L4: Relation to bar resn (Priddy's story) (technical!)

There is another canonical resn, namely the bar resn, which exists for any augmented k-algebra

$$\begin{tikzcd} (k \arrow[r, hook] \arrow[rr, bend right, "1"] & R \arrow[r] & k). \end{tikzcd}$$

(history in alg topology, gp cohomology)

For comparison to Priddy's resn, first do:

Equivalent description of $R^!$ for a quadratic algebra

(also coordinate-free)

Given $W \subseteq V \otimes V = T(V)_2$ k-subspace

have $0 \to W \to V \otimes V \to \text{coker} \to 0$

dual $\curvearrowright$ $0 \to (\text{coker})^* \to V^* \otimes V^* \to W^* \to 0$ (*)

Claim 1: $(\text{coker})^* = W^\perp \leftarrow$ in any coords!

Pf: $W^\perp = \{ p = \sum \lambda_{ij} x_i^* x_j^* \mid \langle w, p \rangle = 0 \ \forall w \in W \}$

whereas

$(\text{coker})^* = \{ f = \sum \lambda_{ij} x_i^* x_j^* \mid f|_W = 0\text{-map} \}$

but for $w = \sum \mu_{ij} x_i x_j$, $f(w) = \sum \lambda_{ij} \mu_{ij} = \langle w, p \rangle$ //

[By (*), this gives $(W^\perp)^* \cong \frac{V \otimes V}{W}$]

Claim 2: $(R^!)^* \cong \bigoplus_n \left(\bigcap_{i+2+j=n} V^{\otimes i} \otimes \underbrace{W}_{\subseteq V \otimes V} \otimes V^{\otimes j} \right) \subseteq \bigoplus_n V^{\otimes n} = T(V)$

Pf: By def, $0 \to \bigoplus_n (V^*)^{\otimes i} \otimes W^\perp \otimes (V^*)^{\otimes j} \to T(V^*) \to R^! \to 0$

dual $\curvearrowright$ $0 \to (R^!)^* \to T(V) \to \bigoplus_n V^{\otimes i} \otimes \underbrace{(W^\perp)^*}_{\cong V \otimes V / W} \otimes V^{\otimes j}$ //

Bar resolution $\supseteq$ Priddy resn

Let R be std grdd k-algebra

The bar resn of any R-mod M is: [Eilenberg-MacLane 53]

twisted $\otimes$ means ∂ involves both factors

$$B(R,R,M)_n \overset{\text{def}}{=} R \overset{\tau}{\underset{k}{\otimes}} B(R)_n \overset{\tau}{\underset{k}{\otimes}} M$$

$$= R \underset{k}{\otimes} \underbrace{\bar{R} \underset{k}{\otimes} \cdots \underset{k}{\otimes} \bar{R}}_{n} \underset{k}{\otimes} M$$

where

$$\bar{R} = R_+ = \bigoplus_{n>0} R_i = \mathfrak{m}$$

$$\partial(r \otimes r_1 \otimes \cdots \otimes r_n \otimes m)$$

$$= \underbrace{rr_1 \otimes r_2 \otimes \cdots \otimes r_n \otimes m}_{\text{call this part of } \partial \; d} \underbrace{- r \otimes r_1 r_2 \otimes \cdots \otimes r_n \otimes m + \cdots + (-1)^i r \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n \otimes m \pm \cdots}_{\text{this part of } \partial \text{ is called } b}$$

$$\underbrace{+ (-1)^n r \otimes r_1 \otimes \cdots \otimes r_{n-1} \otimes}_{\text{call this part } d'} \underbrace{r_n \cdot m}_{=0 \text{ for } M=k}$$

$$\boxed{\text{Thm: } B = B(R,R,M) \xrightarrow[R]{\sim} M \text{ is an } R\text{-free resn of } M}$$

It is very non-min'l: $R \underset{k}{\otimes} \underbrace{\bar{R}^{\otimes_k i}}_{\underline{\underline{\text{huge}}}\ k\text{-v.sp.}} \otimes M$ = huge free R-mod

but $\underline{\underline{\text{universal}}}$ (same form any R, M).

