Problem Set 5: The geometric BGG correspondence, Part 1

Fix $n \geqslant 0$. Let k be a field. Let $S = k[x_0, \dots, x_n]$, and $E = \Lambda_k(e_0, \dots, e_n)$, with gradings given by $\deg(x_i) = 1$ and $\deg(e_i) = -1$.

1. (Short exercise) Using the Čech complex, compute $H^0(\mathbb{P}^1, \mathcal{O}(j))$ for all $j \in \mathbb{Z}$.

Solution. $H^0(\mathbb{P}^1, \mathcal{O}(j))$ is the kernel of

$$S[\frac{1}{x_0}]_j \bigoplus S[\frac{1}{x_1}]_j \xrightarrow{(-1 \;\; 1)} S[\frac{1}{x_0 x_1}]_j. \tag{1}$$

Consider a pair $(f/x_0^{\deg(f)-j}, g/x_1^{\deg(g)-j})$ in the kernel. This implies that $gx_0^{\deg(x)-j} = fx_1^{\deg(y)-j}$, and hence $x_0^{\deg(x)-j}$ divides f (similarly for g). In particular, if $j < 0$, this means that we have $H^0(\mathbb{P}^1, \mathcal{O}(j)) = 0$. If $j \geq 0$, we see that $f = x_0^{\deg(x)-j}m$ and $g = x_1^{\deg(y)-j}m$ for some $m \in S_j$. In other words, the map of vector spaces $S_j \to H^0(\mathbb{P}^1, \mathcal{O}(j))$ sending $m \mapsto (m, m)$ is an isomorphism. Thus, $H^0(\mathbb{P}^1, \mathcal{O}(j)) \cong S_j$.

2. Using the Čech complex, compute $H^1(\mathbb{P}^1, \mathcal{O}(j))$ for all $j \in \mathbb{Z}$.

Note that a similar Čech cohomology calculation (see e.g. Hartshorne III.5) can be used to show that, for any $n \geqslant 0$, we have

$$H^i(\mathbb{P}^n, \mathcal{O}(j)) \cong \begin{cases} S_j, & i = 0 \text{ and } j \geqslant 0; \\ S_{-j-n-1}, & i = n \text{ and } j < -n; \\ 0, & \text{else.} \end{cases}$$

Solution. $H^1(\mathbb{P}^1, \mathcal{O}(j))$ is the cokernel of (1). The vector space $S[1/x_0x_1]_j$ has k-basis given by monomials of the form $x_0^d x_1^e$, with $d, e \in \mathbb{Z}$ and $d + e = j$. If $j \geqslant -1$, each of these is in the image of (1), and so $H^1(\mathbb{P}^1, \mathcal{O}(j)) = 0$ if $j \geq -1$. Suppose $j \leqslant -2$. Certainly if d or e is nonnegative, then $x_0^d x_1^e$ is in the image of (1). In fact, these monomials span the image of (1). Indeed, say $f/x_0^r \in S[1/x_0]_j$, and $g/x_1^s \in S[1/x_1]_j$. Expanding f and g as sums of monomials exhibits f/x_0^r and g/x_1^s as k-linear combinations of monomials of the form $x_0^d x_1^e$ where at least one of d or e is nonnegative.

This implies $H^1(\mathbb{P}^1, \mathcal{O}(j))$ has k-basis given by elements of the form $1/x_0^d x_1^e$ with $d, e \geqslant 1$ and $d + e = -j$. There are exactly $\dim_k S_{-j-2}$ such elements.

