0} \\ \iff \\ Ax - \lambda x &= \mathbf{0} \\ \iff \\ Ax &= \lambda x \end{aligned}$$

**Theorem 9.3.** *A scalar  $\lambda$  is an eigenvalue of  $A : \mathbb{F}^n \rightarrow \mathbb{F}^n \iff \det(A - \lambda I) = 0$*

**Definition.** The polynomial (with variable  $\lambda$ ),  $\det(A - \lambda I)$  is called the **characteristic polynomial** of  $A$ . By the theorem, to find the eigenvalues, compute the characteristic polynomial and find its roots.

**Theorem 9.4. (The Fundamental Theorem of Algebra)** *Any polynomial can be factored across complex roots.*



## 9.2 Working in Arbitrary Bases

**Setup.** Let  $A : V \rightarrow V$  be a linear transformation. Suppose  $\dim(V) = n$ . Let  $\mathcal{B} = \{v_1, \dots, v_n\}$  be a basis for  $V$ .

1.

$$[A]_{\mathcal{B}\mathcal{B}} = \begin{pmatrix} \left| \begin{array}{c} [A(v_1)]_{\mathcal{B}} \\ \vdots \end{array} \right| & \cdots & \left| \begin{array}{c} [A(v_n)]_{\mathcal{B}} \\ \vdots \end{array} \right| \end{pmatrix}$$

2. Find the eigenvalues of  $[A]_{\mathcal{B}\mathcal{B}}$ .

*We will show that  $\det([A]_{\mathcal{B}} - \lambda I) = 0$  is independent of the basis.*

**Definition.** Two square  $n \times n$  matrices  $A$  and  $B$  are **similar** or **conjugate** if there exists an invertible matrix  $Q$  such that,

$$A = SQS^{-1}$$

**Lemma 9.5.** *If  $\mathcal{B}$  and  $\mathcal{C}$  are 2 bases for  $V$  and  $A : V \rightarrow V$  is a linear map, then  $[A]_{\mathcal{B}\mathcal{B}}$  and  $[A]_{\mathcal{C}\mathcal{C}}$  are similar.*

**Lemma 9.6.** *If  $A$  and  $B$  are similar, they have the same determinant. That is,*

$$\det A = \det B$$

**Proposition 9.7.** *The characteristic polynomials of  $[A]_{\mathcal{B}\mathcal{B}}$  and  $[A]_{\mathcal{C}\mathcal{C}}$  are equal.*

*Proof.* Let  $R = [A]_{\mathcal{B}\mathcal{B}}$  and  $T = [A]_{\mathcal{C}\mathcal{C}}$ . Since these are similar matrices, there exists an invertible matrix  $Q$  such that,

$$R = STS^{-1}.$$

To show  $\det(T - \lambda I) = \det(R - \lambda I)$ , it is to show that  $R - \lambda I$  and  $T - \lambda I$  are similar matrices (since similar matrices have the same determinant). It follows that,

$$\begin{aligned} R - \lambda I &= STS^{-1} - \lambda I \\ &= STS^{-1} - \lambda SIS^{-1} \\ &= S(TS^{-1} - \lambda IS^{-1}) \\ &= S(T - \lambda I)S^{-1} \end{aligned}$$

So,  $R - \lambda I$  and  $T - \lambda I$  are similar matrices as desired. □

**Remark.** Consequently, the spectrum  $A : V \rightarrow V$  can be computed with respect to any basis.

## 9.3 Special Cases

**Lemma 9.8.** *The eigenvalues of a triangular matrix are its diagonal entries.*

**Corollary 9.9.** *The eigenvalues of a diagonal matrix are its diagonal entries.*

**Lemma 9.10.** *Let  $A$  be an  $n \times n$  matrix. Let  $\lambda_1, \dots, \lambda_n$  be its complex eigenvalues, listed with multiplicity. Then,*

$$\det A = \lambda_1 \lambda_2 \cdots \lambda_n$$

The following definitions were not covered in class, but were used heavily on homework 9.

