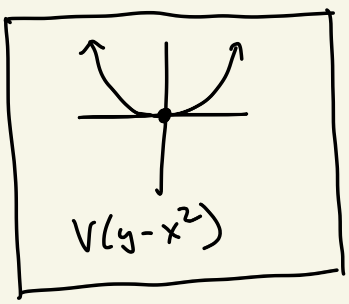
Addendum: kähler differentials, cotangent cx, and André–Quillen (co)homology

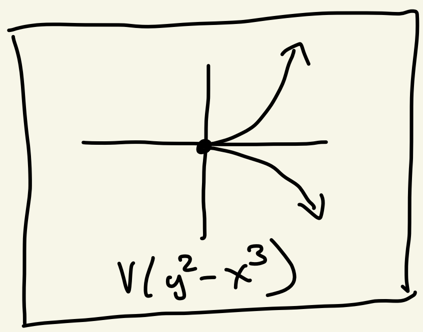
Kähler differentials are the dual of the tangent space:
(= differentials from calculus)

$$R = \frac{k[x_1, \dots, x_n]}{I} = \frac{Q}{I} \quad \text{coord ring of } V(I) \subseteq \mathbb{A}^n_k$$

Ex: $\Omega^1_{R/k} = \dfrac{R\,dx \oplus R\,dy}{df}$ $\quad \left(f = 0 \Rightarrow \underbrace{df}_{f_x dx + f_y dy} = 0 \right)$



vs.



$$\Omega^1 = \frac{R\,dx \oplus R\,dy}{(\underbrace{-2x\,dx + 1\,dy}_{=0})} \cong R\,dx$$

$dy = 2x\,dx$

Ω^1 free
rank 1
smooth

$$\Omega^1 = \frac{R\,dx \oplus R\,dy}{(-3x^2\,dx + 2y\,dy)}$$

near points (x_0, y_0) with $x_0 \neq 0$ or $y_0 \neq 0$
some coeff = unit
$\Rightarrow$ solve eqn for dx or dy

$\Rightarrow \Omega^1 =$ free rank 1

near $(0,0)$
Ω^1 not free!

Def 1: $R = \dfrac{k[x_1, \dots, x_n]}{I}$, $\quad \Omega^1_{R/k} = \dfrac{R\,dx_1 \oplus \cdots \oplus R\,dx_n}{(df \mid f \in I)}$ $\quad \left(\text{so, } \Omega^1_Q = \underbrace{\oplus Q\,dx_i}_{\text{free}} \right)$

Def 2: Let μ be multiplication map: $0 \to \underset{(\text{ker})}{\Delta} \to R \otimes R \xrightarrow{\mu} R \to 0$

(ideal!)
$\hookrightarrow$ gen'd by $\{1 \otimes x_i - x_i \otimes 1\}$

Then

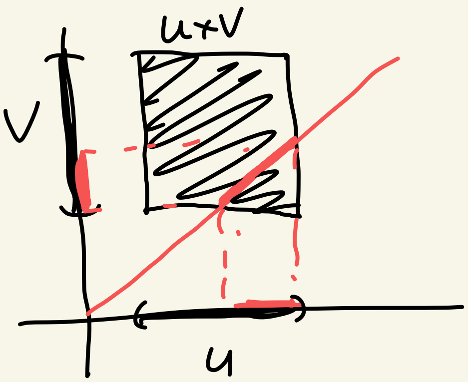
$$\Omega^1_{R/k} \cong \Delta / \Delta^2$$

$$dx_i \longleftrightarrow 1 \otimes x_i - x_i \otimes 1$$

Role of Ω in DAG (derived alg geometry):

$$U \cap V \cong (U \times V) \cap \underset{\text{diagonal}}{\Delta}$$

classic intersection theory
(reduction to the diagonal)



$$\Delta = \ker\left(S \otimes_R S \xrightarrow{\mu} S\right)$$
$$a \otimes b \longmapsto ab$$

$R = k$
$S = Q =$ coordinate ring of ambient

So, it's natural that the tangent & cotangent sheaves of Δ play a role:

$$\Omega_{S/R} = \Delta/\Delta^2$$

derived version

So, $\mathcal{L}_{S/R} = \Omega_{R\langle X\rangle/R} \otimes_{R\langle X\rangle} S$ plays a role in derived intersection thy

why derived natural player here?

