EEE575

ASSIGNMENT 7

Q1

Arthur E. Bryson's paper “Optimal Control: 1950–1985” provides a comprehensive review of the transformative development of optimal control theory during its formative years. This period witnessed the evolution of control theory from abstract mathematical concepts into practical engineering methods, fundamentally establishing the design of modern control systems. Bryson traces the theoretical foundations laid by pioneering researchers whose contributions established optimal control as a rigorous discipline with far-reaching implications.

The paper primarily discusses two groundbreaking theoretical frameworks emerging in the 1950s: Pontryagin's Maximum Principle and Bellman's Dynamic Programming. Developed in 1956, Pontryagin's Maximum Principle established necessary conditions for optimal control by constructing a Hamiltonian and laying the groundwork for “bang-bang” control and singular control strategies. Concurrently, Bellman's dynamic programming approach introduced the Hamilton-Jacobi-Bellman equations and optimality principles, offering an alternative perspective through recursive solution methods. These complementary frameworks address a fundamental problem: how to systematically design controllers that optimize performance criteria while respecting system constraints.

A central theme of Bryson's work is the development and refinement of the Linear Quadratic Regulator (LQR) method, originating from Kalman's contributions in the 1960s. The LQR formulation elegantly addresses control design by minimizing the quadratic cost function $J = \int[x^T Q x + u^T R u]dt$, subject to the linear system dynamics $\dot{x} = Ax + Bu$. This approach provides engineers with a systematic framework to balance state regulation (weighted by Q) and control force (weighted by R), thereby resolving the ad hoc nature of prior controller parameter selection. Through the analytical solution of the algebraic Riccati equation, coupled with guaranteed stability of controllable systems, LQR has become a cornerstone method in modern control theory.

The paper emphasizes that advances in numerical methods and computational tools during the 1960s and 1970s facilitated the transition from theoretical mathematics to practical applications. Optimal control found direct application in aerospace systems, including missile guidance and aircraft autopilot design, where performance optimization is paramount. The method subsequently expanded to process control in chemical plants, robotics, and manufacturing systems. A key limitation noted by Bryson is that while the theoretical framework is elegant, implementation poses computational challenges, requiring substantial expertise to correctly construct weighting matrices and interpret results.

Furthermore, Bryson's historical perspective reveals that developments from 1950 to 1985 laid the foundation for contemporary advanced control techniques. The separation principle—combining linear quadratic regulators (LQR) with Kalman filters for optimal state estimation—demonstrated that control law design and observer design could proceed independently. This insight paved the way for robust control methods, H∞ optimization, and model predictive control (MPC), which dominate modern industrial applications.

This paper emphasizes that while optimal control theory achieved significant theoretical maturity during this period, challenges and issues in practical implementation—such as computational complexity, appropriate problem formulation, and the need for precise system models—remain factors requiring consideration. The enduring influence of the Linear Quadratic Controller approach in contemporary control education and practice validates Bryson's assertion that the period from 1950 to 1985 represents the golden age of control theory, whose principles continue to guide modern controller design.

Q2

(a)

$$G(S) = \frac{s}{s^2 + 4}$$

$$\frac{Y(S)}{U(S)} = \frac{s}{s^2 + 4}$$

$$s^2 Y(s) + 4Y(s) = sU(S)$$

$$\ddot{y} + 4y = \dot{u}$$

$$\ddot{y} + 4y = u'$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -4x_1 + u$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix},, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

(b)

$$s = -2 \pm 2j$$

$$\alpha_c(s) = (s + 2 - 2j)(s + 2 + 2j) = (s + 2)^2 + 4 = s^2 + 4s + 8$$

$$u = -\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 - K_1 & -K_2 \end{bmatrix}$$

$$\det\big(sI - (\mathbf{A} - \mathbf{BK})\big) = \det \begin{bmatrix} s & -1 \\ 4 + K_1 & s + K_2 \end{bmatrix}$$
$$= s(s + K_2) - (-1)(4 + K_1)$$
$$= s^2 + K_2 s + 4 + K_1$$

$$K = \begin{bmatrix} 4 & 4 \end{bmatrix}$$

$$u = -4x_1 - 4x_2$$

Q3

$$\ddot{\theta} = \theta + u$$

$$\ddot{p} = -0.5\theta - u$$

$$x = \begin{bmatrix} \theta & \dot{\theta} & p & \dot{p} \end{bmatrix}$$

$$
\begin{aligned}
\dot{x}_1 &= \dot{\theta} = x_2 \\
\dot{x}_2 &= \ddot{\theta} = \theta + u = x_1 + u \\
\dot{x}_3 &= \dot{p} = x_4 \\
\dot{x}_4 &= \ddot{p} = -0.5\theta - u = -0.5x_1 - u
\end{aligned}
$$

$$
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.5 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, C[0 \quad 0 \quad 1 \quad 0], D = 0
$$

