Problem Set 10: The inverse of the Tate resolution functor

Let k be a field and $S = k[x_0, \ldots, x_n]$, with $\mathbb{Z}$-grading given by $\deg(x_i) = 1$. Let $E = \Lambda_k(e_0, \ldots, e_n)$, with $\mathbb{Z}$-grading given by $\deg(e_i) = -1$.

1. (Short exercise) I'll deviate from the above setting for a moment: say $X = \mathbb{P}^1 \times \mathbb{P}^1$ with Cox ring $S = k[x_0, x_1, y_0, y_1]$ with $\mathbb{Z}^2$-grading $\deg(x_i) = (1, 0)$, $\deg(y_i) = (0, 1)$ for $i = 0, 1$. Just as with weighted projective space, the minimal free resolution of $\mathbf{R}(S)$ is $\mathbf{R}(S)^*$. Show that $\operatorname{cone}(\mathbf{R}(S)^* \xrightarrow{\simeq} \mathbf{R}(S))$ cannot be the Tate resolution of the structure sheaf $\mathcal{O}$.

Solution. $H^i(X, \mathcal{F}) = 0$ for $i > 2$, since X is a surface. On the other hand, we have:

$$\operatorname{cone}(\mathbf{R}(S)^* \xrightarrow{\simeq} \mathbf{R}(S)) = \left(\bigoplus_{d \in \mathbb{Z}^2} \omega_E(-d, 0) \otimes_k S_d \right) \oplus \left(\bigoplus_{d \in \mathbb{Z}^2} \omega_E(d + (2, 2), 3) \otimes_k S_d \right).$$

If this were the Tate resolution, the summands on the right would imply that various twists of $\mathcal{O}$ have nonvanishing H^3, which is impossible. So, taking free resolutions doesn't work to build toric Tate resolutions, in general!

This problem set isn't about toric Tate resolutions; rather, we are returning to the standard graded case to discuss the inverse of Eisenbud-Fløystad-Schreyer's Tate resolution functor.

Throughout, we denote by $\mathfrak{n}$ the maximal ideal of E.

2. Let Ω^i denote the cokernel of the $(i+2)^{\text{th}}$ differential in $K(x_0, \ldots, x_n)$. Prove that the minimal free resolution of $\Omega^i[-i-1]$ is $\mathbf{L}(\omega_E / \mathfrak{n}^{n+1-i} \omega_E)$ for all $0 \leqslant i \leqslant n$. Conclude, using Problem Set 4 Exercise 5, that $\mathbf{L}(\omega_E(i) / \mathfrak{n}^{n+1-i} \omega_E(i))[1]$ is the minimal free resolution of $\Omega^i(i)$.

Comment: the associated sheaf $\widetilde{\Omega^j}$ on $\mathbb{P}^n$ is $\Omega^j_{\mathbb{P}^n}$ for all $0 \leqslant j \leqslant n$, i.e. the j^{th} exterior power of the cotangent bundle on $\mathbb{P}^n$; this follows from Theorem 17.16 in Eisenbud's "Commutative algebra with a view toward algebraic geometry".

Solution. Since $\mathbf{L}(\omega_E)$ is the Koszul complex, a straightforward calculation shows that $\mathbf{L}(\omega_E / \mathfrak{n}^{n+1-i} \omega_E)$ is the portion of the Koszul complex in homological degrees $n+1$ to $i+1$, which is the minimal free resolution of $\Omega^i[-i-1]$.

3. Deduce that $\mathbf{R}(\Omega^i(i))$ is an injective resolution of $\omega_E(i)/\mathfrak{n}^{n+1-i}\omega_E(i)[1]$ for all i.

Solution. Immediate from the previous exercise and the Reciprocity Theorem (Problem Set 4, Exercise 3).
We are now going to use, without proof, the following two facts:

(a) The functor $\mathbf{R}$ induces an equivalence of categories between finitely generated graded S-modules and complexes F of finitely generated free E-modules with finitely generated homology such that F_i is a direct sum of copies of $\omega_E(i)$ (these are linear complexes, in a particularly strong sense). This follows from Proposition 2.1 in Eisenbud-Fløystad-Schreyer's paper.

(b) If F and F' are injective resolutions of finitely generated graded E-modules N and N', and F and F' are *minimal*, then $\mathrm{Hom}_E(F, F') \cong \mathrm{Hom}_E(N, N')$. This follows from standard homological algebra.

Using (a) and (b) along with the above two exercises, we conclude:

$$\mathrm{Hom}_S(\Omega^i(i), \Omega^j(j)) \cong \mathrm{Hom}_E(\omega_E(i)/\mathfrak{n}^{n+1-i}\omega_E(i), \omega_E(j)/\mathfrak{n}^{n+1-j}\omega_E(j)).$$

4. Prove that there is an isomorphism

$$\mathrm{Hom}_E(\omega_E(i), \omega_E(j)) \xrightarrow{\cong} \mathrm{Hom}_E(\omega_E(i)/\mathfrak{n}^{n+1-i}\omega_E(i), \omega_E(j)/\mathfrak{n}^{n+1-j}\omega_E(j)).$$

Hint: use that $\omega_E(i)$ is projective, and apply the universal property of a projective module.

