Research Statement

Zachary Sarver

According to my mother, my answer to, "what do you want to be when you grow up?" was, from the time I was old enough to comprehend the question, "a scientist." I watched *Bill Nye the Science Guy* on TV every single chance I got, and devoured every single science book I could get my hands on. In a very real sense I have always been interested in discovery, although my interests turned to computer science and mathematics only as a young adult.

As an undergraduate, I started out as a computer science major, only picking up mathematics as a second major in my sophomore year. I was fascinated with data structures, particularly trees and graphs, which can have remarkable properties when used wisely. As I progressed in my mathematics classes, I discovered that mathematical structures are also fascinating, and eventually developed an interest in algebraic structures in particular.

1 Undergraduate research

In my undergraduate studies I primarily concentrated on computer science research. I worked as an undergraduate research assistant at Jacksonville State University in fast encoding of multi-reference frame video encoding under Dr. Monica Trifas and Dr. Ming Yang. Briefly, modern video streaming is done by only occasionally transmitting a full video frame, called a reference frame. The rest of the frames are divided into small square blocks and estimated motion vectors from these small blocks to corresponding blocks in the nearest reference frame are transmitted. A multi-reference frame video encoder uses multiple reference frames to estimate these motion vectors, which results in higher-quality video, but encoding that is potentially orders of magnitude slower.

In the last summer at Jacksonville State University, I traveled to Utah State University to work on a summer research project with Dr. Xiaojun Qi in computer vision, specifically image classification. We used a support vector machine to train the computer to classify images based on content, e.g. contains flags vs. contains birds.

2 Lie theory

My dissertation work and published results in the area of Lie theory, particularly in semisimple Lie groups.

A *Lie group* is a group that also has the structure of a smooth manifold such that both multiplication and inversion of group elements are smooth maps. A smooth manifold itself is a topological space that is locally homeomorphic to a Euclidean space $\mathbb{R}^n$, and for any two local homeomorphisms φ_1 and φ_2 whose domains intersect, $\varphi_1 \circ \varphi_2^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ must have partial derivatives of all orders. In my research I have restricted my attention to finite-dimensional Lie groups.

To every Lie group G there is an associated *Lie algebra* $\mathfrak{g}$, isomorphic as a vector space to the tangent space at the identity of G, and having an additional nonassociative algebra structure given by the Lie bracket. Every Lie algebra is isomorphic to a matrix algebra such that $[X, Y] = XY - YX$ where juxtaposition is ordinary matrix multiplication.

A Lie algebra is *simple* if it is nonabelian and its only ideals are itself and the trivial Lie algebra. A Lie algebra is *semisimple* if it is the direct sum of simple Lie algebras. A Lie group is semisimple if its associated Lie algebra is semisimple.

A semisimple Lie group admits a preorder, called Kostant's preorder, and inequalities with Kostant's preorder are primarily where my interest lies. Kostant's preorder has a very lengthy and technical definition involving the Iwasawa decomposition, complete multiplicative Jordan decomposition, and vector log-majorization, and I will omit the definition here. It is, however, worth note that many inequalities involving matrices and vector majorization have been generalized to semisimple Lie groups, and many more seem to be good candidates for doing so.

Inverse limits of Lie groups have been studied, and only recently has a categorial description of generalized inverse limits been given. I am currently very interested in studying this topic.

3 Matrix theory

My current research is in matrix theory. For many mathematicians, a matrix is simply a convenient representation of a linear map. However, various spaces of matrices have rich structure as groups, algebras, topologies, and manifolds.

A complex $n \times n$ matrix is called *positive definite* if it is hermitian and all its eigenvalues are greater than 0. The set $\mathbb{P}_n$ of $n \times n$ positive definite matrices does not form a group, as the product of two positive definite matrices may not be hermitian. $\mathbb{P}_n$ does, however, admit the structure of a Riemannian manifold, and for $A, B \in \mathbb{P}_n$, and the geometric mean $A\sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ is in fact the midpoint of the geodesic connecting A and B. Note that if $n = 1$, i.e. A and B are positive numbers, then $A\sharp B$ is the ordinary geometric mean.