(A)

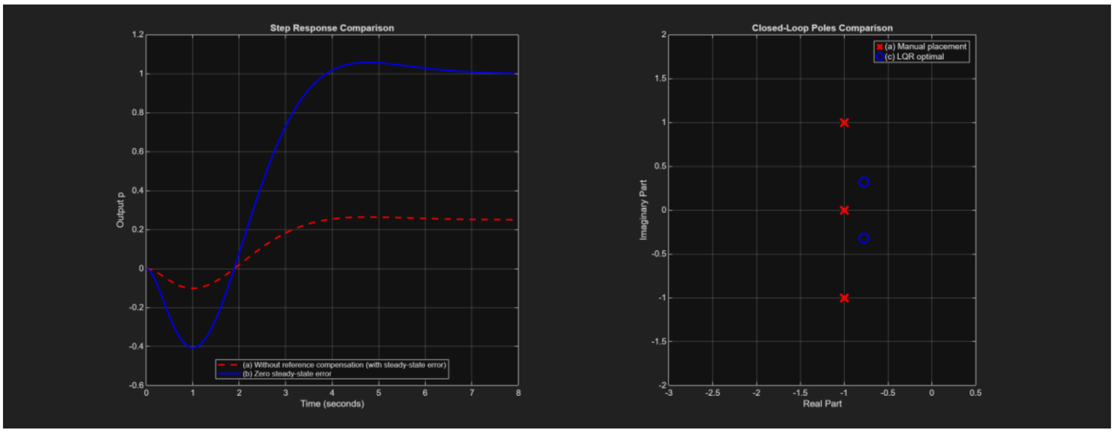
$$s = -1, -1, -1 \pm j$$

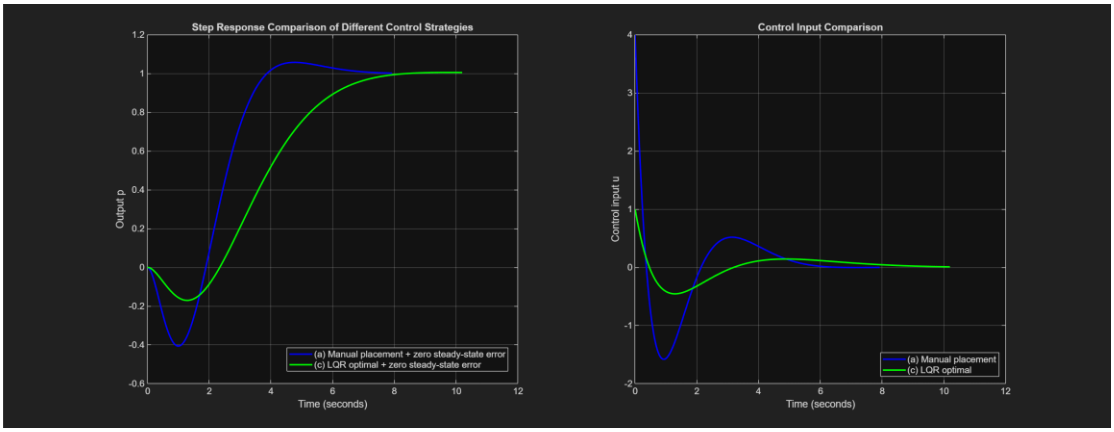
$$\alpha_c(s) = (s+1)^2 \cdot [(s+1)^2 + 1]$$

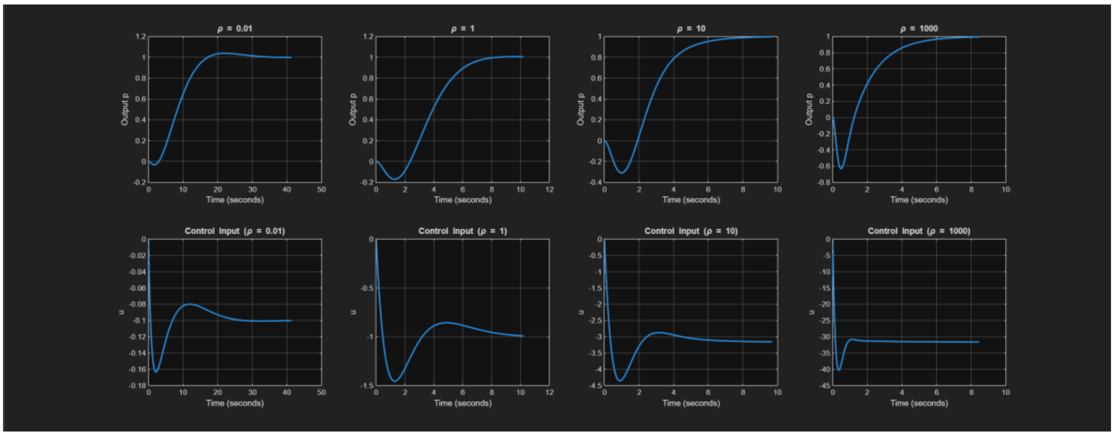
$$\alpha_c(s) = s^4 + 4s^3 + 7s^2 + 6s + 2$$