Solution. Any E-linear map $\omega_E(i) \to \omega_E(j)$ induces an E-linear map

$$\omega_E(i)/\mathfrak{n}^{n+1-i}\omega_E(i) \to \omega_E(j)/\mathfrak{n}^{n+1-j}\omega_E(j),$$

and so we have a natural map

$$\pi\colon \mathrm{Hom}_E(\omega_E(i), \omega_E(j)) \to \mathrm{Hom}_E(\omega_E(i)/\mathfrak{n}^{n+1-i}\omega_E(i), \omega_E(j)/\mathfrak{n}^{n+1-j}\omega_E(j)).$$

Going the other way: given a map $\omega_E(i)/\mathfrak{n}^{n+1-i}\omega_E(i) \to \omega_E(j)/\mathfrak{n}^{n+1-j}\omega_E(j)$, we certainly can obtain an induced map

$$f\colon \omega_E(i) \to \omega_E(j)/\mathfrak{n}^{n+1-j}\omega_E(j)$$

given by precomposing with the natural surjection. Consider the diagram:

$$\begin{tikzcd} & \omega_E(i) \arrow[d, "f"] \\ \omega_E(j) \arrow[r] & \omega_E(j)/\mathfrak{n}^{n+1-j}\omega_E(j). \end{tikzcd}$$

Since $\omega_E(i)$ is projective as an E-module, there is a map $\widetilde{f}\colon \omega_E(i) \to \omega_E(j)$ making the triangle commute. This shows that π is surjective. Now notice that

$$\mathrm{Hom}_E(\omega_E(i), \omega_E(j)) \cong \mathrm{Hom}_E(E(-n-1+i), E(-n-1+j)) \cong E_{j-i}.$$

On the other hand, the monomial basis elements of E_{j-i} induce k-linearly independent maps $\omega_E(i)/\mathfrak{n}^{n+1-i}\omega_E(i) \to \omega_E(j)/\mathfrak{n}^{n+1-j}\omega_E(j)$ given by multiplication. Thus, we have

$$\dim_k \mathrm{Hom}_E(\omega_E(i)/\mathfrak{n}^{n+1-i}\omega_E(i), \omega_E(j)/\mathfrak{n}^{n+1-j}\omega_E(j)) \geqslant \dim_k E_{j-i},$$

and so π must be an isomorphism.

That's it for the exercises: the rest of this document is just a discussion of how Exercise 3 leads to an inverse of the Tate resolution functor. Taking Tate resolutions gives a functor

$$\mathrm{D^b}(\mathbb{P}^n) \to K(E),$$

where $K(E)$ denotes the homotopy category of (unbounded) exact complexes of finitely generated graded free E-modules. We define a functor going the other way by starting with a complex F in $K(E)$, replacing each occurrence of $\omega_E(j)$ with $\Omega^j_{\mathbb{P}^n}(j)$, and using the map

$$\mathrm{Hom}_E(\omega_E(i), \omega_E(j)) \xrightarrow{\cong} \mathrm{Hom}_S(\Omega^i(i), \Omega^j(j)) \to \mathrm{Hom}_{\mathbb{P}^n}(\Omega^i_{\mathbb{P}^n}(i), \Omega^j_{\mathbb{P}^n}(j))$$

to define differentials between the terms. Call this functor $\Omega\colon K(E) \to \mathrm{D^b}(\mathbb{P}^n)$. For instance, since the Tate resolution of a sheaf $\mathcal{F}$ has terms

$$T(\mathcal{F})_i = \bigoplus_{j=0}^{n} H^j(\mathbb{P}^n, \mathcal{F}(-i-j)) \otimes_k \omega_E(i+j),$$

the complex $\Omega(T(\mathcal{F}))$ has terms

$$\Omega(T(\mathcal{F}))_i = \bigoplus_{j=0}^{n} H^j(\mathbb{P}^n, \mathcal{F}(-i-j)) \otimes_k \Omega^{i+j}_{\mathbb{P}^n}(i+j).$$

In Theorem 6.1 of Eisenbud-Fløystad-Schreyer's paper, it is shown that this complex has nonzero homology only in degree 0, and this homology is $\mathcal{F}$; a bit more work shows that Ω is an inverse for T. The complex $\Omega(T(\mathcal{F}))$ is called the *Beilinson monad* of $\mathcal{F}$. Beilinson monads have many applications across algebraic geometry and commutative algebra; a particularly interesting one can be found in Eisenbud-Schreyer's paper "Resultants and Chow forms via exterior syzygies".