L2: DG algebra resns (Tate)

Today: Girin R-alselina S (often S=P/I)
ies ring homom. R &s

build a free resn (usually far from min'le) who DG als stay

Even better, it will be fee as a gold commalgebra (like polynamia) or exterior alg.)

Idea: For each n, if Hn (sofar) +0

· choose rocks { Ziz generating Itn

· atjoin variables { xi} and define a(xi) = 7i so that

def n+1

(6 lin 2)

[let's ree this in action 1st on an example]

 $(e \text{ in } \partial)$ $Su_i = 0$ in handay

Running Example: k field

 $R = \frac{k(ky)}{(k^3+ky^3)} \xrightarrow{TC} > k = \frac{R}{(x_1+y)} = \frac{R}{T}$

Stepo: 0 -> R ->> k -> 0

 e^{2} $H_0 = |cer \pi = (719)$

Adjoin des 1 variables en ez mappins to

eddomn => e/ez=(-1)'eze1 = -eze1 &e;2=0 + extend by R-linea

So, get an exterior algebra!

R[eijez | a(ei)=x, a(ez)=5] => h misis he postul (x)

Step 1: Campute
$$H_1 = \frac{\text{ter } a_1}{\text{in } a_2} \neq 0$$
 since x, y not R -sequence (R not rester)

Generator: $x^2 e_1 t y^2 e_2 \mapsto x^3 t y^3 = 0$

Adjain a des 2 variable T to kill it:

$$R[e_{1,e_{2}},T]$$
 $a(e_{2})=y$ $a(t)=x^{2}e_{1}+y^{2}e_{2}]$ $a(e_{2})=y$ $a(t)=x^{2}e_{1}+y^{2}e_{2}$

Step 2: if check=0:

$$H_2 = --- = 0$$
! tate: emb.ci \Rightarrow $H_{\geq 2} = 0$

done (if check=p) need mechanic: char 2 $= 73(7)$, $= 72$
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Resn looks like:

$$\frac{3}{2} \left\{ \frac{1}{2} \frac{1}{2} \right\} = \frac{3}{2} \left[\frac{1}{2} \frac{1}{2} \frac{1}{2} \right] = \frac{3}{2} \left[\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \right] = \frac{3}{2} \left[\frac{1}{2} \frac$$

(2-periodic!)

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Adjoining rangeles & (to kill adds)
    Let A be a DG alsebra and ZEAn a cocle.
If n even, asjain variable x of odd desnes ntl
               A\langle x \rangle or A[x] = A \oplus A_{x} (x^{2} = 0)
If nodd, eiter
       Datjoin of even des n+1: A[X]=A@AX@Ax20...
polynomial variable
     or @ atjoin divided powers of raisable x : A<x>=A@Ax@Ax@-...
      with ring structure via: X(l) (m) def (etm) x(etm) (etm) (etm)
      [really: X(e) behaves like 1 x would if it existed.]

A-alg 1

Tate 56 imported to recons from Cartan-Inhald for handows of Eilenberg-Moore opaces)
 Define diff l by: extended from 2A and 2(x) = 2
                     via Leibniz nole

2(x^n) = n \ x^{n-1} a(x) if |x| even
                  (and \partial(\chi^{(n)}) = \chi^{(n+1)}\partial(\chi))
Repeat ... set X = all verigibles (Xn = those of).
                                                on J.