**Definition.** Suppose  $A$  is an  $n \times n$  matrix.  $A$  is a **nilpotent matrix** if there exists a *positive integer*  $k$  such that,

$$A^k = 0$$

Note that this is *not* for all  $k$ . Only a specific  $k$  and all the positive integers above that.

**Definition.** Suppose  $A : V \rightarrow V$  is a linear map. Let  $U$  be a subspace of  $V$ . We call  $U$  an invariant under  $A$  if for any arbitrary  $u \in U$ ,  $Au \in U$ . More concisely,

$$\forall u \in U, Au \in U.$$

The following lemma was not explicitly done in class or on homework, but is useful to know.

**Lemma 9.11.** *If  $A$  is an  $n \times n$  matrix, it has  $n$  eigenvalues by the fundamental theorem of algebra.*

## 10 Selected Homework Problems

This section will include homework problems of importance. That is, homework problems that we might be asked to prove on an exam or problems that might be fundamental to proving other, commonly talked about problems. I won't be including the proofs here, but if the proof is somehow important because of the technique used or I thought it was difficult, it will be in the selected proofs section. Note that some of these problems might have already been mentioned in previous parts of this study guide. I'm just including most everything here for completeness' sake.

### 10.1 Homework 3

#### Problem 2:

**Lemma 10.1.** *Let  $V$  and  $W$  be vector spaces. If  $T : V \rightarrow W$  is a linear map, then  $T(\mathbf{0}_V) = \mathbf{0}_W$ .*

#### Problem 4a:

**Lemma 10.2.** *Let  $V$  and  $W$  be vector spaces. Suppose  $v_1, \dots, v_k$  is a set of vectors in  $V$ . If  $Tv_1, \dots, Tv_k$  is a linearly independent list in  $W$ , then  $v_1, \dots, v_k$  is a linearly independent list in  $V$ .*

**Remark.** The converse is not true!

### 10.2 Homework 4

#### Problem 1

**Lemma 10.3.** *Suppose  $A : V \rightarrow W$  is an isomorphism. If  $v_1, \dots, v_k$  is a basis for  $V$ , then  $Av_1, \dots, Av_k$  is a basis for  $W$ .*

#### Problem 2

**Lemma 10.4.** *Suppose  $V$  and  $W$  are vector spaces over  $\mathbb{F}$ . Suppose the set  $v_1, \dots, v_n$  is a basis for  $V$  and the set  $w_1, \dots, w_n$  is a basis for  $W$ . Let  $A : V \rightarrow W$  be a linear map defined by  $A(v_i) = w_i$  for  $i \in \{1, \dots, n\}$ .*

*If  $V$  is a vector space with a basis of  $n$  elements, then  $V$  is isomorphic to  $\mathbb{F}^n$ .*

The following corollary is not from a homework, but is a natural extension of this lemma using a definition from later in the course.

**Corollary 10.5.** *If  $\dim(V) = n$ , then  $V$  is isomorphic to  $\mathbb{F}^n$ .*

#### Problem 3a and 3b

**Lemma 10.6.** *Let  $A : V \rightarrow W$  be a linear transformation. The null space of  $A$  is a subspace of  $V$ .*

**Lemma 10.7.** *Let  $A : V \rightarrow W$  be a linear transformation. The range of  $A$  is a subspace of  $W$ .*

### 10.3 Homework 5

#### Problem 1

**Lemma 10.8.** *Let  $X$  and  $Y$  be vector spaces. Let  $A : X \rightarrow Y$  be a linear transformation.*

*$A$  is invertible **if and only if** for all  $b \in Y$ , the equation  $Ax = b$  has a unique solution for all  $x \in X$ .*

## 10.4 Homework 6

### Problem 1

**Lemma 10.9.** Suppose  $V$  is a vector space and  $v_1, \dots, v_n$  is a **linearly independent** system in  $V$ .  $v_1, \dots, v_n$  is a **basis** **if and only if**  $n = \dim(V)$ .

### Problem 2

**Lemma 10.10.** Suppose  $V$  is a vector space of dimension  $n$ . A system  $v_1, \dots, v_n$  in  $V$  is linearly independent **if and only if** it spans  $V$ .

