Week 2 • Problem Set 5 • Ext algebras

Quick Problem:

Compute bases for $\mathrm{Ext}_R^{\leq 2}(k,k)$ for the algebra $R = k[x,y]/(xy, x^3+y^3)$. In what (internal) degrees do they live? Hint: Write out a *graded* resolution of k.

Problems:

1. Let $R = k[x,y]/(x^2, xy)$, and write the acyclic closure F as $R\langle e_1,\, e_2,\, f_1,\, f_2 \ldots\rangle$ with

$$\partial(e_1) = x, \quad \partial(e_2) = y, \quad \partial(f_1) = xe_1, \text{ and } \partial(f_2) = ye_1,$$

Order the bases of F_1, F_2, and F_3 by

$$\{e_1,\, e_2\}\text{and } \{e_1e_2,\, f_1\, f_2\}. \tag{0.1}$$

Dualizing the acyclic closure and taking homology, one has

$$\mathrm{Ext}_R(k,k) \cong H(\mathrm{Hom}_R(F,k)) \cong \mathrm{Hom}_R(F,k)$$

and so we can represent $\mathrm{Ext}_R^{\leq 2}(k,k)$ as the k-span of duals of the basis elements in 0.1. One also has

$$\mathrm{Ext}_R(k,k) \cong H(\mathrm{Hom}_R(F,k)) \cong H(\mathrm{Hom}_R(F,F))$$

which we use to compute the composition product in Ext. So, by lifting along the quasi-isomorphism $F \to k$, we can find elements in $\mathrm{Hom}_R(F,F)$ whose classes represent the duals of the elements of 0.1, yielding, for example:

$$e_1^* = \ldots \quad \begin{tikzcd}
R^5 \arrow[r, "\begin{bmatrix} 0 & x & 0 & y & x \\ x & y & 0 & 0 & 0 \\ 0 & 0 & x & y & 0 \end{bmatrix}"] \arrow[d, "?"] & R^3 \arrow[r, "\begin{bmatrix} -y & x & y \\ x & 0 & 0 \end{bmatrix}"] \arrow[d, "?"] & R^2 \arrow[r, "\begin{bmatrix} x & y \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 1 & 0 \end{bmatrix}"] & R \arrow[d, "0"] \\
R^3 \arrow[r, "\begin{bmatrix} -y & x & y \\ x & 0 & 0 \end{bmatrix}"'] & R^2 \arrow[r, "\begin{bmatrix} x & y \end{bmatrix}"'] & R \arrow[r] & k
\end{tikzcd}$$

$$e_2^* = \ldots \quad \begin{tikzcd}
R^5 \arrow[r, "\begin{bmatrix} 0 & x & 0 & y & x \\ x & y & 0 & 0 & 0 \\ 0 & 0 & x & y & 0 \end{bmatrix}"] \arrow[d, "?"] & R^3 \arrow[r, "\begin{bmatrix} -y & x & y \\ x & 0 & 0 \end{bmatrix}"] \arrow[d, "?"] & R^2 \arrow[r, "\begin{bmatrix} x & y \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 0 & 1 \end{bmatrix}"] & R \arrow[d, "0"] \\
R^3 \arrow[r, "\begin{bmatrix} -y & x & y \\ x & 0 & 0 \end{bmatrix}"'] & R^2 \arrow[r, "\begin{bmatrix} x & y \end{bmatrix}"'] & R \arrow[r] & k
\end{tikzcd}$$

$$(e_1e_2)^* = \ldots \quad \begin{tikzcd}
R^5 \arrow[r, "\begin{bmatrix} 0 & x & 0 & y & x \\ x & y & 0 & 0 & 0 \\ 0 & 0 & x & y & 0 \end{bmatrix}"] \arrow[d, "?"] & R^3 \arrow[r, "\begin{bmatrix} -y & x & y \\ x & 0 & 0 \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}"] & R^2 \arrow[r, "\begin{bmatrix} x & y \end{bmatrix}"] \arrow[d, "0"] & R \arrow[d] \\
R^2 \arrow[r, "\begin{bmatrix} x & y \end{bmatrix}"'] & R \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

and so on, with f_1^* starting with $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, f_2^* with $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$.

(a) Fill in the lifts above, making sure that each square commutes.

(b) Show that $(e_2^*)^2 = 0$. Thinking of Ext as the quadratic dual of R, with which element of $R_1^!$ may e_2^* be identified with?

(c) Recall that $e_1^2 = 0$. Does this mean that $(e_1^*)^2 = (e_1^2)^* = 0$?

(d) Compute $(e_1)^*(e_2)^*$ and $(e_2)^*(e_1)^*$. How do they relate to $(e_1e_2)^*$?

(e) Is it possible to obtain $(f_2)^*$ as a product (or sum of products) of $(e_1)^*$ and $(e_2)^*$? If so, determine how to express $(f_2)^*$ in such a manner.

(f) Fill in the shifts on the free modules in the diagrams above for e_1^* and e_2^*. Explain the connection between an element of $\mathrm{Ext}_R^i(k,k)$ having internal degree i and the degrees of the entries appearing in its lift to an element of $\mathrm{Hom}_R(F,F)$.

(g) Recall that the quadratic dual of R is $k\langle x^*, y^*\rangle/(y^*)^2$. How does this compare to your calculation above? If you have time, you might compute the first few steps of the Priddy resolution, or you might consider how different choices of images of f_1 and f_2 in the acyclic closure would change how the products are computed.