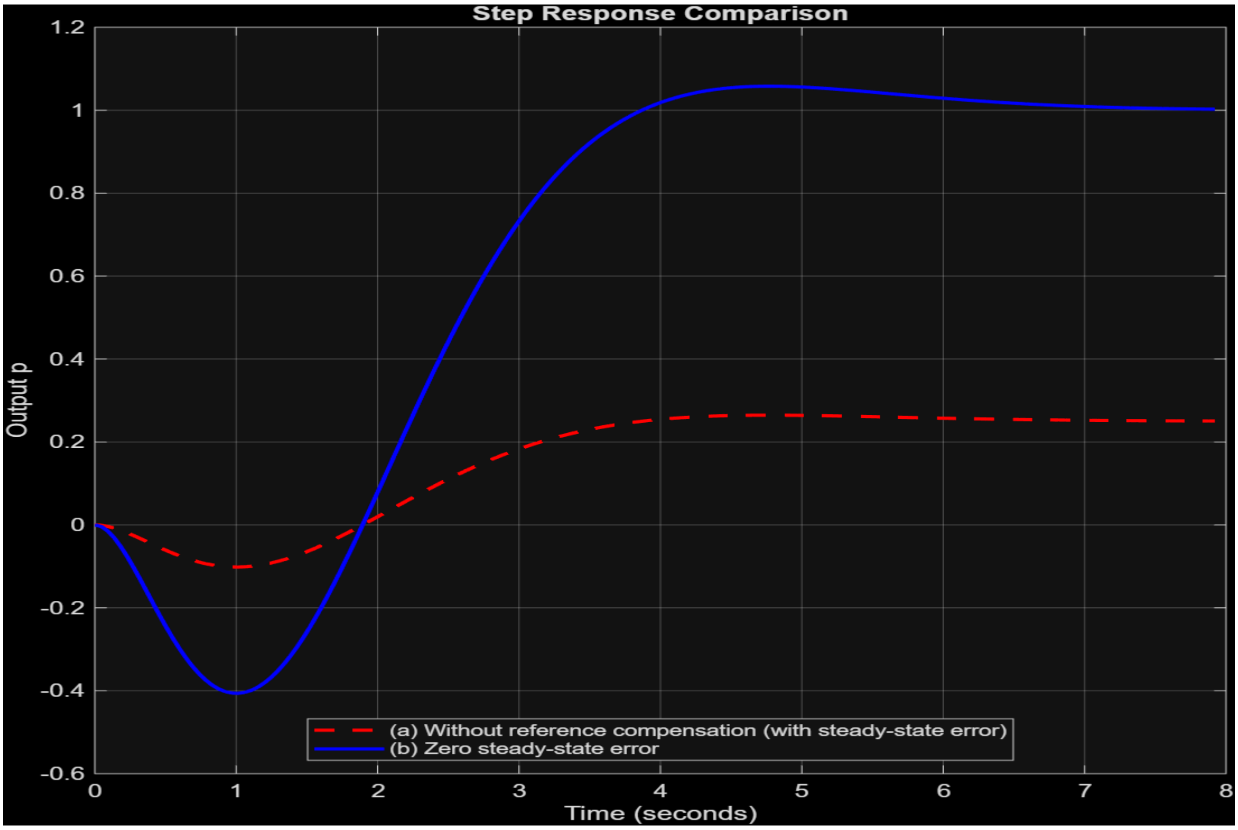
```
 A2
 (a) Pole Placement (Ackermann Formula)
Desired characteristic polynomial coefficients:
     1     4     7     6     2

Rank of controllability matrix:
     4

Feedback gain K =
    12    16     4    12

Actual closed-loop poles:
  -1.0000 + 1.0000i
  -1.0000 - 1.0000i
  -1.0000 + 0.0000i
  -1.0000 + 0.0000i

 (b) Zero Steady-State Error
N_x =
     0     0     1     0

N_u =
     0

N = N_u + K*N_x =
```

