Day 1: Exterior algebras and the Koszul complex

Exterior algebras

R: commutative ring

V: free R-module of rank n

$$T^i(V) = \underbrace{V \otimes_R \cdots \otimes_R V}_{i \text{ copies}}$$

$T(V) = \bigoplus_{i \geq 0} T^i(V)$: the tensor algebra on V.

If $x \in T^i(V)$ and $y \in T^j(V)$, $xy := x \otimes y \in T^{i+j}(V)$. Not commutative!

$I \subseteq T(V)$: 2-sided ideal generated by $\{v^2 : v \in V = T^1(V)\}$

$\Lambda_R(V) := T(V)/I$, the exterior algebra on V.

Properties of $\Lambda_R(V)$:

- Say V has basis $e_1, \ldots, e_n$. For $i \neq j$, we have (in $\Lambda_R(V)$):
  $0 = (e_i + e_j)^2 = e_i^2 + e_i e_j + e_j e_i + e_j^2 = e_i e_j + e_j e_i$
  $\Rightarrow e_i e_j = -e_j e_i$, i.e. $\Lambda_R(V)$ is anticommutative.

- $\Lambda_R(V)$ is a free R-module w/ basis
  $\{e_{i_1} \cdots e_{i_m} : 1 \leq i_1 < \cdots < i_m \leq n,\ 0 \leq m \leq n\}$.

Notation: $\Lambda^m_R(V)$ = free R-module w/ basis $\{e_{i_1} \cdots e_{i_m} : 1 \leq i_1 < \cdots < i_m \leq n\}$

$\Lambda^m_R(V)$ is the m^{th} exterior power of V.

$$\Lambda_R(V) = \bigoplus_{i=0}^{n} \Lambda^i_R(V).$$

rank of $\Lambda^i_R(V)$ as a free R-module: $\binom{n}{i}$.

$\Rightarrow$ rank of $\Lambda_R(V)$ is $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.

Ex: $V = R^3$, w/ basis e_1, e_2, e_3

$\Lambda^0_R(V) = R$

$\Lambda^1_R(V) = Re_1 \oplus Re_2 \oplus Re_3$

$\Lambda^2_R(V) = Re_1e_2 \oplus Re_1e_3 \oplus Re_2e_3$

$\Lambda^3_R(V) = Re_1e_2e_3$.

$$(e_1+e_2)\cdot(e_1e_3 - e_2e_3) = e_1\cancel{e_1}e_3 - e_1e_2e_3 + e_2e_1e_3 - e_2\cancel{e_2}e_3$$
$$= -e_1e_2e_3 - e_1e_2e_3$$
$$= -2\,e_1e_2e_3$$

multiple of $e_1e_2e_3$

Notation: sometimes write $\Lambda_R(V)$ as $\Lambda_R(e_1,\dots,e_n)$.

Koszul complexes:

R, V, $e_1,\dots,e_n$ as above.

$f_1,\dots,f_n \in R$

Def'n: The Koszul complex on $f_1 \dots f_n$ is the complex

$$0 \leftarrow \Lambda^0_R(V) \leftarrow \Lambda^1_R(V) \leftarrow \cdots \leftarrow \Lambda^n_R(V) \leftarrow 0$$

w/ differential $\partial(e_{i_1}\cdots e_{i_m}) = \sum_{\ell=1}^{m} (-1)^{j-1} f_{i_\ell} e_{i_1} \cdots \widehat{e_{i_\ell}} \cdots e_{i_m}$.

Write it as $K(f_1,\dots,f_n)$.

Exercise: $\partial^2 = 0$.

Ex: $n=1$: $K(f_1) = \left[0 \leftarrow R \xleftarrow{f_1} Re_1 \leftarrow 0 \right]$

$n=2$: $K(f_1,f_2) = \left[0 \leftarrow R \xleftarrow{(f_1\ f_2)} Re_1 \oplus Re_2 \xleftarrow{\begin{pmatrix} -f_2 \\ f_1 \end{pmatrix}} Re_1 e_2 \leftarrow 0 \right]$

Observations:

- $K(f_1 \ldots f_n)$ is a complex of free R-modules.
- $H_0 K(f_1, \ldots, f_n) \cong R/(f_1, \ldots, f_n)$.

When is the Koszul cpx a free resolution?

Def'n: $f_1 \ldots f_n \in R$ form a regular sequence if

- f_i is a nzd on $R/(f_1 \ldots f_{i-1})$ $\forall i$, and
- $R/(f_1 \ldots f_n) \neq 0$.

Thm: If $f_1 \ldots f_n$ is a reg. sequence, then $K(f_1 \ldots f_n)$ is a free res'n of $R/(f_1 \ldots f_n)$.

Converse holds if R is Noeth local and module is $\neq 0$.

Ex: If k is a field, and $R = k[x_0, \ldots, x_n]$, then $K(x_0, \ldots, x_n)$ is a free res'n of $R/(x_0 \ldots x_n) \cong k$.

Short exercise: compute $K(f_1, f_2, f_3)$.