

Bar resolution \supseteq Priddy resn

2

Let R be std gdd k -algebra

The bar resn of any R -mod M is: [Eilenberg-Mac Lane 53]

$$B(R, R, M)_n \stackrel{\text{def}}{=} R \overset{\text{twisted } \otimes \text{ means } \otimes \text{ involves both factors}}{\otimes_k} B(R)_n \overset{\text{twisted } \otimes \text{ means } \otimes \text{ involves both factors}}{\otimes_k} M$$

$$= R \otimes_k \underbrace{\bar{R} \otimes_k \cdots \otimes_k \bar{R}}_n \otimes_k M$$

where

$$\bar{R} = R_+ = \bigoplus_{n \geq 0} R_i = m$$

$$\partial(r \otimes r_1 \otimes \cdots \otimes r_n \otimes m)$$

$$= \underbrace{r r_1 \otimes r_2 \otimes \cdots \otimes r_n \otimes m}_{\text{call this part of } \partial} - \underbrace{r \otimes r_1 r_2 \otimes \cdots \otimes r_n \otimes m + \cdots + (-1)^i r \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n \otimes m}_{\text{this part of } \partial \text{ is called } b} \pm \cdots + (-1)^n \underbrace{r \otimes r_1 \otimes \cdots \otimes r_{n-1} \otimes r_n \otimes m}_{\text{call this part } \partial'}$$

$\underbrace{\quad}_{=0 \text{ for } M=k}$

Thm: $B = B(R, R, M) \xrightarrow[\sim]{R} M$ is an R -free resn of M

It is very non-min! $R \otimes_k \underbrace{\bar{R}^{\otimes n}}_{\text{huge } k\text{-v.sp.}} \otimes M = \text{huge free } R\text{-mod}$

but universal (same form any R, M).

Let's take $M = k$ (for simplicity) So, $B(R, R, k)_n = R \otimes_k \underbrace{\bar{R} \otimes_k \cdots \otimes_k \bar{R}}_n \otimes_k k$

Separate B out by graded pieces:

Note 1st:

$$R = \frac{T(V)}{(W)} = \cancel{k} \oplus \underbrace{V}_{R_1} \oplus \underbrace{\frac{V^{\otimes 2}}{W}}_{R_2} \oplus \underbrace{\frac{V^{\otimes 3}}{V \otimes W + W \otimes V}}_{R_3} \oplus \cdots$$

