# L3 : Koszul duality

## First, universal resolutions:

Let $M$ $R$-module.

Priddy's resn $P(R) \to k \to 0$ exact

$\Rightarrow$ ($k$ field) $P(R) \otimes_k M \to \underbrace{\cancel{k \otimes_k} M}_{\cong M} \to 0$ exact

$$\Rightarrow \quad \underbrace{\cdots \to R \otimes_k (R^!_1)^* \otimes_k M \to R \otimes_k \cancel{(R^!_0)^*}^{\,k} \otimes_k M}_{P(R) \otimes_k M \qquad \partial = \partial^{P(R)} \otimes 1_M} \xrightarrow{\mu} M \to 0$$

explicit $R$-free resn

(For $M \neq k$, highly non-min'l : ranks $\infty$ if $\dim_k M = \infty$)

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More generally, there is a duality producing this :

Consider the R-dual of the Priddy / Koszul cx

$$\boxed{\begin{aligned} C &= R \otimes_k R^! \\ \partial &= \sum_i x_i \otimes x_i^* \end{aligned}}$$

$\mathrm{Hom}_R(R \otimes_k R^{!*}, R)$

$\cong \mathrm{Hom}_k(R^{!*}, \cancel{\mathrm{Hom}_R(R,R)}^{\,R}_{\text{flat}/k})$

$\cong R \otimes R^{!**}$

$\cong R \otimes R^!$

($*$ = grdd dual)

or: each $R$-free on $k$-basis of $(R^!_i)^*$ – so dual $R$-free on basis $R^!_i$

Note: $C$ is an $R$-$R^!$-bimodule

**Thm:** (Koszul duality) [Beilinson – Ginsburg – Schechtman & further by Beilinson – Ginsburg – Soergel]

There is an equivalence of cats given by

$$D^b(R^!) \underset{R}{\overset{L}{\rightleftarrows}} D^b(R)$$

$$M \longmapsto L(M) = \mathrm{Tot}\left(C \otimes_{R^!} M\right) \quad \left(\text{as } R\text{-mods}: = R \otimes_k \cancel{R^! \otimes_{R^!}} M \cong R \otimes_k M\right)$$

hence M here

$$R(N) \longleftarrow N$$

$$= \mathrm{Tot}\left(\mathrm{Hom}_R(C, N)\right)$$

$$\left(\text{as } R\text{-mods}: = \mathrm{Hom}_R(R \otimes_k R^!, N) \cong \mathrm{Hom}_k(R^!, N) \cong (R^!)^* \otimes_k N \right) \text{ as } R\text{-mods}$$

totalized by:

- internal degree on $R^!$
- if $M$ is a cx (not just a module), then also the homological deg of $M$ (ie, totalize the bicomplex formed by $\partial^C$ & $\partial^M$)

**Note:**

- $(L, R)$ adjoint functors – classic Hom/$\otimes$
- $R, R^!$ flat/$k$ $\Rightarrow$ $C = R \overset{L}{\otimes}_k R^!$
- $C$ free/$R, R^!$ $\Rightarrow$ $L(M) = C \overset{L}{\otimes}_{R^!} N$, $R(N) = R\mathrm{Hom}_R(C, N)$ } but w/ twisted diff'l
- How to grade/totalize $L(M)$, $R(N)$?

  By grading on $R^!$ and hom'l grading on $M, N$

Idea of pf:

$$RL(M) \cong \underbrace{(R^!)^* \otimes_k R}_{P(R) \simeq k} \otimes_k M \simeq k \otimes_k M \cong M$$

$$LR(N) = \underbrace{R \otimes_k (R^!)^*}_{P(R) \simeq k} \otimes_k M \simeq k \otimes_k N \cong N$$

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[In fact, $LR(N)$ is that universal resn of $N$ we saw.]

Classic BGG:

Consider the case $R = k[x_1, \dots, x_n] = \dfrac{k\langle x_1, \dots, x_n\rangle}{(\{x_i x_j - x_j x_i\})}$

$= \text{symmetric alg } S(V)$

$$R^! = \frac{k\langle x_1, \dots, x_n\rangle}{(\{x_i^2\}, \{x_i x_j + x_j x_i\})} = \text{exterior alg } \Lambda(V) = E$$

$$(R^!)^* = E^* = \mathrm{Hom}_k(E, k) = \omega_E$$

Kostul duality for $S, \Lambda$ recovers the BGG correspondence:

$$\left\{\begin{matrix}\text{coherent}\\ \text{shvs on } \mathbb{P}^n\end{matrix}\right\}$$

$\simeq$

$$\left\{\begin{matrix}\text{gdd f.g.}\\ S\text{-mods}\end{matrix}\right\} \Big/ \begin{matrix}\text{fin length}\\ S\text{-mods}\end{matrix} \quad \underset{\text{duality}}{\overset{\text{kostul}}{\approx}} \quad \left\{\begin{matrix}\text{f.g. gdd}\\ \Lambda\text{-mods}\end{matrix}\right\} \Big/ \mathrm{perf}(\Lambda)$$

Whs do correspond to ?

Exercise (now...!)