L1: Differential graded (dg) algebra resolutions

R = ring

To build (co)homology theories for R-modules, resolve by modules that the functor is exact on:

<u>Def</u>: Given an R-module M, a free (resp. projective) resolution is an exact complex of free (proj) R-mods:

$$\cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} M \to 0$$

$F_2 \twoheadrightarrow K_1 = \ker \nearrow F_1$, $F_1 \twoheadrightarrow K_0 = \ker \varepsilon \nearrow F_0$

$K_1 = \ker$ $(= H(\text{cx}))$

$K_0 = \ker\varepsilon = H(\text{cx so far}) = \dfrac{\ker \varepsilon}{\operatorname{im}(0)}$

<u>Equiv'ly</u>: A free (proj) cx

$$\begin{array}{ccccccc} \cdots \to & F_1 & \to & F_0 & \to 0 \\ & \downarrow 0 & & \downarrow \varepsilon & \\ \cdots \to & 0 & \to & M & \to 0 \end{array}$$

and $\underbrace{\text{quasi-isom.}}_{\text{iso on homology}}$ $F_\bullet \xrightarrow[\varepsilon]{\simeq} M$

$H(F) \xrightarrow{\simeq} H(M) = M$

(Build (co)hom theories – Tor, Ext, ...)

<u>What if want to study commutative rings R?</u>

(or algebras = ring maps $R \to S$)

<u>Pblm</u>: <u>R-mod</u> is abelian cat

<u>Alg</u> is not!

2 levels:

L1 → • put ring structure on free resns

L2... → • build resns w/ free (commut.) ring structure

(Tate, Gulliksen, Assmus, Sjödin, Levin, Avramou, Halperin)

R = a commutative ring

Def: A dg-algebra (dga) is a complex F of R-modules

$$\cdots \xrightarrow{\partial} F_2 \xrightarrow{\partial} F_1 \xrightarrow{\partial} F_0 \to 0$$

equipped with a product making

$$\bigoplus_{i=0}^{\infty} F_i$$

into a $\underbrace{\text{gdd-commutative}}_{(ab = (-1)^{|a||b|} ba)}$ R-algebra

in particular, F_0 ring

so, often $F_0 = R$

s.t. Leibniz rule holds: $\partial(ab) = \partial(a)b + (-1)^{|a|} a\,\partial(b)$

(ie, $F_\bullet \otimes F_\bullet \xrightarrow{\text{mult}} F_\bullet$ is a chain map) ← leave ✓

(Ur-)Example!:

Koszul complex on $f_1, f_2 \in R$:

$$0 \to \underset{2}{R} \xrightarrow{\begin{bmatrix} -f_2 \\ f_1 \end{bmatrix}} \underset{1}{R^2} \xrightarrow{[f_1\ f_2]} \underset{0}{R}$$

can identify w/ exterior powers of $F = R^2$

$$\Lambda^2 F \to \Lambda^1 F \to \Lambda^0 F$$

$$e_1 \longmapsto f_1$$

$$e_2 \longmapsto f_2$$

diff'l ∂

$$e_1 e_2 \longmapsto f_1 e_2 - f_2 e_1$$

and note

$$\bigoplus_{i=0}^{2} \Lambda^i F = \Lambda F = \underbrace{\text{exterior algebra}}_{\text{product}}\ R\langle e_1, e_2\rangle$$

Interaction: $\partial(e_1 e_2) = \partial(e_1)e_2 - e_1\,\partial(e_2)$

(signed) product rule!

(Ur-) Example 2: (algebraic topology)

X topological space

$C^\bullet(X)$ = the chain co-complex for computing its cohomology

Note: The cup product makes $C^\bullet(X)$ into a dga!

$\boxed{\text{Fact: } F_\bullet \text{ dga} \Rightarrow H(F_\bullet) \text{ inherits an alg str.}}$

really $\bigoplus H_i(F)$

Examples:

- Ex 1 $\Rightarrow$ Koszul homology
- Ex 2 $\Rightarrow$ singular cohomology
- cap product (dg module!) $\rightsquigarrow$ $H_*(X)$ homology
- shuffle or concatenation product (on bar resn) $\rightsquigarrow$ HH^* Hochschild cohom.
- A_∞ products $\rightsquigarrow$ Massey operations

more generally

Motivation in CA:

New approach by Buchsbaum-Eisenbud 77 to

B-E-Horrocks Conjecture:

$R = k[x_1, \dots, x_n]$

$M \neq 0$ f.g gdd R-module, $\ell(M) < \infty$

Then min'l gdd free resolution

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

has $\operatorname{rank}_R F_i \geq \binom{n}{i}$

Prop (BE, also Herzog?): If $F_\bullet$ is dga, then BEH holds!

