# Week 2 • Problem Set 2 • Koszul resolution

**Quick Problem:**

Let $R = k[x,y]/(x^2 + y^2)$. Write out the first 2 steps of the Priddy resolution

$$\cdots P_3 \to P_2 \to P_1 \to P_0 \to k \to 0.$$

**Problems:**

**1.** Let $R$ be a graded ring and $M$ a graded $R$-module. Define the Hilbert series $H_M^R(t)$ of $M$ and the diagonal of the Poincaré series $P_M^{R,\Delta}(s,t)$ as

$$P_M^{R,\Delta}(t,s) = \sum_{i \geq 0} \beta_{i,i}(M)(st)^i \qquad H_M^R(t) = \sum_{i \geq 0} \dim_k M_i t^i$$

where the $(i,i)$-th Betti number $\beta_{i,i}$ is the number of basis elements of homological degree $i$ and internal degree $i$ in an $R$-free resolution of $M$. (These definitions apply equal well to $R^!$, using graded resolutions by left free modules.)

Prove the following equalities of power series.

(a) $P_k^{R,\Delta}(t,s) = H_{R^!}(st)$ (here, $R^!$ is considered to be positively graded)

(b) $H_R(t)P_k^R(-t) = 1$ if $R$ is a Koszul algebra. Hint: Think about Priddy's resolution and Euler characteristics (i.e., vector space dimensions along exact sequences).

**2.** The following are all Koszul algebras. Compute the first few steps of the Priddy/Koszul resolution of $k$.

(a) $R = k[x,y]/(x^2, xy)$
(and, if you are combinatorially-minded, find the Poincaré series in closed form sometime)

(b) $R = S = k[x_1, \ldots, x_n]$ the symmetric algebra

(c) $R = \wedge(k\{e_1, \ldots, e_n)\}$ the exterior algebra

(d) $R =$ the trivial algebra over $k$ on $x_1, \ldots, x_n$ (that is, all products of positive degree elements are zero).

**3.** Consider the noncommutative ring $R = k\langle x,y\rangle/(x^2 + y^2, xy + yx)$.

(a) Show that $R$ is Koszul. (Hint: Consider its dual.)

(b) Show that $\operatorname{projdim}_R k = \infty$.

**4.** Let $R$ be a graded quadratic algebra. Prove that $R$ is Koszul if and only if $R^!$ is Koszul.

(You may assume Priddy's theorem.)

Hint: Apply $\operatorname{Hom}_k(-,k)$ and regrade!)

**5.** Let $R$ be a quadratic (commutative) hypersurface $R = Q/(f)$. Show explicitly how the Priddy resolution is isomorphic to Tate's dg algebra resolution from Week 1.

(Note that $Q \to R \to k$ is an embedded complete intersection.)

**6.** Let $\langle -, - \rangle \colon V \otimes V \to k$ be a perfect pairing, where $V$ is the $k$-vector space with basis $\{x, y\}$. Consider the "short Gorenstein algebra" $R = k \oplus V \oplus k$ in degrees 0,1, and 2, with product given by the perfect pairing (and $k$-scalar products).

Show that $R^! = T(V^*)/(q)$ (a noncommutative hypersurface!) where $q$ is the quadratic form of the pairing.