L5: Homotopy Lie algebra and cotangent cx

Fundamental invariant $\pi(R)$ def'd by Avramov (84).

local algebra $\leftrightsquigarrow$ rational homotopy theory

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"Through the Looking Glass"

$\underbrace{\pi(R)}_{\text{gdd Lie algebra}} \hookrightarrow \underbrace{\mathrm{Ext}^R(k,k)}_{\text{its universal embedding alg.}}$ $\qquad \pi^*(X)_{\mathbb{Q}}$

Def: a <u>gdd Lie algebra</u> $/k$ is a gdd k-module L with a k-linear pairing

$$L \times L \longrightarrow L \qquad \text{"Lie bracket"}$$
$$(a,b) \longmapsto \text{"}[a,b]\text{"}$$

s.t. 1) (gdd anti-commut) $[a,b] = -(-1)^{|a||b|}[b,a]$

2) (Jacobi identity) $[a[b,c]] = [[a,b],c] + (-1)^{|a||b|}[b,[a,c]]$

skip (say char 2,3 ...)

3) $|a|$ even $\Rightarrow [a,a] = 0$ and $\exists$ reduced square $a^{[2]}$ (acts like $\frac{[a,a]}{2!}$) char 2

$|a|$ odd $\Rightarrow [a,[a,a]] = 0$ char 3 (else autom.)

Fact: there are functors

$$\left\{\begin{matrix}\text{gdd}\\ k\text{-algs } B\end{matrix}\right\} \underset{U}{\overset{\text{Lie}}{\rightleftarrows}} \left\{\begin{matrix}\text{gdd}\\ \text{Lie algs}/k\end{matrix}\right\}$$

$\mathrm{Lie}(B) = B$ as k-mod w/ bracket $[a,b] \overset{\text{def}}{=} ab - (-1)^{|a||b|}ba$ "gdd commutator"

"universal enveloping algebra" $\quad U(L) = \dfrac{\text{the gdd } k\text{-alg gen'd by basis of } L}{([a,b] - (ab - (-1)^{|a||b|}ba),\ a^{[2]} = a\cdot a \text{ if } |a| \text{ odd})}$ really $= T(L)$ $= k$-span of monomials in elts of L

Similarly: can define a <u>dg</u> Lie algebra (dgla) (with a ∂)

The homotopy Lie algebra $\pi(R)$: (or $\pi^*(R)$)

Version 1: via acyclic closure

$$\begin{tikzcd} & R\langle X\rangle \arrow[dr, "\simeq"] & \\ R \arrow[ur] \arrow[rr] & & k \end{tikzcd}$$

Lemma: $\mathrm{Der}^{\gamma}_R(R\langle X\rangle, R\langle X\rangle) \overset{*}{\subseteq} \mathrm{Lie}(\mathrm{End}_R(R\langle X\rangle))$ is a dg Lie subalgebra ($[d,d']$ = gdd commutator)

Def: $\boxed{\pi(R) = H(\mathrm{Der}^{\gamma}(R\langle X\rangle, R\langle X\rangle))}$ = a Lie alg!

$\cong H(\mathrm{Der}^{\gamma}_k(\cancel{R}\langle X\rangle, k)) \overset{\text{Prop 2}}{=} \mathrm{Hom}_k(kX, k) = kX^*$

Prop: The inclusion $*$ induces an injection (on homology)

$$\pi^*(R) \hookrightarrow \mathrm{Lie}(\mathrm{Ext}^*_R(k,k))$$

giving $U(\pi(R)) \xrightarrow{\cong} \mathrm{Ext}_R(k,k)$ iso (of algebras)

Version 2: via minimal model of min'l Cohen presentation $\hat{R} = Q/I$

$$\begin{tikzcd} & Q[Y] \arrow[dr, "\simeq"] & \\ Q \arrow[ur] \arrow[rr] & & \hat{R} \end{tikzcd}$$

Σ = shift +1 in degree

Def: $\pi^{\geq 2}(R) = (\Sigma kY)^* = \mathrm{Hom}_k(\Sigma kY, k)$

$Q[Y]$ min'l model $\overset{\text{can}}{\underset{\text{show}}{\Rightarrow}}$ $\partial(Y) \subseteq mY + QY^{\geq 2}$ products & powers

mod m

$k[Y]$ has $\partial(Y) \subseteq kY^{\geq 2}$

so $\partial = \partial^{[2]} + \partial^{[3]} + \dots$ where $\partial^{[i]}: kY \to kY^n$

Get $kY \xrightarrow{\partial^{[2]}} kY^2 \longrightarrow kY \times kY$ (fixing an deg-respecting ordering)