Tile to divided
parent
   let: Of A To a DG alsebra)
           a seni-free extr A[X] (seni-free [-extensia A(X))
       is an extra of A obstamed by repealed adjunction of exterior & poly
      (exterior & divided puer) vouriables.
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Def: R[X] is called a minimal model if Xn chosen minimally at each of hill cycles that give minite R(X) is called an acyclic closure if 1. Nok: 1) if S. F. Rals (e.g. R. RX) = RXX = RXX 1 to map to generators of I. Proof: R[X] is called a minimal model if Xn chosen minimally at each of hill cycles generating Hn(R[X=n]). Def: R[X] is called a minimal model if Xn chosen minimally at each of hill cycles that give minite generators of thomology. Nok: 1) if S. F. Rals (e.g. R. RXX = RXX > as does not not still have xn = n! xn) (in RXX). Note: Both can be defined also for the position of time. Note: Both can be defined also for the position of time. Note: Both can be defined also for the position of time. Note: Both can be defined also for the position of time. Note: Both can be defined also for the position of time. Note: Both can be defined also for the position of time. Note: Both can be defined also for the position of time. Note: Both can be defined also for the position of time. Note: Both can be defined also for the position of time. Note: Both can be defined also for the position of time.	Thm & let R POSS be a surjective in map, and set I=ker .
2) I semi-Ree Nextn restm S are R RXX [T quasi-iso] [or if not agentic, 10 atima set Xo of deso st. R[Xo] - 20 S expected.] Proof: As in the example, inductive construction: First, adjoin a set of deg 1 vars X1 to map to generators of I. Then storeach n, adjoin variables of des not to kill cycles generating Hn(R[X=n]). Def: R[X] is called a minimal model if Xn choken minimally at each (to kill cycles that give minly generators of homology (to kill cycles that give minly generators of homology Nok: 1) if S for Raly (eg, R - 20 S viz.) & R Noeth y can take Xn coc 2) if (x = R, then R[X] = R(X) as dges xn - 2 X xn 3) if not y still have xn = n! xn) (in R(X). Note: Both can be defied also for A S DG alsomp T = how (Ao 35) Now: 2(XX) mai gers I not 3(A1) 2(Xx) mai gers I not 4(A1) 2(Xx) mai gers I not 4(A1)	1) I semi-fre extr resolving S ar R
Cor if not appendix, 10 adjan a set Xo of days of the REXO] - so S singerity. Proof: As in the example, inductive construction: First, adjain a set of day 1 vars X1 to map to generators of I. Then, for each n, adjain variables of day not to fill cycles generating Hn(REX=n]). Def: REX] is called a uninimal model if Xn chosen minimally at each (to bill cycles that give minited RXX) is called an acyclic obsure if 11 generators of homology. Note: 1) if Sfg. Raly (eg. R - so S sur) & R Noeth, can take Xn coc 2) if a SR, then REX = RXX as a dgas 3) if not y still have $x^n = n! x^n$ (in RXX). Note: Both can be defined also for a sure of the part of the	$ \begin{array}{ccc} R[X] \\ R & \overline{U} \\ R & \overline{U} \end{array} $ $ \begin{array}{cccc} TU & QURSI \\ R & \overline{U} \end{array} $ $ \begin{array}{cccc} H_1(R[Y]) = \begin{cases} S, i = 0 \\ O, else \end{cases} $
Cor if not aspendie, 10 adjan a set Xo of dus o st. R[Xo] - so S sinjection? Proof: As in the example, inductive construction: First, adjain a set of deg 1 vars X1 to map to generates of I. Then, for each n, adjain variables of deg not to to till cycles generating Hn(R[X=n]). Def: R[X] is called a uniminal model if Xn chosen minimally at each (to kill cycles that give minite R(X) is called an acyclic obsure if 11 (11 11 11 11 11 11 11 11 11 11 11 11	2) Josemi-Lee Mextr redu Saer R
Proof: As in the example, inductive construction: First, adjoin a set of deg 1 vars X, to map to generators of I. Then stor each n, adjoin variables of deg nt1 to kill cycles generating $H_n(R[X \leq n])$. Def: $R[X]$ is called a minimal model if X_n chosen minimally at each step (to kill cycles that give minite $R(X)$ is called an acyclic closure if $H_n(X)$ generators of homology $H_n(X)$ is called an acyclic closure if $H_n(X)$ if $H_n(X)$ is called an acyclic closure if $H_n(X)$ is a dgas Then $H_n(X)$ is a dgas Then $H_n(X)$ is a dgas Then $H_n(X)$ is $H_n(X)$ in $H_n(X)$ in $H_n(X)$. Note: Both can be defined also for $H_n(X)$ in H_n	R(x) π quesi-iso
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generating $H_n(R[X \leq n])$. Def: $R[X]$ is called a minimal model if X_n chosen minimally at each (to kill cycles that give minite $R(X)$ is called an acyclic obsure if I_1 I_2 I_3 I_4 I_4 I_5 I_6 I	Then stor each n, adjoin variables of des not to kill cycles
Note: 1) if Sf.g. Ralg (eg, R-22S sw.) & R Noeth , can take Xn < 00 2) if & = R, then R[X] = R(X) as dgas \[\frac{\text{X}}{\text{N}} \leftrightarrow \text{X} \rightarrow \text{As dgas} \[\frac{\text{X}}{\text{N}} \leftrightarrow \text{X} \rightarrow \text{As dgas} \] 3) if not, still have \[\text{X} = n! \text{X} \rightarrow \text{(in R(X))}. \[\text{Note: Both can be defined also for \text{PG als map} \text{No time} \] \[\text{PG also min gets E nod B(A1)} \\ \text{Now: A(Xn) min gets Hn(A(X))} \]	generating Hn(R[X=n]).
Note: 1) if Sf.g. Ralg (eg, R-22S sw.) & R Noeth , can take Xn < 00 2) if & = R, then R[X] = R(X) as dgas \[\frac{\text{X}}{\text{N}} \leftrightarrow \text{X} \rightarrow \text{As dgas} \[\frac{\text{X}}{\text{N}} \leftrightarrow \text{X} \rightarrow \text{As dgas} \] 3) if not, still have \[\text{X} = n! \text{X} \rightarrow \text{(in R(X))}. \[\text{Note: Both can be defined also for \text{PG als map} \text{No time} \] \[\text{PG also min gets E nod B(A1)} \\ \text{Now: A(Xn) min gets Hn(A(X))} \]	Def: R[X] is called a <u>minimal model</u> if X _n chosen minimally at each (to kill cycles that give minle)
2) if $C \subseteq R$, then $R[X] \cong R(X)$ as dgas $X^n \longrightarrow X^{(n)}$ 3) if not , still have $X^n = n! X^{(n)}$ (in $R(X)$). Note: Both can be defined also for DG als map DG	
3) if not , still have $x^n = n! x^n$ (in $R(x)$). Note: Both can be defined also for DG als map no time $T = \text{ber}(A_0 - S)$ Now: $2(x_1)$ min gers T mad $3(A_1)$ $2(x_{11})$ min gers $H_{\Lambda}(AC(x_{11}))$	Note: 1) if Sf.g. Ralg (eg, R-205 sin) & R Noeth, can take Xn < oc
Note: Both can be defined also for DGrady ving T = ber (Ab S S) Now: 2(Yes) min gens I mad 8(A1) 2(Yes) min gens Har(AEYER)	$\frac{h_1}{x_n} \leftarrow X_{(n)}$
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now: 2(X1) min gens I mod 3(A1) 2(Xny) min gens Hr(A[Xx])	DG als map no time
2(Yng) min gene Hr(AEYEZ)	
	now: 2(X) min gers I mad 8(A1)
	a (Yng) min gets Hn (A [/sn])

For ex: koszul cxs K= R(e1,--, en>
(used in BEH proof to get K--->F)