

1

$$R = \bar{r} \ln g$$

to build (co)homology theories for R -modules,
resolve by modules that the functor is exact on:

Def: Given an R -module M , a free (resp, projective) resolution is an exact complex of free (proj) R -mods:

$$\begin{array}{ccccccc} \dots & \xrightarrow{\alpha_3} & F_2 & \xrightarrow{\alpha_2} & F_1 & \xrightarrow{\alpha_1} & F_0 \xrightarrow{\epsilon} M \rightarrow 0 \\ & & \searrow & & \nearrow & & \nearrow \\ & & & & K_1 = \ker & & K_0 = \ker \epsilon = H(\text{cx so far}) \\ & & & & (= H(\text{cx})) & & = \frac{\ker \epsilon}{\text{im}(0)} \end{array}$$

Equivaly : A free (proj) cx $\dots \rightarrow F_1 \rightarrow F_0 \rightarrow 0$

$$\begin{array}{ccccccc} & & & & \downarrow^0 & \downarrow^\varepsilon & \\ & & & & \dots \rightarrow 0 & \rightarrow \mu & \rightarrow 0 \end{array}$$

and quasi-isom. $F. \xrightarrow[\varepsilon]{\sim} M$

iso on homologs $H(F) \xrightarrow{\cong} H(M) = M$

(Build Cochain theories - Tor, Ext, ...)

What if want to study commutative rings R ?

What I want to study (or algebras = rings maps $R \rightarrow S$)

Prblm: R-mod is abelian cat
Alg is not!

2 levels:

- put ring structure on free resns
- build resns w/ free (commut.) rings structure

(Tate, Gulliksen, Arsmus, Sjodin, Lenin, Avramov, Halperin)

$R = \text{a commutative ring}$

2

Def: A dg-algebra (dga) is a complex F of R -modules

$$\dots \xrightarrow{\partial} F_2 \xrightarrow{\partial} F_1 \xrightarrow{\partial} F_0 \rightarrow 0$$

equipped with a product making

$$\bigoplus_{i=0}^{\infty} F_i$$

into a graded-commutative R -algebra
 $(ab = (-1)^{|a||b|} ba)$

in particular,
 F_0 ring
 so, often
 $F_0 = R$

st. Leibniz rule holds: $\partial(ab) = \partial(a)b + (-1)^{|a|} a \partial(b)$

(ie, $F \otimes F \xrightarrow{\text{mult}} F$ is a chain map) \leftarrow leave \checkmark

(Ur-)Example 1:

Koszul complex on $f_1, f_2 \in R$:

$$0 \rightarrow \underset{2}{R} \xrightarrow{\begin{bmatrix} -f_2 \\ f_1 \end{bmatrix}} \underset{1}{R}^2 \xrightarrow{[f_1 \ f_2]} \underset{0}{R}$$

can identify w/ exterior powers of $F = R^2$

$$\begin{array}{ccccc} \Lambda^2 F & \rightarrow & \Lambda^1 F & \rightarrow & \Lambda^0 F \\ & & e_1 \mapsto f_1 & & \\ & & e_2 \mapsto f_2 & & \\ e_1 e_2 & \mapsto & f_1 e_2 - f_2 e_1 & & \end{array} \quad \left. \vphantom{\begin{array}{ccccc} \Lambda^2 F & \rightarrow & \Lambda^1 F & \rightarrow & \Lambda^0 F \end{array}} \right\} \begin{array}{l} \text{diff!} \\ 2 \end{array}$$

and note

$$\bigoplus_{i=0}^2 \Lambda^i F = \Lambda F = \underbrace{\text{exterior algebra } R\langle e_1, e_2 \rangle}_{\text{product}}$$

Interaction:

$$\partial(e_1 e_2) = \partial(e_1) e_2 - e_1 \partial(e_2)$$

(signed) product rule!

(Ur-) Example 2: (algebraic topology)

3

X topological space

$C^*(X)$ = the chain co-complex for computing its cohomology

Note: The cup product makes $C^*(X)$ into a dga!