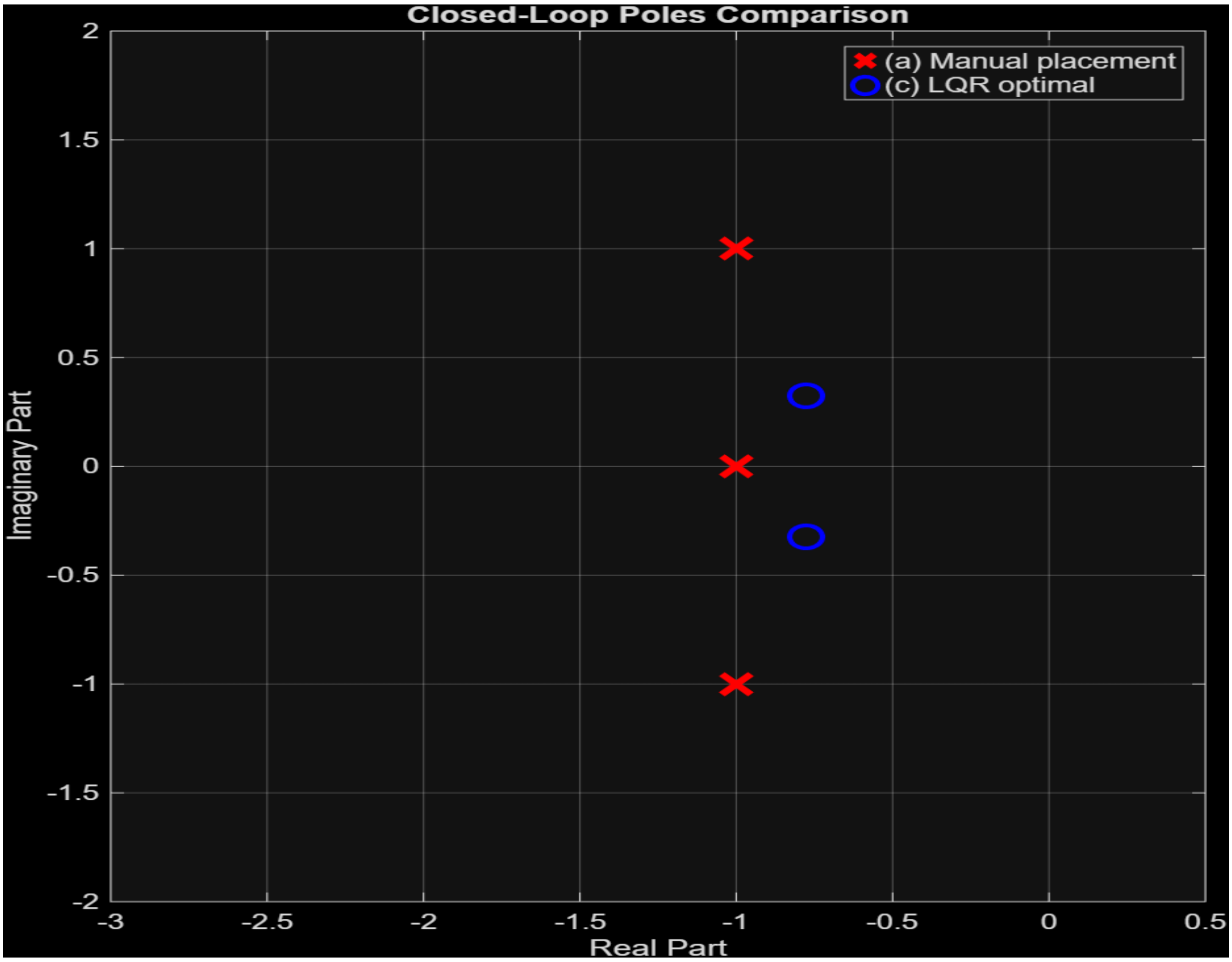
```
     4

(a) Steady-state value: 0.2506 (error: 0.7494)
(b) Steady-state value: 1.0024 (error: -0.0024)
 (c) LQR Optimal Control
Q matrix:
     0     0     0     0
     0     0     0     0
     0     0     1     0
     0     0     0     0

LQR feedback gain K =
    5.8284    7.5023    1.0000    4.3947

LQR closed-loop poles:
  -0.7769 + 0.3218i
  -0.7769 - 0.3218i
  -0.7769 + 0.3218i
  -0.7769 - 0.3218i


n =

     4

 Performance Comparison

Performance Metrics Comparison:
              Rise Time    Overshoot%    Settling Time    Steady-State 
Error
Manual:        1.401s       5.78%        6.353s           0.0024
LQR Optimal:   3.394s       0.60%        7.395s           0.0057

Control Gain Comparison:
Manual placement K = [12.0000, 16.0000, 4.0000, 12.0000]
LQR optimal      K = [5.8284, 7.5023, 1.0000, 4.3947]
LQR Comparison with Different rho Values
rho = 0.01, K = [2.7967, 3.1701, 0.1000, 0.8477]
rho = 1, K = [5.8284, 7.5023, 1.0000, 4.3947]
rho = 10, K = [11.6087, 15.6942, 3.1623, 11.5841]
rho = 1000, K = [76.0767, 106.8793, 31.6228, 97.4502]
```

