Problem Set 6: The geometric BGG correspondence, Part 2

Fix $n \geqslant 0$. Let k be a field, and let $S = k[x_0, \ldots, x_n]$, with grading given by $\deg(x_i) = 1$.

1. (Short exercise) Let $n = 1$ and $M = S/(x_0)$. Compute the regularity of M, and compute the Tate resolution $T(\widetilde{M})$.

2. Use your solution to Exercise 1, along with our formula involving sheaf cohomology for the terms in the Tate resolution, to compute the sheaf cohomology of $\mathcal{F}$. Check your answer against your solution to Exercise 2 on Problem Set 5.

3. Let M be a finitely generated graded S-module. Prove that, if $r > \operatorname{reg}(M)$, then the canonical map

$$M_{\geqslant r} \to \bigoplus_{j \geqslant r} H^0(\mathbb{P}^n, \widetilde{M}(j))$$

of graded S-modules is an isomorphism.

4. The *Serre Vanishing Theorem* states that, if $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n$, then

$$H^i(\mathbb{P}^n, \mathcal{F}(j)) = 0 \text{ for } i > 0 \text{ and } j \gg 0.$$

Prove the following stronger statement. Let M be a finitely generated S-module such that $\widetilde{M} = \mathcal{F}$: if $r \geqslant \operatorname{reg}(M)$, and $i \geqslant 1$, then $H^i(\mathbb{P}^n, \mathcal{F}(j)) = 0$ for $j \geqslant r$. (The reason this is a stronger statement is that we know $\operatorname{reg}(M) < \infty$.)

5. It was commented in Problem Set 5 that

$$H^i(\mathbb{P}^n, \mathcal{O}(j)) \cong \begin{cases} S_j, & i = 0 \text{ and } j \geqslant 0; \\ S_{-j-n-1}, & i = n \text{ and } j < -n; \\ 0, & \text{else.} \end{cases}$$

Use our theorem about Tate resolutions and sheaf cohomology to prove this fact, avoiding the Čech calculations that are typically used to prove this.

6. Use our theorem about Tate resolutions and sheaf cohomology to prove that, for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$, we have $\dim_k H^i(\mathbb{P}^n, \mathcal{F}) < \infty$.