Let's take $M = k$ (for simplicity) So, $B(R,R,k)_n = R \underset{k}{\otimes} \underbrace{\bar{R} \underset{k}{\otimes} \cdots \underset{k}{\otimes} \bar{R}}_{n} \cancel{\underset{k}{\otimes} k}$

Separate B out by graded pieces:

Note 1st:

$$R = \frac{T(V)}{(W)} = \underset{\substack{\| \\ R_0}}{\cancel{k}} \oplus \underset{\substack{\| \\ R_1}}{\boxed{V}} \oplus \underset{\substack{\| \\ R_2}}{\boxed{\frac{V^{\otimes 2}}{W}}} \oplus \underset{\substack{\| \\ R_3}}{\boxed{\frac{V^{\otimes 3}}{V \otimes W + W \otimes V}}} \oplus \cdots$$

delete in $\bar{R}$

cont'd

$B = R \otimes_k$ $B(R)$ portion:

$B = R \otimes_k$

b maps: 0,1's

d maps: min'l (im d ⊆ m⊗...)

$$\begin{tikzcd}
 & & & R_1^{\otimes 4} \arrow[d] \arrow[dl, dashed, "d"'] & B_4 \arrow[l] \\
 & & R_1 \otimes R_1 \otimes R_1 \arrow[d, "b"] \arrow[dl, dashed, "d"'] & \cdots \arrow[d] \arrow[dl, dashed, "d"'] & B_3 \arrow[l] \\
 & R_1 \otimes R_1 \arrow[d, "b"] \arrow[dl, dashed, "d"'] & R_1 \otimes R_2 \oplus R_2 \otimes R_1 \arrow[d, "b"] \arrow[dl, dashed, "d"'] & \cdots \arrow[d] \arrow[dl, dashed, "d"'] & B_2 \arrow[l] \\
R_1 & R_2 \arrow[l, dashed, "d"'] & R_3 & R_4 & B_1 \arrow[l]
\end{tikzcd}$$

The kernel of each top vertical b map forms a subcx:

$R \otimes_k$

$$\begin{tikzcd}
 & & \cdot^{\cdot^{\cdot}} \arrow[dl, dashed] \\
 & \ker b \arrow[dl, dashed] & \\
\ker b \arrow[d, dashed] & & \\
R_1 & &
\end{tikzcd}$$

Note: each $R \otimes R_{i_1} \otimes \cdots \otimes R_{i_s}$ has R-basis sitting in deg $i_1 + \cdots + i_s$ so, $\cong \bigoplus R(-i_1 - \cdots - i_s)$

so, shifts can be read off easily.

In column 2:

$$\ker(R_1 \otimes_k R_1 \xrightarrow{b} R_2) = \ker\left(V \otimes_k V \longrightarrow \frac{V \otimes V}{W}\right) = W \underset{\uparrow}{=} (R_2^!)^*$$

by Claim 2

In general:

$$\ker\Big(\underbrace{R_1 \otimes \cdots \otimes R_1}_{n} \to \bigoplus_{i+j+2=n} \underbrace{R_1 \otimes \cdots \otimes}_{i} R_2 \otimes \underbrace{\cdots \otimes R_1}_{j}\Big) = \bigcap_{i+j+2=n} \underbrace{V \otimes \cdots}_{i} \otimes W \underbrace{\otimes \cdots \otimes V}_{j} = (R_n^!)^*$$

Inherited diff'l: just d

$$d(\alpha) = d(1 \otimes \underbrace{x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n}}_{\wr\downarrow}) = 1 \cdot x_{i_1} \otimes \underbrace{x_{i_2} \otimes \cdots \otimes x_{i_n}}_{\wr\downarrow} = \alpha \bullet (x_{i_1} \otimes x_{i_1}^*) = \partial(\alpha) \text{ in } P(R)!$$

$\uparrow$ basis elt $\longrightarrow$ $x_{i_1}^* x_{i_2}^* \cdots x_{i_n}^* \in (R_n^!)^*$ if basis elt $x_{i_2}^* \cdots x_{i_n}^* \in (R_{n-1}^!)^*$

so, $P(R) \hookrightarrow B = B(R, R, k)$ [Both res'ns of k, but P(R) min'l]