# EEE575

ASSIGNMENT 6

## Q1

By studying the paper 'Modern Control Theory: A Historical Perspective', I gained a deep understanding of the fundamental shift from classical to modern control theory, and of the mathematical framework on which contemporary control system analysis relies.

The state-space representation marks a paradigm shift in control theory methodology by transcending the limitations of classical transfer function approaches. The core principle of this representation is that any finite dynamic system can be represented systematically as a set of first-order ordinary differential equations, typically expressed as $\dot{x} = Ax + Bu$ and $y = Cx + Du$. This mathematical framework reveals the complete internal dynamics through the state vector x, providing unparalleled analytical capabilities for complex system behaviour. Unlike classical approaches, which primarily handle single-input single-output (SISO) systems through frequency-domain analysis, state-space techniques are applicable to multiple-input multiple-output (MIMO) configurations, enabling direct access to the system's internal state.

The non-uniqueness of state-space implementations provides a valuable theoretical insight. A single transfer function can possess multiple canonical representations, each optimised for specific analytical or design objectives. Control regular forms strategically arrange state variables to ensure direct feedback to control inputs, guaranteeing controllability and facilitating controller synthesis. Conversely, observer regular forms serve as mathematical duals, ensuring full observability by feeding outputs back to each state variable, thereby enabling robust state estimation. The modal regular form diagonalises the system matrix, explicitly exposing system poles on the diagonal and facilitating modal decomposition analysis.

The theoretical foundations of controllability and observability represent cornerstone concepts in modern control design. Controllability determines, mathematically, whether arbitrary state transitions can be achieved via appropriate input trajectories. This is verified through the rank condition of the controllability matrix, C = [B, AB, $A^2$, B, ..., A^(n-1)B]. Meanwhile, observability addresses the dual problem of uniquely reconstructing initial states from finite-time input-output measurements, as evaluated via the observability matrix O. These properties are representation-dependent, meaning that different normative implementations of the same transfer function may exhibit distinct controllability and observability characteristics. This highlights the importance of selecting an appropriate system representation.

The shift from classical control theory in the frequency domain to state-space methods in the time domain marked a significant transition, paving the way for sophisticated design approaches such as optimal control, full-state feedback, observer-based compensation, and robust control synthesis. This mathematical framework has become indispensable for advanced applications in aerospace, robotics and modern industrial automation systems.

# Q2

$$G(s) = \frac{s+7}{s(s^2+2s+2)}$$

$$Numerator: s+7$$
$$Denominator: s^3+2s^2+2s$$

Control Canonical Form

$(a_1, a_2, a_3) = (2,2,0)$、$(b_1, b_2, b_3) = (0,1,7)$

$$A_0 = \begin{bmatrix} -2 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C_0 = \begin{bmatrix} 0 & 1 & 7 \end{bmatrix}, D = 0.$$

Model/Modal Canonical Form

$$Denominator: s^3+2s^2+2s$$

Pole

S=0 or s= $-1\pm j$

$$A_m = \begin{bmatrix} -2 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B_m = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$sI - A_m = \begin{bmatrix} s+2 & 2 & 0 \\ -1 & s & 0 \\ 0 & 0 & s \end{bmatrix}$$

$$(sI - A_{2\times2})^{-1} = \frac{1}{s^2+2s+2}\begin{bmatrix} s & -2 \\ 1 & s+2 \end{bmatrix}$$

$$v = (sI - A_m)^{-1}B_m = \begin{bmatrix} \frac{s}{s^2+2s+2} \\ \frac{1}{s^2+2s+2} \\ \frac{1}{s} \end{bmatrix}$$

$$C_m = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}$$

$$G(s) = C_m v = \frac{c_1 s}{s^2+2s+2} + \frac{c_2}{s^2+2s+2} + \frac{c_3}{s}$$

$$\begin{aligned} G(s) &= C_m v = \frac{c_1 s}{s^2+2s+2} + \frac{c_2}{s^2+2s+2} + \frac{c_3}{s} \\ &= \frac{(c_3+c_1)s^2 + (2c_3+c_2)s + 2c_3}{s(s^2+2s+2)} \end{aligned}$$