3. Given a short exact sequence $0 \to M' \to M \to M'' \to 0$ of graded S-modules, there is a short exact sequence $0 \to \widetilde{M'} \to \widetilde{M} \to \widetilde{M''} \to 0$ of associated sheaves. This induces a long exact sequence

$$0 \to H^0(\mathbb{P}^n, \widetilde{M'}) \to H^0(\mathbb{P}^n, \widetilde{M}) \to H^0(\mathbb{P}^n, \widetilde{M''}) \to H^1(\mathbb{P}^n, \widetilde{M'}) \to \cdots$$

on cohomology. Use this long exact sequence to compute $H^i(\mathbb{P}^1, \widetilde{R}(j))$ for $i = 0, 1$ and all $j \in \mathbb{Z}$, where $R = S/(x_0)$. If you like, use `Macaulay2` to check your answer in the following way:

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S = ZZ/101[x_0, x_1]
M = coker matrix{{x_0 - x_1}}
i = 0
j = 2
rank HH^i sheaf(M**S^{{j}})
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Two notes:

(a) You will need to understand the map $H^1(\mathbb{P}^n, \mathcal{O}(j)) \to H^1(\mathbb{P}^n, \mathcal{O}(j+1))$ induced by multiplication by x_0. Identifying this with a map $S_{-j-2} \to S_{-j-3}$, the map is given by "contraction by x_0": that is, it sends a monomial m to m/x_0 if x_0 is a factor of m, and to 0 otherwise. (If you're interested, you can verify this claim using a Čech calculation.)

(b) It is perhaps easier to compute $H^1(\mathbb{P}^n, \widetilde{R}(j))$ using the Čech complex: try it if you're interested.

Solution. The short exact sequence of graded modules

$$0 \to S(j-1) \xrightarrow{x_0} S(j) \to R(j) \to 0$$

induces the long exact sequence of vector spaces (applying the previous exercises):

$$0 \to H^0(\mathbb{P}^1, \mathcal{O}(j-1)) \xrightarrow{x_0} H^0(\mathbb{P}^1, \mathcal{O}(j)) \to H^0(\mathbb{P}^1, \widetilde{R}(j)) \to H^1(\mathbb{P}^1, \mathcal{O}(j-1)) \xrightarrow{x_0} H^1(\mathbb{P}^1, \mathcal{O}(j)) \to H^1(\mathbb{P}^1, \widetilde{R}(j)) \to 0$$

This exact sequence immediately implies the following:

(a) If $j = 0$, then $k \cong S_0 \cong H^0(\mathbb{P}^1, \widetilde{R})$, and $H^1(\mathbb{P}^1, \widetilde{R}) = 0$.

(b) If $j = -1$, then $H^0(\mathbb{P}^1, \widetilde{R}(-1)) \cong H^1(\mathbb{P}^1, \mathcal{O}(-2)) \cong k$, and $H^1(\mathbb{P}^1, \widetilde{R}) = 0$.

(c) If $j \geqslant 1$, then $H^0(\mathbb{P}^1, \widetilde{R}(j)) \cong \operatorname{coker}(H^0(\mathbb{P}^1, \mathcal{O}(j-1)) \xrightarrow{x_0} H^0(\mathbb{P}^1, \mathcal{O}(j))) \cong k \cdot x_1^j \cong k$, and $H^1(\mathbb{P}^1, \widetilde{R}) = 0$.

(d) If $j \leqslant -2$, then $H^0(\mathbb{P}^1, \widetilde{R}(j))$ and $H^1(\mathbb{P}^1, \widetilde{R}(j))$ are isomorphic to the kernel and cokernel of the map $S_{-j-2} \to S_{-j-3}$ given by contraction by x_0 (respectively). This map is surjective, and its kernel is $k \cong k \cdot x_1^{-j-2}$.

We conclude that H^1 vanishes on all twists of $\widetilde{R}$, and H^0 of every twist of $\widetilde{R}$ is k.

The rest of these exercises are homological in nature, with a view toward tomorrow's lecture.

4. Let R be a ring and $f\colon C \to D$ a morphism of complexes of R-modules. The *mapping cone of* f is the complex with terms $\operatorname{cone}(f)_i = D_i \oplus C_{i-1}$ and differential $\begin{pmatrix} \partial_D & f \\ 0 & -\partial_C \end{pmatrix}$. Prove the following:

(a) There is a short exact sequence of complexes

$$0 \to D \to \operatorname{cone}(f) \to C[-1] \to 0,$$

where the first map is the inclusion, and the second is given by $(d, c) \mapsto -c$.

