L2: DG algebra res'ns (Tate)

1

Today: Given R-algebra S (often $S = R/I$)

ie, ring homom. $R \xrightarrow{\phi} S$

build a free res'n (usually far from min'l) w/a DG alg str.

Even better, it will be free as a gdd comm R-algebra (like polynomial or exterior alg)

Idea: For each n, if $H_n(\text{cx built so far}) \neq 0$

- choose cycles $\{z_i\}$ generating H_n
- adjoin variables $\underset{\deg n+1}{\{x_i\}}$ and define $\partial(x_i) = z_i$

so that z_i boundary ($\in \operatorname{im} \partial$) so, $\bar{z}_i = 0$ in homology

[Let's see this in action 1st on an example]

Running Example: k field

$$R = \frac{k[x,y]}{(x^3+y^3)} \xrightarrow{\pi} k = R/(x,y) = R/I$$

Step 0: $0 \to \overset{0}{R} \xrightarrow{\pi} k \to 0$

$$H_0 = \ker \pi = (x,y)$$

R^2, $e_1 \mapsto x$, $e_2 \mapsto y$

Adjoin $\underbrace{\text{deg 1 variables}}$ e_1, e_2 mapping to

+ extend by R-linear & Leibniz

odd comm $\Rightarrow e_1e_2 = (-1)^{1\cdot 1} e_2e_1 = -e_2e_1$

$\& \; e_i^2 = 0$

So, get an exterior algebra!

$$R[e_1, e_2 \mid \partial(e_1) = x, \; \partial(e_2) = y] \xrightarrow{\pi} k$$

This is the Koszul cx!

$$0 \to \overset{2}{R} \xrightarrow{\partial_2} \overset{1}{R^2} \xrightarrow{\partial_1} \overset{0}{R} \xrightarrow{\varepsilon} k \to 0$$

$e_1, e_2 \longmapsto x, y$

$$e_1e_2 \xmapsto{\partial} \partial(e_1)e_2 - e_1\partial(e_2)$$

$(e_1^2=0)$ $\quad = xe_2 - e_1y$

$\quad = xe_2 - ye_1 \quad (|y|=0)$

<u>Step 1</u>: Compute $H_1 = \dfrac{\ker\partial_1}{\mathrm{im}\,\partial_2} \neq 0$ since x, y <u>not</u> R-sequence

(R not reg-tr)

Generator: $\underbrace{x^2e_1+y^2e_2}_{\text{cycle}} \xmapsto{\partial} x^3+y^3=0$ ✓

Adjoin a deg 2 variable T to kill it:

$$R[\overset{\deg 1}{e_1, e_2}, \overset{\deg 2}{T} \mid \partial(e_1)=x,\ \partial(e_2)=y,\ \partial(T)=x^2e_1+y^2e_2]$$

+ extend R-lin + ~~Ba~~ Leibniz

<u>Step 2</u>: if char $k=0$:

$H_2 = \cdots = 0$! Tate: emb.ci $\Rightarrow H_{\geq 2}=0$

(if char $k=p$, need much more: char 2 $T\partial(T)$, T^2)

$\uparrow\partial \qquad \uparrow$

$\cancel{\tfrac{1}{2}T^2}$? $\qquad$?

done ✓

<u>Resn looks like</u>:

$$\cdots \to \underset{e_1e_2T,\,T^2}{R^2} \to \underset{e_1T,\,e_2T}{\overset{3}{R^2}} \xrightarrow{\begin{bmatrix}-y^2 & x^2\\ x & y\end{bmatrix}} \underset{e_1e_2,\,T}{\overset{2}{R^2}} \xrightarrow{\begin{bmatrix}-y & x^2\\ x & y^2\end{bmatrix}} \underset{e_1,e_2}{\overset{1}{R^2}} \to \underset{1}{\overset{0}{R^1}}$$

$$\partial(e_1T) = \partial(e_1)T - \underset{\deg 1}{e_1}\partial(T) = xT - e_1(\cancel{x^2e_1}+y^2e_2)$$

$$= xT - y^2e_1e_2$$

(2-periodic!)

Adjoining variables: (to kill cycles)

Let A be a DG algebra and $z \in A_n$ a cycle.

If n even, adjoin variable x $\underbrace{\text{of odd degree}}_{\text{exterior}}$ $n+1$

$$A\langle x\rangle \text{ or } A[x] = A \oplus Ax \qquad (x^2=0)$$

If n odd, either

① adjoin x $\underbrace{\text{of even deg } n+1}_{\text{polynomial variable}}$: $A[X] = A \oplus Ax \oplus Ax^2 \oplus \cdots$

or ② adjoin divided powers of variable x : $A\langle x\rangle = \underset{\substack{\uparrow \\ x^{(0)}}}{A} \oplus \underset{\substack{\uparrow \\ x^{(1)}}}{Ax} \oplus Ax^{(2)} \oplus \cdots$ not gen'd by x as A-alg!

with ring structure via : $x^{(\ell)} x^{(m)} \overset{\text{def}}{=} \binom{\ell+m}{\ell} x^{(\ell+m)}$

[really: $x^{(\ell)}$ behaves like $\frac{1}{\ell!} x^{\ell}$ would if it existed!]

