L4: Relation to bar resn (Priddy's story) (technical!)

There is another canonical resn, namely the bar resn, which exists for any augmented k-algebra

$$\begin{tikzcd} k \arrow[r, hook] \arrow[rr, bend right, "1"'] & R \arrow[r] & k \end{tikzcd}.$$

(history in alg topology, gp cohomology)

For comparison to Priddy's resn, first:

Equivalent description of $R^!$ for a quadratic algebra
(also coordinate-free)

Given $W \subseteq V \otimes V = T(V)_2$ k-subspace

have $0 \to W \to V \otimes V \to \text{coker} \to 0$

dual $0 \to (\text{coker})^* \to V^* \otimes V^* \to W^* \to 0$ (*)

Claim 1: $(\text{coker})^* = W^\perp$ ← in any coords!

Pf: $W^\perp = \{ p = \sum \lambda_{ij} x_i^* x_j^* \mid \langle w, p \rangle = 0 \ \forall w \in W \}$

whereas

$(\text{coker})^* = \{ f = \sum \lambda_{ij} x_i^* x_j^* \mid f|_W = 0\text{-map} \}$

but for $w = \sum \mu_{ij} x_i x_j$, $f(w) = \sum \lambda_{ij} \mu_{ij} = \langle w, p \rangle$ //

[By (*), this gives $(W^\perp)^* \cong \frac{V \otimes V}{W}$]

Claim 2: $(R^!)^* \cong \bigoplus_n \left(\bigcap_{i+2+j=n} V^{\otimes i} \otimes \underbrace{W}_{\subseteq V \otimes V} \otimes V^{\otimes j} \right)$

Pf: By def, $0 \to \bigoplus_n (V^*)^{\otimes i} \otimes W^\perp \otimes (V^*)^{\otimes j} \to T(V^*) \to R^! \to 0$

dual $0 \to (R^!)^* \to T(V) \to \bigoplus_n V^{\otimes i} \otimes \underbrace{(W^\perp)^*}_{\cong V \otimes V / W} \otimes V^{\otimes j}$ //

Bar resolution $\supseteq$ Priddy resn

Let R be std gdd k-algebra

The bar resn of k is:

$$B(R,R,k)_n \overset{def}{=} R \overset{\tau}{\underset{k}{\otimes}} B(R)_n \cancel{\otimes_k k}$$

$$= R \otimes_k \underbrace{\bar{R} \otimes_k \cdots \otimes_k \bar{R}}_{n} \cancel{\otimes_k k}$$

where

$$\bar{R} = R_+ = \bigoplus_{n>0} R_i = m$$

$$\partial(r \otimes r_1 \otimes \cdots \otimes r_n \cancel{\otimes 1})$$

$$= \underbrace{rr_1 \otimes r_2 \otimes \cdots \otimes r_n}_{\text{this part of } \partial \text{ is called } d} \underbrace{- r \otimes r_1 r_2 \otimes \cdots \otimes r_n + \cdots + (-1)^i r \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n \pm \cdots}_{\text{this part of } \partial \text{ is called } b}$$

[$b^2 \neq 0$, so $B(R)$ not a cx, but satisfies Maurer-Cartan eqn, curved alg···]

Thm: $B = B(R,R,k) \xrightarrow[R]{\sim} k$ is an R-free resn of k

It is very non-min'l: $R \otimes_k \underbrace{\bar{R}^{\otimes_k i}}_{\text{huge } k\text{-v.sp.}}$ = huge free R-mod

but universal (same form any R).

(any M, $B(R,R,M) \xrightarrow[R]{\sim} M$)

Separate B out by graded pieces:

Note 1st:

$$R = \frac{T(V)}{(W)} = \underbrace{\cancel{k}}_{R_0} \oplus \underbrace{V}_{R_1} \oplus \underbrace{\frac{V^{\otimes 2}}{W}}_{R_2} \oplus \underbrace{\frac{V^{\otimes 3}}{V \otimes W + W \otimes V}}_{R_3} \oplus \cdots$$

delete in $\bar{R}$

cont'd

$B = R \otimes_k$ B(R) portion:

$B = R \otimes_k$

b maps: 0,1's

d maps: min'l (ind $\cong m \otimes \ldots$)

$$\begin{tikzcd}
 & & & R_1^{\otimes 4} \arrow[dl, dashed, "d"'] \arrow[d] & \leftarrow B_4 \\
 & & R_1 \otimes R_1 \otimes R_1 \arrow[dl, dashed, "d"'] \arrow[d, "b"] & \cdots \arrow[dl, dashed, "d"'] \arrow[d] & \leftarrow B_3 \\
 & R_1 \otimes R_1 \arrow[dl, dashed, "d"'] \arrow[d, "b"] & R_1 \otimes R_2 \oplus R_2 \otimes R_1 \arrow[dl, dashed, "d"'] \arrow[d, "b"] & \cdots \arrow[dl, dashed, "d"'] \arrow[d] & \leftarrow B_2 \\
R_1 & R_2 \arrow[l, dashed, "d"'] & R_3 & R_4 & \leftarrow B_1
\end{tikzcd}$$

The kernel of each top vertical b map forms a subcx:

$R \otimes_k$

$$\begin{tikzcd}
 & & & \cdots \arrow[dl, dashed] \\
 & & \ker b \arrow[dl, dashed] & \\
 & \ker b \arrow[dl, dashed] & & \\
R_1 & & &
\end{tikzcd}$$

In column 2:

$$\ker(R_1 \otimes_k R_1 \xrightarrow{b} R_2) = \ker\left(V \otimes_k V \longrightarrow \frac{V \otimes V}{W}\right) = W \overset{\text{by Claim 2}}{=} (R_2^!)^*$$

In general:

$$\ker(\underbrace{R_1 \otimes \cdots \otimes R_1}_{n} \to \bigoplus_{\substack{i+j+2\\=n}} \underbrace{R_1 \otimes \cdots \otimes}_{i} R_2 \underbrace{\otimes \cdots \otimes R_1}_{j}) = \bigcap_{\substack{i+j+2\\=n}} \underbrace{V \otimes \cdots \otimes}_{i} W \underbrace{\otimes \cdots \otimes V}_{j} = (R_n^!)^*$$

Inherited diff'l: just d

$$d(\alpha) = d(1 \otimes \underbrace{x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n}}_{\substack{\cong \\ x_{i_1}^* x_{i_2}^* \cdots x_{i_n}^* \in (R_n^!)^*}}) = 1 \cdot x_{i_1} \otimes \underbrace{x_{i_2} \otimes \cdots \otimes x_{i_n}}_{\substack{\cong \\ x_{i_2}^* \cdots x_{i_n}^* \in (R_{n-1}^!)^*}} = \alpha \cdot (x_{i_1} \otimes x_{i_1}^*) = \partial(\alpha) \text{ in } P(R)!$$

α basis elt

if basis elt

So, $P(R) \hookrightarrow B = B(R, R, k)$ [Both resns of k, but $P(R)$ min'l]