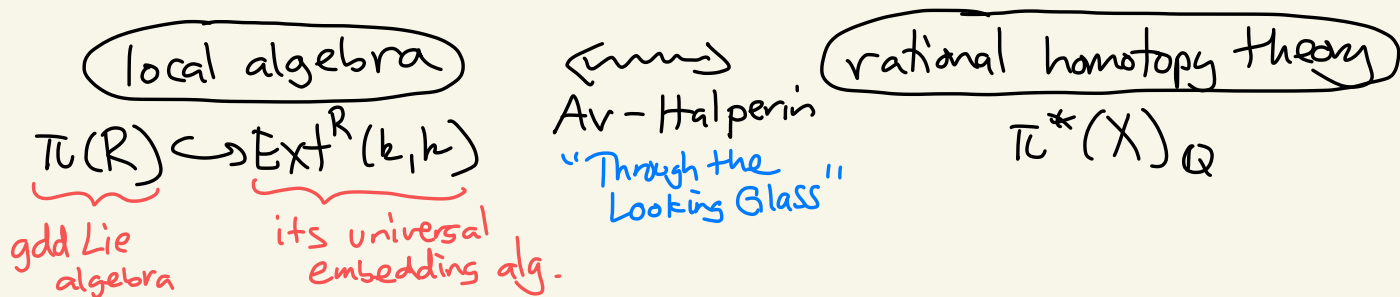


L5: Homotopy Lie algebra and cotangent cx

Fundamental invariant $\pi(R)$ def'd by Avramov (84).



Def: a gdd Lie algebra $/k$ is a gdd k -module L with a k -linear pairing

$$\begin{aligned} L \times L &\longrightarrow L \\ (a, b) &\longmapsto "[a, b]" \end{aligned} \quad \text{"Lie bracket"}$$

st. 1) (anti-commut) $[a, b] = -(-1)^{|a||b|} [b, a]$

2) (Jacobi identity) $[a [b, c]] = [[a, b], c] + (-1)^{|a||b|} [b, [a, c]]$

3) $|a|$ even $\Rightarrow [a, a] = 0$ and \exists reduced square $a^{[2]}$ (acts like $\frac{[a, a]}{2!}$)
 $|a|$ odd $\Rightarrow [a, [a, a]] = 0$ (char 2, char 3 (else autom.))

skip (say char 2, 3)

Fact: there are functors

$$\left\{ \begin{array}{c} \text{gdd} \\ k\text{-algs } B \end{array} \right\} \begin{array}{c} \xrightarrow{\text{Lie}} \\ \xleftarrow{U} \end{array} \left\{ \begin{array}{c} \text{gdd} \\ \text{Lie algs } /k \end{array} \right\}$$

$\text{Lie}(B) = B$ as k -mod w/ bracket $[a, b] \stackrel{\text{def}}{=} ab - (-1)^{|a||b|} ba$
 "gdd commutator"

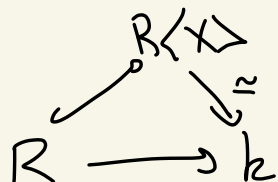
$U(L) = \underline{\text{the gdd } k\text{-alg gen'd by basis of } L}$
 ("universal enveloping algebra")
 $([a, b] - (ab - (-1)^{|a||b|} ba), a^{[2]} = a \cdot a \text{ if } |a| \text{ odd})$
 (really = $T(L)$ = k -span of monomials in elts of L)

Similarly: can define a dg Lie algebra (dgla) (with a ∂)

The homotopy Lie algebra $\pi(R)$: (or $\pi^*(R)$)

2

Version 1: via acyclic closure



Lemma: $\text{Der}_R^*(R\langle X \rangle, R\langle X \rangle) \subseteq \text{Lie}(\text{End}_R(R\langle X \rangle))$
is a dg Lie subalgebra ($[d, d'] = \text{gdd commutator}$)

Def: $\pi(R) = H(\text{Der}^*(R\langle X \rangle, R\langle X \rangle)) = \text{a Lie alg!}$
 $\cong H(\text{Der}^*(R\langle X \rangle, k)) \stackrel{\text{prop 2}}{=} \text{Hom}_k(kX, k) = kX^*$

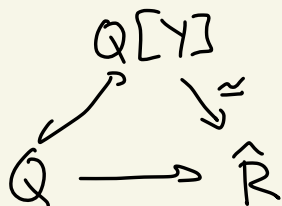
Prop: The inclusion $*$ induces an injection (on homology)

$$\pi^*(R) \hookrightarrow \text{Lie}(\text{Ext}_R^*(k, k))$$

giving $U(\pi(R)) \xrightarrow{\cong} \text{Ext}_R^*(k, k)$ iso (of algebras)

Version 2: via minimal model of min'l Cohen presentation

$$\hat{R} = Q/I$$



$\Sigma = \text{shift} + 1 \text{ in degree}$

Def: $\pi^{\geq 2}(R) = (\Sigma kY)^* = \text{Hom}_k(\Sigma kY, k)$

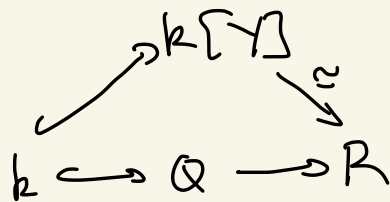
$Q[Y]$ min'l model $\xrightarrow[\text{shw}]{\text{can}}$ $\partial(Y) \subseteq mY + QY^{\geq 2}$
 $k[Y]$ has $\partial(Y) \subseteq kY^{\geq 2}$ (products & powers)

sg $\partial = \partial^{[2]} + \partial^{[3]} + \dots$ where $\partial^{[i]}: kY \rightarrow kY^i$

Get $kY \xrightarrow{\partial^{[2]}} kY^2 \longrightarrow kY \times kY$ (fixing an ordering)
 $y_i y_j \longmapsto y_i \times y_j$ if $y_i > y_j$

Dualizing gives bracket: $kY^* \times kY^* \rightarrow kY^*$
 $a, b \longmapsto [a, b]$

to get π_1 too



now $\gamma = \gamma_0 \cup \gamma_1 \cup \gamma_2 \cup \dots$

↑
to make surjection

let $Q = k[Y_0] \rightarrow R$

and do same thing as above

so, now get $\pi_1(R) = \sum k Y_0^*$ too!

Brackets come from $\mathcal{O}^{[2]}$ as above.

Rks:

- $\dim_k \pi^n(R) = \text{card } X_n = \varepsilon_n(R)$!

- R Golod $\Leftrightarrow \pi \geq 2$ $\text{free}_{\wedge}^{\text{gol}} \text{Lie alg}$ (Anvramov - Löfwall)

- R Koszul algebra $\Rightarrow \pi$ gen'd by π^1

skip

skip (• deformations \rightsquigarrow central elts of $\pi(R)$)

skip

- $\pi^1 \times \pi^1 \rightarrow \pi^2$ via Hessian (I) (Sjödin)

- nilpotent elements in the Lie algs $\pi(R)$
crucial to Briggs' recent proof of Vasconcelos's
Conjecture!