Day 8: The multigraded BGG correspondence.

Tate resolutions have had many applications in CA and AG.

- Algorithm for computing sheaf cohomology over $\mathbb{P}^n$ (Eisenbud-Decker)
- Boij-Söderberg Theory $\Big\}$ (Eisenbud-Schreyer)
- Computing resultants

... many others. Over 200 citations.

Question: does the story extend to multigraded polynomial rings and toric varieties?

Answer: Yes ($\underbrace{\text{Eisenbud-Erman-Schreyer 15}}_{\text{products of proj. spaces}}$, $\underbrace{\text{B-Erman 24}}_{\text{general case}}$)

Story is more complicated.

Multigraded BGG: reference: "Tate resolutions on toric varieties", B-Erman

k a field

Let A be a fg abelian group

Suppose $S = k[x_0, \dots, x_n]$ is A-graded, where $d_i := \deg(x_i) \in A$.

Key example: if X is a projective toric variety, its homogeneous coordinate ring, or Cox ring, is a $Cl(X)$-gdd polynomial ring. More on this later.

For instance: $A = \mathbb{Z}$, $d_i \geq 1$. S = Cox ring of weighted proj. space.

Say $S = k[x_0, x_1]$, $\deg(x_0) = 1$, $\deg(x_1) = 2$. How should BGG work in this case? For instance, what is $\mathbf{R}(S)$?

Naive guess: $\mathbf{R}(S)_i = \omega_E(i) \otimes_k S_{-i}$ w/ differential $\cdot \sum x_i \otimes e_i$.

$$\mathbf{R}(S) = \Big[\begin{tikzcd} \omega_E \arrow[r, "e_0"] \arrow[rr, bend right, "e_1"'] & \omega_E(-1) \arrow[r, "e_0"] \arrow[rr, bend right, "e_1"'] & \omega_E(-2)^2 \arrow[r, shift left, "e_0"] \arrow[r, shift right, "e_0"'] \arrow[rr, bend right, "e_1"'] & \omega_E(-3)^2 \arrow[r, "e_0"] & \cdots \end{tikzcd}\Big]$$

$1 \qquad x_0 \qquad x_1, x_0^2 \qquad x_0x_1, x_0^3$

Not a complex! No way to grade this thing to make it a complex.

Solution: enlarge the category of complexes

$E = \Lambda_k(e_0, \dots, e_n)$, $A \times \mathbb{Z}$ graded w/ $\deg(e_i) = (-d_i, -1)$.

Def'n: A <u>differential</u> <u>E-module</u> is an $A \times \mathbb{Z}$-graded E-module D w/ degree $(0,-1)$ differential ∂ s.t. $\partial^2 = 0$.

Note: Think of E as a dg-algebra w/ internal A-grading and homological $\mathbb{Z}$-grading, and trivial differential. From this perspective, a differential E-module is precisely a dg-E-module.

$Com(S)$: category of complexes of graded S-modules (diff's / morphisms degree 0)

$DM(E)$: " " differential E-modules.

Here, a morphism $D \xrightarrow{\alpha} D'$ of diff. E-modules is a degree 0 map s.t. $\alpha \partial_D = \partial_{D'} \alpha$.)

- The homology of $D \in DM(E)$ is $\ker(\partial : D \to D(0,-1)) / \operatorname{im}(D(0,1) \to D)$.

Thm (Hawwa-Hoffman-Wang, 12) There is an adjunction

$$\mathbb{L} : DM(E) \rightleftarrows Com(S) : \mathbb{R}$$

If $M \in Mod(S)$, $\mathbb{R}(M) = \bigoplus_{d \in A} \omega_E(-d, 0) \otimes_k M_d$, $\partial_{\mathbb{R}} = \sum e_i \otimes x_i$

$\omega_E := Hom_k(E, k) \cong E(-\sum d_i, -n-1)$.

If $C \in Com(S)$, think of each $\mathbb{R}(C_i)$ as a 1-periodic complex, and form a bicomplex. Its totalization is 1-periodic: fold it to a DM to get $\mathbb{R}(C)$.

If $D \in DM(E)$, $\mathbb{L}(D)_i = \bigoplus_{a \in A} S(-a) \otimes_k D_{(a,i)}$ w/ diff.

$$\sum x_i \otimes e_i - id \otimes \partial_D.$$

Ex: $\mathbb{L}(k) = S, \quad \mathbb{R}(k) = \omega_E$

Say $S = k[x_0, x_1]$, $\deg(x_0) = 1$, $\deg(x_1) = 2$.

$$\mathbb{R}(S) = \left[\begin{tikzcd}[column sep=small]
\underset{1}{\omega_E} \arrow[r, bend left, "e_0"] \arrow[drr, bend right, "e_1"'] & \oplus\ \underset{x_0}{\omega_E(-1,0)} \arrow[r, bend left, "e_0"] \arrow[drr, bend right, "e_1"'] & \oplus\ \omega_E(-2,0)\ x_0^2 \arrow[r, bend left, "e_0"] & \oplus\ \omega_E(-3,0)\ x_0^3 & \cdots \\
 & & \oplus\ \omega_E(-2,0)\ x_1 \arrow[r, bend right, "e_0"'] & \oplus\ \omega_E(-3,0)\ x_0 x_1 &
\end{tikzcd}\right]$$

Short exercise: say S is as above. Carefully show that $\mathbb{L}(\omega_E)$ is $K(x_0, \ldots, x_n)$, w/ the appropriate grading twists.