Week 2 • Problem Set 1 • Definition and quadratic dual

Quick Problem:

Let $R = k[x, y]/(x^2 + y^2)$.

(a) Why do we know already that R is a Koszul algebra

(b) Find $R^!$ and determine $\dim_k R^!_3$.

Problems:

1. Find $R^!$ for the following *commutative* rings.

(a) $R = k[x, y, z]/(x^2 - yz)$

(b) $R = k[x, y]$

(c) $R = k[x_1, \ldots, x_n]$, that is, the symmetric algebra $S(V)$ on $V = k\{x_1, \ldots, x_n\}$

In parts (b) and (c), do you recognize $R^!$ as a familiar algebra?

2. Which dual algebras from #1 are Artinian (that is, satisfy $\dim_k R^! < \infty$)?

3. Let R be the exterior algebra on n variables, say $x_1, \ldots, x_n$. Find $R^!$.

4. (a) State the BGG correspondence (from Michael Brown's lectures).

(b) Guess a similar duality theorem for Koszul algebras from what you have seen so far.

5. Let $R = Q/I$ where Q is a polynomial ring and I is a homogeneous ideal. Prove that if I has a linear Q-free resolution then R is Koszul.

Hint: For any graded ring R and R-module M, say with minimal graded free resolution F, writing $F_i = \oplus R(-j)^{\beta_{ij}}$ defines the graded Betti numbers β_{ij}. Putting these as coefficients, one obtains the graded Poincaré series

$$P^R_M(t, s) = \sum_{i,j} \beta_{ij} t^i s^i$$

You may use the following generalized version of a inequality of Golod.

$$P^R_M(t, s) \leq \frac{P^Q_M(t)}{1 - t(P^Q_R(t) - 1)}$$

The inequality is understood to be coefficientwise once one expands the righthand side as a power series using geometric series.

For those of you who are curious, this inequality can be shown using either the spectral sequence

$$XXX \textit{ look up in } IFR$$

or by using the relative bar resolution of Iyengar and Burke.