Problem Set 9: Weighted Tate resolutions

Let $d_0, \dots, d_n \geqslant 0$. Let k be a field and $S = k[x_0, \dots, x_n]$, with $\mathbb{Z}$-grading given by $\deg(x_i) = d_i$. Let $E = \Lambda_k(e_0, \dots, e_n)$, with $\mathbb{Z}^2$-grading given by $\deg(e_i) = (-d_i, -1)$. Let $w \coloneqq \sum_{i=0}^n d_i$.

1. (Short exercise) Show that $\mathbf{R}(S)^*$ is the minimal free resolution of k.

2. Prove that, if M is a graded S-module, then there is an isomorphism $\mathbf{R}(M(i)) \cong \mathbf{R}(M)(i, 0)$ of differential E-modules for all $i \in \mathbb{Z}$.

3. Suppose $n = 1$, $d_0 = 1$, and $d_1 = 2$. Let $M = S/(x_0^2 - x_1)$. Prove the following:

(a) $M_{\geqslant i} \cong M(-i)$ for all $i \geqslant 0$.

(b) $\mathbf{R}(M)$ is not quasi-isomorphic to its homology.

Notice the contrast with the standard graded case, where we have that $\mathbf{R}(M_{\geqslant r})$ is an injective resolution whenever $r \geqslant \operatorname{reg}(M)$.

4. Use your solution to (1) to compute the Tate resolution of $\mathcal{O}$. Then use your solution to recover the following calculation of the cohomology of $\mathcal{O}(j)$ on $\mathbb{P}(\underline{d})$:

$$H^i(\mathbb{P}(\underline{d}), \mathcal{O}(j)) = \begin{cases} S_j, & i = 0 \text{ and } j \geqslant 0; \\ S_{-j-w}, & i = n \text{ and } j \leqslant -w; \\ 0, & \text{else.} \end{cases}$$