Problem Set 2: Graded rings

Let k be a field, and let $R = \bigoplus_{i \in \mathbb{Z}} R_i$ be a graded ring. All modules are left modules, unless assumed otherwise.

Recall that R is a *graded local k-algebra* if R is Noetherian, $R_0 = k$, and either $R_{>0} = 0$ or $R_{<0} = 0$. Given a finitely generated R-module M, recall that M has a minimal graded free resolution

$$0 \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots ,$$

and $\beta_{i,j}(M)$ denotes the number of copies of $S(-j)$ in F_i.

1. (Short exercise) Assume R is a graded local k-algebra, and that $R_{<0} = 0$. Prove the graded version of Nakayama's Lemma: if M is a graded R-module with $M_i = 0$ for $i \ll 0$, and $R_{>0}M = M$, then $M = 0$. Notice: there is no need to assume M is finitely generated. Also, a similar statement holds when $R_{>0} = 0$.

Solution. Assume $R_{<0} = 0$; the case $R_{>0} = 0$ is similar. Suppose $M \neq 0$. Let d be the minimal degree such that $M_d \neq 0$. The minimal possible degree of a nonzero element in $R_{>0}M$ is $d + 1$, in contradiction to $R_{>0}M = M$. Thus, $M = 0$.

2. Assume R is a graded local k-algebra. If M is a finitely generated graded R-module, show that $\dim_k M_i < \infty$ for all i. Suggestion: first prove this for $M = R$.

Solution. Assume $R_{<0} = 0$; the case $R_{>0} = 0$ is similar. Using that R is Noetherian, choose a finite list of homogeneous generators $x_1, \dots, x_m$ for the ideal $R_{>0}$. We argue by induction. Let q be the minimal degree of the x_i. We have $R_i = 0$ for all $0 < i < q$ and R_q is the finite dimensional vector space spanned by those x_i whose degree is q. Now assume $n \geq q$ and that $\dim_k R_n < \infty$ for all $i \leq n$. Let $r \in R_{n+1}$, and write $r = r_1x_1 + \cdots + r_mx_m$ for some homogeneous $r_1, \dots, r_m \in R$. We have either $r_i = 0$ or $\deg(r_i) < \deg(r) = n + 1$. It therefore follows by induction that $\dim_k R_{n+1} < \infty$. This proves the statement for $M = R$. The statement for any finitely generated free R-module follows immediately. Since every finitely generated R-module is a quotient of a finitely generated free R-module, we are done.

3. Prove that, if $I \subseteq R$ is an ideal generated by homogeneous elements, then I is a graded R-module. Conclude that R/I is both a graded ring and a graded R-module.

Solution. For $i \in \mathbb{Z}$, let I_i denote the subset of I given by homogeneous elements of degree i. We must show $I = \bigoplus_{i \in \mathbb{Z}} I_i$ as an abelian group. Choose homogeneous generators $y_1, \dots, y_m$ of I. If $x \in I$, then $x = \sum_{i=1}^m r_iy_i$ for some $r_i \in R$. Writing each r_i as a sum of homogeneous components, we conclude that y is a sum of homogeneous elements of I, i.e. $I = \sum_{i \in \mathbb{Z}} I_i$. This sum must be direct, since the sum $R = \bigoplus_{i \in \mathbb{Z}} R_i$ is direct. It follows that I is a graded R-module. Now, defining $(R/I)_i = R_i/I_i$, we have $R/I = \bigoplus_{i \in \mathbb{Z}}(R/I)_i$, and it is straightforward to check that this gives R/I the structure of both a graded ring and a graded R-module.

4. Let $S = k[x,y]$, and let M denote the S-module $S/(x^2, xy)$. Yesterday, you wrote down the minimal free resolution of M over S. Now, write it as a *graded* free resolution; that is, keep track of the twists of S in the resolution.

Solution. The resolution looks like this:

$$0 \leftarrow S \xleftarrow{\begin{pmatrix} x^2 & xy \end{pmatrix}} S(-2)^2 \xleftarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} S(-3) \leftarrow 0.$$

5. Assume R is *graded commutative*, meaning that, if $r \in R_i$ and $s \in R_j$, then

$$rs = (-1)^{\deg(r)\deg(s)} sr.$$

Prove that, if M is a graded right R-module, then M is also a graded left R-module with left action $rm := (-1)^{\deg(r)\deg(m)} mr$ for homogeneous elements $r \in R$ and $m \in M$.

Solution. Only the associativity axiom needs to be checked. Suppose $r_1 \in R_i$, $r_2 \in R_j$, and $m \in M_t$. We have

$$(r_1 r_2)m = (-1)^{(i+j)t} m r_1 r_2.$$

On the other hand, we have

$$r_1(r_2 m) = (-1)^{i(j+t)}(r_2 m)r_1 = (-1)^{i(j+t)+jt} m r_2 r_1 = (-1)^{i(j+t)+jt+ij} m r_1 r_2.$$

Since $(-1)^{i(j+t)+jt} = (-1)^{(i+j)t}$, we are done.

6. Assume R is a graded local k-algebra. Let M be a finitely generated graded R-module that is generated in degree 0. Prove that M has a linear free resolution (i.e. there is a basis of the free resolution with respect to which each matrix has entries given by 0's or linear forms) if and only if $\beta_{i,j}(M) = 0$ for $i \neq j$.

Solution. Assume M has linear free resolution $(F_\bullet, d)$. Since M is generated in degree 0, we see that $\beta_{0,j} = 0$ for $j \neq 0$. Recall that each d_i is a morphism of graded R-modules. Since each column in a matrix representing d_1 is the image of a basis element in F_1, F_0 is generated in degree 0 and each column has 0's or linear forms, we conclude that each free summand of F_1 is generated in degree 1. Using induction, the same argument implies that each free summand of F_i is generated in degree i for all $i \geqslant 0$. Thus, $\beta_{i,j}(M) = 0$ for $i \neq j$. For the converse, note that a basis element of F_{n+1} maps to a degree $n+1$ homogeneous element in F_n. Since F_n is generated in degree n, this image can be represented by a column of 0's or linear forms.

7 (Do this one only if you're interested, and you have time). We recall that a *graded R-R-bimodule* is a graded left R-module M that is also a graded right R-module and such that $(rm)r' = r(mr')$ for all $r, r' \in R$ and $m \in M$.

Let M and N be finitely generated graded left R-modules. Recall that $\underline{\mathrm{Hom}}_R(M, N)$ denotes the set of all R-linear maps from M to N. Recall that $\underline{\mathrm{Hom}}_R(M, N)$ is a graded abelian group, with $\underline{\mathrm{Hom}}_R(M, N)_i =$ degree i maps. Prove that, if M (resp. N) is an R-R-bimodule, then $\underline{\mathrm{Hom}}_R(M, N)$ is a graded left (resp. right) R-module.

Aside: the module M in Problem 5 is in fact a graded R-R-bimodule: prove this if you're interested.

Solution. Suppose M is an R-R-bimodule. If $f \in \underline{\mathrm{Hom}}_R(M, N)$ and $r \in R$, then define $r \cdot f$ as follows: for $m \in M$, set $(r \cdot f)(m) = f(mr)$. It is easily checked that this map is a morphism of left R-modules. Associativity also holds since $((rr')f)(m) = f(mrr') = (r(r'f))(m)$. Moreover, this is clearly a *graded* R-module. The proof when N is an R-R-bimodule is similar.