Day 7: Geometric BGG, Part 3

Let $\mathcal{F} \in \mathrm{coh}(\mathbb{P}^n)$.

Recall: the Tate resolution $T(\mathcal{F})$ is defined as follows:

Choose M s.t. $\widetilde{M} = \mathcal{F}$.
Let $r > \mathrm{reg}(M)$.
Let F = min'l free res'n of $H_{-r}\mathbb{R}(M_{\geq r})$.

$T(\mathcal{F}) = \mathrm{cone}(F \xrightarrow{\simeq} \mathbb{R}(M_{\geq r}))$.

Thm: (1) $T(\mathcal{F})$ doesn't depend on choice of M.

(2) $T(\mathcal{F})$ is a min'l, exact complex of f.g. free modules.

(3) $T(\mathcal{F})_i = \bigoplus_{j=0}^{n} H^j(\mathbb{P}^n, \mathcal{F}(-i-j)) \otimes_k \omega_E(i+j)$

To prove it, we will need:

lemma (EFS Lemma 3.5) Let F be a bicomplex of modules over some ring

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & F_{i-1,j} \arrow[l] \arrow[d] & F_{i,j} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & F_{i-1,j-1} \arrow[l] \arrow[d] & F_{i,j-1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & \vdots & \vdots &
\end{tikzcd}$$

such that: (1) The columns are bounded above.

(2) The columns split, i.e. $F_{i,j} = H_{i,j} \oplus G_{i,j} \oplus d^{vert}(G_{i,j+1})$, where $\ker(d^{vert}_{i,j}) = H_{i,j} \oplus B_{i,j}$, and $G_{i,j+1} \xrightarrow{d^{vert}} d^{vert}(G_{i,j+1})$ is an isom.

$\mathrm{Tot}(F)$ is homotopy equivalent to a complex C w/ terms

$C_i = \bigoplus_j H_{i-j,j}$ and diff. $\partial = \sum_{\ell \geq 0} \sigma (d^{hor}\pi)^{\ell} d^{hor}$, where

$\sigma: F_{i,j} \twoheadrightarrow H_{i,j}$ is the projection, and $\pi: F_{i,j} \longrightarrow G_{i-1,j}$ is the

Pf: Omitted.

composition $F_{i,j} \twoheadrightarrow d^{\text{very}}(G_{i,j+1}) \xrightarrow{\cong} G_{i,j+1}$.

Pf of Thm: By an exercise, the complex

$$\cdots \leftarrow \mathcal{F}(i) \otimes_k \omega_E(-i) \leftarrow \mathcal{F}(i-1) \otimes_k \omega_E(-i+1) \leftarrow \cdots,$$

hom degs: $-i$ $\quad$ $-i+1$

w/ diff. given by left mult by $\sum_{i=0}^{n} x_i \otimes e_i$, is exact.

Take a Čech resolution of this complex to get a bicomplex B:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & \\ \cdots & \check{C}^0(\mathcal{F}(i)) \otimes_k \omega_E(-i) \arrow[l] \arrow[d] & \check{C}^0(\mathcal{F}(i-1)) \otimes_k \omega_E(-i+1) \arrow[l] \arrow[d] & \cdots \arrow[l] \\ \cdots & \check{C}^1(\mathcal{F}(i)) \otimes_k \omega_E(-i) \arrow[l] \arrow[d] & \check{C}^1(\mathcal{F}(i-1)) \otimes_k \omega_E(-i+1) \arrow[l] \arrow[d] & \cdots \arrow[l] \\ & \vdots & \vdots & \end{tikzcd}$$

Think of B as a bicomplex of E-modules. The columns are bounded above, and they split E-linearly.

Rows exact $\Rightarrow$ $\mathrm{Tot}(B)$ exact.

Lemma $\Rightarrow$ we get an exact complex T' w/ terms

$$T'_i = \bigoplus_{j=0}^{n} H^j(\mathbb{P}^n, \mathcal{F}(-i-j)) \otimes_k \omega_E(i+j) \quad \text{and diff.} \quad \sum_{\ell} \sigma(d^h \pi)^{\ell} d^h,$$

where d^h = left mult. by $\sum_{i=0}^{n} x_i \otimes e_i$.

Serre Vanishing $\Rightarrow$ $T'_i = H^0(\mathbb{P}^n, \mathcal{F}(-i)) \otimes_k \omega_E(i)$ for $i \leq -r$.

Now, let M be a fg gdd S-module s.t. $\widetilde{M} = \mathcal{F}$, and fix $r > \mathrm{reg}(M)$. Construct $T(\mathcal{F})$ using M and r.

It suffices to show $T(\mathcal{F}) \cong T'$.

Short exercise: $T(\mathcal{F}) \cong T'$ in homological degrees $\leq -r$.

Hint: use one of yesterday's exercises.

Theorem follows b uniqueness of min'l free res'ns. $\square$