Fact: $F. \text{ dga} \Rightarrow H(F_\bullet)$ inherits an alg. str.
 \uparrow really $\oplus H_i(F)$

Examples:

- $Ex1 \Rightarrow$ Koszul homology
- $Ex2 \Rightarrow$ singular cohomology
- cap product (dg module!) $\rightsquigarrow H_*(X)$ homology
- shuffle or concatenation product $\rightsquigarrow HH^*$ Hochschild cohom.
(on bar resn)
- A_∞ products \rightsquigarrow Massey operations
more genly

Motivation in CA:

4

New approach by Buchsbaum-Eisenbud 77 to

B-E-Homology Conjecture:

$$R = k[x_1, \dots, x_n]$$

$M \neq 0$ f.g. gdd R -module, $\ell(M) < \infty$

Then min'l gdd free resolution

$$0 \rightarrow F_n \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

$$\text{has } \text{rank}_R F_i \geq \binom{n}{i}$$

Prop (BE, also Herzog): If F_0 is dga, then BETH holds!

Pf: • Take max'l R -reg seq f_1, \dots, f_n in $\text{ann}_R M$.

• Form Koszul cx $K = R\langle e_1, \dots, e_n \rangle$
 $\alpha(e_i) = f_i$

• K "free" dga $\Rightarrow \exists K \xrightarrow{\phi} F$
(gdd cannot)

• ϕ injective [show $H_n(\phi)$ inj $\Rightarrow \phi_n \Rightarrow$ all ϕ_i]

Weaker Conj: Total Rank Conjecture ($\sum \text{rk } F_i \geq 2^n$) Walker
VanderBout
-Walker

Positive results (\exists dga):

- $pd\ R/\mathbb{I} \leq 3$ - Herzog, Buchsbaum-Eisenbud
- $pd\ R/\mathbb{I} \leq 4$, Gor - Kustin-Miller
- + combinatorial exs + ... (powers max ideal in poly - Srinivasan)
- (Taylor, Eliashberg-Kervaire)

SKIP IF NEEDED ...
 (just summarize
 in genl \exists dga [Avramov 74])

Negative (\nexists dga):

Avramov

74, 81
 dash.
 $pd\ R/\mathbb{I} = 4$

Srinivasan

92 96
 $pd\ R/\mathbb{I} = 5$, Gor

Kathan

18
 monom! !

What instead?

- $R/\mathbb{I} \times R/\mathbb{I} \xrightarrow{M} R/\mathbb{I}$ always lifts to a chain map!

Comparison then { $F \otimes F \xrightarrow{M} F$
 but may fail to be associative,
 but is assoc. up to htpy
 :
 give a M_3

$\Rightarrow F_0$ is A_{∞} algebra
 ($\Rightarrow H(F)$ still an algebra + Massey ops!)

$F^{\otimes 3} \xrightarrow{M \otimes 1 - 1 \otimes M} F$
 $a \otimes b \otimes c \xrightarrow{\quad} (ab)c - a(bc)$
 So, ≈ 0 w/ htpy M_3

\nearrow Leibniz rule

lifts 0-map on R/\mathbb{I}

- There is always a free resn of R/\mathbb{I} that has dga structure (Tate's beautiful 1956 paper)

[Also: bar resn]

non-min! (usually)
 L2!

Solution to the Quick Problem:

k field, $R = k[x, y]$, $I = (x^2, xy)$
 Let's put DS abs. on resn

$$0 \rightarrow R \xrightarrow{\begin{bmatrix} -y \\ x \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} x^2 & xy \end{bmatrix}} R \rightarrow R/I \rightarrow 0$$

F_2 F_1 F_0
 g e_1, e_2 1

$$F_0 \times F_i \xrightarrow{\mu} F_i \quad \text{usual } R\text{-mod str.}$$

\uparrow
 R

$$F_1 \times F_2 \xrightarrow{\mu} F_3 = 0$$

(the only non-obvious product)

$$F_1 \times F_1 \xrightarrow{\mu} F_2$$

$$e_1 e_2 = ??$$

well, all we know is that it needs to satisfy Leibniz!

$$\begin{aligned} \text{Leibniz: } d(e_1 e_2) &= d(e_1) e_2 + (-1)^{|e_1|} e_1 d(e_2) \\ &= x^2 e_2 - xy e_1 \\ &= x(xe_2 - ye_1) \end{aligned}$$

so, let's define

$$e_1 e_2 = xg$$

(unique here since d_2 injective!)

but in gen'l for larger CXS, not unique