Problem Set 2: Graded rings

Let k be a field, and let $R = \bigoplus_{i \in \mathbb{Z}} R_i$ be a graded ring. All modules are left modules, unless assumed otherwise.

Recall that R is a *graded local k-algebra* if R is Noetherian, $R_0 = k$, and either $R_{>0} = 0$ or $R_{<0} = 0$. Given a finitely generated R-module M, recall that M has a minimal graded free resolution

$$0 \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots,$$

and $\beta_{i,j}(M)$ denotes the number of copies of $S(-j)$ in F_i.

1. (Short exercise) Assume R is a graded local k-algebra, and that $R_{<0} = 0$. Prove the graded version of Nakayama's Lemma: if M is a graded R-module with $M_i = 0$ for $i \ll 0$, and $R_{>0}M = M$, then $M = 0$. Notice: there is no need to assume M is finitely generated. Also, a similar statement holds when $R_{>0} = 0$.

2. Assume R is a graded local k-algebra. If M is a finitely generated graded R-module, show that $\dim_k M_i < \infty$ for all i. Suggestion: first prove this for $M = R$.

3. Prove that, if $I \subseteq R$ is an ideal generated by homogeneous elements, then I is a graded R-module. Conclude that R/I is both a graded ring and a graded R-module.

4. Let $S = k[x, y]$, and let M denote the S-module $S/(x^2, xy)$. Yesterday, you wrote down the minimal free resolution of M over S. Now, write it as a *graded* free resolution; that is, keep track of the twists of S in the resolution.

5. Assume R is *graded commutative*, meaning that, if $r \in R_i$ and $s \in R_j$, then

$$rs = (-1)^{\deg(r)\deg(s)} sr.$$

Prove that, if M is a graded right R-module, then M is also a graded left R-module with left action $rm := (-1)^{\deg(r)\deg(m)} mr$ for homogeneous elements $r \in R$ and $m \in M$.

6. Assume R is a graded local k-algebra. Let M be a finitely generated graded R-module that is generated in degree 0. Prove that M has a linear free resolution (i.e. there is a basis of the free resolution with respect to which each matrix has entries given by 0's or linear forms) if and only if $\beta_{i,j}(M) = 0$ for $i \neq j$.

7 (Do this one only if you're interested, and you have time). We recall that a *graded R-R-bimodule* is a graded left R-module M that is also a graded right R-module and such that $(rm)r' = r(mr')$ for all $r, r' \in R$ and $m \in M$.

Let M and N be finitely generated graded left R-modules. Recall that $\underline{\mathrm{Hom}}_R(M, N)$ denotes the set of all R-linear maps from M to N. Recall that $\underline{\mathrm{Hom}}_R(M, N)$ is a graded abelian group, with $\underline{\mathrm{Hom}}_R(M, N)_i =$ degree i maps. Prove that, if M (resp. N) is an R-R-bimodule, then $\underline{\mathrm{Hom}}_R(M, N)$ is a graded left (resp. right) R-module.

Aside: the module M in Problem 5 is in fact a graded R-R-bimodule: prove this if you're interested.