Problem Set 7: The geometric BGG correspondence, Part 3

Fix $n \geqslant 0$. Let k be a field. Let $S = k[x_0, \dots, x_n]$, and $E = \Lambda_k(e_0, \dots, e_n)$, with gradings given by $\deg(x_i) = 1$ and $\deg(e_i) = -1$.

1. (Short exercise) Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$ and M a finitely generated graded S-module such that $\widetilde{M} = \mathcal{F}$. Choose $r > \operatorname{reg}(M)$, and use it to construct the Tate resolution $T(\mathcal{F})$. Prove that the complex T' from the proof of theorem discussed today coincides with the Tate resolution $T(\mathcal{F})$ in homological degrees $\leqslant -r$.

Solution Fix a homological degree $i \leq -r$. We have

$$T'_i = \bigoplus_{j=0}^{n} H^j(\mathbb{P}^n, \mathcal{F}(-i-j)) \otimes_k \omega_E(i+j).$$

For $1 \leq j \leq n$, we have an isomorphism $H^j(\mathbb{P}^n, \mathcal{F}(-i-j)) \cong H^{j+1}_{\mathfrak{m}}(M)_{-i-j}$, which is 0 since $i \leqslant -r$. Thus, $T'_i = H^0(\mathbb{P}^n, \mathcal{F}(-i)) \otimes_k \omega_E(i)$. By an earlier exercise, we know that $H^0(\mathbb{P}^n, \mathcal{F}(-i)) \cong M_{-i}$ when $i \leqslant -r$. Thus, the terms of the complexes in question are the same. For degree reasons, the only summand in the differential on T' that is nonzero is the one corresponding to $\ell = 0$, and this coincides with the BGG differential.

The goal of this problem set is to guide you through an alternative proof of our theorem stating that the ranks of the terms of Tate resolutions compute sheaf cohomology. In fact, this was the original approach of Eisenbud-Fløystad-Schreyer to proving this theorem. The key technical tool we will use is an important result called Local Duality.

2. This exercise is annoying and technical, but you need it for Exercise 3. I therefore suggest taking it for granted and skipping to Exercise 3; if you're interested, come back to this one at the end. Let N be a finitely generated graded E-module with minimal free resolution F. Prove, via the following steps, that the Betti numbers of the minimal free resolution of N are equal to the Betti numbers of $\mathbf{R}(\bigoplus_{i \in \mathbb{Z}} H_i \mathbf{L}(N^*)[i])^*$. Recall that $(-)^* := \underline{\operatorname{Hom}}_k(-, k)$.

(a) Prove that $(\mathbf{R}(H_i\mathbf{L}(N^*)[i])^*)_j = (\mathbf{R}(H_i\mathbf{L}(N^*))^*)_{j-i} = (\mathbf{R}(H_i(\mathbf{L}(N^*)))_{i-j})^*$.

(b) Show that $(\mathbf{R}(H_i(\mathbf{L}(N^*)))_{i-j})^* = \operatorname{Ext}^j_E(k, N^*)_{j-i} \otimes_k E(j-i)$.

(c) Show that F^* is an injective resolution of N. Letting $\beta_{i,j}(N)$ denote the Betti numbers of N, conclude that $(\mathbf{R}(H_i\mathbf{L}(N^*)[i])^*)_j = E(j-i)^{\beta_{j,i-j}(N)}$.

(d) Now sum over all i to conclude that the Betti numbers of $\mathbf{R}(\bigoplus_{i \in \mathbb{Z}} H_i\mathbf{L}(N^*)[i])^*$ coincide with the Betti numbers of N.

Solution. For Part (a), we have:

$$(\mathbf{R}(H_i\mathbf{L}(N^*)[i])^*)_j = (\mathbf{R}(H_i\mathbf{L}(N^*))^*)[-i]_j,$$

since, by Problem Set 4 Exercise 5(b), $\mathbf{R}$ commutes with homological shifts; and also, given a complex C, we have $(C[i])^* = (C^*)[-i]$. The second equality in part (a) holds since, for any complex C, we have $(C^*)_i = C^*_{-i}$. For Part (b), we have

$$\begin{aligned}\mathbf{R}(H_i(\mathbf{L}(N^*)))_{i-j} &= \omega_E(i-j) \otimes_k H_i(\mathbf{L}(N^*))_{j-i}\\ &\cong \omega_E(i-j) \otimes_k \operatorname{Ext}^j_E(k, N^*)_{j-i}.\end{aligned}$$