Serre's intersection multiplicity defined from derived $\otimes$:

$$\chi(S/I, S/J) \overset{\text{def}}{=} \sum_{i \geq 0} (-1)^i \dim_k (\mathrm{Tor}_i^S(S/I, S/J))$$

coordinate rings
of $V(I), V(J) \subseteq \operatorname{Spec} S$

Cotangent complex & André-Quillen (co)homology

Let $R \to S$ (S R-alg)

The Kähler differentials $\Omega_{S/R}$ measure smoothness of the map.

To get derived functor of $\Omega_{-/R}$:

take acyclic closure $R\langle X\rangle \xrightarrow{\simeq} S$ (resn!)

(In char p, need a simplicial resn instead.)

Def: (char 0) The cotangent complex of φ is:

$$\mathcal{L}_{S/R} \stackrel{\text{def}}{=} \underbrace{\Omega_{R\langle X\rangle/R}}_{\text{"derived Kähler diff"}} \otimes_{R\langle X\rangle} S$$

semi free / $R\langle X\rangle$

(the univ. derivation)

with $\partial(dx) = d(\partial(x))$ where $d: R\langle X\rangle \to \Omega_{R\langle X\rangle/S}$, $x \longmapsto dx$

Note: $\mathcal{L}$ is cx free S-mods on $\{dx \mid x \in X\}$ w/ $\operatorname{rank}_S(\mathcal{L}_{S/R})_i = \# X_i = \varepsilon_i(R)$

Def: (char 0) the AQ (co)homology is:

$$\left\{\begin{array}{l} D_n(S/R;N) = H_n(\mathcal{L}_{S/R} \otimes_S N) \\ D^n(S/R;N) = H^n(\operatorname{Hom}_S(\mathcal{L}_{S/R}, N)) \end{array}\right\} \text{coeffs in } S\text{-mod } N$$

Concretely: Let $A = R\langle X\rangle$, $\Delta = \ker A \otimes_R A \xrightarrow{\text{mult}} A$
(or $J = \ker A \otimes_R S \xrightarrow{\text{mult}} S$)

- $\mathcal{L} = \Omega_{A/R} \otimes_A S = \Delta/\Delta^2 \otimes_R S = J/J^2 = \operatorname{Ind}^{\gamma}_R R\langle X\rangle$!!
- R, S Noeth, S f.g R-alg $\Rightarrow$ $\mathcal{L}$ is cx of f.g. free S-mods

Quillen's Conjectures...

Lastly, in geometric setting for $V(I) \subseteq \mathbb{A}^n_k$

$$\boxed{k \hookrightarrow \underbrace{Q}_{k[x_1,\dots,x_n]} \twoheadrightarrow \underbrace{R}_{Q/I} \twoheadrightarrow k}$$

Can use: $\Omega_{R/k}$ (p.4), $\underbrace{\Omega_{Q/k} \text{ (p.5)}}_{\text{trivial}}$, $\Omega_{k/R}$, $\Omega_{R/Q}$

so, actually use

They are all related!

Prop/Thm: [Jacobi-Zariski sequence]

Given ring maps $A \to B \to C$, have

- exact seq: $\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$
- distinguished Δ: $\mathcal{L}_{B/A} \otimes_B C \to \mathcal{L}_{C/A} \to \mathcal{L}_{C/B} \to \mathcal{L}_{B/A} \otimes_B C\,[1]$

Apply to:

$k \to Q \to R$ or $Q \to R \to k$