Week 1 • Problem Set 1 • DG algebras

Quick Problem:

Let $Q = k[x, y]$ and $I = (x^2, xy)$.
Find an explicit DG algebra structure on the minimal graded Q-free resolution of $R = Q/I$.

Note: This is a particular example of a Taylor resolution, which exists for any monomial ideal in a polynomial ring, however is not always minimal. In fact, every Taylor resolution is known to have a DG algebra structure.

Problems:

Recall we assume DG algebras to be graded commutative, that is, $ab = (-1)^{|a||b|}ba$ for all a, b.

1. Prove that the Koszul complex is a DG algebra. (You may assume the exterior algebra is an algebra, and so you need only verify that it satisfies the Leibniz rule.)

2. If A is a DG algebra, then show that its homology $H(A)$ inherits an algebra structure. (Of course, it inherits graded commutativity, meaning that $ab = (-1)^{|a||b|}ba$ for all $a, b \in A$.)

3. (a) Show that $\operatorname{End}_R X = \operatorname{Hom}_R(X, X)$ is a (not necessarily commutative) dg algebra for any complex X. Here the differential is the graded commutator with ∂^X. That is, $\partial^{\operatorname{End}_R X}(\alpha) = \partial^X \alpha - (-1)^{|\alpha|}\alpha\partial^X$.

[Do not spend long verifying this; just think about it briefly.]

(b) Apply part (b) to $\operatorname{Hom}_R(P, P)$ for a projective resolution P of an R-module M to get algebra structure on $\operatorname{Ext}_R(M, M)$.

(c) More generally, show that $\operatorname{Hom}_R(X, Y)$ is a DG $\operatorname{End}_R Y$-$\operatorname{End}_R X$-bimodule (and deduce that $\operatorname{Ext}_R(M, N)$ is a graded $\operatorname{Ext}_R(N, N)$-$\operatorname{Ext}_R(M, M)$-bimodule)

[Do not spend long verifying this; just think about it briefly.]