Day 6 : Geometric BGG, Part 2

Local cohomology

Let M be a fg graded S-module.

Recall, $\check{C}(\tilde{M}) = \left[\bigoplus_{i=0}^{n} M[\frac{1}{x_i}]_0 \xrightarrow{\partial^0} \cdots \right]$

So, $\bigoplus_{d \in \mathbb{Z}} \check{C}(\tilde{M}(d)) = \left[\bigoplus_{i=0}^{n} M[\frac{1}{x_i}] \xrightarrow{\partial^0} \cdots \right] \in \mathrm{Com}(S)$.

There is a natural map $M \longrightarrow \ker(\partial^0) \subseteq \bigoplus_{i=1}^{n} M[\frac{1}{x_i}]$.

We arrive at an "augmented" Čech complex

$\check{C}(M) = \left[M \longrightarrow \bigoplus_{i=0}^{n} M[\frac{1}{x_i}] \longrightarrow \cdots \right]$ (just tack M onto the beginning

cohom. degree: 0   1   ...   of $\bigoplus_{d \in \mathbb{Z}} \check{C}(\tilde{M}(d))$)

Let $m = (x_0, \ldots, x_n) \subseteq S$.

Def'n: The <u>local cohomology of M</u> is $H^i_m(M) := H^i \check{C}(M)$.

Ex: $H^0_m(M) = \ker\left(M \longrightarrow \bigoplus_{i=0}^{n} M[\frac{1}{x_i}]\right) \cong \{ y \in M : m^j y = 0 \text{ for some } j \geq 1 \}$.

(the "m-torsion" of M).

Prop: Let M be a gdd S-module.

① $H^i(\mathbb{P}^n, \tilde{M}(d)) \cong H^{i+1}_m(M)_d$ for $i \geq 1$.

② There is an exact sequence of graded S-modules

$$0 \longrightarrow H^0_m(M) \longrightarrow M \longrightarrow \bigoplus_{d \in \mathbb{Z}} H^0(\mathbb{P}^n, \tilde{M}(d)) \longrightarrow H^1_m(M) \longrightarrow 0.$$

Castelnuovo - Mumford regularity

Def'n: A graded S-module M is <u>r-regular</u> if $H^i_m(M)_j = 0$ for $i \geq 0$ and $j > r-i$.

The <u>regularity</u> of M, $\mathrm{reg}(M)$, is the smallest r s.t. M is r-regular.

If there is no such r, $\mathrm{reg}(M) = \infty$.

Thm (Eisenbud-Goto, 84) Let M be a fg graded S-module.

① M is r-regular iff $\mathrm{Tor}_i^S(M,k)_j = 0$ for $i \geq 0$ and $j > r+i$.

② If $r \geq \mathrm{reg}(M)$, then the module $M_{\geq r} = \bigoplus_{i \geq r} M_i$ is generated in degree r and has a linear free res'n.

Notice: ① makes it clear that $\mathrm{reg}(M) < \infty$.

Want a BGG-type functor

$\mathrm{coh}(\mathbb{P}^n) \longrightarrow \mathrm{Com}(E)$. Could extend to complexes of sheaves, but choosing not to for simplicity.

Let $\mathcal{F} \in \mathrm{coh}(\mathbb{P}^n)$.

Choose a fg gdd S-module M s.t. $\mathcal{F} = \widetilde{M}$.

Let $r > \mathrm{reg}(M)$.

$M_{\geq r}$ satisfies $\mathrm{Tor}_i^S(M_{\geq r}, k)_j = 0$ unless $j = i+r$. What does this say about $\mathbb{R}$? Friday's exercise, but let's go through it.

We know $\mathrm{Tor}_{i+j}^S(M_{\geq r}, k)_j = H_i\, \mathbb{R}(M_{\geq r})_j$

$\Rightarrow \mathrm{Tor}_i^S(M_{\geq r}, k)_j = H_{i-j}\, \mathbb{R}(M_{\geq r})_j$,

$\Rightarrow \mathbb{R}(M_{\geq r})$ has homology only in homol. degree $-r$.

Let F be the min'l free res'n of $H_{-r}\mathbb{R}(M_{\geq r})$.

$\Rightarrow$ There is a quasi-isomorphism $F \xrightarrow{\simeq} \mathbb{R}(M_{\geq r})$.

Define $T(\mathcal{F}) := \mathrm{cone}(F \xrightarrow{\simeq} \mathbb{R}(M_{\geq r}))$, the <u>Tate resolution of $\mathcal{F}$</u>.

Thm: Let $\mathcal{F} \in \mathrm{coh}(\mathbb{P}^n)$

① $T(\mathcal{F})$ does not depend on the choice of M.

② $T(\mathcal{F})$ is an exact, minimal complex of fg free E-modules.

③ $T(\mathcal{F})_i = \bigoplus_{j=0}^{n} H^j(\mathbb{P}^n, \mathcal{F}(-i-j)) \otimes_k \omega_E(i+j)$

$\Rightarrow H^j(\mathbb{P}^n, \mathcal{F}(-i-j)) = \#$ of copies of $\omega_E(j+i)$ in $T(\mathcal{F})_i$.

Short exercise: Let $n = 1$ and $M = S/(x_0)$. Compute $\operatorname{reg}(M)$, and compute $T(\tilde{M})$.