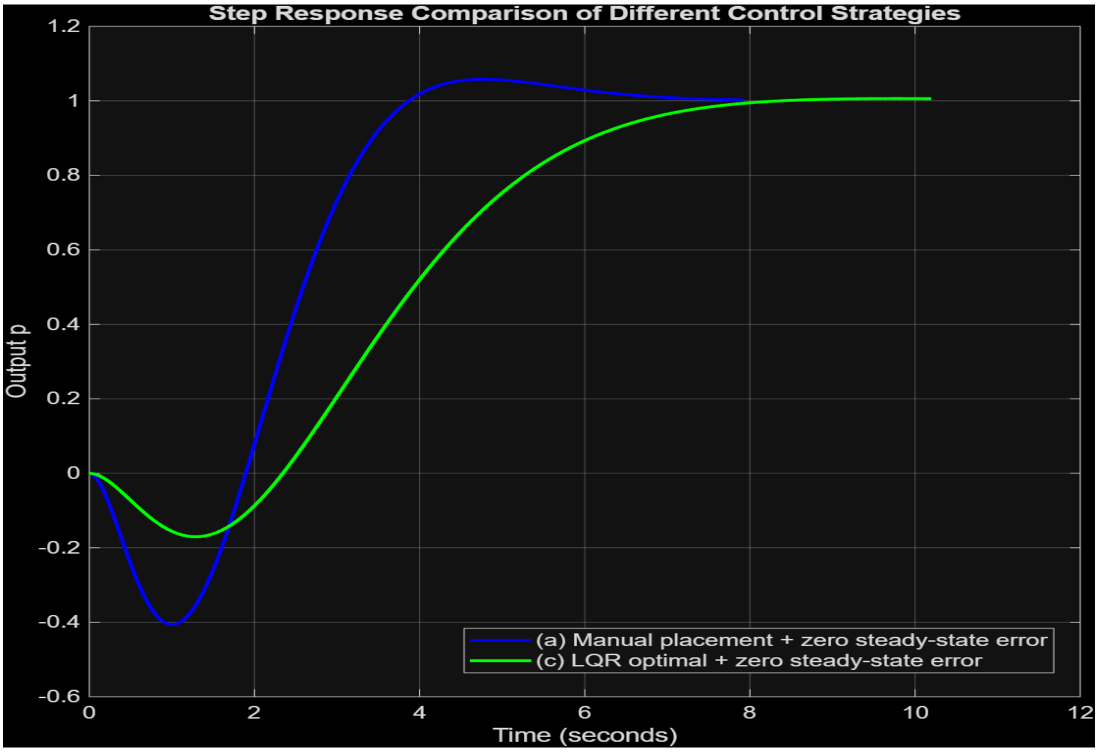

Step Response Comparison
Output p
Time (seconds)
(a) Without reference compensation (with steady-state error)
(b) Zero steady-state error
Closed-Loop Poles Comparison
Imaginary Part
Real Part
(a) Manual placement
(c) LQR optimal

