L4: Closer look at acyclic closure (technical!)

(R, m, k) local

$$\begin{tikzcd} & R\langle X\rangle \arrow[dl] \arrow[dr, "\simeq"] & \\ R \arrow[rr] & & k = R/m \end{tikzcd}$$

$R\langle X\rangle$ = acyclic closure (min'l choices of X)

Today we prove:

Thm [Gulliksen, Schoeller (indep)]
via derivations, Hopf algebras

If (R, m) local, the acyclic closure of k over R is a min'l complex ($\partial(F_i) \subseteq m F_{i-1}$)

We give Avramov's version of [Gu].

key: Focus on a smaller complex, which can be thought of as its skeleton/backbone:

Def: For semifree Γ-extn $R \hookrightarrow R\langle X\rangle$ (assume $X_0 = \emptyset$) the complex of Γ-indecomposables is:

$$\mathrm{Ind}_R^{\gamma} R\langle X\rangle = \frac{R\langle X\rangle}{R + IX + RX^{(\geq 2)}}$$

($X^{(\geq 2)}$ products, d. powers "decomposable")
(R/I w/ set S)

where $I = \ker(R \twoheadrightarrow S)$, $S = H_0(R\langle X\rangle) = R/I$

and $X^{(\geq 2)}$ = set of Γ-monomials $x_{\alpha_1}^{(i_1)} \cdots x_{\alpha_n}^{(i_n)}$ w/ $\sum_j i_j \geq 2$

(if $X_0 \neq \emptyset$, $J = \ker(R\langle X\rangle \twoheadrightarrow S)$ $\rightsquigarrow$ $\mathrm{Ind} = \dfrac{R\langle X\rangle}{R + JX + RX^{(\geq 2)}}$)

So: $\cdots \to \underbrace{SX_{n+1}}_{\text{free } S\text{-mod on } X_{n+1}} \to \underbrace{SX_n}_{\text{etc}} \to \cdots$

Running ex: $R = \frac{k[x,y]}{x^3+y^3} \to k = S$

$$R\langle X\rangle = R\langle \overset{1}{e_1}, \overset{1}{e_2}, \overset{2}{T}\rangle \xrightarrow{\simeq} k$$

$$e_1 \mapsto x, \quad e_2 \mapsto y, \quad T \mapsto x^2e_1 + y^2e_2$$

$$\mathrm{Ind}^\gamma R\langle X\rangle = (0 \to kT \xrightarrow{0} ke_1 \oplus ke_2 \to 0)$$

Uses for Ind:

Prop 1: (recognizing acyclic closures)

just write Ind

Then $R\langle X\rangle \xrightarrow{\simeq} S$ acyclic closure $\iff$ $\mathrm{Ind}^\gamma_R R\langle X\rangle$ is min'l cx S-mods

(min choices for X) (min'l differentials, im $\partial \subseteq mF$)

Proof of Thm:

Take $R\langle X\rangle \xrightarrow{\simeq} k$: acyclic closure

$$\cdots \to \underset{e_ie_je_k}{\underbrace{RX_3 \oplus RX_1X_2 \oplus RX_1^3}} \xrightarrow{e_iT_j} \underset{e_ie_j}{\underbrace{RX_2 \oplus RX_1^2}} \xrightarrow{T_j\text{'s}} \underset{e_i}{RX_1} \to R \to k \to 0$$

θ_x: $x \mapsto 1$, $y \neq x \mapsto 0$

For each $x \in X_n$, $\exists$ chain map derivation θ_x of deg $-n$: $\theta_x(y) = \begin{cases} 1 & y = x \\ 0 & y \neq x, \text{ in } X_n \\ ? & y \in X_{>n} \end{cases}$

(How? Use Ind...)

Γ-monomials (exterior + divided)

If $\partial(x) = \sum r_m \cdot m$ $\left(m = x_1^{(i_1)} \cdots x_r^{(i_r)} : \sum i_j = n-1, x_1 > x_2 > \dots > x_r\right)$

with some $r_{m_0} \notin m$. Take the greatest such (in lexicographic order)

$$\partial(\underbrace{\theta_{x_r}^{i_r} \circ \cdots \circ \theta_{x_1}^{i_1}}_{\text{deg } 1})(x) \underset{\text{chain map}}{=} (\theta\text{'s})(\textstyle\sum r_m m) = \underbrace{r_{m_0} \cdot 1 + \cancel{\sum_{m<m_0} r_m \cdot 0} + \sum_{r_m \in m} r_m \cdot (?)}_{\notin m, \text{ contradicts im}(\partial_1) \subseteq m} \quad //$$