### Problem 4

**Theorem 10.11.** Let  $V$  be a subspace of a vector space  $W$  with  $\dim(W) < \infty$ . Then,

- (a)  $V$  is finite dimensional.
- (b)  $\dim(V) \leq \dim(W)$ , and
- (c) if  $\dim(V) = \dim(W)$ , then  $V = W$ .

## 10.5 Homework 7

### Problem 2

**Lemma 10.12.** Let  $A$  be a matrix. The pivot columns in  $A$  form a basis for  $\text{Ran}(A)$ .

**Remark.** You can substitute  $A$  for  $A_{re}$  and this is still true (since  $A_{re}$  is just another matrix). Talking in terms of  $A_{re}$  is exactly problem 1 of homework 7. In a proof, it might be easier to prove something about  $A_{re}$  first, then talk about the original matrix.

### Problem 7

**Theorem 10.13. (General Rank's Theorem.)**

Suppose  $V$  and  $W$  are finite-dimensional vector spaces. Let  $A : V \rightarrow W$  be a linear map. Then,

$$\dim(V) = \dim(\text{null}(A)) + \dim(\text{Ran}(A)).$$

### Problem 8

**Lemma 10.14.** If  $A : X \rightarrow Y$  is a linear map and  $V$  is a subspace of  $X$ , then  $\dim(AV) \leq \dim(V)$ .

## 10.6 Homework 8

### Problem 1

**Lemma 10.15.** Let  $T : V \rightarrow W$  be a linear transformation. Let  $\mathcal{A} = \{v_1, \dots, v_n\}$  be a basis for  $V$ . Let  $\mathcal{B} = \{w_1, \dots, w_m\}$  be a basis for  $W$ . Finally, let  $[T]_{\mathcal{A}\mathcal{B}}$  be the matrix of linear transformation  $T$  with respect to  $\mathcal{A}$  and  $\mathcal{B}$ . If  $v \in V$ , then

$$[T]_{\mathcal{A}\mathcal{B}}[v]_{\mathcal{A}} = [T(v)]_{\mathcal{B}}$$

### Problem 2

**Lemma 10.16.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be bases for vector spaces  $V$ . Let  $[I]_{\mathcal{B}\mathcal{A}}$  be the change in coordinates matrix from basis  $\mathcal{A}$  to basis  $\mathcal{B}$ .

$$[I]_{\mathcal{A}\mathcal{B}} = ([I]_{\mathcal{B}\mathcal{A}})^{-1}$$

## 10.7 Homework 9

### Problem 1b

**Lemma 10.17.** *Suppose matrices  $A$  and  $B$  are similar matrices. Then,*

$$\det A = \det B.$$

### Problem 2

**Lemma 10.18.** *Let  $A \in \mathbb{M}_{n \times n}^{\mathbb{C}}$ . The determinant of  $A$  is the product of its eigenvalues. That is, if  $\{\lambda_1, \dots, \lambda_n\}$  are its eigenvalues, then*

$$\det A = \lambda_1 \lambda_2 \cdots \lambda_n.$$

There was a hint for this problem, which can be expressed as the following lemma.

**Lemma 10.19.** *Let  $A \in \mathbb{M}_{n \times n}^{\mathbb{C}}$ . Then,*

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda),$$

Where  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the eigenvalues of  $A$ , listed with multiplicity.

### Problem 3a and 3b

**Lemma 10.20.** *Let  $A : V \rightarrow V$  be a linear map. Let  $U$  be a subspace of  $V$ . If  $U \subset \text{null}(A)$ , then  $U$  is invariant under  $A$ .*

**Lemma 10.21.** *Let  $A : V \rightarrow V$  be a linear map. Let  $U$  be a subspace of  $V$ . If  $\text{Ran}(A) \subset U$ , then  $U$  is invariant under  $A$ .*

## 11 Selected Proofs

This section will include proofs that I found difficult, cool, or especially important. These are here for my own reference and easy access so I can learn more about these proofs and proof techniques.