Week 1 • Problem Set 5 • Homotopy Lie algebra and cotangent complex

Quick Problem:

Consider the running example $R = k[[x,y]]/(x^3+y^3)$.

Compute $\pi(R)$ and the cotangent complex $\mathcal{L}_{R|k} = \mathbb{L}_{R|k} = L_{R|k}$. .

Problems:

1. Let k be a field of characteristic zero. Consider again the running example from the lectures:

$$\text{Let } R = k[[x,y]]/(f) \text{ with } f = x^3+y^3, \text{ and set } S = k = R/(x,y).$$

Recall that in the quick problem at the end of class today, we showed that all the brackets of $\pi(R)$ are zero. What does this imply about $\operatorname{Ext}_R(k,k) = U(\pi(R))$?

2. Let $R = k[x,y]/(x^2, xy)$ with k of characteristic 0. Then from Macaulay2 we see that a minimal model for R over k has the form $k[Y] = k[x,y,t_1,t_2,\dots]$ with

$$\partial(t_1) = x^2,\ \partial(t_2) = xy,\ \partial(t_3) = xy_2 - yt_1,$$

$$\partial(t_4) = -t_1t_2 + xt_3,\ \partial(t_5) = -t_2t_3 + yt_4,\ \partial(t_6) = -t_1t_3 + xt_4,$$

$$\partial(t_7) = -t_2t_4 + xt_5,\ \partial(t_8) = -t_3^2 - 2t_2t_4 + 2yt_6,\ \partial(t_9) = -t_1t_4 + xt_6, \dots$$

(a) Write ordered bases for the sets Y_0, Y_1, Y_2, Y_3, Y_4, and Y_5 of variables of degrees $0, 1, \dots$.

(b) Using lexicographic ordering, order the sets Y_0^2, Y_1Y_0, $Y_1^2 \oplus Y_2Y_0$, $Y_2Y_1 \oplus Y_3Y_0$, and $Y_2Y_2 \oplus Y_3Y_1 \oplus Y_4Y_0$. Use the resulting bases to write matrices for the quadratic part $\partial^{[2]}\colon kT \to kY^2$. *hint: Based on the formulas for the differentials, how do ∂ and $\partial^{[2]}$ compare in this example?*

(c) Transposing these matrices, one gets matrices describing brackets of $\pi(R)$. Use this to determine some of the relations of $\pi(R)$.

(d) Based on your work above, it $\operatorname{Ext}_R(k,k)$ graded-commutative?

3. Let $R \to R[Y]$ be a semi-free extension with no variables of degree 0. Define its module of indecomposables as

$$\operatorname{Ind}_R R[Y] = R[Y]/(R + IY + Y^2).$$

Assume that $R[Y]$ resolves S.

(a) Prove that $R[Y]$ is a minimal model for $R \to S$ if and only if the differential is *decomposable:* setting Y_0 equal to a minimal set of generators for $\mathfrak{m}$, one has $\partial(Y_{n+1}) \subseteq \sum_{i+j=n} RY_iY_j$ for all $n \geq 0$.

(b) Deduce that $R[Y]$ is a minimal model for $R \to S$ if and only if the complex $\operatorname{Ind}_R R[Y]$ is minimal.

(c) Use part (a) and lifting lemmas to show that minimal models are unique up to isomorphism of dg algebras.

Note 1: One can also define $\operatorname{Ind}_R R[Y]$ *when* $Y_0 \neq \emptyset$*, by also modding out by the entire degree 0 piece of* $R[Y]$*, but that complicates the notation introduced in part (a) above.*

Note 2: Some people define both $\operatorname{Ind}^{\gamma}_{R} R\langle X\rangle$ *and* $\operatorname{Ind}_{R} R[Y]$ *by going modulo the maximal ideal* $\mathfrak{m}$ *of* R *(or* $\mathfrak{n}$ *of* S*, that is, applying* $-\otimes_S k$*), so then the differentials in these complexes would vanish for an acyclic closure* $R\langle X\rangle$*, respectively a minimal model* $R[Y]$*.*