Week 2 • Problem Set 1 • Definition and quadratic dual

Quick Problem:

Let $R = k[x, y]/(x^2 + y^2)$.

(a) Why do we know already that R is a Koszul algebra?

(b) Find $R^!$ and determine $\dim_k R^!_3$.

Problems:

1. Find $R^!$ for the following *commutative* rings.

(a) $R = k[x, y, z]/(x^2 - yz)$

(b) $R = k[x, y]$

(c) $R = k[x_1, \ldots, x_n]$, that is, the symmetric algebra $S(V)$ on $V = k\{x_1, \ldots, x_n\}$

In parts (b) and (c), do you recognize $R^!$ as a familiar algebra?

2. Which dual algebras from #1 are Artinian (that is, satisfy $\dim_k R^! < \infty$)?

3. Let R be the exterior algebra on n variables, say $x_1, \ldots, x_n$. Find $R^!$.

4. (a) State the BGG correspondence (from Michael Brown's lectures).

(b) Guess a similar duality theorem for Koszul algebras from what you have seen so far.

5. Let $R = Q/I$ where Q is a polynomial ring and I is a homogeneous ideal. Prove that if I has a linear Q-free resolution then R is Koszul.

Hint: For any graded ring R and graded R-module M, say with minimal graded free resolution F, writing $F_i = \oplus R(-j)^{\beta_{ij}}$ defines the graded Betti numbers β_{ij}. Putting these as coefficients, one obtains the graded Poincaré series

$$P^R_M(t, s) = \sum_{i,j} \beta_{ij} t^i s^i$$

You may use the following generalized version of a inequality of Golod [3] (also sometimes credited to Serre) which holds for each graded module M.

$$P^R_M(t, s) \leq \frac{P^Q_M(t, s)}{1 - t(P^Q_R(t, s) - 1)}$$

The inequality is understood to be coefficientwise once one expands the righthand side as a power series using geometric series.

For those of you who are curious, this inequality can be shown using either the "Avramov" spectral sequence

$${}^2E_{p,q} = \mathrm{Tor}_p^{\mathrm{Tor}^Q(R,k)}(\mathrm{Tor}^Q(M, k), k)_q \Rightarrow \mathrm{Tor}^R_{p+q}(M, k)$$

(see [1, pp. 3.2.4, 3.3.2] or by using the relative bar resolution of Iyengar [4] and Burke [2]).

References

[1] Luchezar L. Avramov. "Infinite free resolutions". In: *Six lectures on commutative algebra*. Mod. Birkhäuser Class. Birkhäuser Verlag, Basel, 2010, pp. 1–118. DOI: 10.1007/978-3-0346-0329-4_1. URL: https://doi.org/10.1007/978-3-0346-0329-4_1.

[2] Jesse Burke. *Higher homotopies and Golod rings*. 2015. arXiv: 1508.03782 [math.AC].

[3] E. S. Golod. "Homologies of some local rings". In: *Dokl. Akad. Nauk SSSR* 144 (1962), pp. 479–482. ISSN: 0002-3264.

[4] Srikanth Iyengar. "Free resolutions and change of rings". In: *J. Algebra* 190.1 (1997), pp. 195–213. ISSN: 0021-8693,1090-266X. DOI: 10.1006/jabr.1996.6901. URL: https://doi.org/10.1006/jabr.1996.6901.