EEE575

ASSIGNMENT 8

Q1

The paper *"Early Developments in Nonlinear Control"* by Derek P. Atherton provides a concise historical review of the evolution of nonlinear control theory and its foundational methods. The author emphasizes that nonlinear control emerged from the limitations of classical linear approaches, which dominated industrial practice until the mid-20th century. Real physical systems—such as aircraft, servomechanisms, and electrical machines—exhibited inherent nonlinearities including saturation, dead zones, friction, and backlash. These effects could not be accurately captured by linear models, motivating researchers to explore new analytical frameworks.

The paper identifies three major milestones in the early development of nonlinear control.

First, the **phase-plane method** introduced a geometric tool for visualizing the trajectories of second-order systems, allowing engineers to assess equilibrium stability, limit cycles, and qualitative behavior without solving differential equations.

Second, **Lyapunov stability theory** established a rigorous mathematical basis for analyzing nonlinear dynamics. The indirect method uses linearization and eigenvalue analysis to infer local stability, while the direct method employs an energy-like Lyapunov function $V(x)$ to prove asymptotic stability when $\dot{V}(x) < 0$.

Third, the **describing-function approach** extended frequency-domain analysis to systems with hard nonlinearities such as relays or saturation, enabling approximate prediction of self-excited oscillations using graphical Nyquist-type criteria.

Atherton concludes that these classical techniques collectively transformed control engineering from an empirical art into a theoretically grounded discipline capable of addressing complex nonlinear phenomena. Their enduring influence persists in modern methodologies such as adaptive control, feedback linearization, and sliding-mode control. Understanding these early developments highlights that nonlinear control is not merely an extension of linear theory but represents a fundamental shift in modeling, analysis, and design philosophy for real-world dynamic systems.

Q2

$$\dot{x} = -4x + x^3$$

At the equilibrium point, the system no longer changes, that is, $\dot{x} = 0$:

$$\dot{x} = -4x + x^3 = 0$$

$$-4x + x^3 = 0$$

$$x(x-2)(x+2) = 0$$

$$x = 0, \pm 2$$

So the three singular points are: $x_1 = -2$, $x_2 = 0$, $x_3 = 2$

$$\frac{d\dot{x}}{dx} = -4 + 3x^2$$

Equilibrium	Derivative	Stability
(x=0)	(-4<0)	Stable (node)
(x=+2)	(-4+12=8>0)	Unstable
(x=-2)	(-4+12=8>0)	Unstable

Thus the origin is asymptotically stable, and the outer two points are unstable.
The phase portrait is a single trajectory that starts near −2 or 2 and flows toward 0.

Q3

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \sin\ x_1 - a(1+x_2)^5 + a \end{bmatrix}$$

a>0,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_1 = 0$$

$$\dot{x}_2 = \sin\ x_1 - a(1 + x_2)^5 + a$$

$$\dot{x}_2 = \sin\ 2\pi - a(1 + 0)^5 + a = 0$$

$$[2\pi, 0]^T\ is\ indeed\ the\ equilibrium\ point\ of\ the\ system.$$

$$\begin{aligned} f(x_1, x_2) &= x_2 \\ f(x_1, x_2) &= \sin\ x_1 + a(1 + a)^5 + a \end{aligned}$$

$$\begin{aligned} \frac{\partial f_1}{\partial x_1} &= \frac{\partial}{\partial x_1}(x_2) = 0 \\ \frac{\partial f_1}{\partial x_2} &= \frac{\partial}{\partial x_2}(x_2) = 1 \end{aligned}$$

$$\begin{aligned} \frac{\partial f_2}{\partial x_1} &= \frac{\partial}{\partial x_1}[\sin\ x_1 - a(1 + x_2)^5 + a] \\ &= \cos x_1 - 0 + 0 \\ &= \cos x_1 \\ \frac{\partial f_2}{\partial x_2} &= \frac{\partial}{\partial x_2}[\sin\ x_1 - a(1 + x_2)^5 + a] \\ &= 0 - a \cdot 5(1 + x_2)^4 \cdot 1 + 0 \\ &= -5a(1 + x_2)^4 \end{aligned}$$