Audenaert recently proved in [Aud15] that for k-many commuting pairs of positive definite matrices A_i and B_i and for any unitarily invariant norm $\|\cdot\|$,

$$\left\|\sum_{i=1}^{k} A_i B_i\right\| \leq \left\|\left(\sum_{i=1}^{k} A_i^{1/2} B_i^{1/2}\right)^2\right\| \leq \left\|\left(\sum_{i=1}^{k} A_i\right)\left(\sum_{i=1}^{k} B_i\right)\right\|$$

It has been conjectured that a similar inequality holds in the noncommuting case when matrix multiplication is replaced with the geometric mean.

4 Topological groups

Much like a Lie group is a group that is a differentiable manifold, a topological group is a group with a simpler structure: it must also be a topological space with multiplication and inversion continuous. A topological space, and hence topological group, is locally compact if every point has a compact neighborhood.

In [Aco96], it is proved that every locally compact topological group has something resembling a differential structure, in that a derivative may be defined, four of them, in fact, that behaves remarkably similar to the ordinary derivative of calculus.

I lead a two-semester seminar, along with my friend and colleague Alan Bertl, to investigate the subject of integration in topological groups, during which we worked through a large swath of Pontryagin's classic text [Pon86].

Much about integration in topological groups remains intriguing and mysterious to me, so I am keenly interested in pushing the boundaries of this field.

5 Multilinear algebra

A multilinear map $\alpha : V_1 \times \cdots \times V_m \to W$ is a map that is linear in each coordinate. The difficulty in studying multilinear maps is that they are not linear maps, but they are easier to work with when viewed through the lens of tensors. There are several constructions of the tensor product that are all isomorphic for vector spaces over $\mathbb{R}$ or $\mathbb{C}$, so I will give a more concise one here. Let $T : V_1 \cdots \times \ldots V_m \to W$ be a multilinear map satisfying $\text{span}(\text{Im } T) = \prod_{i=1}^{m} \dim V_i$. Then $\bigotimes_{i=1}^{m} V_i = \text{span}(\text{Im } T)$. An element of the form $v_1 \otimes \cdots \otimes v_m$ with $v_i \in V_i$ for all i is called a *pure tensor*, and a general element of $\bigotimes_{i=1}^{m} V_i$ is a linear combination of pure tensors, hence we may conveniently define things on pure tensors and extend linearly.

Let S_m be the symmetric group on m symbols. Let G be a subgroup of S_m, and let G act on pure tensors by $P(\sigma)(v_1 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}$, where σ acts on $1, \ldots, m$ by permutation. Extend this action linearly to get a group action, which we denote by the operator $P(\sigma)$, on the entire tensor space.

Let χ be an irreducible character of G. The operator $T(G;\chi) = \frac{\chi(e)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P(\sigma)$ is called the *symmetrizer with respect to G and χ*. Symmetrizers are projections, and the image of a symmetrizer is called a *symmetry class of tensors*.

Symmetry classes of tensors are of varied combinatorial and algebraic interest. I have written routines for computations involving symmetry classes of tensors in the programming language GAP. Although there are many open problems in multilinear algebra, I am currently most interested in porting these routines to a speedier general purpose programming library to make them readily available.

Tensors have seen a wide range of applications in the last decade or so, most notably in bioinformatics, computer vision, and numerical linear algebra. Of particular interest is the rank r of a tensor, the lowest r such that a tensor may be expressed as $\sum_{i=1}^{r} \alpha_i v_1^i \otimes \cdots \otimes v_m^i$. There are a number of different kinds of rank that are also of interest, and currently tensor rank is best studied with the methods of algebraic geometry.

6 Other interests

Although I have not done any work or research in the following areas, I am also interested in the following:

- Category theory
- Computability
- Quantum computation
- Programming language design

References

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