L3: Deviations

First, a theorem of Tate (same paper)

Def: R is complete intersection (c.i.)

if $R = Q/(f_1,\dots,f_c)$ local (or std gd) s.t. • Q = regular local (or std gd poly)

• $f_1,\dots,f_c$ is regular sequence

(more gen'ly if $\hat{R}$ is such)

Def: I is an embedded c.i. ideal if

$$Q/(f_1,\dots,f_c) \twoheadrightarrow Q/(g_1,\dots,g_m)$$

Tate's process terminates at deg 2!

Write each $f_j = \sum_{i=1}^{m} r_{ij} g_i$

> **Thm [Tate]**: With the set-up above,
> the semifree Γ-extn
>
> $$A = R\left\langle e_1,\dots,e_m, T_1,\dots,T_c \;\middle|\; \begin{array}{l} \partial(e_i) = g_i \\ \partial(T_j) = \sum_{i=1}^{m} r_{ij} e_i \end{array} \right\rangle$$
>
> resolves R/I (obvious augmentation $A \xrightarrow{\simeq} R/I$)

Ex: For $R = \dfrac{k[[x,y,z]]}{(x^4, y^4 + xz^3)}$ ($k[[x,y,z]] = Q$), resolve $k = R/m = Q/(x,y,z)$.

(x,y,z) c.i. in Q

c.i. in Q: $f_1 = x^3 \cdot x$, $f_2 = y^3 \cdot y + z^3 \cdot x$

$$\cdots \longrightarrow R^5 \longrightarrow R^3 \xrightarrow{\partial} R \xrightarrow{\varepsilon} k$$

$e_1 e_2, \dots$ ↦ koszul cycles

$e_1 \to x$, $e_2 \to y$, $e_3 \to z$

done! $T_1 \mapsto x^3 e_1$, $T_2 \mapsto y^3 e_2 + z^3 e_1$

$R\langle e_1, e_2, e_3, T_1, T_2\rangle \xrightarrow{\simeq} k$

More gen'ly:

- if R is a local c.i., $k = R/m$ is always an embedded c.i.
- c.i.'s rich "dg" topic (Shamash, Eisenbud, Gulliksen, Avramov, Buchweitz ... (won't cover))

II. Deviations:

Let (R, m) be local and $k = R/m$.

We discuss the "acyclic closure of R" (ie, of k):

$$\begin{tikzcd} & R\langle X\rangle \arrow[dr, "\pi", "\simeq"'] & \\ R \arrow[ur] \arrow[rr, "\phi"] & & k \end{tikzcd}$$

exterior + divided powers w/ minimal choices...

and what invariants it yields.

Running ex: $R = \frac{k[x,y]}{(x^3+y^3)} \twoheadrightarrow S = k = R/(x,y)$

$$R\langle X\rangle = R\langle \overset{\deg 1}{e_1, e_2}, \overset{\deg 2}{T}\rangle$$
$$e_1 \overset{\partial}{\mapsto} x,\quad e_2 \overset{\partial}{\mapsto} y,\quad T \overset{\partial}{\mapsto} x^2e_1 + y^2e_2$$

Def: The nth deviation of R is

$$\varepsilon_n = \varepsilon_n(R) \overset{def}{=} \text{card } X_n = \#\text{ variables of deg } n \quad (< \infty \text{ since } R \text{ Noeth})$$

Running ex: $\varepsilon_1(R) = 2$, $\varepsilon_2(R) = 1$, $\varepsilon_{\geq 3}(R) = 0$

and $F = R\langle X\rangle$ is clearly a min'l cx $\left(\partial(X) \subseteq mF \overset{\text{Leibniz}}{\Longrightarrow} \partial(F) \subseteq mF\right)$

$$\cdots \to \underset{e_1e_2T,\ T^{(2)}}{F_4} \to \underset{e_1T,\ e_2T}{F_3} \to \underset{e_1e_2,\ T}{F_2} \to \underset{e_1, e_2}{F_1} \to \underset{1}{F_0}$$

Questions: what can one ask?

- finite vs infinite # vars? can't skip a degree?
- if finite, stop at 2 or can it stop later?
- only finite for c.i.? (sort of - for k)
- when is $\underbrace{\text{acyc. closure}}_{\text{huge!}}$ = min'l resn (as cx)?

they ask!

Poincaré series

As we'll see soon the acyclic closure $R\langle X\rangle \xrightarrow{\simeq} k$ of k over R is always a <u>minimal</u> resolution!