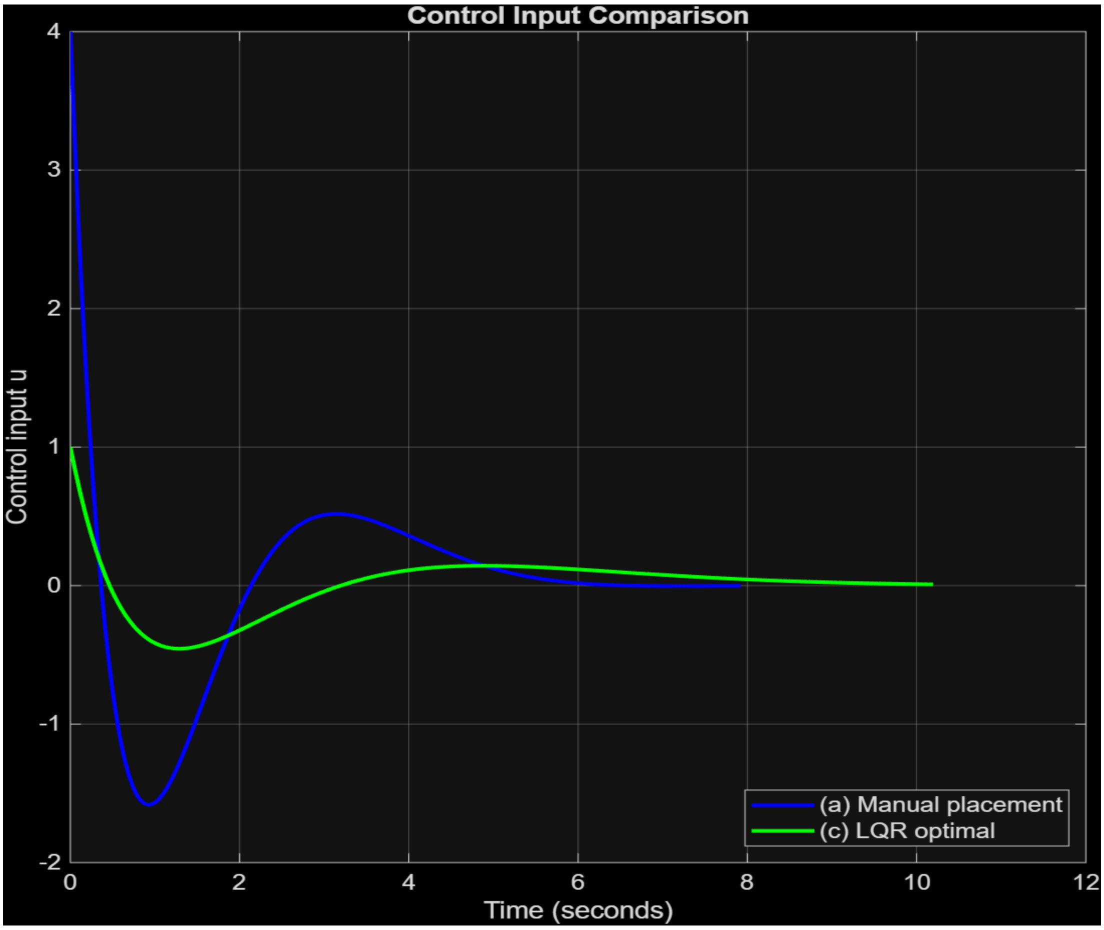

Step Response Comparison of Different Control Strategies
Output p
Time (seconds)
(a) Manual placement + zero steady-state error
(c) LQR optimal + zero steady-state error
Control Input Comparison
Control input u
Time (seconds)
(a) Manual placement
(c) LQR optimal

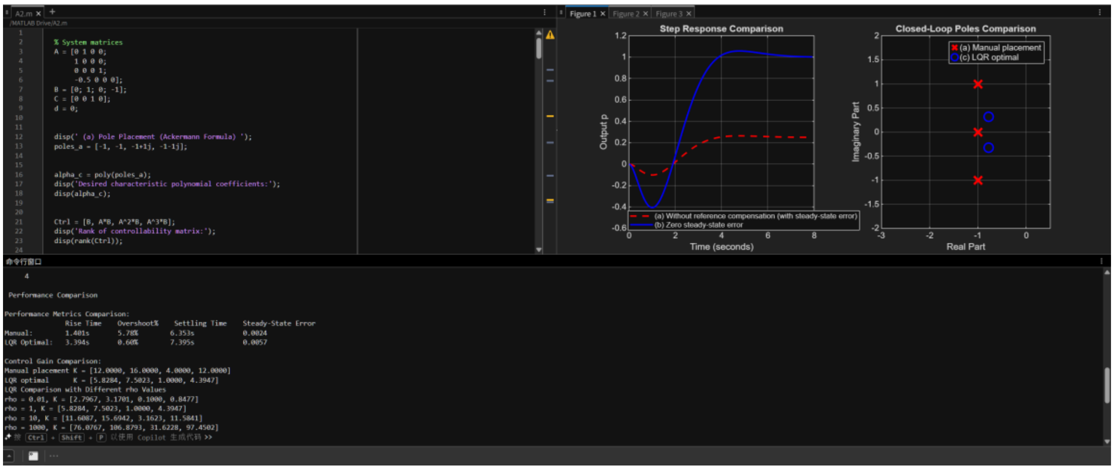

ρ = 0.01
ρ = 1
ρ = 10
ρ = 1000
Output p
Time (seconds)
Control Input (ρ = 0.01)
Control Input (ρ = 1)
Control Input (ρ = 10)
Control Input (ρ = 1000)
u
Time (seconds)

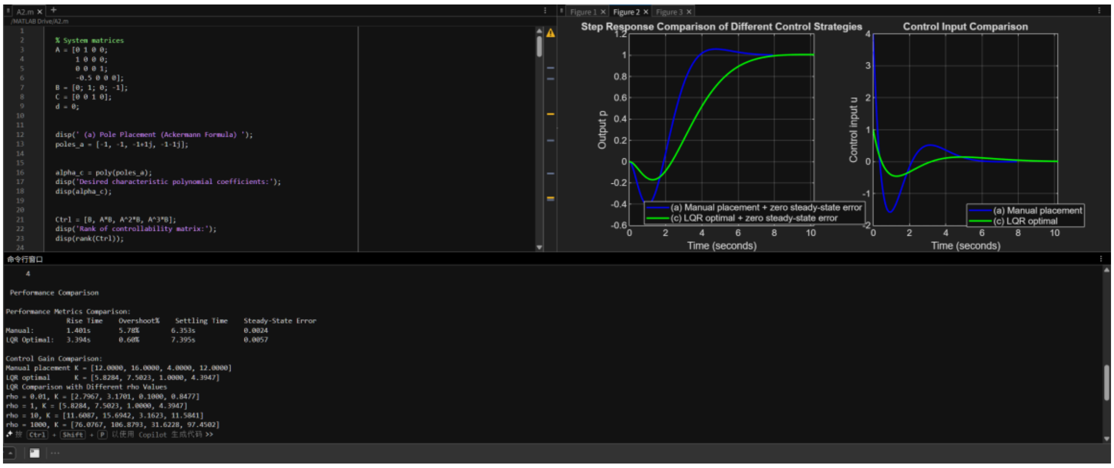

Step Response Comparison
Output p
Time (seconds)
(a) Without reference compensation (with steady-state error)
(b) Zero steady-state error

