Problem Set 1: Exterior algebras and the Koszul complex

Let R be a commutative ring, V a free R-module with basis $e_1, \ldots, e_n$, and $f_1, \ldots, f_n \in R$. Recall that the *Koszul complex on* $f_1, \ldots, f_n$, denoted $K(f_1, \ldots, f_n)$, is the complex

$$0 \leftarrow \Lambda_R^0 V \xleftarrow{\partial_1} \Lambda_R^1 V \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_n} \Lambda_R^n V \leftarrow 0$$

with differential

$$\partial_j(e_{i_1} \cdots e_{i_j}) = \sum_{\ell=1}^{j} (-1)^{\ell-1} f_{i_\ell} e_{i_1} \cdots \widehat{e_{i_\ell}} \cdots e_{i_j}.$$

1. (Short exercise) Compute $K(f_1, f_2, f_3)$.

Solution.

$$0 \leftarrow R \xleftarrow{\begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix}} R^3 \xleftarrow{\begin{pmatrix} -f_2 & -f_3 & 0 \\ f_1 & 0 & -f_3 \\ 0 & f_1 & f_2 \end{pmatrix}} R^3 \xleftarrow{\begin{pmatrix} f_3 \\ -f_2 \\ f_1 \end{pmatrix}} R \leftarrow 0,$$

where the bases have been ordered lexicographically with respect to e_1, e_2, e_3.

2. Prove that $\partial_{j-1}\partial_j = 0$ for all j.

Solution. Fix j and a basis element $e_{i_1} \cdots e_{i_j} \in \Lambda_R^j V$. Applying $\partial_{j-1}\partial_j$ to $e_{i_1} \cdots e_{i_j}$ gives:

$$\sum_{\ell=1}^{j} (-1)^{\ell-1} f_{i_\ell} \left(\sum_{1 \leqslant k < \ell} (-1)^{k-1} f_{i_k} e_{i_1} \cdots \widehat{e_{i_k}} \cdots \widehat{e_{i_\ell}} \cdots e_{i_j} + \sum_{\ell < k \leqslant j} (-1)^{k-2} f_{i_k} e_{i_1} \cdots \widehat{e_{i_\ell}} \cdots \widehat{e_{i_k}} \cdots e_{i_j} \right),$$

which is equal to

$$\sum_{1 \leq k < \ell \leq j} \left((-1)^{\ell+k-2} f_{i_\ell} f_{i_k} e_{i_1} \cdots \widehat{e_{i_k}} \cdots \widehat{e_{i_\ell}} \cdots e_{i_j} + (-1)^{\ell+k-3} f_{i_k} f_{i_\ell} e_{i_1} \cdots \widehat{e_{i_k}} \cdots \widehat{e_{i_\ell}} \cdots e_{i_j} \right) = 0.$$

3. Given two complexes C and D of R-modules with differentials d_C and d_D, their *tensor product* $C \otimes_R D$ has terms $(C \otimes D)_m = \bigoplus_{i+j=m} C_i \otimes_R D_j$ and differential that sends an element $c \otimes d \in C_i \otimes D_j \subseteq (C \otimes D)_m$ to $d_C(c) \otimes d + (-1)^i c \otimes d_D(d)$. Prove that, for all $n \geqslant 1$, there is an isomorphism

$$K(f_1, \ldots, f_n) \cong K(f_1) \otimes_R \cdots \otimes_R K(f_n).$$

Solution. We argue by induction on n. This is trivial for $n = 1$. Assume $n > 1$ and that $K(f_1, \dots, f_k) \cong K(f_1) \otimes_R \cdots \otimes_R K(f_k)$ for all $k \leq n-1$. We will give an isomorphism

$$K(f_1, \dots, f_{n-1}) \otimes_R K(f_n) \xrightarrow{\cong} K(f_1, \dots, f_n).$$

