

## Problem Set 4: The BGG correspondence, Part 2

Fix  $n \geq 0$ . Let  $k$  be a field. Let  $S = k[x_0, \dots, x_n]$ , and  $E = \Lambda_k(e_0, \dots, e_n)$ , with gradings given by  $\deg(x_i) = 1$  and  $\deg(e_i) = -1$ . Recall that  $\omega_E := E^* \cong E(-n-1)$ .

1. (Short exercise) Prove that  $\mathbf{R}(S)$  is an injective resolution of  $k$ . Conclude that

$$\mathbf{R}(S)^\vee(-n-1) := \underline{\mathrm{Hom}}_E(\mathbf{R}(S), E)(-n-1)$$

is a free resolution of  $k$ .

Aside: the free resolution  $\mathbf{R}(S)^\vee$  is an example of the *Priddy resolution* of the residue field over a Koszul algebra, which you will see in general later in Claudia's lectures.

**Solution.** Recall from the lecture that  $H_i(\mathbf{R}(S))_j = \mathrm{Tor}_{i+j}^S(k, S)_j$ . The latter object is nonzero if and only if  $i = j = 0$ ; and if  $i = j = 0$ , it is precisely  $k$ . Thus,  $H_i(\mathbf{R}(S)) = 0$  for all  $i \neq 0$ ,  $H_0(\mathbf{R}(S))_j = 0$  for  $j \neq 0$ , and  $H_0(\mathbf{R}(S))_0 = k$ . Since  $E$  is injective, it follows that  $k \rightarrow \mathbf{R}(S)$  is an injective resolution. The complex  $\mathbf{R}(S)^\vee(-n-1)$  is a free resolution of  $k$  since  $\underline{\mathrm{Hom}}_E(-, E)$  is exact, and  $\underline{\mathrm{Hom}}_E(k, E) \cong k(n+1)$ .

2. Suppose  $n = 1$ , and let  $F$  denote the following complex of free  $E$ -modules:

$$\omega_E \xrightarrow{\begin{pmatrix} e_0 \\ e_1 \end{pmatrix}} \omega_E(-1)^2 \xrightarrow{\begin{pmatrix} 0 & e_1 \end{pmatrix}} \omega_E(-2) \xrightarrow{e_1} \omega_E(-3) \xrightarrow{e_1} \dots$$

Find a graded  $S$ -module  $M$  such that  $\mathbf{R}(M) \cong F$ . If you like, you can check your answer using `Macaulay2` (see the last problem for a demo of how to compute with  $\mathbf{R}$  in `Macaulay2`).

**Solution.** The module  $M = S/(x_0^2, x_0x_1)$  satisfies  $\mathbf{R}(M) = F$ . To get here, notice that

$$\dim_k M_i = \begin{cases} 1, & i = 0 \text{ or } i \geq 2; \\ 2, & \text{else.} \end{cases}$$

Also,  $x_0$  annihilates  $M_d$  for  $d \geq 1$ .

3. Prove Eisenbud-Fløystad-Schreyer's *Reciprocity Theorem*, which is stated as follows. Let  $M$  be a graded  $S$ -module and  $N$  a graded  $E$ -module. Prove that  $\mathbf{L}(N)$  is a free resolution of  $M$  if and only if  $\mathbf{R}(M)$  is an injective resolution of  $N$ . Hint:  $\mathbf{L}$  and  $\mathbf{R}$  are exact, and so they preserve quasi-isomorphisms.

**Solution** Assume  $\mathbf{L}(N) \xrightarrow{\sim} M$  is a free resolution of  $M$ . Since  $\mathbf{R}$  is exact, we have a quasi-isomorphism  $\mathbf{R}\mathbf{L}(N) \xrightarrow{\sim} \mathbf{R}(M)$ . Recall from the lecture that there is a quasi-isomorphism  $N \rightarrow \mathbf{R}\mathbf{L}(N)$ ; composing, we see that  $\mathbf{R}(M)$  is an injective resolution of  $N$ . The other direction is similar.

4. Let  $M$  be a finitely generated graded  $S$ -module and  $N$  a finitely generated graded  $E$ -module. Let  $M^*$  (resp.  $N^*$ ) denote the graded  $S$ -module  $\underline{\text{Hom}}_k(M, k)$  (resp. graded  $E$ -module  $\underline{\text{Hom}}_k(N, k)$ ). The left module actions are defined in the same way as the left module action on  $E^*$  we discussed earlier (except there is no sign over  $S$ ).

- (a) Prove that the complexes  $\mathbf{L}(N^*)$  and  $\mathbf{L}(N)^\vee$  have isomorphic terms.
- (b) If you're feeling ambitious, prove that there is an isomorphism  $\mathbf{L}(N^*) \cong \mathbf{L}(N)^\vee$  of complexes.

Conclude from part (b) that  $\mathbf{L}(\omega_E) = K(x_0, \dots, x_n)$ .

Note: one similarly shows that there is an isomorphism  $\mathbf{R}(M^*) \cong \mathbf{R}(M)^\vee := \underline{\text{Hom}}_E(\mathbf{R}(M), E)$  of complexes of graded  $E$ -modules.

