Starting w/ def'n of $\mathbb{R}$, which is written in Lecture 8 notes.

Day 9: Weighted Tate resolutions

Weighted projective space

Let $d_0, \ldots, d_n \geq 1$. $\underline{d} = (d_0, \ldots, d_n)$.

$$\mathbb{P}(\underline{d}) := k^{n+1} \setminus \{0\} / \sim,$$

$$(a_0, \ldots, a_n) \sim (\lambda^{d_0} a_0, \ldots, \lambda^{d_n} a_n).$$

Sheaves of modules and sheaf cohomology over $\mathbb{P}(\underline{d})$ are defined just w/ $\mathbb{P}^n$. Let $S = k[x_0, \ldots, x_n]$, $\deg(x_i) = d_i$.

Write $w := \sum_{i=0}^{n} d_i$.

Ex: $$H^i(\mathbb{P}(\underline{d}), \mathcal{O}(j)) = \begin{cases} S_j, & i = 0 \text{ and } j \geq 0 \\ S_{-j-w}, & i = n \text{ and } j \leq -w. \\ 0, & \text{else.} \end{cases}$$

Every coherent sheaf on $\mathbb{P}(\underline{d})$ is $\widetilde{M}$ for some graded S-module M, where $\widetilde{M}$ is defined exactly as before.

Tate resolutions

Recall the recipe for building the Tate res'n of a sheaf $\mathcal{F}$ on $\mathbb{P}^n$:

1. Choose M s.t. $\widetilde{M} = \mathcal{F}$
2. Choose $r > \operatorname{reg}(M)$.
3. Let F be the min'l free res'n of $H_{-r}\mathbb{R}(M_{\geq r})$.

$T(\mathcal{F}) = \operatorname{cone}(F \xrightarrow{\simeq} \mathbb{R}(M_{\geq r}))$.

Problem: in the weighted case, there are modules M s.t., for all r, $\mathbb{R}(M_{\geq r})$ is not quasi-isomorphic to its homology.

Ex: $\underline{d} = (1,2)$, so $S = k[x_0, x_1]$ w/ $\deg(x_0) = 1$, $\deg(x_1) = 2$.

$M = S/(x_0^2 - x_1)$. $\implies E = \Lambda_k(e_0, e_1)$, $\deg(e_0) = (-1,-1)$, $\deg(e_1) = (-2,-1)$.

Exercises: $M_{\geq i} \cong M(i)$ $\forall i \geq 0$,

$\mathbb{R}(M(i)) \cong \mathbb{R}(M)(i, 0)$.

$\mathbb{R}(M)$ is not quasi-isom. to its homology.

So, weighted Tate resolutions are a little different.

Let $\mathcal{F} \in \mathrm{coh}(\mathbb{P}(\underline{d}))$.

1. Choose M s.t. $\widetilde{M} = \mathcal{F}$.
2. Compute a $\underbrace{\text{min'l free res'n } F \text{ of the diff. } E\text{-module}}_{\text{Need to define this!}}$ $\mathbb{R}(M)$.

$T(\mathcal{F}) = \mathrm{cone}(F \xrightarrow{\simeq} \mathbb{R}(M))$.

Here, if $f : D \longrightarrow D'$ is a morphism in $DM(E)$,

$\mathrm{cone}(f) = D' \oplus D(0,-1)$ w/ diff. $\begin{pmatrix} \partial' & f \\ 0 & -\partial \end{pmatrix}$.

Thm: (B-Erman, 24) If $\mathcal{F} \in \mathrm{coh}(\mathbb{P}(\underline{d}))$, then

$\dim_k H^i(\mathbb{P}(\underline{d}), \mathcal{F}(j)) = \#$ of copies of $\omega(-j, i)$ in $T(\mathcal{F})$.

Minimal free res'ns of diff. E-modules:

Let $D \in DM(E)$, and assume $H(D)$ is fg.

Def'n: A <u>min'l</u> <u>free</u> <u>flag</u> <u>res'n</u> of D is a quasi-isom $F \xrightarrow{\simeq} D$

$F = \bigoplus_{i \geq 0} F_i$, where F_i is free, $\partial_F(F_i) \subseteq \bigoplus_{j < i} F_j$, and $\partial_F(F) \subseteq m_E F$,

where $m_E = (e_0, \ldots, e_n) \subseteq E$.

Same as a min'l semifree res'n.

Thm: If $D \in DM(E)$, and $H(D)$ fg, then D admits a

min'l free flag res'n, and it is unique up to isom.

Reference: "Minimal free res'ns of diff modules" B-Erman.
Full power not necessary.

Short exercise: Write $\mathbb{R}(S)^* := \underline{Hom}_k(\mathbb{R}(S), k)$, w/ induced differential. Show that $\mathbb{R}(S)^*$ is the min'l free flag res'n of k.

Use the following: If M is a graded S-module,

$$H(\mathbb{R}(M))_{(a,j)} \cong \mathrm{Tor}_j^S(M, k)_a.$$