L3 : Koszul duality

First, universal resolutions:

Let M R-module.

Priddy's resn $P(R) \to k \to 0$ exact

$\Rightarrow$ $P(R) \overset{\tau}{\otimes}_k M \to \underbrace{\cancel{k} \otimes_k M}_{\cong M} \to 0$ exact

k field

if one adds a "twisting" term ∂' to the diff'l, namely: on $R \otimes (R^!_j)^* \otimes M$

$\partial' =$ left multipl. by $1 \otimes \sum_{i=1}^n x_i^* \otimes x_i$

$$\Rightarrow \underbrace{\cdots \to R \otimes_k (R^!_1)^* \otimes_k M \to R \otimes_k \cancel{(R^!_0)^*} \otimes_k M}_{P(R) \overset{\tau}{\otimes}_k M} \xrightarrow{\mu} M \to 0$$

k

$\partial = \partial^{P(R)} \otimes 1_M + 1_R \otimes \partial'$

explicit R-free resn

(For $M \neq k$, highly non-min'l : ranks $\overset{\text{even}}{\wedge}$ ∞ if $\dim_k M = \infty$!)

More generally, there is a duality producing this:

Consider the R-dual of the Priddy / Koszul cx

$$\boxed{\begin{aligned} C &= R \otimes_k R^! \\ \partial &= \sum_i x_i \otimes x_i^* \end{aligned}}$$

$\mathrm{Hom}_R(R \otimes_k R^{!*}, R)$

$\cong \mathrm{Hom}_k(R^{!*}, \cancel{\mathrm{Hom}_R(R,R)}\ R)$ R flat/k

$\cong R \otimes R^{!**}$

$\cong R \otimes R^!$

($*$ = gd'd dual)

or: each R-free on k-basis of $(R^!_i)^*$ — so dual R-free on basis $R^!_i$

Note: C is an R-$R^!$-bimodule

Thm: (Koszul duality) [Beilinson – Ginsburg – Schechtman & further by Beilinson – Ginsburg – Soergel]

There is an equivalence of cats given by

$$\mathcal{D}^b(R^!) \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}^b(R)$$

$$M \longmapsto L(M) = \mathrm{Tot}\left(C \underset{R^!}{\otimes} M\right) \quad \text{as } R\text{-mods: } \left(= R \underset{k}{\otimes} \cancel{R^! \underset{R^!}{\otimes}} M \;\cong R \underset{k}{\otimes} M\right)$$

hence M here

$$R(N) \longleftarrow N$$

$$= \mathrm{Tot}\left(\mathrm{Hom}_R(C, N)\right)$$

as R-mods:

$$\left(\begin{aligned} &= \mathrm{Hom}_R\left(R \underset{k}{\otimes} R^!, N\right) \\ &\cong \mathrm{Hom}_k(R^!, N) \\ &\cong (R^!)^* \underset{k}{\otimes} N \end{aligned}\right) \Bigg\} \text{ as } R\text{-mods}$$

totalized by:

- internal degree on $R^!$
- if M is a cx (not just a module), then also the homological deg of M (ie, totalize the bicomplex formed by ∂^C & ∂^M)

Note:

- (L, R) adjoint functors – classic Hom/$\otimes$
- $R, R^!$ flat$/k$ $\Rightarrow$ $C = R \overset{L}{\underset{k}{\otimes}} R^!$
- C free$/R, R^!$ $\Rightarrow$ $\left.\begin{aligned} L(M) &= C \overset{L}{\underset{R^!}{\otimes}} N \\ R(N) &= R\mathrm{Hom}_R(C, N) \end{aligned}\right\}$ but w/twisted diff'l
- How to grade/totalize $L(M)$, $R(N)$?

  By grading on $R^!$ and hom'l grading on M, N

Idea: [but not actually so easy b/c ∂'s don't correspond immediately to those of $P(R)$! ... !]

$$\underset{\substack{\uparrow \\ R^!\text{-mod} \\ (\text{or cx})}}{RL(M)} \cong \underbrace{(R^!)^* \otimes_k R}_{P(R)\, \simeq\, k} \otimes_k M \simeq k \otimes_k M \cong M$$

$$\underset{\substack{\uparrow \\ R\text{-mod} \\ (\text{or cx})}}{LR(N)} = \underbrace{R \otimes_k (R^!)^*}_{P(R)\, \simeq\, k} \otimes_k M \simeq k \otimes_k N \cong N$$

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[In fact, $LR(N)$ is that universal resn of N we saw.]

Classic BGG:

Consider the case $R = k[x_1, \dots, x_n] = \dfrac{k\langle x_1, \dots, x_n\rangle}{(\{x_i x_j - x_j x_i\})}$

$= \text{symmetric alg } S(V)$

$$R^! = \frac{k\langle x_1, \dots, x_n\rangle}{(\{x_i^2\}, \{x_i x_j + x_j x_i\})} = \text{exterior alg } \Lambda(V) = E$$

$$(R^!)^* = E^* = \mathrm{Hom}_k(E, k) = \omega_E \cong E(-n)$$

we have only n variables in my lectures

Koszul duality for S, Λ recovers the BGG correspondence:

$$\begin{array}{ccc} & & \left\{\begin{array}{l}\text{coherent} \\ \text{shvs on } \mathbb{P}^n\end{array}\right\} \\ & & \wr\wr \\ \left\{\begin{array}{l}\text{f.g. gdd} \\ E\text{-mods}\end{array}\right\} \Big/ \mathrm{perf}(E) & \underset{R}{\overset{L}{\rightleftarrows}} \ \text{Koszul duality} \ \simeq & \left\{\begin{array}{l}\text{gdd f.g.} \\ S\text{-mods}\end{array}\right\} \Big/ \mathrm{fl}(S) \end{array}$$

$\mathrm{perf}(E)$: things built from E via cones (exact seq's) equiv, from $E^* = \omega_E = E(-n)$

ie, E-mods of finite projective dimension

$\mathrm{fl}(S)$: finite length S-mods (built from k via cones, equiv, exact seqs)

Why $\mathrm{Perf}(E) \approx \mathrm{fl}(S)$?
Exercise (now...!)