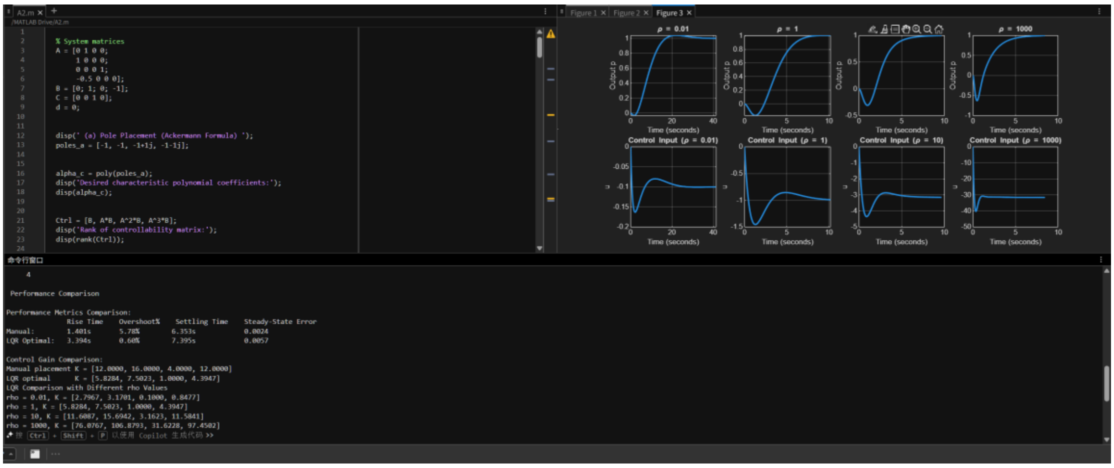

Closed-Loop Poles Comparison
Imaginary Part
Real Part
(a) Manual placement
(c) LQR optimal


Step Response Comparison of Different Control Strategies
Output p
Time (seconds)
(a) Manual placement + zero steady-state error
(c) LQR optimal + zero steady-state error


Control Input Comparison
Control input u
Time (seconds)
(a) Manual placement
(c) LQR optimal







