Week 2 • Problem Set 5 • Ext algebras

Quick Problem:

Compute bases for $\operatorname{Ext}_R^{\leq 2}(k,k)$ for the algebra $R = k[x,y]/(xy, x^3+y^3)$. In what (internal) degrees do they live? Hint: Write out a *graded* resolution of k.

Problems:

1. Let $R = k[x,y]/(x^2, xy)$, and write the acyclic closure F as $R\langle e_1,\, e_2,\, f_1,\, f_2,\, e_3, \ldots\rangle$ with

$$\partial(e_1) = x,\quad \partial(e_2) = y,\quad \partial(f_1) = xe_1,\quad \partial(f_2) = ye_2,\ \text{, and}\quad \partial(e_3) = xe_1e_2.$$

Order the bases of F_1, F_2, and F_3 by

$$\{e_1,\, e_2\}, \{e_1e_2,\, f_1\, f_2\}, \text{ and } \{e_1f_1,\, e_2f_1,\, e_1f_2, e_2f_2, e_3\}. \tag{0.1}$$

Dualizing the acyclic closure and taking homology, one has

$$\operatorname{Ext}_R(k,k) \cong H(\operatorname{Hom}_R(F,k)) \cong \operatorname{Hom}_R(F,k)$$

and so we can represent $\operatorname{Ext}_R^{\leq 3}(k,k)$ as the k-span of duals of the basis elements in 0.1. One also has

$$\operatorname{Ext}_R(k,k) \cong H(\operatorname{Hom}_R(F,k)) \cong H(\operatorname{Hom}_R(F,F))$$

which we use to compute the composition product in Ext. So, by lifting along the quasi-isomorphism $F \to k$, we can find elements in $\operatorname{Hom}_R(F,F)$ whose classes represent the duals of the elements of 0.1, yielding, for example:

$$e_1^* = \ldots \quad \begin{tikzcd}[ampersand replacement=\&]
R^5 \arrow[r, "{\begin{bmatrix} 0 & x & 0 & y & x \\ x & y & 0 & 0 & 0 \\ 0 & 0 & x & y & 0 \end{bmatrix}}"] \arrow[d, "?"] \& R^3 \arrow[r, "{\begin{bmatrix} -y & x & y \\ x & 0 & 0 \end{bmatrix}}"] \arrow[d, "?"] \& R^2 \arrow[r, "{\begin{bmatrix} x & y \end{bmatrix}}"] \arrow[d, "{\begin{bmatrix} 1 & 0 \end{bmatrix}}"] \& R \arrow[d, "0"] \\
R^3 \arrow[r, "{\begin{bmatrix} -y & x & y \\ x & 0 & 0 \end{bmatrix}}"'] \& R^2 \arrow[r, "{\begin{bmatrix} x & y \end{bmatrix}}"'] \& R \arrow[r] \& k
\end{tikzcd}$$

$$e_2^* = \ldots \quad \begin{tikzcd}[ampersand replacement=\&]
R^5 \arrow[r, "{\begin{bmatrix} 0 & x & 0 & y & x \\ x & y & 0 & 0 & 0 \\ 0 & 0 & x & y & 0 \end{bmatrix}}"] \arrow[d, "?"] \& R^3 \arrow[r, "{\begin{bmatrix} -y & x & y \\ x & 0 & 0 \end{bmatrix}}"] \arrow[d, "?"] \& R^2 \arrow[r, "{\begin{bmatrix} x & y \end{bmatrix}}"] \arrow[d, "{\begin{bmatrix} 0 & 1 \end{bmatrix}}"] \& R \arrow[d, "0"] \\
R^3 \arrow[r, "{\begin{bmatrix} -y & x & y \\ x & 0 & 0 \end{bmatrix}}"'] \& R^2 \arrow[r, "{\begin{bmatrix} x & y \end{bmatrix}}"'] \& R \arrow[r] \& k
\end{tikzcd}$$

$$(e_1e_2)^* = \ldots \quad \begin{tikzcd}[ampersand replacement=\&]
R^5 \arrow[r, "{\begin{bmatrix} 0 & x & 0 & y & x \\ x & y & 0 & 0 & 0 \\ 0 & 0 & x & y & 0 \end{bmatrix}}"] \arrow[d, "?"] \& R^3 \arrow[r, "{\begin{bmatrix} -y & x & y \\ x & 0 & 0 \end{bmatrix}}"] \arrow[d, "{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}"] \& R^2 \arrow[r, "{\begin{bmatrix} x & y \end{bmatrix}}"] \arrow[d, "0"] \& R \arrow[d] \\
R^2 \arrow[r, "{\begin{bmatrix} x & y \end{bmatrix}}"'] \& R \arrow[r] \& k \arrow[r] \& 0
\end{tikzcd}$$

and so on, with f_1^* starting with $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, f_2^* with $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, and e_3^* with $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

(a) Fill in the lifts above, making sure that each square commutes.

(b) Show that $(e_2^*)^2 = 0$. Thinking of Ext as the quadratic dual of R, with which element of $R_1^!$ may e_2^* be identified with?

(c) Recall that $e_1^2 = 0$. Does this mean that $(e_1^*)^2 = (e_1^2)^* = 0$?

(d) Compute $(e_1)^*(e_2)^*$ and $(e_2)^*(e_1)^*$. How do they relate to $(e_1e_2)^*$?

(e) Recall that $\partial(f_2) = xy$. Does this mean that $(f_2)^*(e_2)^* = 0$?

(f) Put shifts in the diagrams above for e_1^* and e_2^*. Explain the connection between an element of $\mathrm{Ext}_R^i(k,k)$ having internal degree i and the degrees of the entries appearing in its lift to an element of $\mathrm{Hom}_R(F,F)$.