Problem Set 3: The BGG correspondence, Part 1

Fix $n \geqslant 0$. Let k be a field. Let $S = k[x_0, \dots, x_n]$, and $E = \Lambda_k(e_0, \dots, e_n)$, with gradings given by $\deg(x_i) = 1$ and $\deg(e_i) = -1$.

1. (Short exercise) Prove that $E^* := \underline{\mathrm{Hom}}_k(E, k)$ is isomorphic, as a graded left E-module, to the free module $E(-n-1)$.

Solution. We construct an explicit isomorphism. Recall that the left action of E on $E^* := \underline{\mathrm{Hom}}_k(E, k)$ is as follows: let $f \in E^*$ and $e \in E$ be homogeneous. We have

$$(e \cdot f)(y) = (-1)^{\deg(e)\deg(f)} f(ey).$$

Given an exterior monomial m, let $\alpha_m \in \underline{\mathrm{Hom}}_k(E, k)$ be the map that sends m to 1 and every other exterior monomial to 0. Notice that the degree of α_m is $-\deg(m)$. Moreover, the maps α_m form a k-basis of E^*, called the *dual basis* associated to the monomial basis of E. Define a map $\varphi \colon E(-n-1) \to E^*$ by sending 1 to $\alpha_{e_0 \cdots e_n}$ and extending E-linearly; observe that φ is a degree 0 map. We will show that θ identifies the monomial basis of $E(-n-1)$ with the above dual basis. Suppose m is an exterior monomial. Choose an exterior monomial $m' \in E$ such that $mm' = e_0 \cdots e_n$. For any exterior monomial m'', we have

$$(\theta(m))\,(m'') = \alpha(mm'') = \begin{cases} 1, & \text{if } m'' = m' \\ 0, & \text{else} \end{cases}$$

This shows that $\theta(m) = \alpha_{m'}$. Thus, θ identifies the monomial basis of $E(-n-1)$ with the dual basis of E^*, and so it is an isomorphism.

2. Let R be a graded local ring that is an algebra over a field k. Prove that $R^* := \underline{\mathrm{Hom}}_k(R, k)$ is an injective graded left R-module. Conclude that the exterior algebra E is injective as a graded left E-module.

Hint: recall that an R-module I is injective if and only if the functor $\mathrm{Hom}_R(-, I)$ is exact. You may also want to use Hom-tensor adjunction: that is, if R and S are graded rings, then:

$$\mathrm{Hom}_R(N, \underline{\mathrm{Hom}}_S(N', N'')) \cong \mathrm{Hom}_S(N' \otimes_R N, N'').$$

for any left R-module N, left S-module N'', and S-R-bimodule N'.

3. Let N be a finitely generated graded E-module. Recall that the i^{th} term of $\mathbf{L}(N)$ is $S(-i) \otimes_k N_i$. Prove that the differential $\partial_{\mathbf{L}}$ on $\mathbf{L}(N)$, which is given by $s \otimes y \mapsto \sum_{i=0}^{n} x_i s \otimes e_i y$, squares to zero.

4. Suppose $n = 1$, and let F be the complex

$$0 \leftarrow S(1) \xleftarrow{x_0 - x_1} S \leftarrow 0,$$

where S is in homological degree 0. Find a graded E-module N such that $\mathbf{L}(N) = F$.

5. This isn't really an exercise: it's just a demo for how to compute with the BGG functor $\mathbf{L}$ in `Macaulay2`. Unfortunately, there is no functionality for $\mathbf{L}$ in the `Macaulay2` package `BGG` (it can only handle $\mathbf{R}$). However, the newer package `MultigradedBGG` (which can also handle more exotic gradings, as we will see next week), can compute the functor $\mathbf{L}$. Try running the following code to check that $\mathbf{L}(E)$ is the dual of the Koszul complex.

```
needsPackage "MultigradedBGG"
n = 3;
S = ZZ/101[x_0..x_n]
E = dualRingToric S
toricLL(E^1)
```

We can also apply the BGG functor $\mathbf{L}$ to E/I, where I is a random monomial ideal, and see what it looks like. Here is how to do this:

```
needsPackage "MultigradedBGG"
n = 5;
S = ZZ/101[x_0..x_n]
E = dualRingToric S
needsPackage "RandomIdeals"
I = randomMonomialIdeal({-3, -4}, E)
N = coker matrix entries gens I
C = toricLL(N)
```

You can take a look at the full differential of C by typing `C.dd`, or individual differentials by typing `C.dd_i` for various i. If you do this in our example above, you'll find that the entries of the matrices are always *linear forms*. Indeed, this is true for $\mathbf{L}$ applied to *any* exterior module; this follows from the definition of $\mathbf{L}$. In fact, we can say more: the entries in the above example are just *variables*, up to a sign; this is because the ideal I above is generated by *monomials*.