So one can determine the Betti numbers

$$\boxed{\beta_n \overset{\text{def}}{=} \operatorname{rank} F_n = \dim_k \operatorname{Tor}_n^R(k,k)}$$

or equiv'ly the Poincaré series

$$\boxed{P_k^R(t) = \sum_{n=0}^{\infty} \beta_n t^n}$$

from the ε's:

a basis of $F_n = \{$ $\underbrace{\text{normal } \Gamma\text{-monomials}}$ $x_{\alpha_1}^{(i_1)} \cdots x_{\alpha_n}^{(i_n)}$ $\}$

- $x_{\alpha_1} < \cdots < x_{\alpha_n}$ (fixing a total, degree-preserving order on X)
- $x_{\alpha_j}^{(1)}$ means x_{α_j}
- if $|x_{\alpha_j}|$ odd, $i_j = 1$

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So,
$$\beta_0 = 1$$
$$\beta_1 = \varepsilon_1$$
$$\beta_2 = \binom{\varepsilon_1}{2} + \varepsilon_2$$
$$\beta_3 = \varepsilon_1\varepsilon_2 + \binom{\varepsilon_1}{3} + \varepsilon_3$$
$$\vdots$$

<u>More slickly:</u>

$$R\langle X\rangle = \bigotimes_{x\in X} R\langle x\rangle$$

where each $R\langle x\rangle = R \oplus Rx \oplus \underbrace{Rx^{(2)} \oplus \cdots}_{\text{only if } |x| \text{ even}}$

But the gen'g fn of ranks for each tensor factor is:

$$\sum_{n=0}^{\infty} \operatorname{rank}(R\langle x\rangle)_n t^n = \begin{cases} 1+t^{2i+1}, & \text{if } |x| = 2i+1 \\ 1+t^{2i}+t^{4i}+\cdots = \frac{1}{1-t^{2i}} & \text{if } |x| = 2i \end{cases}$$

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So,
$$\boxed{P_k^R(t) = \frac{\prod_{n \text{ odd}} (1+t^n)^{\varepsilon_n}}{\prod_{n \text{ even}} (1-t^n)^{\varepsilon_n}}}.$$

Conversely,

$$\varepsilon_1 = \beta_1$$

$$\varepsilon_2 = \beta_2 - \binom{\varepsilon_1}{2} = \beta_2 - \binom{\beta_1}{2}$$

$$\vdots$$

However, the minus signs can cause problems:

bounds for ε's $\Rightarrow$ bounds for β's

but $\not\Leftarrow$

The deviations are both a more compact and a more fundamental way of packaging the information.

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Some history: (R, m, k) local

Note: ① $R = k$ field $\iff \varepsilon_n(R) = 0 \ \forall n \geq 1 \iff$ for $n=1$

② R regular $\iff \varepsilon_n(R) = 0 \ \forall n \geq 2 \iff$ for $n=2$

(see ex on $\varepsilon_2(R) \ldots$)

Thm: TFAE

(i) R c.i.

(ii) $\varepsilon_3 = 0$

(iii) $\varepsilon_n = 0 \ \forall n \geq 3$ ⟸ Assmus [15] + new pf by Avram via min'l models

(iv) $\varepsilon_n = 0 \ \forall n \gg 0$ ⟸ Gulliksen [78]

(v) $\varepsilon_n = 0 \ \forall$ even $n \gg 0$ ⟸ " [82]

** (vi) $\varepsilon_n = 0$ some $n \geq 1$ ⟸ Halperin [84]

no time

- ~~Extended to $\varepsilon_n(\phi)$ for relative situation $R \xrightarrow{\phi} S$ eft (essentially of finite type)~~
  - ~~by Avramov: $\mathrm{pd}_S R < \infty$ [99]~~
  - ~~by Avramov-Iyengar: for retracts $R \rightleftarrows S$ [2000's]~~
  - ~~by Briggs-Iyengar, not assuming $\mathrm{pd}_R S < \infty$, or retract, if $\varepsilon_n = 0$ for ∞ly many even & odd [recent!]~~
- ~~Phrased in terms of André-Quillen (co)homology~~

~~there are conjectures of Quillen!~~

** Theorem (Avramov 84) [23] If R is not a complete intersection, then $\exists$ subseq $\{\varepsilon_{n_i}\}_{i=0}^{\infty}$ of the deviations that has exponential growth:

$$\exists \alpha > 1 : \quad \varepsilon_{n_i} \geq \alpha^{n_i} \quad \forall i$$

(so, Betti #'s grow exp.'lly too)