L1: Koszul algebras

not necess. commutative!

Def (Priddy): a Koszul algebra is a standard gdd k-alg

$$\left(R = \bigoplus_{i\geq 0} R_i,\ R_0 = k,\quad R = R_0[R_1],\ m = R_+ = \bigoplus_{i>0} R_i\right)$$

Priddy: more gen'l setting

such that the min'l gdd R-free resn of $k = R/m$ is linear:

$$\cdots \to R(-i)^{b_i} \to \cdots \to R(-2)^{b_2} \to R(-1)^{b_1} \to R \xrightarrow{\varepsilon} k \to 0$$

Ie, matrices have entries linear (deg 1)!

$R(-j)$ = shifted R
$(R(-j))_n = R_{-j+n}$
(so, $1 \in R_0 = R(-j)_j$)
in degree j

Equiv: gdd Betti numbers:

$$\beta_{ij} \ (= \dim_k \operatorname{Tor}_i(k,k)_j) = \begin{cases} \text{some } b_i & i = j \\ 0 & i \neq j \end{cases}$$

$$\operatorname{Tor}_i(k,k) = \bigoplus_{i\geq 0} \operatorname{Tor}_i(k,k)_i = \text{“diagonal part”}$$

(same for $\operatorname{Ext}_i(k,k)$)

Ex: $R = k[x_1, \dots, x_n]$ poly ring

resn of $k = R/(x_1,\dots,x_n)$ is the Koszul cx $K(x_1,\dots,x_n)$ ✓

eg, $R = k[x,y] \rightsquigarrow$ $0 \to R(-2) \xrightarrow{\begin{bmatrix} -y \\ x\end{bmatrix}} R(-1)^2 \xrightarrow{[x\ y]} R \xrightarrow{\varepsilon} k \to 0$

Ex: $R = \dfrac{k[x,y]}{(x^2,y^2)}$

from $x^2, y^2 = 0$

$$?? \cdots \to R^3 \xrightarrow{\begin{bmatrix} x & 0 & -y \\ 0 & y & x\end{bmatrix}} R^2 \xrightarrow{[x\ y]} R \xrightarrow{\varepsilon} k \to 0$$

$R(-2)^3$ Koszul reln   really $R(-1)^2$

Need whole resn!

$R \to k$ embedded c.i. $\overset{\text{Tate}}{\Longrightarrow}$ $R\langle e_1^{1}, e_2^{1}, T_1^{2}, T_2^{2}\rangle \xrightarrow[R]{\simeq} k$

$R = Q/(x^2,y^2)$, $k = Q/(x,y)$

∂: $e_1 \mapsto x$, $e_2 \mapsto y$, $T_1 \mapsto xe_1$, $T_2 \mapsto ye_2$

Leibniz + linear coeffs $\Rightarrow$ resn linear ✓

<u>Note</u>: $R = Q/I$, $Q = $ poly

R koszul $\underset{\not\Leftarrow}{\Rightarrow}$ R quadratic alg (I gen'd by deg 2)
(else ∂_2 not linear!)

<u>Examples of koszul algs</u>:

1) quadratic c.i.'s (Tate resn!)

2) I = quadratic monomial ideal

R * 3) I has quadratic Gröbner basis
"G-koszul" (in(I) quadratic monomial $\overset{\text{deform}}{\rightsquigarrow}$ I)

eg, determinantal 2×2 (eg $\dfrac{k\begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}}{I_2(\text{''})}$)

4) I linear resn – see pblms

<u>But</u>: no <u>finite</u> characterization known! (open)

<u>Roos</u>: $\exists$ examples, $\forall n$, s.t. resn of k linear for n steps but not ∂_{n+1} (n unrelated to dim or edim)

<u>Why important?</u>

1) duality theory ...