Let ∂_K denote the Koszul differential. The differential for the complex on the left acts as follows:

$$e_{i_1} \cdots e_{i_j} \otimes 1 \mapsto \partial_K(e_{i_1} \cdots e_{i_j}) \otimes 1,$$

and

$$e_{i_1} \cdots e_{i_j} \otimes e_n \mapsto \partial(e_{i_1} \cdots e_{i_j}) \otimes e_n + (-1)^j e_{i_1} \cdots e_{i_j} \otimes f_n.$$

One now checks that the map

$$K(f_1, \dots, f_{n-1}) \otimes_R K(f_n) \to K(f_1, \dots, f_n)$$

given by $e_{i_1} \cdots e_{i_j} \otimes 1 \mapsto e_{i_1} \cdots e_{i_j}$ and $e_{i_1} \cdots e_{i_j} \otimes e_n \mapsto e_{i_1} \cdots e_{i_j} \cdot e_n$ is an isomorphism of complexes.

4. Recall that $T(V)$ denotes the tensor algebra on V, and the exterior algebra $\Lambda_R(V)$ is the quotient of $T(V)$ by the two-sided ideal I generated by v^2 for all $v \in V$. Prove that, when $2 \in R$ is invertible, we have $\Lambda_R(V) \cong T(V)/J$, where J is the two-sided ideal generated by $vw + wv$ for $v, w \in V$.

Solution. In fact, we have $I = J$. Given $v \in V$, we have $2v^2 = vv + vv \in J$, and so, since $2 \in R$ is invertible, we conclude $v^2 \in J$. On the other hand, for $v, w \in V$, we have $vw + wv = (v+w)^2 - v^2 - w^2 \in I$.

5. Suppose R is a field, which we will denote by k. Assume $n = 1$, so that $\Lambda_k(V)$ is isomorphic to $k[x]/(x^2)$. Prove that this ring has the following property: every finitely generated $\Lambda_k(V)$-module is either free or has infinite projective dimension.

Hint: use the structure theorem for finitely generated modules over PID's.

In fact, this is true for *any* exterior algebra over a field. If you want a more challenging exercise, try and prove this. Hint: as we will see later, E is injective as an E-module; you can take this fact for granted for now. It follows that the functor $\mathrm{Hom}_E(-, E)$ is exact. Therefore, given a finite length minimal free resolution F of a finitely generated E-module N, the quasi-isomorphism $F \xrightarrow{\simeq} N$ induces a quasi-isomorphism $\mathrm{Hom}_E(N, E) \xrightarrow{\simeq} \mathrm{Hom}_E(F, E)$. Prove that this forces N to be free.

Solution. The only nonzero proper ideal of $E = k[x]/(x^2)$ is generated by the class of x. By the structure theorem of finitely generated modules over PIDs, the only indecomposable modules over E are therefore E and k. Now observe that k has the minimal free resolution

$$0 \leftarrow E \xleftarrow{x} E \xleftarrow{x} E \leftarrow \cdots \tag{1}$$

Thus, any indecomposable module is either free or has infinite projective dimension, and so the same is true for any finitely generated module.

(Note: this is also immediate from the Auslander-Buchsbaum formula.)

As for the case of a general exterior algebra: say N is a finitely generated E-module with finite minimal free resolution F. By way of contradiction, suppose N is not free, so that F has the form $0 \to F_n \to \cdots \to F_0 \to 0$, where $n > 0$. Since $\mathrm{Hom}_E(-, E)$ is exact, we conclude that the induced map

$$\mathrm{Hom}_E(F_{n-1}, E) \to \mathrm{Hom}_E(F_n, E)$$

is surjective. But this map takes values in $\mathfrak{m} \cdot \mathrm{Hom}_E(F_n, E)$, where $\mathfrak{m}$ denotes the homogeneous maximal ideal of E. By Nakayama's Lemma, we conclude that $\mathrm{Hom}_E(F_n, E) = 0$, i.e. $F_n = 0$, a contradiction.