

Week 2 • Problem Set 2 • Koszul resolution

Quick Problem:

Let $R = k[x, y]/(x^2 + y^2)$. Write out the first 3 steps of the Priddy resolution

$$\cdots P_3 \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow k \rightarrow 0.$$

Problems:

1. Let R be a graded ring and M a graded R -module. Define the Poincaré and Hilbert series of M as

$$P_M^R(t) = \sum_{i \geq 0} \beta_i(M) t^i \quad H_M^R(t) = \sum_{i \geq 0} \dim_k M_i t^i$$

where the i th Betti number β_i is the rank of F_i in an R -free resolution of M . (These definitions apply equal well to $R^!$.)

Prove the following equalities of power series.

- (a) $P_k^R(t) = H_{R^!}(t)$
 - (b) $H_R(t) P_k^R(-t) = 1$ Hint: Think about Priddy's resolution and Euler characteristics (i.e., vector space dimensions along exact sequences).
2. "Compute" the Priddy/Koszul resolution of k for the following algebras.
 - (a) $R = k[x, y, z]/(x^2, xy)$
(and, if you are combinatorially-minded, find the Poincaré series in closed form sometime)
 - (b) $R = S = k[x_1, \dots, x_n]$ the symmetric algebra
 - (c) $R = \wedge(k\{e_1, \dots, e_n\})$ the exterior algebra
 - (d) $R =$ the trivial algebra over k on x_1, \dots, x_n (that is, all products of positive degree elements are zero).
 3. Consider the noncommutative ring $R = k\langle x, y \rangle / (x^2 + y^2, xy + yx)$.
 - (a) Show that R is Koszul. (Hint: Consider its dual.)
 - (b) Show that $\text{projdim}_R k < \infty$.
 4. Let R be a graded quadratic algebra. Prove that R is Koszul if and only if $R^!$ is Koszul. (You may assume Priddy's theorem.)
Hint: Apply $\text{Hom}_k(-, k)$ and regrade!
 5. Let R be a quadratic (commutative) hypersurface $R = Q/(f)$. Show explicitly how the Priddy resolution is isomorphic to Tate's dg algebra resolution from Week 1.
(Note that $Q \rightarrow R \rightarrow k$ is an embedded complete intersection.)
 6. Let $\langle -, - \rangle: V \otimes V \rightarrow k$ be a perfect pairing, where V is the k -vector space with basis $\{x, y\}$. Consider the "short Gorenstein algebra" $R = k \oplus V \oplus k$ in degrees 0, 1, and 2, with product given by the perfect pairing (and k -scalar products).
Show that $R^! = T(V^*)/(q)$ (a noncommutative hypersurface!) where q is the quadratic form of the pairing.