Prop 2: (constructing derivations from $R\langle X\rangle$ via chain maps from Ind)

$$\begin{tikzcd}
R\langle X\rangle \arrow[r, dashed, "\exists \tilde{d}", "\substack{\text{chain } \Gamma\text{-derivation} \\ \deg -n}"'] \arrow[d, two heads] & \cancel{U}^{\,R\langle X\rangle} = \text{dg } R\langle X\rangle\text{-mod} \arrow[d, "\simeq", "\text{qu-iso}"'] \\
\text{Ind} \arrow[r, "\forall d", "\substack{\text{chain map} \\ \deg -n}"'] & \underset{k}{\cancel{V}} = \text{cx } S\text{-mods}
\end{tikzcd}$$

For us, Ind is min'l !!!

$$\begin{tikzcd}
\text{so, Ind} = \cdots \arrow[r] & kX_{n+1} \arrow[r, "0"] \arrow[d] & kX_n \arrow[r, "0"] \arrow[d, "\substack{x \mapsto 1 \\ y \neq x \mapsto 0}"] & kX_{n-1} \arrow[r, "0"] \arrow[d] & \cdots \\
V = \cdots \arrow[r] & 0 \arrow[r] & k \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

So, Prop 2 $\Rightarrow \exists\, \theta_x : R\langle X\rangle \circlearrowleft$ as desired

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Usually $R\langle X\rangle \xrightarrow{\simeq} S$ (acyclic closure) not min'l, but for some maps it is (called "closed" $R \to S$)

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skip

Def: an R-linear Γ-derivation ("signed derivative-like")

$$\theta : R\langle X\rangle \longrightarrow U, \text{ an } R\langle X\rangle\text{-module}$$

is a gdd map st.

- θ is R-linear (equiv, $\theta(R) = 0$)
- * $\theta(ab) = \theta(a)b + (-1)^{|\theta||a|} a\,\theta(b) \quad \forall$ hgs $a, b \in R\langle X\rangle$
- $\theta(x^{(n)}) = x^{(n-1)}\theta(x) \quad \forall x \in X_{\text{even}}, |x| > 0$

Extra topic: $\underset{\substack{\text{regular}\\\text{local}}}{Q} \twoheadrightarrow \underset{\substack{\shortparallel \\ Q/I}}{R} \twoheadrightarrow k$ w/ $I \subseteq \mathfrak{m}_Q^2$

(or $Q \twoheadrightarrow \hat{R}$ min'l Cohen pres.)

Thm (Avramov 84):

Consider $\underset{\text{min'l model}}{Q[Y] \xrightarrow{\simeq} R}$, $\underset{\text{acyclic closure}}{R\langle X\rangle \xrightarrow{\simeq} k}$

Then $\operatorname{card} Y_n = \operatorname{card} X_{n+1}$ $(= \varepsilon_{n+1})$

and $\exists$ seq qu-isos

$$R\langle X_{\leq n}\rangle \xleftarrow{\simeq} \; ? \; \xrightarrow{\simeq} \underbrace{\frac{Q[Y]}{(Y_{<n})}}_{\text{"} k[Y_{\geq n}] \text{"}}$$

where $Y_0 \stackrel{\text{def}}{=} \mathfrak{m}_Q$ also

(The equality of ranks was previously shown by Wolffhardt if R contains a field and then Avramov in general via Hopf algebra techniques.)

Pf: base case $n=1$ ...

inductive step:

Given $\boxed{R\langle X_{\leq n}\rangle \xleftarrow[\varphi_n]{\simeq} Q[Y]\langle X_{\leq n}\rangle \xrightarrow[\pi_n]{\simeq} k[Y_{\geq n}]}$

Now $\partial(X_{n+1})$ = set of cycles gen $H_n(R\langle X_{\leq n}\rangle)$ $\overset{\varphi \text{ surj. quiso here}}{\Longrightarrow}$ lift to a set of cycles in $Q[Y]\langle X_{\leq n}\rangle$

... $\rightsquigarrow$ $\boxed{R\langle X_{\leq n+1}\rangle \xleftarrow[\varphi_{n+1}=\varphi_n\langle X_{n+1}\rangle]{\simeq} Q[Y]\langle X_{\leq n+1}\rangle \xrightarrow[\pi_{n+1}=\pi_n\langle X_{n+1}\rangle]{\simeq} k[Y_{\geq n}]\langle X_{n+1}\rangle}$

$\downarrow \simeq$

but $\partial(X_{n+1})$ $\overset{\text{min'l}}{\text{generates}}$ $H_n(k[Y_{\geq n}]) \overset{\text{calculation}}{=} Y_n$ $\Rightarrow$ $\boxed{\frac{k[Y_{\geq n}]}{(Y_n)} = k[Y_{n+1}]}$