$\text{delete in } \bar{R}$

cont'd

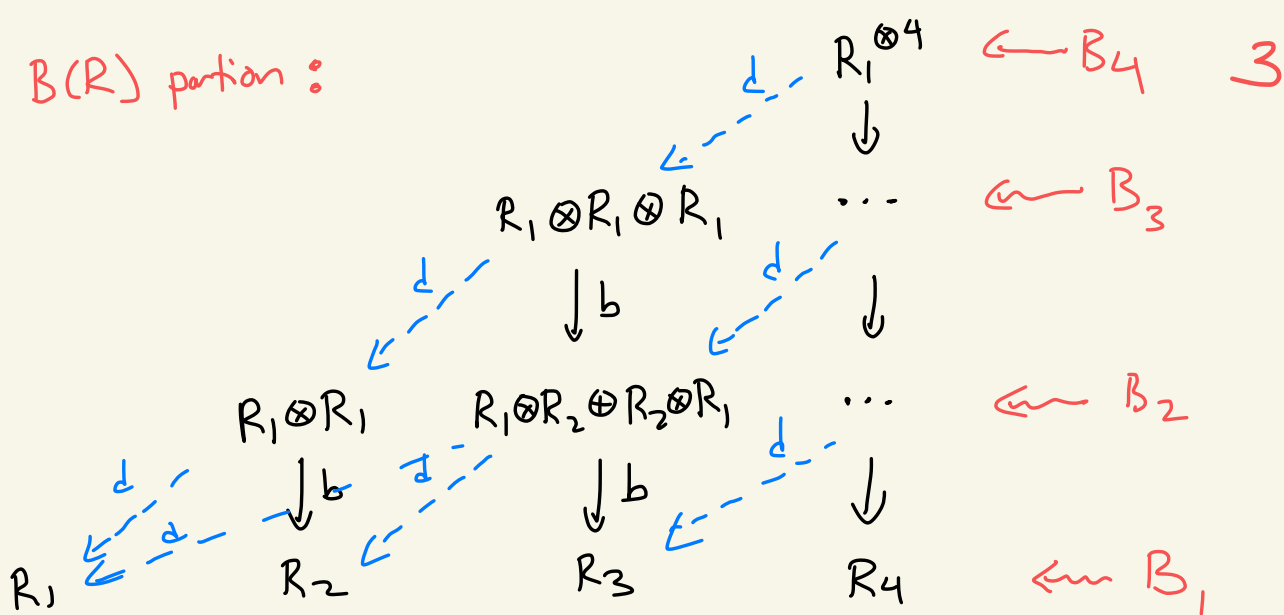
$$B = R \otimes_k R$$

$B(R)$ portion:

$$B = R \otimes_k R$$

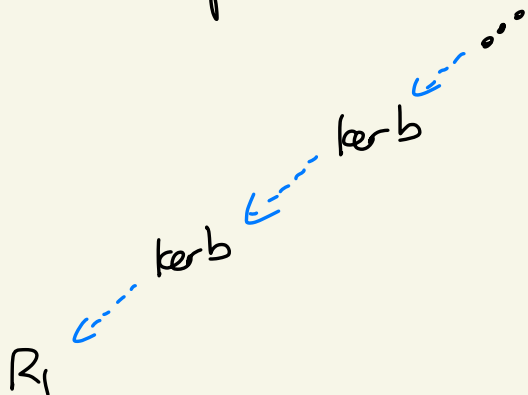
b maps: 0, 1's

d maps:
min'l
(ind = 0...)



The kernel of each top vertical b map forms a subcx:

$$R \otimes_k R$$



Note: each $R \otimes R_{i_1} \otimes \dots \otimes R_{i_s}$
has R -basis sitting in
deg $i_1 + \dots + i_s$
so, $\cong \bigoplus R(-i_1 - \dots - i_s)$

so, shifts can be read off easily.

In column 2:

$$\ker(R_1 \otimes_k R_1 \xrightarrow{b} R_2) = \ker(V \otimes_k V \rightarrow \frac{V \otimes V}{W}) = W = (R_2^!)^*$$

by Claim 2

In general:

$$\ker(\underbrace{R_1 \otimes \dots \otimes R_1}_n \rightarrow \bigoplus_{\substack{i+j+2 \\ =n}} \underbrace{R_1 \otimes \dots \otimes R_2 \otimes \dots \otimes R_1}_{i+j}) = \bigcap_{\substack{i+j+2 \\ =n}} \underbrace{V \otimes \dots \otimes W \otimes \dots \otimes V}_{i+j} = (R_n^!)^*$$

Inherited diff'l: just d

$$= \alpha \cdot (x_{i_1} \otimes x_{i_1}^*) = 2(\alpha) \text{ in } P(R)!$$

$$d(\alpha) = d(1 \otimes x_{i_1} \otimes x_{i_2} \otimes \dots \otimes x_{i_n}) = 1 \cdot x_{i_1} \otimes x_{i_2} \otimes \dots \otimes x_{i_n}$$

↑
basis
elt

$$\xrightarrow{\text{basis elt}} x_{i_1}^* x_{i_2}^* \dots x_{i_n}^* \in (R_n^!)^*$$

$$\text{if basis elt} \quad x_{i_2}^* \dots x_{i_n}^* \in (R_{n-1}^!)^*$$

so, $P(R) \hookrightarrow B = B(R, R, k)$ [Both recs of k , but $P(R)$ min'l]