$y_i y_j \longmapsto y_i \times y_j$ if $y_i > y_j$

Dualizing gives bracket: $kY^* \times kY^* \longrightarrow kY^*$

$a, b \longmapsto [a,b]$

to get π^1 too

↳ OR:
(Version 2′)

$$\begin{tikzcd} & k[Y] \arrow[dr, "\simeq"] & \\ k \arrow[ur] \arrow[r, hook] & Q \arrow[r] & R \end{tikzcd}$$

now $Y = Y_0 \cup Y_1 \cup Y_2 \cup \cdots$

↑ to make surjection let $Q = k[Y_0] \twoheadrightarrow R$

and do same thing as above

so, now get $\pi_1(R) = \sum k Y_0^*$ too!

Brackets come from $\partial^{[2]}$ as above.

Rks:

- $\dim_k \pi^n(R) = \operatorname{card} X_n = \varepsilon_n(R)$!
- R Golod $\iff$ $\pi^{\geq 2}$ free gdd Lie alg (Avramov - Löfwall)
- R Koszul algebra $\Rightarrow$ π gen'd by π^1
- skip • deformations $\rightsquigarrow$ central elts of $\pi(R)$?!
- skip • $\pi^1 \times \pi^1 \to \pi^2$ via Hessian (I) (Sjödin)
- nilpotent elements in the Lie alg $\pi(R)$ crucial to Briggs' recent proof of Vasconcelos's Conjecture!

Cotangent complex & André–Quillen (co)homology

Let $R \to S$ (S R-alg)

The Kähler differentials $\Omega_{S/R}$ measure smoothness of the map.

To get derived functor of $\Omega_{-/R}$:

take acyclic closure $R\langle X\rangle \xrightarrow{\simeq} S$ (resn!)

(In char p, need a simplicial resn instead.)

Def: (char 0) The cotangent complex of φ is:

$$\mathcal{L}_{S/R} \overset{\text{def}}{=} \underbrace{\Omega_{R\langle X\rangle/R}}_{\substack{\text{"derived Kähler diff"} \\ \text{semifree}/R\langle X\rangle}} \otimes_{R\langle X\rangle} S \quad = \text{cx free } S\text{-mods}$$

(the univ. derivation)

with $\partial(dx) = d(\partial(x))$ where $d: R\langle X\rangle \to \Omega_{R\langle X\rangle/S}$, $x \longmapsto dx$

Def: (char 0) the AQ (co)homology is:

$$\left\{\begin{array}{l} D_n(S/R; N) = H_n(\mathcal{L}_{S/R} \otimes_S N) \\ D^n(S/R; N) = H^n(\mathrm{Hom}_S(\mathcal{L}_{S/R}, N)) \end{array}\right\} \text{ coeffs in } S\text{-mod } N$$

Concretely: Let $A = R\langle X\rangle$, $\Delta = \ker A \otimes_R A \xrightarrow{\text{mult}} A$
(or $J = \ker A \otimes_R S \xrightarrow{\text{mult}} S$)

- $\mathcal{L} = \Omega_{A/R} \otimes_A S = \Delta/\Delta^2 \otimes_R S = J/J^2 = \mathrm{Ind}_R^S R\langle X\rangle$!!

- R, S Noeth, S f.g R-alg $\Rightarrow$ $\mathcal{L}$ is cx of f.g. free S-mods

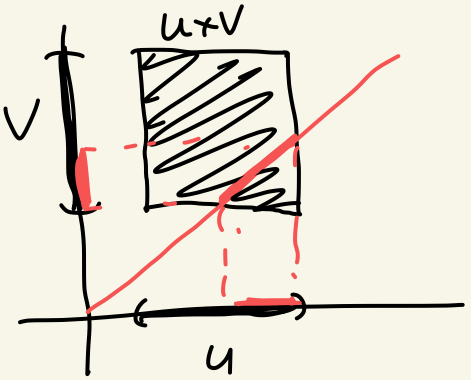
Quillen's Conjectures...

Role of $\mathcal{L}$ in DAG (derived alg geometry):

$$U \cap V \cong (U \times V) \cap \underset{\text{diagonal}}{\Delta}$$

classic intersection theory

redn to diagonal



$$\Delta = \ker\left(S \otimes_R S \xrightarrow{\mu} S \right)$$
$$a \otimes b \longmapsto ab$$

$R = k$

S = coordinate ring of ambient

$$\Omega_{S/R} = \Delta / \Delta^2$$

$$\rightsquigarrow$$

so, $\mathcal{L}_{S/R} = \Omega_{R\langle X\rangle / R} \otimes_{R\langle X\rangle} S$ plays a role in derived intersection thy