CODE

```
% System matrices
A = [0 1 0 0;
     1 0 0 0;
     0 0 0 1;
```

```
     -0.5 0 0 0];
B = [0; 1; 0; -1];
C = [0 0 1 0];
d = 0;


disp(' (a) Pole Placement (Ackermann Formula) ');
poles_a = [-1, -1, -1+1j, -1-1j];


alpha_c = poly(poles_a);
disp('Desired characteristic polynomial coefficients:');
disp(alpha_c);


Ctrl = [B, A*B, A^2*B, A^3*B];
disp('Rank of controllability matrix:');
disp(rank(Ctrl));


n = length(A);
alpha_A = alpha_c(n+1) * eye(n);
for i = 1:n
    alpha_A = alpha_A + alpha_c(n+1-i) * A^i;
end


e_n = [0; 0; 0; 1];
K_a = e_n' * inv(Ctrl) * alpha_A;

disp('Feedback gain K =');
disp(K_a);


poles_closed = eig(A - B*K_a);
disp('Actual closed-loop poles:');
disp(poles_closed);


disp(' (b) Zero Steady-State Error ');
N_bar = inv([A B; C d]) * [zeros(4,1); 1];
N_x = N_bar(1:4);
N_u = N_bar(5);
N = N_u + K_a * N_x;

disp('N_x =');
```

```
disp(N_x');
disp('N_u =');
disp(N_u);
disp('N = N_u + K*N_x =');
disp(N);


sys_a = ss(A - B*K_a, B, C, d);
sys_b = ss(A - B*K_a, B*N, C, d);


[Y_a, T_a, X_a] = step(sys_a);
[Y_b, T_b, X_b] = step(sys_b);


figure('Position', [100 100 1200 500]);

subplot(1,2,1);
plot(T_a, Y_a, 'r--', 'LineWidth', 2);
hold on;
plot(T_b, Y_b, 'b-', 'LineWidth', 2);
legend('(a) Without reference compensation (with steady-state
error)', ...
       '(b) Zero steady-state error', 'Location', 'best');
title('Step Response Comparison');
xlabel('Time (seconds)');
ylabel('Output p');
grid on;
set(gca, 'FontSize', 11);


fprintf('(a) Steady-state value: %.4f (error: %.4f)\n', Y_a(end), 1-
Y_a(end));
fprintf('(b) Steady-state value: %.4f (error: %.4f)\n', Y_b(end), 1-
Y_b(end));


disp(' (c) LQR Optimal Control');
rho = 1;
Q = rho * C' * C;
R = 1;

disp('Q matrix:');
disp(Q);
```

```
K_c = lqr(A, B, Q, R);

disp('LQR feedback gain K =');
disp(K_c);

poles_c = eig(A - B*K_c);
disp('LQR closed-loop poles:');
disp(poles_c);


subplot(1,2,2);
plot(real(poles_a), imag(poles_a), 'rx', 'MarkerSize', 15, 'LineWidth',
3);
hold on;
plot(real(poles_c), imag(poles_c), 'bo', 'MarkerSize', 12, 'LineWidth',
2);
grid on;
xlabel('Real Part');
ylabel('Imaginary Part');
legend('(a) Manual placement', '(c) LQR optimal', 'FontSize', 11);
title('Closed-Loop Poles Comparison');
axis equal;
xlim([-3 0.5]);
ylim([-2 2]);
set(gca, 'FontSize', 11);


N_bar_c = inv([A B; C d]) * [zeros(4,1); 1];
N_x_c = N_bar_c(1:4);
N_u_c = N_bar_c(5);
N_c = N_u_c + K_c * N_x_c;

sys_c = ss(A - B*K_c, B*N_c, C, d);


[Y_c, T_c, X_c] = step(sys_c);


figure('Position', [100 100 1200 500]);

subplot(1,2,1);
plot(T_b, Y_b, 'b-', 'LineWidth', 2);
hold on;
plot(T_c, Y_c, 'g-', 'LineWidth', 2);
legend('(a) Manual placement + zero steady-state error', ...
```

```
       '(c) LQR optimal + zero steady-state error', 'FontSize', 12);
title('Step Response Comparison of Different Control Strategies');
xlabel('Time (seconds)');
ylabel('Output p');
grid on;
set(gca, 'FontSize', 12);

subplot(1,2,2);
u_b = -K_a * X_b' + N * ones(1, length(T_b));
u_c = -K_c * X_c' + N_c * ones(1, length(T_c));

plot(T_b, u_b, 'b-', 'LineWidth', 2);
hold on;
plot(T_c, u_c, 'g-', 'LineWidth', 2);
legend('(a) Manual placement', '(c) LQR optimal', 'FontSize', 12);
title('Control Input Comparison');
xlabel('Time (seconds)');
ylabel('Control input u');
grid on;
set(gca, 'FontSize', 12);

n
disp(' Performance Comparison ');


info_b = stepinfo(sys_b);
info_c = stepinfo(sys_c);

fprintf('\nPerformance Metrics Comparison:\n');
fprintf('              Rise Time    Overshoot%%    Settling
Time    Steady-State Error\n');
fprintf('Manual:       %.3fs      %.2f%%       %.3fs           %.4f\n',
...
    info_b.RiseTime, info_b.Overshoot, info_b.SettlingTime, abs(1-
Y_b(end)));
fprintf('LQR
Optimal:   %.3fs      %.2f%%       %.3fs           %.4f\n', ...
    info_c.RiseTime, info_c.Overshoot, info_c.SettlingTime, abs(1-
Y_c(end)));


fprintf('\nControl Gain Comparison:\n');
fprintf('Manual placement K = [%.4f, %.4f, %.4f, %.4f]\n', K_a);
fprintf('LQR optimal      K = [%.4f, %.4f, %.4f, %.4f]\n', K_c);
```

```
disp('LQR Comparison with Different rho Values');

rho_values = [0.01, 1, 10, 1000];
figure('Position', [100 100 1200 900]);

for idx = 1:4
    rho = rho_values(idx);
    Q = rho * C' * C;
    K_lqr = lqr(A, B, Q, R);


    N_bar_temp = inv([A B; C d]) * [zeros(4,1); 1];
    N_temp = N_bar_temp(5) + K_lqr * N_bar_temp(1:4);

    sys_lqr = ss(A - B*K_lqr, B*N_temp, C, d);
    [Y_lqr, T_lqr, X_lqr] = step(sys_lqr);
    u_lqr = -K_lqr * X_lqr';


    subplot(2,4,idx);
    plot(T_lqr, Y_lqr, 'LineWidth', 2);
    title(sprintf('\\rho = %g', rho));
    xlabel('Time (seconds)');
    ylabel('Output p');
    grid on;


    subplot(2,4,idx+4);
    plot(T_lqr, u_lqr, 'LineWidth', 2);
    title(sprintf('Control Input (\\rho = %g)', rho));
    xlabel('Time (seconds)');
    ylabel('u');
    grid on;

    fprintf('rho = %g, K = [%.4f, %.4f, %.4f, %.4f]\n', rho, K_lqr);
end
```