Problem Set 5: The geometric BGG correspondence, Part 1

Fix $n \geqslant 0$. Let k be a field. Let $S = k[x_0, \ldots, x_n]$, and $E = \Lambda_k(e_0, \ldots, e_n)$, with gradings given by $\deg(x_i) = 1$ and $\deg(e_i) = -1$.

1. (Short exercise) Using the Čech complex, compute $H^0(\mathbb{P}^1, \mathcal{O}(j))$ for all $j \in \mathbb{Z}$.

2. Using the Čech complex, compute $H^1(\mathbb{P}^1, \mathcal{O}(j))$ for all $j \in \mathbb{Z}$.

Note that a similar Čech cohomology calculation (see e.g. Hartshorne III.5) can be used to show that, for any $n \geqslant 0$, we have

$$H^i(\mathbb{P}^n, \mathcal{O}(j)) \cong \begin{cases} S_j, & i = 0 \text{ and } j \geqslant 0; \\ S_{-j-n-1}, & i = n \text{ and } j < -n; \\ 0, & \text{else.} \end{cases}$$

3. Given a short exact sequence $0 \to M' \to M \to M'' \to 0$ of graded S-modules, there is a short exact sequence $0 \to \widetilde{M'} \to \widetilde{M} \to \widetilde{M''} \to 0$ of associated sheaves. This induces a long exact sequence

$$0 \to H^0(\mathbb{P}^n, \widetilde{M'}) \to H^0(\mathbb{P}^n, \widetilde{M}) \to H^0(\mathbb{P}^n, \widetilde{M''}) \to H^1(\mathbb{P}^n, \widetilde{M'}) \to \cdots$$

on cohomology. Use this long exact sequence to compute $H^i(\mathbb{P}^1, \widetilde{R}(j))$ for $i = 0, 1$ and all $j \in \mathbb{Z}$, where $R = S/(x_0)$. If you like, use `Macaulay2` to check your answer in the following way:

```
S = ZZ/101[x_0, x_1]
M = coker matrix{{x_0 - x_1}}
i = 0
j = 2
rank HH^i sheaf(M**S^{{j}})
```

Two notes:

(a) You will need to understand the map $H^1(\mathbb{P}^n, \mathcal{O}(j)) \to H^1(\mathbb{P}^n, \mathcal{O}(j+1))$ induced by multiplication by x_0. Identifying this with a map $S_{-j-2} \to S_{-j-3}$, the map is given by "contraction by x_0": that is, it sends a monomial m to m/x_0 if x_0 is a factor of m, and to 0 otherwise. (If you're interested, you can verify this claim using a Čech calculation.)

(b) It is perhaps easier to compute $H^1(\mathbb{P}^n, \widetilde{R}(j))$ using the Čech complex: try it if you're interested.

The rest of these exercises are homological in nature, with a view toward tomorrow's lecture.

4. Let R be a ring and $f\colon C \to D$ a morphism of complexes of R-modules. The *mapping cone of* f is the complex with terms $\operatorname{cone}(f)_i = D_i \oplus C_{i-1}$ and differential $\begin{pmatrix} \partial_D & f \\ 0 & -\partial_C \end{pmatrix}$. Prove the following:

(a) There is a short exact sequence of complexes

$$0 \to D \to \operatorname{cone}(f) \to C[-1] \to 0,$$

where the first map is the inclusion, and the second is given by $(d, c) \mapsto -c$.

(b) The morphism f is a quasi-isomorphism if and only if $\operatorname{cone}(f)$ is exact.

5. Let M be a graded S-module that is generated in a single degree d. Prove that M has a linear free resolution if and only if $H_i(\mathbf{R}(M)) = 0$ for $i \neq -d$.

6. Let K denote the complex

$$\cdots \to S(i) \otimes_k \omega_E(-i) \to S(i+1) \otimes_k \omega_E(-i-1) \to \cdots$$

with $S \otimes_k \omega_E$ in homological degree 0, and differential given by $s \otimes y \mapsto \sum_{i=0}^{n} x_i s \otimes e_i y$.

(a) Prove that K is isomorphic, as a complex of S-modules, to $\bigoplus_{i \in \mathbb{Z}} K(x_0, \ldots, x_n)(i)[i]$.

(b) Let M be a finitely generated graded S-module. We consider the complex $M \otimes_S K$, i.e. the complex

$$\cdots \to M(i) \otimes_k \omega_E(-i) \to M(i+1) \otimes_k \omega_E(-i-1) \to \cdots$$

with differential $m \otimes y \mapsto \sum_{i=0}^{n} x_i m \otimes e_i y$. Prove that $H_i(M \otimes_S K)$ is a finite dimensional k-vector space for all $i \in \mathbb{Z}$.

(c) Conclude that associated complex

$$\cdots \to \widetilde{M}(i) \otimes_k \omega_E(-i) \to \widetilde{M}(i+1) \otimes_k \omega_E(-i-1) \to \cdots$$

of sheaves on $\mathbb{P}^n$ is exact.

(Tensoring the sheaf $\widetilde{M}(i)$ over k with $\omega_E(-i)$ may look weird, but just think of it as taking a direct sum of $\dim_k \omega_E(-i)$ copies of the sheaf $\widetilde{M}(i)$.)