Day 10: Tate resolutions on toric varieties

k a field.

Defn: A **toric variety** over k is an algebraic variety X w/ a dense open subset that is isomorphic to an algebraic torus T (i.e. $\mathrm{Spec}(k[t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}])$ for some $n \geq 1$, if k is alg. closed) and such that the group action of T on itself extends to all of X.

Examples of projective toric varieties:
Weighted proj. space, products of (weighted) proj. spaces, projective bundles over projective spaces.

Say X is a projective toric variety. X has an associated homogeneous coordinate ring (or Cox ring)
$S = k[x_0, \dots, x_n]$ graded by its divisor class group $Cl(X)$.
Here, $n+1 = \mathrm{rank}\, Cl(X) + \dim X$. (See CLS ch. 5)

Ex: • $X =$ (weighted) proj. space $\mathbb{P}(d_0, \dots, d_n)$: $Cl(X) \cong \mathbb{Z}$, $\dim X = n$
$S = k[x_0, \dots, x_n]$, $\deg(x_i) = d_i$.

• $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$: $\dim X = \sum n_i$, $Cl(X) \cong \mathbb{Z}^t$.

$S = k[x_{10}, \dots, x_{1n_1}, \dots, x_{t0}, \dots, x_{tn_t}]$, $\deg(x_{ij}) = (0, \dots, 1, \dots, 0) \in \mathbb{Z}^t$
$\uparrow$
i

e.g.: $X = \mathbb{P}^1 \times \mathbb{P}^1$, $S = k[x_0, x_1, y_0, y_1]$, $\deg(x_i) = (1,0)$,
$\deg(y_i) = (0,1)$.

Toric varieties also have an irrelevant ideal $B \subseteq S = k[\underline{x}]$.

Ex: $X = \mathbb{P}(d_0, \dots, d_n) \rightsquigarrow B = (x_0, \dots, x_n)$.

Ex: $X = \mathbb{P}^1 \times \mathbb{P}^1 \rightsquigarrow B = (x_0, x_1) \cap (y_0, y_1)$.

$D^b_{V(B)}(S) \subseteq D^b(S)$: complexes w/ support in $V(B)$.

key fact: $D^b(X) \simeq D^b(S) / D^b_{V(B)}(S)$.

Given a proj toric variety X w/ $Cl(X)$-gdd Cox ring $S = k[x_0, \ldots, x_n]$, we have a BGG correspondence

$$D^b(S) \simeq D^b_{DM}(E)$$

$\left[E = \Lambda_k(e_0 \ldots e_n),\ \deg(e_i) = (-\deg(x_i), -1) \right]$

in $Cl(X)$

What does geometric BGG look like now?

Note: every sheaf on X corresponds to a $Cl(X)$-gdd S-module, as before.

The correct Tate resolution of $\mathcal{F} = \widetilde{M} \in coh(X)$ can't be just given by taking a free res'n of $\mathbb{R}(M)$, because this does not involve B: it ignores the toric geometry. There are multiple toric varieties w/ the same Cox ring.

Thm (B-Erman) There is a functor $T : coh(X) \longrightarrow DM(E)$ such that, if $\mathcal{F} \in coh(X)$

1. $T(\mathcal{F})$ is exact
2. $T(\mathcal{F})$ is free as an E-module. Moreover,

$$T(\mathcal{F}) = \bigoplus H^i(X, \mathcal{F}(a)) \otimes_k \omega_E(-a, i).$$

$a \in Cl(X)$

(3) Let $P = (x_{i_1}, \ldots, x_{i_m})$ be a primary component of B. $T(\mathcal{F})/I$ is exact for $I = (e_t : t \neq i_j \ \forall\ 1 \leq j \leq m)$.

($T(\mathcal{F})$ is not only exact... it's really exact.)

Conj: (B-Erman, 24) The functor T induces an equivalence
$$D^b(X) \longrightarrow K^B_{DM}(E),$$
where

$K^B_{DM}(E)$ = homotopy cat. of $Cl(X) \times \mathbb{Z}$-gdd diff. E-modules D

s.t. (1) $\dim_k D_{(a,i)} < \infty \quad \forall (a,i) \in Cl(X) \times \mathbb{Z}$, and

(2) D has the exactness properties in part 3 of the Thm.

Short exercise. Say $X = \mathbb{P}^1 \times \mathbb{P}^1$, S its Cox ring, so $S = k[x_0, x_1, y_0, y_1]$, $\mathbb{Z}^2$-gdd by $\deg(x_i) = (1,0)$, $\deg(y_i) = (0,1)$. $\mathbb{R}(S)$ has min'l free res'n $\mathbb{R}(S)^*$, as before. (Prove this if you like). Show cone $(\mathbb{R}(S)^* \longrightarrow \mathbb{R}(S))$ can't be $T(G)$ by showing it doesn't encode all the sheaf coh. Hint: $H^i(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{F}) = 0 \quad \forall i > 2$, since $\mathbb{P}^1 \times \mathbb{P}^1$ is a surface.