# L2: Koszul resolutions

Recall $R$ is a Koszul alg if $k$ has a linear resn.

In fact, then one can precisely give it:

$$R = \frac{T(V)}{(W)} = \frac{k\langle x_1,\dots,x_n\rangle}{(W)} \quad (\underbrace{W \subseteq V^{\otimes 2}}_{\text{quadratic}}), \qquad R^! = \frac{T(V^*)}{(W^\perp)}$$

**Thm** (Priddy): If $R$ Koszul, then the following is a min'l gdd $R$-free resn of $k$:

$$\boxed{P(R):} \quad \cdots \xrightarrow{\partial} R \otimes_k \underbrace{(R^!_2)^*}_{k\text{ v.sp}} \xrightarrow{\partial} R \otimes_k \underbrace{(R^!_1)^*}_{k\text{ v.sp}} \xrightarrow{\partial} \underbrace{R \otimes_k \underbrace{\overset{k}{\cancel{(R^!_0)^*}}}_{k\text{ v.sp.}}}_{R} \to k \to 0$$

Priddy cx/resn
or
Koszul resn

$$\partial = \sum_{i=1}^{n} x_i \otimes x_i^* \quad \text{(right mult. by "trace elt")}$$

where $x_i^*$ induced from $R^!_{m-1} \xrightarrow{x_i^*} R^!_m$

$\mathrm{Hom}_k(-,k)$

$(R^!_m)^* \xrightarrow{x_i^*} (R^!_{m-1})^*$

ie, transposed matrix

So, $\partial(1 \otimes \overline{x_{i_1}^* x_{i_2}^* \cdots x_{i_m}^*})$

$$= \sum_{i=1}^{n} x_i \otimes \overline{x_{i_1}^* x_{i_2}^* \cdots x_{i_m}^*} \cdot x_i^*$$

linear coeffs ✓

dual (transpose) action

Sort of: removes an $x_i^*$ factor (after rewriting using relns of $R^!$)

see explanation on page 4

Note: $\beta_i(k) = \operatorname{rank}_R P(R)_i$

$= \operatorname{rank}_R R \otimes_k (R^!_i)^*$

$= \dim_k R^{!*}_i = \dim_k R^!_i$

growth = growth gdd pieces of $R^!$

$R \otimes_k k^n \cong (R \otimes_k k)^n \cong R^n$

Equivalent conditions: (assume $R = \frac{T(V)}{(W)}$ quadratic alg)

Note: 1) $(R^!)^! = R$ for any quadratic alg
(obvious: $(W^\perp)^\perp = W$) ← orthogonal complements

2) one can still form Priddy's cx for quadratic alg
and it's linear

$\boxed{\text{Priddy's Thm says: } R \text{ koszul} \Longleftrightarrow P(R) \xrightarrow{\simeq} k \text{ resn (ie, exact)}}$
(k has linear resn)

$$\boxed{\text{Thm: } R \text{ koszul} \Rightarrow \underbrace{\mathrm{Ext}_R(k,k)}_{\bigoplus \mathrm{Ext}^i_R(k,k)} \cong R^!_\bullet \text{ as } k\text{-algebras}}$$

Pf: Let $P \xrightarrow[R]{\simeq} k$ be Priddy's free resn

Then $\mathrm{Ext}_R(k,k) = H(\mathrm{Hom}_R(P,P))$ ← dga under composition (hw 1)
so, Ext is (assoc) alg ✓

$\cong H(\mathrm{Hom}_R(P,k))$

$= H(\mathrm{Hom}_R(R \otimes_k (R^!_\bullet)^*, k))$

$= \cancel{H}(\mathrm{Hom}_k((R^!_\bullet)^*, k))$ $\partial = 0$

$= (R^!_\bullet)^{**} = R^!$

[actually agrees w/ Yoneda product up to $\pm$ (not obvious)]

where $R^!$ alg $\Rightarrow (R^!)^*$ "co-alg" $\Rightarrow P$ (dg) "co-alg"
not quite satisfying some properties (but has a co-multiplication)

$\Rightarrow \mathrm{Hom}_R(P,k)$ alg

⇑ is

$\mathrm{Hom}_R(P,P)$ alg under comp ↑ can check!

Converse ?

