Problem Set 1: Exterior algebras and the Koszul complex

Let R be a commutative ring, V a free R-module with basis $e_1, \ldots, e_n$, and $f_1, \ldots, f_n \in R$. Recall that the *Koszul complex on* $f_1, \ldots, f_n$, denoted $K(f_1, \ldots, f_n)$, is the complex

$$0 \leftarrow \Lambda_R^0 V \xleftarrow{\partial_1} \Lambda_R^1 V \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_n} \Lambda_R^n V \leftarrow 0$$

with differential

$$\partial_j(e_{i_1} \cdots e_{i_j}) = \sum_{\ell=1}^{j} (-1)^{\ell-1} f_{i_\ell} e_{i_1} \cdots \widehat{e_{i_\ell}} \cdots e_{i_j}.$$

1. (Short exercise) Compute $K(f_1, f_2, f_3)$.

2. Prove that $\partial_{j-1}\partial_j = 0$ for all j.

3. Given two complexes C and D of R-modules with differentials d_C and d_D, their *tensor product* $C \otimes_R D$ has terms $(C \otimes D)_m = \bigoplus_{i+j=m} C_i \otimes_R D_j$ and differential that sends an element $c \otimes d \in C_i \otimes D_j \subseteq (C \otimes D)_m$ to $d_C(c) \otimes d + (-1)^i c \otimes d_D(d)$. Prove that, for all $n \geqslant 1$, there is an isomorphism

$$K(f_1, \ldots, f_n) \cong K(f_1) \otimes_R \cdots \otimes_R K(f_n).$$

4. Recall that $T(V)$ denotes the tensor algebra on V, and the exterior algebra $\Lambda_R(V)$ is the quotient of $T(V)$ by the two-sided ideal I generated by v^2 for all $v \in V$. Prove that, when $2 \in R$ is invertible, we have $\Lambda_R(V) \cong T(V)/J$, where J is the two-sided ideal generated by $vw + wv$ for $v, w \in V$.

5. Suppose R is a field, which we will denote by k. Assume $n = 1$, so that $\Lambda_k(V)$ is isomorphic to $k[x]/(x^2)$. Prove that this ring has the following property: every finitely generated $\Lambda_k(V)$-module is either free or has infinite projective dimension.

Hint: use the structure theorem for finitely generated modules over PID's.

In fact, this is true for *any* exterior algebra over a field. If you want a more challenging exercise, try and prove this. Hint: as we will see later, E is injective as an E-module; you can take this fact for granted for now. It follows that the functor $\mathrm{Hom}_E(-, E)$ is exact. Therefore, given a finite length free resolution F of a finitely generated E-module N, we have that $\mathrm{Hom}_E(F, E)$ is exact. Prove that this forces N to be free.