2) arise naturally in many places:

- alg geom (Segre, Veronese, toric, ...)
- alg topology (Steenrod algebra, cohom alg of $K(\pi,1)$-spaces, holonomy algebras of supersolvable hyperplane arrangements
  also: $\left\{\begin{matrix}\text{perverse sheaves} \\ \text{on } \Delta\text{'d space}\end{matrix}\right\} \approx \left\{\begin{matrix}\text{modules over} \\ \text{a koszul alg}\end{matrix}\right\}$
- noncommut. geom. (nat'l condition on an exceptional collection, noncommut. deformations of $\mathbb{P}^n$)
- # theory (Milnor K-theory $\otimes \mathbb{Z}/p\mathbb{Z}$ of fields $\leftsquigarrow$ conjecturally) (stronger than Bloch-Kato)

operadic stuff...
relative version...

Quadratic / Koszul dual: (to prepare for Koszul duality)

Def: $k\langle x_1, \dots, x_n\rangle$ = the noncommutative poly ring
= {polys in noncommuting variables}

$$x_1 x_2 \neq x_2 x_1$$

Our algs are quotients:

Ex: $k[x_1, \dots, x_n] = \dfrac{k\langle x_1, \dots, x_n\rangle}{(\{x_i x_j - x_j x_i\})}$ still quadratic ✓

Ex: $\dfrac{k[x,y]}{(x^2, xy)} = \dfrac{k\langle x, y\rangle}{(x^2, xy, \cancel{xy} - yx)}$ quad ✓

Equiv: $V = k\{x_1, \dots, x_n\}$ v. space

The tensor algebra on V:

$$T = T(V) = k \oplus V \oplus (V \otimes_k V) \oplus \cdots \oplus (\overbrace{V \otimes_k \cdots \otimes_k V}^{i}) \oplus \cdots$$

$$= \bigoplus_{i \geq 0} V^{\otimes_k i} \cong k\langle x_1, \dots, x_n\rangle$$

eg, $\left.\begin{array}{l} x_1 \otimes x_2 \longmapsto x_1 x_2 \\ x_2 \otimes x_1 \longmapsto x_2 x_1 \end{array}\right\} \neq$

Pairing: $T(V)_2 \otimes_k T(V^*)_2 \longrightarrow k$

$$x_i x_j \otimes x_k^* x_\ell^* \longmapsto \begin{cases} 1 & (i,j) = (k,\ell) \\ 0 & \text{else} \end{cases}$$

[$T(V^*) \cong T(V)^*$ really evaluation on dual basis $(x_\ell x_k)^*$]

Ex: $(\underbrace{2x_1x_2}_{=0} + \underbrace{3x_2^2}_{=1}) \cdot ((x_2^*)^2) = 3$

<u>Def</u>: For a quadratic alg

$$R = \frac{k\langle x_1, \dots, x_n \rangle}{(W)} = \frac{T(V)}{(W)} \quad \text{where } W \overset{\text{v.space}}{\subseteq} T(V)_2$$

its <u>quadratic dual</u> is

$$R^! = \frac{k\langle x_1^*, \dots, x_n^* \rangle}{(W^\perp)} = \frac{T(V^*)}{(W^\perp)}$$

where $W^\perp$ = orthogonal complement of W under the pairing

$$= \{ f \in (V^*)^{\otimes 2} \mid \alpha f = 0 \ \forall \alpha \in W \}$$

<u>Ex</u>: $R = \dfrac{k[x,y]}{(x^2, xy+y^2)} \overset{!}{=} \dfrac{k\langle x,y \rangle}{(\underbrace{xy-yx}_{\text{commut.}}, x^2, xy+y^2)} = \dfrac{T(V)}{(W)}$

note $\dim_k T_2 = \dim k\{x^2, xy, yx, y^2\} = 4$

$\dim_k W = 3$

so $\dim_k W^\perp = 1$

$$R^! = \frac{k\langle x^*, y^* \rangle}{(x^*y^* + y^*x^* - (y^*)^2)} = \frac{T(V^*)}{(W^\perp)}$$

(lazy: call x^* just x, y^* " y $\quad \dfrac{k\langle x,y \rangle}{(xy+yx-y^2)}$)

$$R^! = \underbrace{k1}_{R^!_0} \oplus \underbrace{k\{x,y\}}_{R^!_1} \oplus \underbrace{k\{x^2, xy, yx, \cancel{y^2}\}}_{R^!_2} \oplus \cdots$$

noncommut. ring