Pf: • Take max'l R-reg seq $f_1, \dots, f_n$ in $\operatorname{ann}_R M$.

• Form Koszul cx $K = R\langle e_1, \dots, e_n \rangle$, $\partial(e_i) = f_i$

• K "free" dga (gdd commut) $\Rightarrow \exists\, K \xrightarrow{\phi} F$

• ϕ injective [show $H_n(\phi)$ inj $\rightsquigarrow \phi_n \rightsquigarrow$ all ϕ_i]

Weaker conj: Total Rank Conjecture $\left(\sum \operatorname{rk} F_i \geq 2^n\right)$ Walker, VandeBogert-Walker

Positive results ($\exists$ dga):

$\operatorname{pd} R/I \leq 3$ – Herzog, Buchsbaum–Eisenbud

$\operatorname{pd} R/I \leq 4$, Gor – Kustin–Miller

+ combinatorial exs + ... (powers max ideal in polys – Srinivasan)
(Taylor, Eliahou–Kervaire)

SKIP IF NEEDED ...
(just summarize
in gen'l $\not\exists$ dga [Avramov 74]

Negative ($\not\exists$ dga):

Avramov	Srinivasan	Katthan
74, 81	92 96	18
$\operatorname{pd}_R R/I = 4$ (distr.)	$\operatorname{pd} R/I = 5$, Gor	monomial !

What instead?

- $R/I \times R/I \xrightarrow{\mu} R/I$ always lifts to a chain map:

Leibniz rule

Comparison Thm: $F_\bullet \otimes F_\bullet \xrightarrow{\mu} F_\bullet$
but may fail to be associative,
but is assoc. up to htpy
given a m_3
⋮

(b/c $F^{\otimes 3} \xrightarrow{\mu(\mu\otimes 1 - 1\otimes\mu)} F$
$a\otimes b\otimes c \longrightarrow (ab)c - a(bc)$
lifts 0-map on R/I
so, $\simeq 0$ w/ htps m_3)

$\Rightarrow F_\bullet$ is A_∞ algebra
($\Rightarrow H(F)$ still an algebra + Massey ops!)

- There is always a free resn of R/I that has dga structure
(Tate's beautiful 1956 paper)

non-min'l (usually)
L2!

[Also: bar resn]

Solution to the Quick Problem:

k field, $R = k[x,y]$, $I = (x^2, xy)$

Let's put DG alg. on resn

$$\begin{array}{ccccccccc}
 & & F_2 & & F_1 & & F_0 & & \\
0 & \longrightarrow & R & \xrightarrow{\begin{bmatrix} -y \\ x \end{bmatrix}} & R^2 & \xrightarrow{[x^2 \ \ xy]} & R & \longrightarrow & \left(R/I \to 0 \right) \\
 & & g & & e_1, e_2 & & 1 & &
\end{array}$$

$$\underset{\underset{R}{\parallel}}{F_0} \times F_i \xrightarrow{\mu} F_i \quad \text{usual } R\text{-mod str.}$$

$$F_1 \times F_2 \xrightarrow{\mu} F_3 = 0$$

(the only non-obvious product) $\longrightarrow$ $F_1 \times F_1 \xrightarrow{\mu} F_2$

$$\underbrace{e_1 e_2 = ??}_{}$$

well, all we know is that it needs to satisfy Leibniz!

<u>Leibniz</u>: $\partial(e_1 e_2) \overset{\text{wait}}{=} \partial(e_1)e_2 + (-1)^{\overset{1}{|e_1|}} e_1 \partial(e_2)$

$$= x^2 e_2 - xy\, e_1$$

$$= x(xe_2 - ye_1)$$

$$= \partial(xg)$$

so, let's define

$$e_1 e_2 = xg$$

(unique here since ∂_2 injective!)

but in gen'l for longer cxs, not unique