Dualizing gives the result, noting that for any graded E-module L, $L(i)^* = L^*(-i)$. For Part (c): F^* is an injective resolution because E is injective as an E-module, and taking k-duals is exact. For the second statement in (c), we compute:

$$\begin{aligned}(\mathbf{R}(H_i\mathbf{L}(N^*)[i])^*)_j &\cong \operatorname{Ext}^j_E(k, N^*)_{j-i} \otimes_k E(j-i)\\ &\cong \operatorname{Hom}_E(k, (F^*)_{-j})_{j-i} \otimes_k E(j-i).\end{aligned}$$

So we must show that $\dim_k \operatorname{Hom}_E(k, F^*_{-j})_{j-i} = \beta_{j,i-j}(N)$. Write $F_j = \bigoplus_\ell E(-\ell)^{\beta_{j,\ell}}$ We have $(F^*)_{-j} = (F_j)^* = \bigoplus_\ell \omega_E(\ell)^{\beta_{j,\ell}}$. Moreover,

$$\operatorname{Hom}_E(k, \omega_E(\ell))_{j-i} = 0 \quad \text{unless } \ell = i-j.$$

Part (c) follows. Finally, (d) is immediate from (c): the rank of $E(-\ell)$ in $\mathbf{R}(\bigoplus_{i\in\mathbb{Z}} H_i\mathbf{L}(N^*)[i])^*_j$ is exactly $\beta_{j,\ell}(N)$.

For the rest of this problem set, let M be a finitely generated graded S-module with a free resolution of the form $\mathbf{L}(N)$ for some (necessarily finitely generated) graded E-module N.

3. Prove that there is an isomorphism $\operatorname{Ext}^*_S(M, S) \cong H_*(\mathbf{L}(N^*))$, where $\operatorname{Ext}^i_S(M, S)$ is interpreted as living in homological degree $-i$. Conclude that the Betti numbers of the minimal free resolution of N are equal to the Betti numbers of $\mathbf{R}(\bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}^i_S(M, S)[i])^*$. Hint: use Exercise 4(a) from Problem Set 4.

Solution. Recall that $\mathbf{L}(N^*) \simeq \mathbf{L}(N)^\vee$ as complexes of graded S-modules. This gives $H_*(\mathbf{L}(N^*)) \cong H_*(\mathbf{L}(N)^\vee)$. The latter object is precisely $\operatorname{Ext}^*_S(M, S)$, and hence the first assertion follows. By Problem 2, the Betti numbers of N are equal to the Betti numbers of $\mathbf{R}(\bigoplus_{i\in\mathbb{Z}} H_i(\mathbf{L}(N^*))[i])^* \cong \mathbf{R}(\bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}^i_S(M, S)[i])^*$.

4. Given a finitely generated graded S-module P, prove that the complexes $\mathbf{R}(P^*)$ and $\mathbf{R}(P)^*(-n-1)$ have isomorphic terms.

Solution. We have $\mathbf{R}(P^*)_i = \omega_E(i) \otimes_k P^*_{-i} = \omega_E(i) \otimes_k P_i$. On the other hand, we compute:

$$\begin{aligned}(\mathbf{R}(P)^*(-n-1))_i &= (\mathbf{R}(P)_{-i})^*(-n-1)\\ &= \omega_E(-i)^*(-n-1) \otimes_k P_i\\ &= E(i-n-1) \otimes_k P_i\\ &= \omega_E(i) \otimes_k P_i.\end{aligned}$$

5. The *Local Duality Theorem* states that, for any finitely generated graded S-module L, there is an isomorphism

$$H^i_{\mathfrak{m}}(L) \cong \operatorname{Ext}_S^{n+1-i}(L, S(-n-1))^*.$$

for all $0 \leqslant i \leqslant n+1$. Use this theorem to conclude that the Betti numbers of our module N coincide with the Betti numbers of $\bigoplus_{i \in \mathbb{Z}} \mathbf{R}(H^i_{\mathfrak{m}}(M))[i]$.

