L1: Koszul algebras

not necess. commutative!

Def (Priddy): a Koszul algebra is a standard gdd k-alg

$$\left(R = \bigoplus_{i \geq 0} R_i,\ R_0 = k,\ R = R_0[R_1],\ m = R_+ = \bigoplus_{i > 0} R_i\right)$$

Priddy: more genl setting

such that the min'l gdd R-free resn of $k = R/m$ is linear:

$$\cdots \to R(-i)^{b_i} \to \cdots \to R(-2)^{b_2} \to R(-1)^{b_1} \to R \to k \to 0$$

Ie, matrices have entries linear (deg 1)!

$R(-j)$ = shifted R

$(R(-j)_n = R_{-j+n})$

(so, $1 \in R_0 = R(-j)_j$ in degree j)

Equiv: gdd Betti numbers:

$$\beta_{ij}\ (= \dim_k \operatorname{Tor}_i(k,k)_j) = \begin{cases} \overset{\text{some}}{b_i} & i = j \\ 0 & i \neq j \end{cases}$$

$$\operatorname{Tor}_i(k,k) = \bigoplus_{i \geq 0} \operatorname{Tor}_i(k,k)_i = \text{“diagonal part”}$$

(same for $\operatorname{Ext}_i(k,k)$)

Ex: $R = k[x_1, \ldots, x_n]$ poly ring

resn of $k = R/(x_1, \ldots, x_n)$ is the Koszul cx $K(x_1, \ldots, x_n)$ ✓

eg, $R = k[x,y] \rightsquigarrow 0 \to R(-2) \xrightarrow{\begin{bmatrix} -y \\ x \end{bmatrix}} R(-1)^2 \xrightarrow{[x\ y]} R \xrightarrow{\varepsilon} k \to 0$

Ex: $R = \dfrac{k[x,y]}{(x^2, y^2)}$

from $x^2, y^2 = 0$

$$??\ \cdots \to \underset{R(-2)^3}{R^3} \xrightarrow{\begin{bmatrix} x & 0 & -y \\ 0 & y & x \end{bmatrix}} \underset{\text{really } R(-1)^2}{R^2} \xrightarrow{[x\ y]} R \xrightarrow{\varepsilon} k \to 0$$

Koszul reln

Need whole resn!

$$R \longrightarrow k \quad \text{embedded ci.} \quad \overset{\text{Tate}}{\Longrightarrow} \quad R\langle \overset{1}{e_1, e_2}, \overset{2}{T_1, T_2} \rangle \xrightarrow[R]{\simeq} k$$

$$R = Q/(x^2, y^2) \qquad k = Q/(x,y)$$

$$\partial:\ e_1 \mapsto x,\quad e_2 \mapsto y,\quad T_1 \mapsto x e_1,\quad T_2 \mapsto y e_2$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{Leibniz + linear coeffs}} \Rightarrow \text{resn linear ✓}$$

Note: $R = Q/I$, Q = poly

R koszul $\Rightarrow$ R quadratic alg (I gen'd by deg 2)
$\not\Leftarrow$ (else ∂_2 not linear!)

Examples of koszul algs:

1) quadratic c.i.'s (Tate resn!)
2) I = quadratic monomial ideal

R $*$ 3) I has quadratic Gröbner basis
"G-koszul"
($\mathrm{in}(I)$ quadratic monomial $\overset{\text{deform}}{\rightsquigarrow}$ I)

eg, determinantal 2×2 $\left(\text{eg } \dfrac{k\begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}}{I_2('')}\right)$

4) I linear resn – see pblms

But: no finite characterization known! (open)

Roos: $\exists$ examples, $\forall n$, st. resn of k linear for n steps but not ∂_{n+1} (n unrelated to dim or edim)

Why important?

