Day 2: Graded rings

Def'n: Suppose R is a ring, and $R = \bigoplus_{i \in \mathbb{Z}} R_i$ as an abelian group. R is a <u>graded ring</u> if, when $r_i \in R_i$ and $r_j \in R_j$, we have $r_i r_j \in R_{i+j}$.

If $r \in R_i$, r is called <u>homogeneous</u> <u>of degree i</u>, and we write $\deg(r) = i$.

Key examples: let k be a field.

$S = k[x_0, \ldots, x_n]$ is graded with S_i = sums of monomials of degree i.

$E = \Lambda_k(e_0, \ldots, e_n)$ is graded with E_{-i} = sums of exterior monomials w/ i factors.

our convention

Ex: $x_0 x_1 + x_2^2 \in S_2$

$x_0^2 + x_1^3$: not homog.

$e_0 e_1 + e_1 e_2 \in E_{-2}$

$e_0 + e_1 e_2$: not homog.

Def'n: If R is a graded ring, a <u>graded left R-module</u> is a left R-module M s.t. $M = \bigoplus_{i \in \mathbb{Z}} M_i$ as an abelian group, and $r_i m_j \in M_{i+j}$ when $r_i \in R_i$ and $m_j \in M_j$.

All modules left modules unless noted otherwise.

Exercise: If $I \subseteq R$ is an ideal generated by homog. elements, then I is a graded R-module, and R/I is a graded ring (and R-module).

<u>Grading twists</u>:

If M is a graded R-module, and $j \in \mathbb{Z}$, then $M(j)$ is equal to M as an R-module, w/ grading $M(j)_i = M_{i+j}$.

Ex: $R(j)$ is a free module generated in degree $-j$.

An R-linear map $f: M \to N$ of graded R-modules is homog. of degree d (or just degree d) if $f(M_i) \subseteq N_{i+d}$ $\forall i \in \mathbb{Z}$.

Note: if $f: M \to N$ is degree d, then $f: M \to N(d)$ and $f: M(-d) \to N$ are degree 0.

Notation: If M, N are gdd R-modules, then $\mathrm{Hom}_R(M,N)$ = deg. 0 maps $M \to N$

Need M, N finitely generated here. $\underline{\mathrm{Hom}}_R(M,N)$ = all R-linear maps $M \to N$.

$\underline{\mathrm{Hom}}_R(M,N)$ is a graded abelian group w/ $\underline{\mathrm{Hom}}_R(M,N)_i$ = deg i maps.

If M (resp. N) is an R-R bimodule (e.g. if R is gdd comm.), then $\underline{\mathrm{Hom}}_R(M,N)$ is a graded left (resp. right) module.

Def'n: R is graded local if R is Noetherian, $R_0 = k$, and either $R_{>0} = 0$ or $R_{<0} = 0$.

Graded local rings have a unique homog. max'l ideal (either $R_{>0}$ or $R_{<0}$).

Examples: $k[x_0 \ldots x_n]$ and $\Lambda_k(e_0 \ldots e_n)$.

If R is graded local, then every fg gdd R-module has a graded min'l free res'n, and it is unique up to chain isomorphism.

$$M \leftarrow \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{0j}} \xleftarrow{\partial_1} \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{1j}} \xleftarrow{\partial_2} \cdots$$ differentials are degree 0.

The ∂_i are matrices of homog. elements w/ $\boxed{\text{nonzero}}$ degree.

The β_{ij} are the Betti numbers of M, written $\beta_{ij}(M)$.

Ex: k is a graded $S = k[x_0, x_1]$-module. The Koszul complex

$$k \leftarrow S \xleftarrow{(x_0\ x_1)} S(-1)^2 \xleftarrow{\begin{pmatrix} -x_1 \\ x_0 \end{pmatrix}} S(-2) \leftarrow 0$$

is its min'l graded free resolution.

$\beta_{00}(k) = \beta_{22}(k) = 1$, $\beta_{11}(k) = 2$.

Short exercise: Assume R is graded local, and $R_{<0} = 0$. Prove that if M is a gdd R-module w/ $M_i = 0$ for $i \ll 0$, and $R_{>0} M = M$, then $M = 0$. (a similar statement holds if $R_{>0} = 0$).

This is the graded version of Nakayama's lemma. Notice: there is no need to assume M is fg.