## Day 5 Geometric BGG Part 1

Graded $S$-modules induce sheaves on $\mathbb{P}^n$.

Eisenbud - Fløystad - Schreyer, 03: There is a version of BGG for sheaves on $\mathbb{P}^n$, and the functors can be computed explicitly.

Crash course on sheaves and sheaf cohomology:

Projective $n$-space:

$\mathbb{P}^n = k^{n+1} \setminus \{0\} / \sim$ , $(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$, $\lambda \in k \setminus \{0\}$.

Def'n: (Sketch) A <u>(coherent)</u> <u>sheaf of modules</u> on $\mathbb{P}^n$ is a collection of (fg) $S[\frac{1}{x_i}]_0$-modules $M_i$ for $0 \le i \le n$ satisfying:

$S[\frac{1}{x_i x_j}]_0$ - linear

(1) $M_i \otimes_{S[\frac{1}{x_i}]_0} S[\frac{1}{x_i x_j}]_0 \cong M_j \otimes_{S[\frac{1}{x_j}]_0} S[\frac{1}{x_i x_j}]_0$ for all $i,j$

(2) The isomorphisms in (1) are "compatible". (cocycle condition)

Idea: a sheaf of modules on $\mathbb{P}^n$ is the result of gluing $n+1$ modules over a polynomial ring.

Analogy: sheaves are to $\mathbb{P}^n$ as modules are to $S$.

Any (fg) graded $S$-module $M$ determines a (coherent) sheaf $\widetilde{M}$ on $\mathbb{P}^n$: $M[\frac{1}{x_0}]_0, \ldots, M[\frac{1}{x_n}]_0$.

Every coherent sheaf on $\mathbb{P}^n$ arises this way. But $M \mapsto \widetilde{M}$ is not a bijection.

Ex: If $\dim_k M < \infty$, then $M[\frac{1}{x_i}] = 0$ $\forall i$, so $\widetilde{M} = 0$.

Notation: $\mathcal{O}(i) := \widetilde{S(i)}$.
$\widetilde{M}(j) := \widetilde{M} \otimes \mathcal{O}(j) (\cong \widetilde{M(j)})$.
$\mathrm{coh}(\mathbb{P}^n)$: category of coherent sheaves on $\mathbb{P}^n$.

Exact sequences of sheaves make sense in $\mathrm{coh}(\mathbb{P}^n)$.

If $M' \to M \to M''$ is an exact sequence of gdd $S$-modules,

$\widetilde{M'} \to \widetilde{M} \to \widetilde{M''}$ is exact.

$\Rightarrow$ If $M' \subseteq M$, and $\dim_k M/M' < \infty$, then $\widetilde{M'} \cong \widetilde{M}$.

Sheaf cohomology: Let $M \in \mathrm{Mod}(S)$. The Čech complex of $\widetilde{M}$ is

the complex $\check{C}(\widetilde{M})$ of $k$-vector spaces w/ $i^{th}$ term

$$\check{C}^i(\widetilde{M}) = \bigoplus_{0 \le j_0 < \cdots < j_i \le n} M\left[\frac{1}{x_{j_0} \cdots x_{j_i}}\right]_0 \quad (\text{for } 0 \le i \le n).$$

Write an element $\alpha \in \check{C}^i(\widetilde{M})$ as a tuple $(\alpha_{j_0, \ldots, j_i})$.
The differential $\check{C}(\widetilde{M})^i \longrightarrow \check{C}(\widetilde{M})^{i+1}$ sends $\alpha$ to the tuple $d\alpha$
such that $(d\alpha)_{j_0 \cdots j_{i+1}} := \sum_{\ell=0}^{i+1} (-1)^\ell \alpha_{j_0, \ldots, \widehat{j_\ell}, \ldots, j_{i+1}}$.

(This does not depend on the choice of $M$).

Ex: $n=1$, $M = S(j)$, $j \in \mathbb{Z}$.

$$\check{C}(\mathcal{O}(j)) = S\left[\tfrac{1}{x_0}\right]_j \oplus S\left[\tfrac{1}{x_1}\right]_j \xrightarrow{(-1 \ \ 1)} S\left[\tfrac{1}{x_0 x_1}\right]_j$$

Def'n: The $i^{th}$ cohomology of $\widetilde{M}$ is: $H^i(\mathbb{P}^n, \widetilde{M}) := H^i \check{C}(\widetilde{M})$

Short exercise: compute $H^0(\mathbb{P}^1, \mathcal{O}(j))$, $\forall j \in \mathbb{Z}$.