

Problem Set 10: The inverse of the Tate resolution functor

Let k be a field and $S = k[x_0, \dots, x_n]$, with \mathbb{Z} -grading given by $\deg(x_i) = 1$. Let $E = \Lambda_k(e_0, \dots, e_n)$, with \mathbb{Z} -grading given by $\deg(e_i) = -1$.

1. (Short exercise) I'll deviate from the above setting for a moment: say $X = \mathbb{P}^1 \times \mathbb{P}^1$ with Cox ring $S = k[x_0, x_1, y_0, y_1]$ with \mathbb{Z}^2 -grading $\deg(x_i) = (1, 0)$, $\deg(y_i) = (0, 1)$ for $i = 0, 1$. Just as with weighted projective space, the minimal free resolution of $\mathbf{R}(S)$ is $\mathbf{R}(S)^*$. Show that $\text{cone}(\mathbf{R}(S)^* \xrightarrow{\sim} \mathbf{R}(S))$ cannot be the Tate resolution of the structure sheaf \mathcal{O} .

Solution. $H^i(X, \mathcal{F}) = 0$ for $i > 2$, since X is a surface. On the other hand, we have:

$$\text{cone}(\mathbf{R}(S)^* \xrightarrow{\sim} \mathbf{R}(S)) = \left(\bigoplus_{d \in \mathbb{Z}^2} \omega_E(-d, 0) \otimes_k S_d \right) \oplus \left(\bigoplus_{d \in \mathbb{Z}^2} \omega_E(d + (2, 2), 3) \otimes_k S_d \right).$$

If this were the Tate resolution, the summands on the right would imply that various twists of \mathcal{O} have nonvanishing H^3 , which is impossible. So, taking free resolutions doesn't work to build toric Tate resolutions, in general!

This problem set isn't about toric Tate resolutions; rather, we are returning to the standard graded case to discuss the inverse of Eisenbud-Fløystad-Schreyer's Tate resolution functor.

Throughout, we denote by \mathfrak{n} the maximal ideal of E .

2. Let Ω^i denote the cokernel of the $(i+2)^{\text{th}}$ differential in $K(x_0, \dots, x_n)$. Prove that the minimal free resolution of $\Omega^i[-i-1]$ is $\mathbf{L}(\omega_E/\mathfrak{n}^{n+1-i}\omega_E)$ for all $0 \leq i \leq n$. Conclude, using Problem Set 4 Exercise 5, that $\mathbf{L}(\omega_E(i)/\mathfrak{n}^{n+1-i}\omega_E(i))[1]$ is the minimal free resolution of $\Omega^i(i)$.

Comment: the associated sheaf $\widetilde{\Omega^j}$ on \mathbb{P}^n is $\Omega_{\mathbb{P}^n}^j$ for all $0 \leq j \leq n$, i.e. the j^{th} exterior power of the cotangent bundle on \mathbb{P}^n ; this follows from Theorem 17.16 in Eisenbud's "Commutative algebra with a view toward algebraic geometry".

3. Deduce that $\mathbf{R}(\Omega^i(i))$ is an injective resolution of $\omega_E(i)/\mathfrak{n}^{n+1-i}\omega_E(i)[1]$ for all i .

We are now going to use, without proof, the following two facts:

- (a) The functor \mathbf{R} induces an equivalence of categories between finitely generated graded S -modules and complexes F of finitely generated free E -modules with finitely generated homology such that F_i is a direct sum of copies of $\omega_E(i)$ (these are linear complexes, in a particularly strong sense). This follows from Proposition 2.1 in Eisenbud-Fløystad-Schreyer's paper.

- (b) If F and F' are injective resolutions of finitely generated graded E -modules N and N' , and F and F' are *minimal*, then $\mathrm{Hom}_E(F, F') \cong \mathrm{Hom}_E(N, N')$. This follows from standard homological algebra.

Using (a) and (b) along with the above two exercises, we conclude:

$$\mathrm{Hom}_S(\Omega^i(i), \Omega^j(j)) \cong \mathrm{Hom}_E(\omega_E(i)/\mathfrak{n}^{n+1-i}\omega_E(i), \omega_E(j)/\mathfrak{n}^{n+1-j}\omega_E(j)).$$

4. Prove that there is an isomorphism

$$\mathrm{Hom}_E(\omega_E(i), \omega_E(j)) \xrightarrow{\cong} \mathrm{Hom}_E(\omega_E(i)/\mathfrak{n}^{n+1-i}\omega_E(i), \omega_E(j)/\mathfrak{n}^{n+1-j}\omega_E(j)).$$

Hint: use that $\omega_E(i)$ is projective, and apply the universal property of a projective module.

That's it for the exercises: the rest of this document is just a discussion of how Exercise 3 leads to an inverse of the Tate resolution functor. Taking Tate resolutions gives a functor

$$D^b(\mathbb{P}^n) \rightarrow K(E),$$

where $K(E)$ denotes the homotopy category of (unbounded) exact complexes of finitely generated graded free E -modules. We define a functor going the other way by starting with a complex F in $K(E)$, replacing each occurrence of $\omega_E(j)$ with $\Omega_{\mathbb{P}^n}^j(j)$, and using the map

$$\mathrm{Hom}_E(\omega_E(i), \omega_E(j)) \xrightarrow{\cong} \mathrm{Hom}_S(\Omega^i(i), \Omega^j(j)) \rightarrow \mathrm{Hom}_{\mathbb{P}^n}(\Omega_{\mathbb{P}^n}^i(i), \Omega_{\mathbb{P}^n}^j(j))$$

to define differentials between the terms. Call this functor $\Omega: K(E) \rightarrow D^b(\mathbb{P}^n)$. For instance, since the Tate resolution of a sheaf \mathcal{F} has terms

$$T(\mathcal{F})_i = \bigoplus_{j=0}^n H^j(\mathbb{P}^n, \mathcal{F}(-i-j)) \otimes_k \omega_E(i+j),$$

the complex $\Omega(T(\mathcal{F}))$ has terms

$$\Omega(T(\mathcal{F}))_i = \bigoplus_{j=0}^n H^j(\mathbb{P}^n, \mathcal{F}(-i-j)) \otimes_k \Omega_{\mathbb{P}^n}^{i+j}(i+j).$$

In Theorem 6.1 of Eisenbud-Fløystad-Schreyer's paper, it is shown that this complex has nonzero homology only in degree 0, and this homology is \mathcal{F} ; a bit more work shows that Ω is an inverse for T . The complex $\Omega(T(\mathcal{F}))$ is called the *Beilinson monad* of \mathcal{F} . Beilinson monads have many applications across algebraic geometry and commutative algebra; a particularly interesting one can be found in Eisenbud-Schreyer's paper "Resultants and Chow forms via exterior syzygies".