L5: Homotopy Lie algebra and cotangent cx

Fundamental invariant $\pi(R)$ def'd by Avramov (84).

local algebra $\longleftrightarrow$ Av–Halperin "Through the Looking Glass" rational homotopy theory

$\underbrace{\pi(R)}_{\text{gdd Lie algebra}} \hookrightarrow \underbrace{\mathrm{Ext}^R(k,k)}_{\text{its universal embedding alg.}}$ $\qquad\qquad$ $\pi^*(X)_{\mathbb{Q}}$

<u>Def</u>: a <u>gdd Lie algebra</u> / k is a gdd k-module L

with a k-linear pairing

$$L \times L \longrightarrow L \qquad \text{"Lie bracket"}$$
$$(a,b) \longmapsto \text{"}[a,b]\text{"}$$

s.t. 1) (gdd anti-commut) $[a,b] = -(-1)^{|a||b|}[b,a]$

2) (Jacobi identity) $[a[b,c]] = [[a,b],c] + (-1)^{|a||b|}[b,[a,c]]$

skip (say char 2,3 ...) 3) $|a|$ even $\Rightarrow [a,a]=0$ and $\exists$ reduced square $a^{[2]}$ (acts like $\frac{[a,a]}{2!}$) ← char 2

$|a|$ odd $\Rightarrow [a,[a,a]] = 0$ ← char 3 (else autom.)

<u>Fact</u>: there are functors

$$\left\{\begin{matrix}\text{gdd}\\ k\text{-algs } B\end{matrix}\right\} \underset{U}{\overset{\text{Lie}}{\rightleftarrows}} \left\{\begin{matrix}\text{gdd}\\ \text{Lie algs}/k\end{matrix}\right\}$$

$\mathrm{Lie}(B) = B$ as k-mod w/ bracket $[a,b] \overset{\text{def}}{=} ab - (-1)^{|a||b|}ba$ "gdd commutator"

"universal enveloping algebra" $U(L) = \dfrac{\text{the gdd } k\text{-alg gen'd by basis of } L}{\left([a,b]-(ab-(-1)^{|a||b|}ba),\ a^{[2]} = a\cdot a \text{ if } |a| \text{ odd}\right)}$ ← really $=T(L)$ $= k$-span of monomials in elts of L

<u>Similarly</u>: can define a <u>dg</u> Lie algebra (dgla) (with a ∂)

The homotopy Lie algebra $\pi(R)$: (or $\pi^*(R)$)

Version 1: via acyclic closure

$$\begin{tikzcd} & R\langle X\rangle \arrow[dr, "\simeq"] & \\ R \arrow[ur, hook] \arrow[rr] & & k \end{tikzcd}$$

Lemma: $\mathrm{Der}^\gamma_R(R\langle X\rangle, R\langle X\rangle) \overset{*}{\subseteq} \mathrm{Lie}(\mathrm{End}_R(R\langle X\rangle))$ is a dg Lie subalgebra ($[d,d'] =$ gdd commutator)

Def: $\boxed{\pi(R) = H(\mathrm{Der}^\gamma(R\langle X\rangle, R\langle X\rangle))}$ = a Lie alg!

$\cong H(\mathrm{Der}^\gamma(\cancel{R}_k\langle X\rangle, k))$ $\overset{\text{Prop 2}}{=}$ $\mathrm{Hom}_k(kX, k) = kX^*$

Prop: The inclusion $*$ induces an injection (on homology)

$$\pi^*(R) \hookrightarrow \mathrm{Lie}(\mathrm{Ext}^*_R(k,k))$$

giving $U(\pi(R)) \xhookrightarrow{\cong} \mathrm{Ext}_R(k,k)$ iso (of algebras)

Version 2: via minimal model of min'l Cohen presentation $\hat{R} = Q/I$

$$\begin{tikzcd} & Q[Y] \arrow[dr, "\simeq"] & \\ Q \arrow[ur, hook] \arrow[rr] & & \hat{R} \end{tikzcd}$$

Σ = shift +1 in degree

Def: $\pi^{\geq 2}(R) = (\Sigma kY)^* = \mathrm{Hom}_k(\Sigma kY, k)$

$Q[Y]$ min'l model $\overset{\text{can}}{\underset{\text{show}}{\Longrightarrow}}$ $\partial(Y) \subseteq mY + QY^{\geq 2}$ products & powers

mod m

$k[Y]$ has $\partial(Y) \subseteq kY^{\geq 2}$

so $\partial = \partial^{[2]} + \partial^{[3]} + \dots$ where $\partial^{[i]}: kY \to kY^n$

Get $kY \xrightarrow{\partial^{[2]}} kY^2 \longrightarrow kY \times kY$ (fixing an deg-respecting ordering)

$y_i y_j \longmapsto y_i \times y_j$ if $y_i > y_j$

to get π^1 too

Dualizing gives bracket: $kY^* \times kY^* \longrightarrow kY^*$

$a, b \longmapsto [a,b]$

↳ OR: (Version 2')

$$\begin{tikzcd} & k[Y] \arrow[dr, "\simeq"] & \\ k \arrow[ur] \arrow[r, hook] & Q \arrow[r] & R \end{tikzcd}$$

now $Y = Y_0 \cup Y_1 \cup Y_2 \cup \cdots$

↑ to make surjection let $Q = k[Y_0] \twoheadrightarrow R$

and do same thing as above

so, now get $\pi_1(R) = \sum k Y_0^*$ too!

Brackets come from $\partial^{[2]}$ as above.

Rks:

- $\dim_k \pi^n(R) = \operatorname{card} X_n = \varepsilon_n(R)$!
- R Golod $\iff$ $\pi^{\geq 2}$ free odd Lie alg (Avramov-Löfwall)
- R Koszul algebra $\Rightarrow$ π gen'd by π^1
- skip • deformations $\rightsquigarrow$ central elts of $\pi(R)$?!
- skip • $\pi^1 \times \pi^1 \to \pi^2$ via Hessian (I) (Sjödin)
- nilpotent elements in the Lie alg $\pi(R)$ crucial to Briggs' recent proof of Vasconcelos's Conjecture!