L2: Koszul resolutions

Recall R is a Koszul alg if k has a linear resn.
In fact, then one can precisely give it:

$$R = \frac{T(V)}{(W)} = \frac{k\langle x_1, \dots x_n\rangle}{(W)} \quad (\underbrace{W \subseteq V^{\otimes 2}}_{\text{quadratic}}), \quad R^! = \frac{T(V^*)}{(W^\perp)}$$

Thm (Priddy): If R Koszul, then the following is a min'l gdd R-free resn of k:

$\boxed{P(R):}$

Priddy cx/resn or Koszul resn

$$\cdots \xrightarrow{\partial} R \otimes_k \underbrace{(R^!_2)^*}_{k \text{ v.sp}} \xrightarrow{\partial} R \otimes_k \underbrace{(R^!_1)^*}_{k \text{ v.sp}} \xrightarrow{\partial} \underbrace{R \otimes_k \underbrace{\cancel{(R^!_0)^*}}_{k \text{ v.sp.}}}_{R} \to k \to 0$$

$$\partial = \sum_{i=1}^n x_i \otimes x_i^* \quad \text{(right mult. by "trace elt")}$$

where x_i^* induced from $R^!_{m-1} \xrightarrow{x_i^*} R^!_m$ $\Big) \mathrm{Hom}_k(-,k)$

$$(R^!_m)^* \xrightarrow{x_i^*} (R^!_{m-1})^*$$

i.e, removes an x_i^* factor (but after possible rewriting via relns in $R^!$)

i.e, transpose matrix

So, $\partial(1 \otimes \overline{x_{i_1}^* x_{i_2}^* \cdots x_{i_m}^*})$

$$= \sum_{i=1}^n x_i \otimes \overline{x_{i_1}^* x_{i_2}^* \cdots x_{i_m}^*} \cdot x_i^*$$

linear coeffs ✓

remove if matches (possibly rewrite) (sum!)

Note: $\beta_i(k) = \mathrm{rank}_R P(R)_i$
$= \mathrm{rank}_R R \otimes_k (R^!_i)^*$
$= \dim_k R^{!*}_i = \dim_k R^!_i$

growth = growth gdd pieces of $R^!$

$R \otimes_k k^n \cong (R \otimes_k k)^n \cong R^n$

Equivalent conditions: (assume $R = \frac{T(V)}{(W)}$ quadratic alg)

Note: 1) $(R^!)^! = R$ for any quadratic alg
(obvious: $(W^\perp)^\perp = W$) ← orthogonal complements

2) one can still form Priddy's cx for quadratic alg and it's linear

$\boxed{\text{Priddy's Thm says: } R \text{ koszul} \iff P(R) \xrightarrow{\simeq} k \text{ resn (ie, exact)}}$
(k has linear resn)

$$\boxed{\text{Thm: } R \text{ koszul} \Rightarrow \underbrace{\mathrm{Ext}_R(k,k)}_{\bigoplus \mathrm{Ext}_R^i(k,k)} \cong R^!_\bullet \text{ as } k\text{-algebras}}$$

Pf: Let $P \xrightarrow[R]{\simeq} k$ be Priddy's free resn

Then $\mathrm{Ext}_R(k,k) = H(\mathrm{Hom}_R(P,P))$ ← dga under composition (hw 1) so, Ext is (assoc) alg ✓

$$\cong H(\mathrm{Hom}_R(P,k))$$

$$= H(\mathrm{Hom}_R(R \otimes_k (R^!_\bullet)^*, k))$$

$$= \cancel{H}(\mathrm{Hom}_k((R^!_\bullet)^*, k)) \quad \partial = 0$$

$$= (R^!_\bullet)^{**} = R^!$$

[actually agrees w/ Yoneda product up to ± (not obvious)]

where $R^!$ alg $\Rightarrow (R^!)^*$ co-alg $\Rightarrow P$ (dg) co-alg

$\Rightarrow \mathrm{Hom}_R(P,k)$ alg ↑ can check!

$\Uparrow$ is

$\mathrm{Hom}_R(P,P)$ alg under comp

Converse ?

Thm: For any quadratic alg R,

$$R^! = \underbrace{\text{"diagonal part" } \mathrm{Ext}_R^{\Delta}(k,k)}_{\substack{\overset{\text{def}}{=} \bigoplus_{i \geq 0} \mathrm{Ext}_R^i(k,k)_i \\ = \bigoplus_i k^{\beta_{ii}}}} \text{ of } \mathrm{Ext}_R(k,k)$$

as algebras!

Pf: k has linear resn $\iff \beta_{ij} = 0,\ i \neq j$

$\iff \mathrm{Ext} = \mathrm{Ext}^{\Delta}$

Cor: R koszul $\iff \mathrm{Ext}_R(k,k) \cong R^!$ as k-algs

Pf: $\Rightarrow$ ✓

$\Leftarrow$ by Thm, $R^! \xrightarrow{\cong} \mathrm{Ext}^{\Delta}(k,k) \overset{i}{\hookrightarrow} \mathrm{Ext}(k,k)$

but $\mathrm{Ext}_R^i(k,k)_j = k^{\beta_{ij}} \leftarrow$ gdd Betti #

so, i iso $\Rightarrow \beta_{ij} = 0\quad i \neq j$

In particular: R koszul $\Rightarrow \mathrm{Ext}(k,k) = k\langle \mathrm{Ext}^1(k,k)_1 \rangle$
$$= k\langle \mathrm{Ext}^1(k,k) \rangle$$

since $R^!$ is so.

Thm: R koszul $\iff R^!$ koszul

Pf: exercise today...

Note: we have, defining $E(R) = \mathrm{Ext}_R(k,k)$ for any stdgdd k-alg R

$$R \text{ koszul} \iff E(E(R)) \cong R$$

(Warning: in some sources, this (or other conditions above) is given as the def of a koszul alg!)