$x_1 = 2\pi$, $x_2 = 0$

$$J(2\pi, 0) = \begin{bmatrix} 0 & 1 \\ \cos(2\pi) & -5a(1 + 0)^4 \end{bmatrix}$$

$$J(2\pi, 0) = \begin{bmatrix} 0 & 1 \\ 1 & -5a \end{bmatrix}$$

$$det(\lambda I - J) = 0$$

$$\lambda I - J = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & -5a \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ -1 & \lambda + 5a \end{bmatrix}$$

$$det(\lambda I - J) = \lambda(\lambda + 5a) - (-1)(-1) = \lambda^2 + 5a\lambda - 1$$

$$\lambda^2 + 5a\lambda - 1 = 0$$

$$\lambda = \frac{-5a \pm \sqrt{25a^2 + 4}}{2}$$

$$\Delta = 25a^2 + 4$$

a>0

$$\Delta > 0$$

$$\begin{aligned} \lambda_1 &= \frac{-5a + \sqrt{25a^2 + 4}}{2} > 0 \\ \lambda_2 &= \frac{-5a - \sqrt{25a^2 + 4}}{2} < 0 \end{aligned}$$

5.1 Lyapunov's First Method Criteria

For a linearized system

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ \sin\ x_1 - a(1 + x_2)^5 + a \end{bmatrix}$$

Asymptotically stable: all eigenvalues have real parts < 0

Unstable: at least one eigenvalue has real part > 0

Critically stable: the real parts of the eigenvalues are ⩽ 0, and the eigenvalues on the imaginary axis are singletful

Conclusion

Since $\lambda_1 > 0$ (positive real eigenvalues), according to Lyapunov's first method:

The equilibrium point [2π, 0] is unstable for all cases a > 0.

Q4

$$G(s) = \frac{K}{(s + 4)(s - 1)}$$

$$T(s) = \frac{G(S)}{1 + G(S)} = \frac{K}{K + (s + 4)(s - 1)}$$

$$s^2 + 3s + (K - 4) = 0$$

$$\ddot{x} + 3\dot{x} + (K-4)x = 0$$

$$A = \begin{bmatrix} 0 & 1 \\ -(K-4) & -3 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -K+4 & -3 \end{bmatrix}$$

$$A^T P + PA = -Q = -I$$

Let $P = \begin{bmatrix} m & b \\ b & n \end{bmatrix}$

$$A^T P \quad = \begin{bmatrix} 0 & -(K-4) \\ 1 & -3 \end{bmatrix} \begin{bmatrix} m & b \\ b & n \end{bmatrix} = \begin{bmatrix} -(K-4)b & -(K-4)n \\ m-3b & b-3n \end{bmatrix}$$

$$PA \quad = \begin{bmatrix} m & b \\ b & n \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -(K-4) & -3 \end{bmatrix} = \begin{bmatrix} -(K-4)b & m-3b \\ -(K-4)n & b-3n \end{bmatrix}$$

$$A^T P + PA \quad = \begin{bmatrix} -2(K-4)b & m-3b-(K-4)n \\ m-3b-(K-4)n & 2b-6n \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$-2(K-4)b = -1 \to b = \frac{1}{2(K-4)}$$

$$m - 3b - (K-4)n = 0$$

$$2b - 6n = -1 \to n = \frac{2b+1}{6} = \frac{\frac{1}{K-4}+1}{6} = \frac{K-3}{6(K-4)}$$

$$m - 3b - (K-4)n = 0$$

$$b = \frac{9+(k-3)(k-4)}{6(k-4)} = \frac{k^2-7k+21}{6(k-4)}$$

So $b > 0$

requires $K^2 - 7K + 21 > 0$ and $K > 4$

$$\det(P) > 0$$

$$mn - b^2 > 0$$

$$\begin{aligned} \det(P) &= mn - b^2 \\ &= \frac{K^2-7K+21}{6(K-4)} \cdot \frac{K-3}{6(K-4)} - \left[\frac{1}{2(K-4)}\right]^2 \\ &= \frac{(K^2-7K+21)(K-3)}{36(K-4)^2} - \frac{1}{4(K-4)^2} \\ &= \frac{(K^2-7K+21)(K-3)-9}{36(K-4)^2} \\ &= \frac{K^2-6K+18}{36(K-4)} \end{aligned}$$

$$\det(P) = \frac{K^2 - 6K + 18}{36(K-4)}$$

$$\frac{K^2 - 6K + 18}{36(K-4)} > 0$$

$$K > 4$$

$$s^2 + 3s + (K-4) = 0$$

Row	s^2	s^0
s^2	1	K-4
s^1	3	0
s^0	K-4	

$$K - 4 > 0$$

$$K > 4$$

The Lyapunov equation has a positive solution P when K > 4, indicating that the system is asymptotically stable. This is consistent with Routh's stability criterion: the stability condition is

$$K > 4$$

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