Solution. One checks that the inclusion and projection maps in each homological degree are morphisms of complexes. Exactness in each homological degree is immediate.

(b) The morphism f is a quasi-isomorphism if and only if $\operatorname{cone}(f)$ is exact.

Solution This follows from the long exact sequence in homology corresponding to the above short exact sequence.

5. Let M be a graded S-module that is generated in a single degree d. Prove that M has a linear free resolution if and only if $H_i(\mathbf{R}(M)) = 0$ for $i \neq -d$.

Solution. A straightforward generalization of a previous exercise gives that M has a linear free resolution if and only if $\beta_{i,j}(M) = 0$ unless $j - i = d$. On the other hand, $H_i(R(M))_j \simeq k^{\oplus \beta_{i+j,j}(M)}$. Thus, the two assertions in the question are equivalent.

6. Let K denote the complex

$$\cdots \to S(i) \otimes_k \omega_E(-i) \to S(i+1) \otimes_k \omega_E(-i-1) \to \cdots$$

with $S \otimes_k \omega_E$ in homological degree 0, and differential given by $s \otimes y \mapsto \sum_{i=0}^{n} x_i s \otimes e_i y$.

(a) Prove that K is isomorphic, as a complex of S-modules, to $\bigoplus_{i \in \mathbb{Z}} K(x_0, \dots, x_n)(i)[i]$.

Solution. Fix $j \in \mathbb{Z}$. Consider the S-linear subcomplex of K given by

$$\cdots \to S(i) \otimes_k \omega_E(-i)_{-j} \to S(i+1) \otimes_k \omega_E(-i-1)_{-j} \to \cdots,$$

which is equal to

$$\cdots \to S(i) \otimes_k (\omega_E)_{-j-i} \to S(i+1) \otimes_k (\omega_E)_{-j-i-1} \to \cdots.$$

Observe that this complex is equal to $\mathbf{L}(\omega_E(-j)) \cong \mathbf{L}(\omega_E)(j)[j] \cong K(x_0, \dots, x_n)(j)[j]$. Now notice that summing over all j gives K.

(b) Let M be a finitely generated graded S-module. We consider the complex $M \otimes_S K$, i.e. the complex

$$\cdots \to M(i) \otimes_k \omega_E(-i) \to M(i+1) \otimes_k \omega_E(-i-1) \to \cdots$$

with differential $m \otimes y \mapsto \sum_{i=0}^{n} x_i m \otimes e_i y$. Prove that $H_i(M \otimes_S K)$ is a finite dimensional k-vector space for all $i \in \mathbb{Z}$.

Solution. For each j, $H_j(M \otimes_S K) \simeq \bigoplus_{i \in \mathbb{Z}} H_j((K(x_0, \ldots, x_n) \otimes_S M)(i)[i])$. There are only finitely many i for which $(K(x_0, \ldots, x_n) \otimes_S M)(i)[i]$ is supported in homological degree j, and the total homology of $(K(x_0, \ldots, x_n) \otimes_S M)(i)[i]$ is (a twist and shift of) $\operatorname{Tor}_*^S(M, k)$, which is a finite dimensional k-vector space.

(c) Conclude that associated complex

$$\cdots \to \widetilde{M}(i) \otimes_k \omega_E(-i) \to \widetilde{M}(i+1) \otimes_k \omega_E(-i-1) \to \cdots$$

of sheaves on $\mathbb{P}^n$ is exact.

Solution. This follows from (b), the fact that taking associated sheaves is an exact functor, the fact that exact functors commute with taking homology, and that the associated sheaf of a finite dimensional vector space is the zero sheaf.

(Tensoring the sheaf $\widetilde{M}(i)$ over k with $\omega_E(-i)$ may look weird, but just think of it as taking a direct sum of $\dim_k \omega_E(-i)$ copies of the sheaf $\widetilde{M}(i)$.)