Problem Set 6: The geometric BGG correspondence, Part 2

Fix $n \geqslant 0$. Let k be a field, and let $S = k[x_0, \dots, x_n]$, with grading given by $\deg(x_i) = 1$.

1. (Short exercise) Let $n = 1$ and $M = S/(x_0)$. Compute the regularity of M, and compute the Tate resolution $T(\widetilde{M})$.

Solution. Set $M = S/(x_0)$. Note that M is generated in degree 0 and has a linear resolution, so $\operatorname{reg}(M) = 0$. Thus, $\mathbf{R}(M)$ has homology concentrated in degree 0 and looks like

$$0 \to \omega_E \xrightarrow{e_1} \omega_E(-1) \xrightarrow{e_1} \omega_E(-2) \xrightarrow{e_1} \cdots$$

Taking a minimal free resolution of $H_0(\mathbf{R}(M))$ and taking the cone into the above complex gives $T(\mathcal{F})$:

$$\dots \xrightarrow{e_1} \omega_E(i+1) \xrightarrow{e_1} \omega_E(i) \xrightarrow{e_1} \omega_E(i-1) \xrightarrow{e_1} \cdots$$

where the term $\omega_E(i)$ is in homological degree i.

2. Use your solution to Exercise 1, along with our formula involving sheaf cohomology for the terms in the Tate resolution, to compute the sheaf cohomology of $\mathcal{F}$. Check your answer against your solution to Exercise 2 on Problem Set 5.

Solution. From the formula for sheaf cohomology involving Tate resolutions, we see that $H^0(\mathbb{P}^1, \mathcal{F}(m))$ is the vector space with dimension equal to the number of copies of $\omega_E(-m)$ in $T(\mathcal{F})_{-m}$ for all m. Thus, for all m, $H^0(\mathbb{P}^1, \mathcal{F}(m)) \cong k$. Similarly, $\dim_k H^1(\mathbb{P}^1, \mathcal{F}(m))$ is the number of copies of $\omega_E(-m)$ in $T(\mathcal{F})_{-m-1}$. Thus, $H^1(\mathbb{P}^1, \mathcal{F}(m)) = 0$ for all m.

3. Let M be a finitely generated graded S-module. Prove that, if $r > \operatorname{reg}(M)$, then the canonical map

$$M_{\geqslant r} \to \bigoplus_{j \geqslant r} H^0(\mathbb{P}^n, \widetilde{M}(j))$$

of graded S-modules is an isomorphism.

Solution. Consider the exact sequence of graded S-modules

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to \bigoplus_{d \in \mathbb{Z}} H^0(\mathbb{P}^n, \widetilde{M}(d)) \to H^1_{\mathfrak{m}}(M) \to 0.$$

In degrees $\geq r$, the outer terms vanish, giving the necessary isomorphism.

4. The *Serre Vanishing Theorem* states that, if $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n$, then

$$H^i(\mathbb{P}^n, \mathcal{F}(j)) = 0 \text{ for } i > 0 \text{ and } j \gg 0.$$

Prove the following stronger statement. Let M be a finitely generated S-module such that $\widetilde{M} = \mathcal{F}$: if $r \geqslant \operatorname{reg}(M)$, and $i \geqslant 1$, then $H^i(\mathbb{P}^n, \mathcal{F}(j)) = 0$ for $j \geqslant r$. (The reason this is a stronger statement is that we know $\operatorname{reg}(M) < \infty$.)

Solution. This follows by noting that for $i \geq 1$, $H^i(\mathbb{P}^n, \mathcal{F}(j)) \simeq H^{i+1}_{\mathfrak{m}}(M)_j$ and that the object on the right vanishes by definition, whenever $r \geq \operatorname{reg}(M)$.

5. It was commented in Problem Set 5 that

$$H^i(\mathbb{P}^n, \mathcal{O}(j)) \cong \begin{cases} S_j, & i = 0 \text{ and } j \geqslant 0; \\ S_{-j-n-1}, & i = n \text{ and } j < -n; \\ 0, & \text{else.} \end{cases}$$

Use our theorem about Tate resolutions and sheaf cohomology to prove this fact, avoiding the Čech calculations that are typically used to prove this.

Solution. Note that the Tate resolution of $\mathcal{O}$ is $T(\mathcal{O}) = \operatorname{cone}(\mathbf{R}(S)^\vee(-n-1) \to \mathbf{R}(S))$ where the morphism is the natural quasi-isomorphism. From the formula for sheaf cohomology, we have $\dim_k H^i(\mathbb{P}^n, \mathcal{O}(j))$ is the number of copies of $\omega_E(-j)$ in $T(\mathcal{O})_{-j-i}$. Note that the difference between the twists on ω_E and the homological degree of $T(\mathcal{O})$ is either 0 or n. This immediately gives $H^i(\mathbb{P}^n, \mathcal{O}(j)) = 0$ for all $0 < i < n$ and all j. The number of copies of $\omega_E(-j)$ in $T(\mathcal{O})_{-j}$ is non-zero if and only if $j \geq 0$. If it is non-zero, the number of copies is precisely $\dim_k S_j$. This completes the calculation for $H^0(\mathbb{P}^n, \mathcal{O}(j))$. If the difference between the twists on ω_E and the homological degree of $T(\mathcal{O})$ is n, then the minimum possible twist on ω_E is $n+1$. This implies, $H^n(\mathbb{P}^n, \mathcal{O}(j)) = 0$ for all $j > -n-1$. Finally, for $j \leq -n-1$, it is evident that the number of copies of $\omega(E)(-j)$ in $T(\mathcal{F})_{-j-n}$ is $\dim_k S_{-j-n-1}$. Thus, $H^n(\mathbb{P}^n, \mathcal{O}(j)) \cong S_{-j-n-1}$ for $j \leq -n-1$.

6. Use our theorem about Tate resolutions and sheaf cohomology to prove that, for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$, we have $\dim_k H^i(\mathbb{P}^n, \mathcal{F}) < \infty$.

Solution. This follows from the theorem about Tate resolutions and sheaf cohomology, along with the following two observations:

(a) If M is a finitely generated graded S-module, then $\dim_k M_i < \infty$ (this was a previous exercise). Thus, choosing a finitely generated M with regularity r such that $\widetilde{M} = \mathcal{F}$, we have that, for all $j \leqslant -r$, $T(\mathcal{F})_j = \mathbf{R}(M_{\geqslant r})_j = \omega_E(j) \otimes_k M_{-j}$ is a finite rank free E-module.

(b) In particular, $H_{-r}\mathbf{R}(M_{\geqslant r})$ is a finitely generated E-module, and so each term in its minimal free resolution is a finite rank free E-module.