L5 : ~~Ext~~, formality, topology

skip in lecture

Actually replaced this lecture completely with Addendum to dg lecture 5.

(I) Ext (skip this section)

Last time : $\mathrm{Tor}^R(k,k) = H\left(k \cancel{\otimes_R} \underbrace{\cancel{R} \overset{\tau}{\otimes} B(R)}_{B}\right) = H(B(R), b)$

$\mathrm{Ext}_R(k,k) = H\left(\mathrm{Hom}_{\cancel{R}}(\cancel{R} \overset{\tau}{\otimes} B(R), k)\right) = H(B(R)^*, b)$

(R koszul $\iff$ these are diagonal (internal deg = hom'l deg))

In gen'l : $\widehat{\mathrm{Ext}_R}(k,k) = \langle \mathrm{Ext}^1(k,k) \rangle = R^!$

Löfwall

pf uses $\mathrm{Ext}_R(k,k) = U(\pi(R))$, $\pi(R)$ = htpy Lie alg

and so just show $\pi(R)$ is diagonal

Use min'l model to compute...

By the way: [If R koszul, $Q[X] \xrightarrow{\simeq} R$ has $\partial = \partial^{[2]}$!
In gen'l, use $\partial^{[2]}$ part to compute $\pi(R)$]

I might post this proof here later.

II (co)formality

Thm 1 [Keller]

R local k-alg (not nec. commut), $R/\mathfrak{m}_R \cong k$

There is an equiv. of cats

$$D^{fl}(R) \simeq D^{perf}(\underbrace{R\mathrm{Hom}_R(k,k)}_{\text{dga } (= \mathrm{Hom}_R(F,F),\ F \xrightarrow{\simeq} k)})$$

Thm 2 [Guggenheim–May / Keller / Berglund / Priddy]

R koszul $\iff$ $R\mathrm{Hom}_R(k,k)$ formal

(ie, $\exists$ zig-zag of quasi-isos as dgas

$$R\mathrm{Hom}_R(k,k) \xrightarrow{\simeq} \cdot \xleftarrow{\simeq} H(R\mathrm{Hom}_R(k,k))$$
$$= \mathrm{Ext}_R(k,k) \stackrel{\text{def}}{=} E(R)$$

we say R is "coformal".

Cor of Thms 1 & 2: (R gdd $\rightsquigarrow$ localize, equiv)

If R koszul, $D^{fl}(R) \simeq D^{perf}(\underbrace{\mathrm{Ext}_R(k,k)}_{=R^!})$

(hence also $D^{fl}(R^!) \simeq D^{perf}(R)$ since $R^!$ is also koszul then).

We have seen the Cor:

$$D^b(R^!) \underset{R}{\overset{L}{\rightleftarrows}} D^b(R)$$

$$k \longmapsto R \otimes_k k = R$$

$$\underbrace{k}_{\text{gens } D^{fl}(R^!)} \simeq (R^!)^* \otimes_k R \longleftarrow\!\!| \underbrace{R}_{\text{gens } D^{perf}(R)}$$

so also get $\boxed{D^b(R^!)/D^{fl}(R^!) \simeq D^b(R)/D^{perf}(R)}$ $\longleftarrow$ gives BGG if $R = S$, $R^! = E$, $(R^!)^* = \omega_E$

III Topology

let X = topological space.

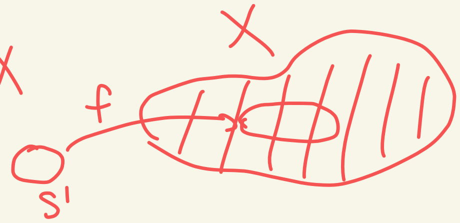
If $\underbrace{A = C^*(X, \mathbb{Q})}_{\text{a dga!}}$ is the cochain cx for rational (coeff in $\mathbb{Q}$) singular cohomology $(H(A) = H^*(X; \mathbb{Q}))$

then its Koszul dual (as dga) is:

$$A^! = C_*(\Omega X; \mathbb{Q})$$

(unbased)≠(free) loop space on X

$\Omega X = \mathrm{Map}(S^1, X)$



Berglund: (2014 TAMS)

Def: X is a Koszul space if it's simply connected and both A is formal & $\underbrace{\text{coformal}}_{\text{ie, } A^! \text{ is formal}}$

(a formal cx is one w/a zig-zag of quasi-isos to its homology)

IV Koszulity for dgas

Def: a dg algebra A is called Koszul if A is $\underbrace{\text{formal}}_{A \simeq H(A)}$ and $\underbrace{\text{coformal}}$

$A^! \overset{\text{def}}{=} R\mathrm{Hom}_A(k, k)$ is formal

ie, $A^! \simeq \mathrm{Ext}_A(k, k)$

V Relative notion:

$R \xrightarrow{\varphi} S$ local homom iso. on residue fields

Def (Briggs-Cameron-Letz-Pollitz, 2024)

φ is Koszul if the derived fiber $F (= k \otimes^L_R S)$ is formal and coformal (dga)

($\overset{\text{Thm 2}}{\underset{\text{today}}{\Longrightarrow}}$ ∃ grading s.t. $H(F)$ is a Koszul alg)