1) duality theory ...
2) arise naturally in many places:
   - alg geom (Segre, Veronese, toric, ...)
   - alg topology (Steenrod algebra, cohom alg of $K(\pi,1)$-spaces holonomy algebras of supersolvable hyperplane arrangements
     also: $\left\{\begin{matrix}\text{perverse sheaves}\\ \text{on } \Delta\text{'d space}\end{matrix}\right\} \approx \left\{\begin{matrix}\text{modules over}\\ \text{a koszul alg}\end{matrix}\right\}$
   - noncommut. geom. (nat'l condition on an exceptional collection, noncommut. deformations of $\mathbb{P}^n$)
   - # theory (Milnor K-theory $\otimes \mathbb{Z}/p\mathbb{Z}$ of fields $\leftsquigarrow$ conjecturally)
     (stronger than Bloch-Kato)

operadic stuff...
relative version...

Quadratic / Koszul dual: (to prepare for Koszul duality)

Def: $k\langle x_1,\dots,x_n\rangle$ = the noncommutative poly ring
= {polys in noncommuting variables}

$$x_1x_2 \neq x_2x_1$$

Our algs are quotients:

Ex: $k[x_1,\dots,x_n] = \dfrac{k\langle x_1,\dots,x_n\rangle}{(\{x_ix_j - x_jx_i\})}$ still quadratic ✓

Ex: $\dfrac{k[x,y]}{(x^2,xy)} = \dfrac{k\langle x,y\rangle}{(x^2,\ xy,\ \cancel{xy}-yx)}$ quad ✓

Equiv: $V = k\{x_1,\dots,x_n\}$ v.space

The tensor algebra on V:

$$T = T(V) = k \oplus V \oplus (V \otimes_k V) \oplus \cdots \oplus \overbrace{(V \otimes_k \cdots \otimes_k V)}^{i} \oplus \cdots$$

$$= \bigoplus_{i \geq 0} V^{\otimes_k i} \cong k\langle x_1,\dots,x_n\rangle$$

eg, $\left.\begin{array}{l} x_1 \otimes x_2 \longmapsto x_1x_2 \\ x_2 \otimes x_1 \longmapsto x_2x_1 \end{array}\right\} \neq$

Pairing: $T(V)_2 \times T(V^*)_2 \xrightarrow{\langle\, ,\, \rangle} k$

$$(x_ix_j\ ,\ x_k^*x_\ell^*) \longmapsto \begin{cases} 1 & (i,j)=(k,\ell) \\ 0 & \text{else} \end{cases}$$

really: $(\alpha \otimes \beta\ ,\ f \otimes g) \longmapsto f(\alpha) \otimes g(\beta)$

for any $\alpha,\beta \in V$ for any $f,g \in V^*$

Ex: $(\underbrace{2x_1x_2}_{=0} + \underbrace{3x_2^2}_{=1}) \cdot ((x_2^*)^2) = 3$

Def: For a quadratic alg

$$R = \frac{k\langle x_1,\dots,x_n\rangle}{(W)} = \frac{T(V)}{(W)} \quad \text{where } \overset{\text{v.space}}{W} \subseteq T(V)_2$$

its quadratic dual is

$$R^! = \frac{k\langle x_1^*,\dots,x_n^*\rangle}{(W^\perp)} = \frac{T(V^*)}{(W^\perp)}$$

where $W^\perp$ = orthogonal complement of W under the pairing

$$= \{ f \in (V^*)^{\otimes 2} \mid \alpha f = 0 \ \forall \alpha \in W \}$$

Ex: $$R = \frac{k[x,y]}{(x^2, xy+y^2)} \overset{!}{=} \frac{k\langle x,y\rangle}{(\underbrace{xy-yx}_{\text{commut.}}, x^2, xy+y^2)} = \frac{T(V)}{(W)}$$

note $\dim_k T_2 = \dim k\{x^2, xy, yx, y^2\} = 4$

$\dim_k W = 3$

so $\dim_k W^\perp = 1$

$$R^! = \frac{k\langle x^*, y^*\rangle}{(x^*y^* + y^*x^* - (y^*)^2)} = \frac{T(V^*)}{(W^\perp)}$$

(lazy: call x^* just x $\quad \dfrac{k\langle x,y\rangle}{(xy+yx-y^2)}$)
" y^* " y

$$R^! = \underbrace{k1}_{R^!_0} \oplus \underbrace{k\{x,y\}}_{R^!_1} \oplus \underbrace{k\{x^2, xy, yx, \cancel{y^2}\}}_{R^!_2} \oplus \cdots$$

noncommut. ring