Solution. By Exercises 3 and 4, Exercise 5(b) on Problem Set 4, and Local Duality; the Betti numbers of N agree with the Betti numbers of:

$$\begin{aligned}
\bigoplus_{i \in \mathbb{Z}} \mathbf{R}(\operatorname{Ext}_S^i(M,S)[i])^* &= \bigoplus_{i \in \mathbb{Z}} \mathbf{R}(\operatorname{Ext}_S^i(M,S))^*[-i] \\
&= \bigoplus_{i \in \mathbb{Z}} \mathbf{R}(\operatorname{Ext}_S^i(M,S)^*)(n+1)[-i] \text{ (Exercise 4)} \\
&= \bigoplus_{i \in \mathbb{Z}} \mathbf{R}(\operatorname{Ext}_S^i(M,S(-n-1))^*)[n+1-i] \text{ (Exercise 5(b), Problem Set 4)} \\
&= \bigoplus_{i \in \mathbb{Z}} \mathbf{R}(H^{n+1-i}_{\mathfrak{m}}(M))[n+1-i] \\
&= \bigoplus_{i \in \mathbb{Z}} \mathbf{R}(H^{i}_{\mathfrak{m}}(M))[i].
\end{aligned}$$

6. Exercise 5 leads quickly to an alternative proof of a result we proved today, namely that the Betti numbers of the Tate resolution of a coherent sheaf on $\mathbb{P}^n$ compute the ranks of the sheaf cohomologies of the twists of $\mathcal{F}$. If you're up for a somewhat more challenging exercise, try and carry out this alternative proof. You will need the following tools:

(a) Our four-term exact sequence involving 0^{th} sheaf cohomology along with $H^0_{\mathfrak{m}}$ and $H^1_{\mathfrak{m}}$, along with the identification $H^i(\mathbb{P}^n, \widetilde{(-)}(j)) = H^{i+1}_{\mathfrak{m}}(-)_j$.

(b) The following fact, which we haven't discussed in lecture: if a finitely generated graded S-module M generated in a single degree has a linear free resolution, then this free resolution is of the form $\mathbf{L}(N)$ for some finitely generated graded E-module N. Use this without proof.

Solution. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. We want to show

$$T(\mathcal{F})_i = \bigoplus_{j=0}^{n} H^j(\mathbb{P}^n, \mathcal{F}(-i-j)) \otimes_k \omega_E(i+j). \tag{1}$$

Let M' be a graded S-module such that $\widetilde{M'} = \mathcal{F}$. Let $r > \operatorname{reg}(M')$, and set $M = M'_{\geq r}$. We have $\mathcal{F} \cong \widetilde{M}$, and M has a linear free resolution, which, by fact (b) above, must be of the form $\mathbf{L}(N)$ for some finitely generated graded E-module N. Moreover, by the Reciprocity Theorem, N is the homology of $\mathbf{R}(M)$ (concentrated in degree $-r$).

By Problems 3 and 4 from yesterday, we know:

(a) $M_j \cong H^0(\mathbb{P}^n, \mathcal{F}(j))$ for $j \geqslant r$, and

(b) $H^i(\mathbb{P}^n, \mathcal{F}(j)) = 0$ for $i > 0$ and $j \geqslant r$.

Combining these points immediately implies that (1) holds when $i \leqslant -r$.

Let F be the minimal free resolution of N, where N is considered as concentrated in homological degree $-r$. Thus, F is concentrated in homological degrees $\geqslant -r$. Recall that F_i is, by definition, $T(\mathcal{F})_{i+1}$ for $i \geqslant 0$. We therefore must show that for $i \geqslant -r+1$,

$$F_i = \bigoplus_{j=0}^{n} H^j(\mathbb{P}^n, \mathcal{F}(-i-1-j)) \otimes_k \omega_E(i+1+j).$$

By the previous exercise, along with our results relating sheaf cohomology and local cohomology, we have:

$$F_i = \bigoplus_{j \in \mathbb{Z}} \omega_E(i+j) \otimes_k H^j_{\mathfrak{m}}(M)_{-i-j} = \omega_E(i+j) \otimes_k H^{j-1}(\mathbb{P}^n, \mathcal{F}(-i-j)). \tag{2}$$

In a bit more detail: we have $H^0_{\mathfrak{m}}(M) = 0$ due to our choice of r. Using the short exact sequence

$$0 \to M \to \bigoplus_{j \in \mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{F}(j)) \to H^1_{\mathfrak{m}}(M) \to 0,$$

we conclude that $H^0(\mathbb{P}^n, \mathcal{F}(j)) = H^1_{\mathfrak{m}}(M)_j$ when $j < r$. And of course, $H^i(\mathbb{P}^n, \mathcal{F}(j)) = H^{i+1}_{\mathfrak{m}}(M)_j$ for all $i \geqslant 1$ and all $j \in \mathbb{Z}$.

Reindexing the sum (2) gives the desired result.