**Thm:** For any quadratic alg $R$,

$$R^! \underset{\substack{\text{as} \\ \text{algebras!}}}{=} \underbrace{\text{"diagonal part" } \operatorname{Ext}^{\Delta}_R(k,k)} \text{ of } \operatorname{Ext}_R(k,k)$$

$$\stackrel{\text{def}}{=} \bigoplus_{i \geq 0} \operatorname{Ext}^i_R(k,k)_i$$

$$= \bigoplus_i k^{\beta_{ii}}$$

Pf: $k$ has linear resn $\iff \beta_{ij} = 0$, $i \neq j$

$\iff \operatorname{Ext} = \operatorname{Ext}^{\Delta}$

**Cor:** $R$ koszul $\iff \operatorname{Ext}_R(k,k) \cong R^!$ as $k$-algs

Pf: $\Rightarrow$ ✓

$\Leftarrow$ by Thm, $R^! \xrightarrow{\cong} \operatorname{Ext}^{\Delta}(k,k) \xhookrightarrow{i} \operatorname{Ext}(k,k)$

but $\operatorname{Ext}^i_R(k,k)_j = k^{\beta_{ij}}$ ← gdd Betti #

so, $i$ iso $\Rightarrow \beta_{ij} = 0$ $i \neq j$

In particular: $R$ koszul $\Rightarrow \operatorname{Ext}(k,k) = k\langle \operatorname{Ext}^1(k,k)_1 \rangle$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overset{\shortparallel}{\phantom{x}}$

$= k\langle \operatorname{Ext}^1(k,k) \rangle$

since $R^!$ is so.

**Thm:** $R$ koszul $\iff R^!$ koszul

Pf: exercise today...

Note: we have, defining $E(R) = \operatorname{Ext}_R(k,k)$ for any std gdd $k$-alg $R$

$$R \text{ koszul} \iff E(E(R)) \cong R$$

(Warning: in some sources, this (or other conditions above) is given as the def of a koszul alg!)

# Concrete computation of $\partial$ in Priddy resn:

$$\cdots \xrightarrow{\partial} R \otimes_k \underbrace{(R^!_m)^*}_{k\text{ v.sp}} \xrightarrow{\partial} R \otimes_k \underbrace{(R^!_{m-1})^*}_{k\text{ v.sp}} \xrightarrow{\partial} \cdots$$

Recall $\partial =$ right multiplication by $\sum_{i=1}^n x_i \otimes x_i^*$

acts on R factor by mult ✓

how acts on $(R^!_m)^*$ factor?

Recall $R^! = \dfrac{k\langle x_1^*, \ldots, x_n^*\rangle}{(W^\perp)}$

$$\boxed{R^!_{m-1} \xrightarrow{x_i^*} R^!_m \quad \overset{\text{dualize}}{\rightsquigarrow} \quad (R^!_m)^* \xrightarrow{x_i^*} (R^!_{m-1})^*}$$

How to compute concretely ?!

First: Choose bases for $R^!_{m-1}$ and $R^!_m$.

Method 1: Compute matrix of mult map $R^!_{m-1} \xrightarrow{x_i^*} R^!_m$ in terms of the given bases!

Transpose the matrix to get $(R^!_m)^* \xrightarrow{x_i^*} (R^!_{m-1})^*$ (in terms of dual bases)

Method 2: The map $(R^!_m)^* \xrightarrow{\cdot x_i^*} (R^!_{m-1})^*$ = right mult. by $x_i^*$

(same really!) is really $\mathrm{Hom}(R^!_m, k) \to \mathrm{Hom}(R^!_{m-1}, k)$ given by pre-composition with left mult. by $x_i^*$.

Writing in terms of dual bases, we can compute this.

For example $(R^!_2)^* \xrightarrow{t = x_2^*} (R^!_1)^*$ for $R^! = \dfrac{k\langle s,t\rangle}{(st+ts,\ s^2-t^2)}$ (here we renamed $x_1^* = s$, $x_2^* = t$)

basis $R^!_m$ is $\{s^m, s^{m-1}t\}$

$$\begin{array}{ccc} \text{basis} & s^2 & st \\ & \downarrow & \downarrow \\ & 1 & 0 \end{array} \qquad R^!_2 \xleftarrow{t\cdot} R^!_1, \quad R^!_2 \xrightarrow{(s^2)^*} k$$

So, composition $(s^2)^* t$ is: $(= (x_1^*)^2 x_2^*)$

basis:

$-st = ts \longleftarrow s$, $ts \mapsto 0$

$t^2 = s^2 \longleftarrow t$, $s^2 \mapsto 1$

this is $t^*$ $(= x_2^*)$

whereas: $(st)^* t$ takes

$ts = -st \longleftarrow s$, $-st \mapsto -1$

$t^2 \longleftarrow t$, $t^2 \mapsto 0$

so: equals $-t^*$