**Solution.** Part (a) follows from the identifications

$$\mathbf{L}(N^*)_i = S(-i) \otimes_k N_i^* = S(-i) \otimes_k N_{-i} = (\mathbf{L}(N)_{-i})^\vee = (\mathbf{L}(N)^\vee)_i. \quad (1)$$

Let's now prove (b). We must check that the differentials are identified via (1). Fix  $i \in \mathbb{Z}$ . Choose  $k$ -bases  $y_1, \dots, y_\ell$  and  $z_1, \dots, z_m$  for  $N_{-i}$  and  $N_{-i+1}$ , respectively. Notice that (1) sends dual basis elements to dual basis elements; that is, it sends  $1 \otimes y_\alpha^*$  to  $(1 \otimes y_\alpha)^\vee$ .

The differential  $\mathbf{L}(N)_{-i+1} \rightarrow \mathbf{L}(N)_{-i}$  is a matrix  $A$  with respect to our bases, and its transpose  $A^T$  is the differential  $\mathbf{L}(N)_i^\vee \rightarrow \mathbf{L}(N)_{i-1}^\vee$ . The  $(\alpha, \beta)$  entry of the matrix  $A$  is  $\sum_{t=0}^n c_t x_t$ , where  $c_t$  is the coefficient of  $y_\beta$  in  $e_t z_\alpha$ . Thus, the  $(\alpha, \beta)$  entry of the matrix  $A^T$  is  $\sum_{t=0}^n c_t x_t$ , where  $c_t$  is the coefficient of  $y_\alpha$  in  $e_t z_\beta$ . In other words,  $A^T$  sends the dual basis element  $(1 \otimes y_\alpha)^\vee$  to the column vector with  $\beta^{\text{th}}$  entry  $\sum_{t=0}^n c_t x_t$ , where  $c_t$  is the coefficient of  $y_\alpha$  in  $e_t z_\beta$ . Now check that the differential on  $\mathbf{L}(N^*)$  acts on  $1 \otimes y_\alpha^*$  in exactly the same way (up to a sign, depending on the parity of  $i$ , since the left  $E$ -action on  $N^*$  involves a sign).

Finally, it follows from (b) and our observation that  $\mathbf{L}(E) = K(x_0, \dots, x_n)^\vee$  that  $\mathbf{L}(\omega_E) = K(x_0, \dots, x_n)$ .

5. Let  $C$  be a complex of graded  $S$ -modules. The *homological shift*  $C[i]$  of the complex  $C$  is the complex with  $C[i]_j = C_{i+j}$  and  $d_{C[i]} := (-1)^i d_C$ .

- (a) Prove that, if  $M \in \text{Mod}(S)$ , considered as a complex concentrated in homological degree 0, then  $\mathbf{R}(M[i]) \cong \mathbf{R}(M)[i]$  for all  $i \in \mathbb{Z}$ .
- (b) Prove also that  $\mathbf{R}(M(i)) \cong \mathbf{R}(M)(i)[-i]$  for all  $i \in \mathbb{Z}$ .

Note: these statements extend to complexes verbatim, but you need not prove this. One can prove in the same way that, given a complex  $C$  of graded  $E$ -modules, we have  $\mathbf{L}(C[i]) \cong \mathbf{L}(C)[i]$ , and  $\mathbf{L}(C(i)) \cong \mathbf{L}(C)(i)[-i]$ . If you have time, prove these identities, and/or verify them via some examples in Macaulay2.

**Solution.** We have  $\mathbf{R}(M[i])_j = \bigoplus_{s+t=j} \mathbf{R}(M[i]_s)_t = \mathbf{R}(M)_{j+i} = \mathbf{R}(M)[i]_j$ . The differentials also agree up to sign, and so the complexes are isomorphic. Similarly,  $\mathbf{R}(M(i))_j = \omega_E(j) \otimes_k M(i)_{-j} = \omega_E(j) \otimes_k M_{i-j} = \mathbf{R}(M)_{j-i}(i) = (\mathbf{R}(M)[-i])_j(i)$ . Once again, the differentials agree up to a sign, and so the complexes must be isomorphic.

**6.** As a demo of how to compute with the functor  $\mathbf{R}$  in `Macaulay2`, let's verify computationally that  $\mathbf{R}(S)$  is an injective resolution of  $k$ .

(a) First, we need to load two packages:

```
needsPackage "BGG"
needsPackage "Complexes"
```

(b) Next, we build our polynomial ring and exterior algebra. I will work with four variables, but you can toggle this choice.

```
n = 3
Edegrees = for i from 0 to n list -1
S = ZZ/101[x_0..x_n]
E = ZZ/101[e_0..e_n, Degrees => Edegrees, SkewCommutative => true]
```

Notice that we add an optional input in the last line to make the degrees of the exterior variables  $-1$ . The default is to make the degree of each variable  $1$ .

(c) Now we build the maps in  $\mathbf{R}(S)$ . The function `bgg(i, M, E)` builds the  $(-i)^{\text{th}}$  differential in  $\mathbf{R}(M)$ . Let's make a list of the first few differentials in  $\mathbf{R}(S)$ .

```
L = for i from -5 to 0 list bgg(-i, S^1, E);
```

(d) Finally, we build a complex out of this list of matrices, and we compute its homology.

```
I = complex(L, Base => -6) ** E^{{-n-1}}
presentation HH_0 I
for i from -5 to -1 do print (HH_i I == 0)
```

Couple things here: the optional input `Base => -6` makes our complex live in the right homological degrees, and tensoring with  $E(-n-1) \cong E^*$  makes it live in the correct internal degrees.