

Problem Set 3: The BGG correspondence, Part 1

Fix $n \geq 0$. Let k be a field. Let $S = k[x_0, \dots, x_n]$, and $E = \Lambda_k(e_0, \dots, e_n)$, with gradings given by $\deg(x_i) = 1$ and $\deg(e_i) = -1$.

1. (Short exercise) Prove that $E^* := \underline{\text{Hom}}_k(E, k)$ is isomorphic, as a graded left E -module, to the free module $E(-n-1)$.

Solution. We construct an explicit isomorphism. Recall that the left action of E on $E^* := \underline{\text{Hom}}_k(E, k)$ is as follows: let $f \in E^*$ and $e \in E$ be homogeneous. We have

$$(e \cdot f)(y) = (-1)^{\deg(e) \deg(f)} f(e y).$$

Given an exterior monomial m , let $\alpha_m \in \underline{\text{Hom}}_k(E, k)$ be the map that sends m to 1 and every other exterior monomial to 0. Notice that the degree of α_m is $-\deg(m)$. Moreover, the maps α_m form a k -basis of E^* , called the *dual basis* associated to the monomial basis of E . Define a map $\varphi: E(-n-1) \rightarrow E^*$ by sending 1 to $\alpha_{e_0 \dots e_n}$ and extending E -linearly; observe that φ is a degree 0 map. We will show that θ identifies the monomial basis of $E(-n-1)$ with the above dual basis. Suppose m is an exterior monomial. Choose an exterior monomial $m' \in E$ such that $mm' = e_0 \cdots e_n$. For any exterior monomial m'' , we have

$$(\theta(m))(m'') = \alpha(mm'') = \begin{cases} 1, & \text{if } m'' = m' \\ 0, & \text{else} \end{cases}$$

This shows that $\theta(m) = \alpha_{m'}$. Thus, θ identifies the monomial basis of $E(-n-1)$ with the dual basis of E^* , and so it is an isomorphism.

2. Let R be a graded local ring that is an algebra over a field k . Prove that $R^* := \underline{\text{Hom}}_k(R, k)$ is an injective graded left R -module. Conclude that the exterior algebra E is injective as a graded left E -module.

Hint: recall that an R -module I is injective if and only if the functor $\text{Hom}_R(-, I)$ is exact. You may also want to use Hom-tensor adjunction: that is, if R and S are graded rings, then:

$$\text{Hom}_R(N, \underline{\text{Hom}}_S(N', N'')) \cong \text{Hom}_S(N' \otimes_R N, N'').$$

for any left R -module N , left S -module N'' , and S - R -bimodule N' .

Solution. We need to show that the functor $\text{Hom}_R(-, R^*)$ is exact. By Hom-tensor adjunction, we have

$$\text{Hom}_R(-, R^*) = \text{Hom}_R(-, \underline{\text{Hom}}_k(R, k)) \cong \text{Hom}_k(R \otimes_R -, k) \cong \text{Hom}_k(-, k)$$

Moreover, each of these isomorphisms is *natural*; that is, they are isomorphisms of functors. Since taking k -duals is exact, we are done.

3. Let N be a finitely generated graded E -module. Recall that the i^{th} term of $\mathbf{L}(N)$ is $S(-i) \otimes_k N_i$. Prove that the differential $\partial_{\mathbf{L}}$ on $\mathbf{L}(N)$, which is given by $s \otimes y \mapsto \sum_{i=0}^n x_i s \otimes e_i y$, squares to zero.

Solution. We have:

$$\partial_{\mathbf{L}}^2(s \otimes y) = \sum_{i,j=0}^n x_i x_j s \otimes e_i e_j y = \sum_{0 \leq i < j \leq n} x_i x_j s \otimes (e_i e_j + e_j e_i) y + \sum_{i=0}^n x_i^2 s \otimes e_i^2 y = 0.$$

4. Suppose $n = 1$, and let F be the complex

$$0 \leftarrow S(1) \xleftarrow{x_0 - x_1} S \leftarrow 0,$$

where S is in homological degree 0. Find a graded E -module N such that $\mathbf{L}(N) = F$.

Solution. Take $N = E/(e_0 + e_1)$.

5. This isn't really an exercise: it's just a demo for how to compute with the BGG functor \mathbf{L} in Macaulay2. Unfortunately, there is no functionality for \mathbf{L} in the Macaulay2 package BGG (it can only handle \mathbf{R}). However, the newer package MultigradedBGG (which can also handle more exotic gradings, as we will see next week), can compute the functor \mathbf{L} . Try running the following code to check that $\mathbf{L}(E)$ is the dual of the Koszul complex.

```
needsPackage "MultigradedBGG"
n = 3;
S = ZZ/101[x_0..x_n]
E = dualRingToric S
toricLL(E^1)
```

We can also apply the BGG functor \mathbf{L} to E/I , where I is a random monomial ideal, and see what it looks like. Here is how to do this:

```
needsPackage "MultigradedBGG"
n = 5;
S = ZZ/101[x_0..x_n]
E = dualRingToric S
needsPackage "RandomIdeals"
I = randomMonomialIdeal({-3, -4}, E)
N = coker matrix entries gens I
C = toricLL(N)
```

You can take a look at the full differential of C by typing `C.dd`, or individual differentials by typing `C.dd_i` for various i . If you do this in our example above, you'll find that the entries of the matrices are always *linear forms*. Indeed, this is true for \mathbf{L} applied to *any* exterior module; this follows from the definition of \mathbf{L} . In fact, we can say more: the entries in the above example are just *variables*, up to a sign; this is because the ideal I above is generated by *monomials*.