Problem Set 9: Weighted Tate resolutions

Let $d_0, \dots, d_n \geqslant 0$. Let k be a field and $S = k[x_0, \dots, x_n]$, with $\mathbb{Z}$-grading given by $\deg(x_i) = d_i$. Let $E = \Lambda_k(e_0, \dots, e_n)$, with $\mathbb{Z}^2$-grading given by $\deg(e_i) = (-d_i, -1)$. Let $w \coloneqq \sum_{i=0}^n d_i$.

1. (Short exercise) Show that $\mathbf{R}(S)^*$ is the minimal free resolution of k.

Solution. We have $H(\mathbf{R}(S))_{(a,j)} = \operatorname{Tor}_j^S(S,k)_a$. Moreover, $\operatorname{Tor}_j^S(S,k)_a = 0$ unless $a = j = 0$, in which case it is k. Thus $H(\mathbf{R}(S)) = k$ concentrated in bidegree $(0,0)$. It follows that the composition of maps of differential modules $k \to \omega_E \hookrightarrow \mathbf{R}(S)$ is a quasi-isomorphism. Since dualizing over k is exact, the induced map $\mathbf{R}(S)^* \to k$ is a quasi-isomorphism. Note that $F := \mathbf{R}(S)^* \simeq \bigoplus_{i\geq 0} E(i,0) \otimes_k S_i$ with the induced differential. Since $\partial(\mathbf{R}(S)) \subseteq \mathfrak{m}_E \mathbf{R}(S)$ and $(-)^* = \operatorname{Hom}_E(-,E)(-n-1)$, it follows that $\partial(\mathbf{R}(S)^*) \subseteq \mathfrak{m}_E \mathbf{R}(S)^*$. Setting $F_i := \omega_E(i,0) \otimes_k S_i$, we have $F = \bigoplus_{i\geq 0} F_i$ and $\partial(F_i) \subseteq \bigoplus_{j<i} F_i$.

2. Prove that, if M is a graded S-module, then there is an isomorphism $\mathbf{R}(M(i)) \cong \mathbf{R}(M)(i,0)$ of differential E-modules for all $i \in \mathbb{Z}$.

Solution. We have

$$\begin{aligned}
\mathbf{R}(M(i)) &= \bigoplus_{d\in\mathbb{Z}} \omega_E(-d,0) \otimes_k M(i)_d \\
&= \bigoplus_{d\in\mathbb{Z}} \omega_E(-d,0) \otimes_k M_{i+d} \\
&= \bigoplus_{d\in\mathbb{Z}} \omega_E(-d-i,0)(i,0) \otimes_k M_{i+d} \\
&= \mathbf{R}(M)(i,0)
\end{aligned}$$

as graded E-modules, and the differentials are the same.

3. Suppose $n = 1$, $d_0 = 1$, and $d_1 = 2$. Let $M = S/(x_0^2 - x_1)$. Prove the following:

(a) $M_{\geqslant i} \cong M(-i)$ for all $i \geqslant 0$.

(b) $\mathbf{R}(M)$ is not quasi-isomorphic to its homology.

Notice the contrast with the standard graded case, where we have that $\mathbf{R}(M_{\geqslant r})$ is an injective resolution whenever $r \geqslant \operatorname{reg}(M)$.

Solution. (a) We have $\dim_k(M_i) = 1$ for all $i \geq 0$. The S-linear map of graded S-modules $M(-i) \to M_{\geq i}$ sending 1 in degree i to x_0^i is an isomorphism. (b) Note that from the formula $H(\mathbf{R}(M))_{(a,j)} = \operatorname{Tor}_j^S(M,k)_a$, we see that $H(\mathbf{R}(M))_{(a,j)} = k$ if $(a,j) = (0,0)$ or $(a,j) = (2,1)$

and 0 otherwise (to compute this Tor, resolve M over S via the Koszul complex on $x_0^2 - x_1$). Thus, $\mathbf{L}(H(\mathbf{R}(M)))$ is isomorphic to a complex of the form

$$0 \leftarrow S \xleftarrow{\ell} S(-2) \leftarrow 0.$$

The output of the functor $\mathbf{L}$ is always a *linear* complex, in the sense that the differentials must be matrices of linear forms. Thus, ℓ is either 0 or a linear form of degree 2, i.e. a k-multiple of x_1. We conclude that $\mathbf{L}(H(\mathbf{R}(M))$ is quasi-isomorphic to either $S \oplus S(-2)[-1]$ or $S/(x_1)$. If there is a quasi-isomorphism between $H(\mathbf{R}(M))$ and $\mathbf{R}(M)$, then, since $\mathbf{L}$ is exact (we didn't observe this in the multigraded case, but it's true and not too hard to prove), there exists a quasi-isomorphism between $\mathbf{LR}(M)$ and $\mathbf{L}(H(\mathbf{R}(M)))$ and hence between $\mathbf{L}(H(\mathbf{R}(M)))$ and M. (Once again, we didn't observe in the multigraded case that there are quasi-isomorphisms $\mathbf{LR}(C) \xrightarrow{\simeq} C$ and $D \xrightarrow{\simeq} \mathbf{RL}(D)$ for any $C \in \mathrm{Com}(S)$ and $D \in \mathrm{DM}(E)$, but this is true). Since M is not isomorphic to either $S \oplus S(-2)[-1]$ or $S/(x_1)$, we conclude that $H(\mathbf{R}(M))$ is not quasi-isomorphic to $\mathbf{R}(M)$.

4. Use your solution to (1) to compute the Tate resolution of $\mathcal{O}$. Then use your solution to recover the following calculation of the cohomology of $\mathcal{O}(j)$ on $\mathbb{P}(\underline{d})$:

$$H^i(\mathbb{P}(\underline{d}), \mathcal{O}(j)) = \begin{cases} S_j, & i = 0 \text{ and } j \geqslant 0; \\ S_{-j-w}, & i = n \text{ and } j \leqslant -w; \\ 0, & \text{else.} \end{cases}$$

Solution. We have $T(\mathcal{O}) = \mathrm{cone}(\mathbf{R}(S)^* \xrightarrow{\varepsilon} \mathbf{R}(S))$, where ε is the evident quasi-isomorphism. Thus, as graded E-modules, $T(\mathcal{O})$ is isomorphic to $(\bigoplus_{d \geq 0} \omega_E(w+d, n) \otimes_k S_d) \bigoplus (\bigoplus_{d \geq 0} \omega_E(-d, 0) \otimes_k S_d)$. Recall that $\dim_k H^i(\mathbb{P}(\underline{d}), \mathcal{O}(j))$ is the number of copies of $\omega_E(-j, i)$ in $T(\mathcal{O})$. From the description at hand of $T(\mathcal{O})$, the desired conclusions are easily read off.