Match

$$G(s) = \frac{s+7}{s(s^2+2s+2)}$$

$$c_1 = -\frac{7}{2}, c_2 = -6, c_3 = \frac{7}{2}$$

$$A_m = \begin{bmatrix} -2 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_m = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, C_m = \begin{bmatrix} -\frac{7}{2} & -6 & \frac{7}{2} \end{bmatrix}, D = 0.$$

Observer Canonical Form

$$AB = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, D = 0$$

$(a_1, a_2, a_3) = (2,2,0)$、$(b_1, b_2, b_3) = (0,1,7)$:

$$AB = \begin{bmatrix} -2 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 7 \end{bmatrix}, C = [1 \quad 0 \quad 0], D = 0$$

# Q3

$$A = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, C = [1 \quad 0], D = [0]$$

$$sI - A = \begin{bmatrix} s+2 & -1 \\ 2 & s \end{bmatrix}$$

$$G(s) = C(sI - A)^{-1}B = \frac{s+3}{s^2 + 2s + 2}$$

OR

$$C = [B, AB] = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$$

$$C^{-1} = -\frac{1}{5}\begin{bmatrix} -2 & -1 \\ -3 & 1 \end{bmatrix}$$

$$t_2 = [0\ 1]C^{-1},\ t_1 = t2A$$

$$T^{-1} = [t1 \quad t2]$$

$$T^{-1} = \frac{1}{5}\begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$$

$$T^{-1}AT = A_c, T^{-1}B = B_c, CT = C_c.$$

$$A_c = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}, B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C_c = [1 \quad 3], D = 0$$

# Q4

## (A)

$$G_1 = \frac{1}{s+4} G_2 = \frac{1}{2s} H_1 = \frac{1}{s} H_2 = \frac{1}{s}$$

$$G_1 = \frac{1}{s+4}$$

$$\frac{X_4}{U - x_3} = \frac{1}{s+4}$$
$$\rightarrow (s+4)X_4 = U - x_3$$
$$\rightarrow sX_4 = -4x_4 + U - x_3$$
$$\rightarrow \dot{x}_4 = -4x_4 - x_3 + u$$

$$G_2 = \frac{1}{2s}$$

$$\frac{X^1}{x^4 - x^2} = \frac{1}{2s}$$
$$\rightarrow 2sX^1 = x^4 - x^2$$
$$\rightarrow \dot{X}_1 = \left(\frac{1}{2}\right)x^4 - \left(\frac{1}{2}\right)x^2$$

$$H_1 = \frac{1}{s}$$

$$\dot{X}_2 = X_1$$

$$H_2 = \frac{1}{s}$$

$$\dot{X}_3 = X_4$$

$$\begin{aligned}
\dot{X}_1 &= 0 \cdot X_1 - \left(\frac{1}{2}\right) \cdot X_2 + 0 \cdot X_3 + \left(\frac{1}{2}\right) \cdot X_4 + 0 \cdot u \\
\dot{X}_2 &= 1 \cdot X_1 + 0 \cdot X_2 + 0 \cdot X_3 + 0 \cdot X_4 + 0 \cdot u \\
\dot{X}_3 &= 0 \cdot X_1 + 0 \cdot X_2 + 0 \cdot X_3 + 1 \cdot X_4 + 0 \cdot u \\
\dot{X}_4 &= 0 \cdot X_1 + 0 \cdot X_2 - 1 \cdot X_3 - 4 \cdot X_4 + 1 \cdot u
\end{aligned}$$

$$X = [X_1 \quad X_2 \quad X_3 \quad X_4]^T$$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -4 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u, y = \underbrace{[1 \quad 0 \quad 0 \quad 0]}_{C} x, D = 0.$$

(B)

$$G(s) = \frac{Y}{U}$$

$$G(s) = \frac{Y}{U} = \frac{G_2}{1 + G_2 H_1} \frac{G_1}{1 + G_1 H_2} = \frac{s}{2s^2 + 1} \frac{s}{s^2 + 4s + 1}$$
$$= \frac{s^2}{2s^4 + 8s^3 + 3s^2 + 4s + 1} = \frac{0.5s^2}{s^4 + 4s^3 + 1.5s^2 + 2s + 0.5}$$

$$A_c = \begin{bmatrix} -4 & -\frac{3}{2} & -2 & -\frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, C_c = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \end{bmatrix}, D = 0.$$