Day 3: The BGG correspondence (Part 1)

Reference: Sheaf cohomology and free resolutions over exterior algebras, by Eisenbud-Fløystad-Schreyer.

Recall: k a field

$$S = k[x_0, \dots, x_n], \quad \deg(x_i) = 1$$

$$E = \Lambda_k(e_0, \dots, e_n), \quad \deg(e_i) = -1$$

$\mathrm{Mod}(S)$ = categ. of fg gdd S-modules (w/ morphisms degree 0 maps)

$\mathrm{Com}(S)$ = categ. of cpx's of fg gdd S-modules (diff's and morphisms degree 0)

$\mathrm{Mod}(E)$, $\mathrm{Com}(E)$: defined similarly. We work w/ left E-modules.

Thm: (Bernstein-Gel'fand-Gel'fand, 78) There are adjoint functors

$$\mathbb{L} : \mathrm{Com}(E) \rightleftarrows \mathrm{Com}(S) : \mathbb{R}$$

come back to what this means later

that induce an equivalence on bdd derived categories.

Surprising. finite dim $\longleftrightarrow$ finite global dim
∞ gl. dim $\longleftrightarrow$ ∞ dim

$\mathbb{L} : \mathrm{Com}(E) \longrightarrow \mathrm{Com}(S)$

N a gdd E-module (conc. in homol. degree 0)

$$\mathbb{L}(N)_i = S(-i) \otimes_k N_i, \qquad \partial_{\mathbb{L}} : \mathbb{L}(N)_i \longrightarrow \mathbb{L}(N)_{i-1}$$

Internal induces homological. $\qquad s \otimes y \longmapsto \sum_{i=0}^{n} x_i s \otimes e_i y.$

Check: $\partial_{\mathbb{L}}^2 = 0$.

If $C \in \mathrm{Com}(E)$, $\mathbb{L}(C)$ is the totalization of the bicomplex:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & \mathbb{L}(C_{i-1})_j \arrow[l] \arrow[d, "\partial_{\mathbb{L}}"] & \mathbb{L}(C_i)_j \arrow[l, "\partial_C"'] \arrow[d, "\partial_{\mathbb{L}}"] & \cdots \arrow[l] \\
\cdots & \mathbb{L}(C_{i-1})_{j-1} \arrow[l] \arrow[d] & \mathbb{L}(C_i)_{j-1} \arrow[l, "\partial_C"'] \arrow[d] & \cdots \arrow[l] \\
 & & &
\end{tikzcd}$$

Examples:

$\mathbb{L}(k) = S$

What is $\mathbb{L}(E)$? Say $n=1$, so $S = k[x_0, x_1]$, $E = \Lambda_k(e_0, e_1)$

$E_0 = k \cdot 1$, $E_{-1} = k \cdot e_0 \oplus k \cdot e_1$, $E_{-2} = k \cdot e_0 e_1$

$$\mathbb{L}(E) = \left[S(2) \otimes k \cdot e_0 e_1 \xleftarrow{(-x_1\ x_0)} \begin{matrix} S(1) \otimes k \cdot e_0 \\ \oplus \\ S(1) \otimes k \cdot e_1 \end{matrix} \xleftarrow{\begin{pmatrix} x_0 \\ x_1 \end{pmatrix}} S \otimes k \cdot 1 \leftarrow 0 \right]$$

Hom degrees: $\quad -2 \qquad -1 \qquad 0$

In general, $\mathbb{L}(E) = K(x_0, \ldots, x_n)^{\vee} := \underline{\mathrm{Hom}}_S(K(x_0, \ldots, x_n), S)$.

Short exercise: Prove that $E^* := \underline{\mathrm{Hom}}_k(E, k)$ is isomorphic, as a graded left E-module, to $E(-n-1)$.

Here, E^* is a left E-module via the following action:

If $f \in (E^*)_i$, and $e \in E_j$, then $(ef)(y) = (-1)^{ij} f(ey)$, for all $y \in E$.