Week 2 • Problem Set 2 • Koszul resolution

Quick Problem:

Let $R = k[x,y]/(x^2+y^2)$. Write out the first 3 steps of the Priddy resolution

$$\cdots P_3 \to P_2 \to P_1 \to P_0 \to k \to 0.$$

Problems:

1. Let R be a graded ring and M a graded R-module. Define the Poincaré and Hilbert series of M as

$$P_M^R(t) = \sum_{i\geq 0} \beta_i(M) t^i \qquad H_M^R(t) = \sum_{i \geq 0} \dim_k M_i t^i$$

where the ith Betti number β_i is the rank of F_i in an R-free resolution of M. (These definitions apply equal well to $R^!$.)

Prove the following equalities of power series.

(a) $P_k^R(t) = H_{R^!}(t)$ (here, $R^!$ is considered to be positively graded)

(b) $H_R(t) P_k^R(-t) = 1$ Hint: Think about Priddy's resolution and Euler characteristics (i.e., vector space dimensions along exact sequences).

2. The following are all Koszul algebras. Compute the first few steps of the Priddy/Koszul resolution of k.

(a) $R = k[x,y,z]/(x^2, xy)$
(and, if you are combinatorially-minded, find the Poincaré series in closed form sometime)

(b) $R = S = k[x_1, \dots, x_n]$ the symmetric algebra

(c) $R = \wedge(k\{e_1, \dots, e_n)\}$ the exterior algebra

(d) $R =$ the trivial algebra over k on $x_1, \dots, x_n$ (that is, all products of positive degree elements are zero).

3. Consider the noncommutative ring $R = k\langle x, y\rangle/(x^2+y^2, xy+yx)$.

(a) Show that R is Koszul. (Hint: Consider its dual.)

(b) Show that $\operatorname{projdim}_R k < \infty$.

4. Let R be a graded quadratic algebra. Prove that R is Koszul if and only if $R^!$ is Koszul.

(You may assume Priddy's theorem.)

Hint: Apply $\operatorname{Hom}_k(-,k)$ and regrade!)

5. Let R be a quadratic (commutative) hypersurface $R = Q/(f)$. Show explicitly how the Priddy resolution is isomorphic to Tate's dg algebra resolution from Week 1.

(Note that $Q \to R \to k$ is an embedded complete intersection.)

6. Let $\langle -, -\rangle \colon V \otimes V \to k$ be a perfect pairing, where V is the k-vector space with basis $\{x, y\}$. Consider the "short Gorenstein algebra" $R = k \oplus V \oplus k$ in degrees 0,1, and 2, with product given by the perfect pairing (and k-scalar products).

Show that $R^! = T(V^*)/(q)$ (a noncommutative hypersurface!) where q is the quadratic form of the pairing.