Day 4: BGG Part 2

We now define:

$\mathbb{R} : \mathrm{Com}(S) \longrightarrow \mathrm{Com}(E)$

Write $\omega_E := E^*$. Recall: $\omega_E \cong E(-n-1)$.

M a graded S-module.

$\mathbb{R}(M)_i = \omega_E(i) \otimes_k M_{-i}$, $\quad \partial_{\mathbb{R}}(y \otimes m) = \sum_{i=0}^{n} e_i y \otimes x_i m$.

Extend $\mathbb{R}$ to complexes in the same way as w/ $\mathbb{L}$.

Ex: $\mathbb{R}(k) = \omega_E$

Exercise: $\mathbb{R}(S)$ is an injective res'n of k.

$n=0$: $S_i = k \cdot x_0^i$ for $i \geq 0$

$$\Rightarrow \mathbb{R}(S) = \Big[\underset{0}{\omega_E} \xrightarrow{e_0} \underset{-1}{\omega_E(-1)} \xrightarrow{e_0} \underset{-2}{\omega_E(-2)} \xrightarrow{e_0} \cdots \Big]$$

We have: $H_i\, \mathbb{R}(S) = \begin{cases} k, & i=0 \\ 0, & \text{else} \end{cases}$.

Thm (BGG, 78) The functors $\mathbb{L} : \mathrm{Com}(E) \rightleftarrows \mathrm{Com}(S) : \mathbb{R}$ form an adjoint pair.

That is, there are natural isomorphisms

$$\mathrm{Hom}_S(\mathbb{L}(C), D) \cong \mathrm{Hom}_E(C, \mathbb{R}(D)) \quad \forall\, C \in \mathrm{Com}(E),\ D \in \mathrm{Com}(S).$$

Pf: Not hard, but technical and annoying, so we omit it. $\square$

Thm: There are natural quasi-isomorphisms

$$C \xrightarrow{\simeq} \mathbb{R}\mathbb{L}(C), \quad \mathbb{L}\mathbb{R}(D) \xrightarrow{\simeq} D \qquad \text{unit / counit of adjunction.}$$

Pf: Interesting, but too long, so we omit it. See EFS Thm 2.6. $\square$

Ex: $\mathbb{L}\mathbb{R}(k) = \mathbb{L}(\omega_E) \overset{\text{Exercise}}{=} \text{Kosz. complex} \simeq k$

$\mathbb{R}\mathbb{L}(k) = \mathbb{R}(S) \overset{\text{Exercise}}{=} \text{inj res'n of } k$ (exercise)

Hand-wavy explanation of theorem: $\mathbb{R}$ " = " $- \otimes_k E$, $\mathbb{L}$ " = " $- \otimes_k S$ } along w/ differentials

$$\Longrightarrow \mathbb{L}\mathbb{R} = \underbrace{- \otimes_k E \otimes_k S}_{\text{Koszul complex}} \overset{\swarrow \text{quasi-isom}}{\simeq} - \otimes_k k = \mathrm{id}.$$ Similarly for $\mathbb{R}\mathbb{L}$.

Being very vague about gradings and differentials, which is the whole difficulty, but this is the right intuition.

Cor: The derived categories of S and E are equivalent.

Prop: Let M be a gdd S-module and N a gdd E-module.

(a) $H_i\, \mathbb{R}(M)_j \cong \mathrm{Tor}^S_{i+j}(k, M)_j$

(b) $H_i\, \mathbb{L}(N)_j \cong \mathrm{Ext}_E^{i+j}(k, N)_j$.

Note: This extends to complexes as well.

Sketch of pf of (a): Write $K = K(x_0, \dots, x_n)$.

$$(K_{i+j} \otimes_S M)_j \cong \Big(\underbrace{(\omega_E)_{i+j}}_{\text{degree } 0} \otimes_k S(-i-j) \otimes_S M \Big)_j$$

$$\cong \Big((\omega_E)_{i+j} \otimes_k M(-i-j) \Big)_j$$

$$\cong (\omega_E)_{i+j} \otimes_k M_{-i}$$

$$= \Big(\underbrace{\omega_E(i)}_{\text{degree } 0} \otimes_k M_{-i} \Big)_j = \big(i^{\text{th}} \text{ term of } \mathbb{R}(M) \big)_j$$

Now show differential is preserved. $\square$

Short exercise: Prove that $\mathbb{R}(S)$ is an injective res'n of k.

Conclude that $\mathbb{R}(S)^{\vee}(-n-1) := \underline{\mathrm{Hom}}_E(\mathbb{R}(S), E)(-n-1)$ is a free res'n of k.

(Recall from yesterday's exercises: E is injective as an E-module.)