Problem Set 8 (just a short exercise): The multigraded BGG correspondence

Let $d_0, \ldots, d_n \geqslant 0$. Let k be a field and $S = k[x_0, \ldots, x_n]$, with $\mathbb{Z}$-grading given by $\deg(x_i) = d_i$. Let $E = \Lambda_k(e_0, \ldots, e_n)$, with $\mathbb{Z}^2$-grading given by $\deg(e_i) = (-d_i, -1)$.

1. (Short exercise) Let $n = 1$ and $\deg(x_0) = 1$, $\deg(x_1) = 2$. Show that $\mathbf{L}(\omega_E)$ is the Koszul complex on x_0, x_1, graded appropriately.

Solution. We have $\omega_E \cong E(-3, -2)$. The module ω_E lives in the bidegrees $(0, 0)$, $(1, 1)$, $(2, 1)$ and $(3, 2)$; the k-dimension of ω_E in each of these degrees is 1. Using the identification $\omega_E \cong E(-3, -2)$, we compute $\mathbf{L}(\omega_E)$ as follows:

$$0 \leftarrow S \xleftarrow{\begin{pmatrix} x_0 & -x_1 \end{pmatrix}} S(-1) \oplus S(-2) \xleftarrow{\begin{pmatrix} x_1 \\ x_0 \end{pmatrix}} S(-3) \leftarrow 0,$$

where the module S lives in homological degree 0. This is (isomorphic to) the Koszul complex $K(x_0, x_1)$.

Note: the signs appearing in this complex aren't the typical ones that appear in the Koszul complex. If you think of ω_E as the k-dual of E, rather than identifying it with $E(-3, -2)$, you will get the usual signs.