Tate 56 imported to rings from Cartan-introd for homology of Eilenberg-Moore spaces)

Define diff'l by: extended from ① $\boxed{\partial A}$ and ② $\boxed{\partial(x) \overset{\text{def}}{=} z}$

$\Downarrow$

via Leibniz rule $\qquad \partial(x^n) = n\, x^{n-1}\partial(x)$ if $|x|$ even

(and ③ $\boxed{\partial(x^{(n)}) = x^{(n-1)}\partial(x)}$ if $|x|$ even)

Repeat... set $X =$ all variables adjoined ($X_n =$ those of deg n).

Def: If A is a DG algebra,

a semi-free extn $A[X]$ (semi-free Γ-extension $A\langle X\rangle$)

Γ, γ refers to divided powers

is an extn of A obtained by repeated adjunction of exterior & poly

(exterior & divided power) variables.

[but polys in deg 0]

Thm: let $R \overset{\phi}{\twoheadrightarrow} S$ be a surjective ring map, and set $I = \ker\phi$.

1) $\exists$ semi-free extn resolving S over R

$$\begin{tikzcd} & R[X] \arrow[dr, two heads, "\pi", "\simeq"'] & \\ R \arrow[ur, hook] \arrow[rr, "\phi"] & & S \end{tikzcd}$$

$\left[\pi \text{ quasi-iso } \left(H_i(R[Y]) = \begin{cases} S, & i=0 \\ 0, & \text{else} \end{cases}\right)\right]$

2) $\exists$ semi-free Γ-extn resolving S over R

$$\begin{tikzcd} & R\langle X\rangle \arrow[dr, two heads, "\pi", "\simeq"'] & \\ R \arrow[ur, hook] \arrow[rr, "\phi"] & & S \end{tikzcd}$$

$[\pi$ quasi-iso$]$

[or if not surjective, 1st adjoin a set X_0 of deg 0 st. $R[X_0] \twoheadrightarrow S$ surjective]

Proof: As in the example, inductive construction:
First, adjoin a set of deg 1 vars X_1 to map to generators of I.
Then, for each n, adjoin variables of deg $n+1$ to kill cycles generating $H_n(R[X_{\leq n}])$.

Def: $R[X]$ is called a minimal model if X_n chosen minimally at each step (to kill cycles that give min'l generators of homology)

$R\langle X\rangle$ is called an acyclic closure if " " " " " "

Note: 1) if S f.g. R-alg (eg, $R \twoheadrightarrow S$ surj) & R Noeth, can take $|X_n| < \infty$

2) if $\mathbb{Q} \subseteq R$, then $R[X] \cong R\langle X\rangle$ as dgas

$$\frac{x^n}{n!} \longleftrightarrow x^{(n)}$$

3) if not, still have $x^n = n!\, x^{(n)}$ (in $R\langle X\rangle$).

~~Note: Both can be defined also for~~

$$\cancel{\underset{\text{DG alg}}{A} \longrightarrow \underset{\text{ring}}{S} \quad \text{DG alg map}}$$

~~$I = \ker(A_0 \to S)$~~

~~now: $\partial(X_1)$ min gens I mod $\partial(A_1)$~~

~~$\partial(X_{n+1})$ min gens $H_n(A[X_{\leq n}])$ or $A\langle X_{\leq n}\rangle$, resp.~~

no time

Let A be DG alg, z cycle in A

consider the semi-free Γ-extn $A\langle x \mid \partial(x) = z\rangle$

<u>lemma</u>: $H_i(A\langle x\rangle \text{ or } A[x]) = \begin{cases} H_i(A), & i < n = |z| \\ H_n(A)/\langle z\rangle, & i = n \\ ??, & i > n \end{cases}$

Next one shows how efficient divided powers can be.

<u>Lemma</u>: (Tate)

z is nzd (odd) on $H(A) \iff H(A\langle x\rangle) \cong H(A)/zH(A)$ (via map induced by $A \hookrightarrow A\langle x\rangle$)

($|z|$ even : z nzd
$|z|$ odd : $z\alpha = 0 \Rightarrow \alpha \in (\bar{z})$)

no time

<u>Lifting Lemma 1</u> ($\rightsquigarrow$ uniqueness of min'l model up to htpy)

If $A[X]$ semi-free extn of a dga A, then in the $\underbrace{\text{category of dg } A\text{-algs}}$

ob: dgas
mor: algebra chain maps

$$\begin{tikzcd} & B \arrow[d, "g", "\simeq"'] \\ A[X] \arrow[r, "f"] \arrow[ur, dashed, "\exists h"] & C \end{tikzcd}$$

$\forall g$ surjective, quasi-iso and $\forall f$, $\exists h$ (unique up to A-linear htpy)

<u>Lifting Lemma 2</u>: there's a similar one for semi-free Γ-extns, but need to work in cat of DG Γ-algebras (DG algs w/ divided power structure)

[These make resns unique up to htpy! And the construction functorial.]

<u>For ex</u>: Koszul cxs $K = R\langle e_1, \dots, e_n\rangle$

(used in BEH proof to get $K \dashrightarrow F$)